SERIES EXPANSIONS FOR ENCOUNTER-TYPE SOLUTIONS OF HILL'S PROBLEM

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Abstract. Hill's problem is defined as the limiting case of the planar three-body problem when two of the masses are very small. This paper describes analytic developments for encounter-type solutions, in which the two small bodies approach each other from an initially large distance, interact for a while, and separate. It is first pointed out that, contrary to prevalent belief, Hill's problem is not a particular case of the restricted problem, but rather a different problem with the same degree of generality. Then we develop series expansions which allow an accurate representation of the asymptotic motion of the two small bodies in the approach and departure phases. For small impact distances, we show that the whole orbit has an adiabatic invariant, which is explicitly computed in the form of a series. For large impact distances, the motion can be approximately described by a perturbation theory, originally due to Goldreich and Tremaine and rederived here in the context of Hill's problem.

1. Introduction

Consider the planar problem of three bodies M_1, M_2, M_3, in the case where M_2 and M_3 have a much smaller mass than M_1. In general, the mutual attraction of M_2 and M_3 can then be neglected, and the problem reduces in a fair approximation to a superposition of two independent two-body problems. However, if the distance between M_2 and M_3 becomes sufficiently small, their mutual attraction becomes of the same order as the differential attraction from M_1 and can no longer be ignored. This is known as *Hill's problem* (Hill, 1878).

Hill's problem has been somewhat neglected, by comparison with its illustrious cousin, the restricted problem. This is surprising, because it is of comparable interest. First, a number of problems in celestial mechanics can be adequately approximated by Hill's equations. Examples are: the Sun-Earth-Moon problem, which was the motivation of Hill's original work; the interaction between particles in planetary rings, and between satellites on nearby orbits; the accretion of particles by a planet or proto-planet; the distribution of particles around the Earth; the temporary capture of a comet by a planet. Second, on the mathematical side, Hill's problem can be considered as the simplest non-integrable case of the N-body problem: the equations are even simpler than those of the restricted problem, and

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contain no parameter. Yet its solutions exhibit the inexhaustible richness which is characteristic of non-integrable systems in general (see for instance Hénon 1969, 1970).

The present paper arose from a study of the gravitational interaction of particles in Saturn's rings (Petit and Hénon, 1985). This problem is an almost perfect case for Hill's equations: for instance, a particle with a radius of the order of 1 meter has a mass of the order of 10^6 g, which is smaller than Saturn's mass by a factor 10^{24}; and the orbits of the particles are coplanar and circular with better than 10^{-6} accuracy. This study required some analytic developments, which apparently were not to be found in the literature. Since these developments might be useful in other applications, we present them here as a separate paper.

It will be convenient to call M_1 the *planet*, M_2 and M_3 the *satellites*. It should be kept in mind, however, that the results are applicable to other situations.

In Section 2, we present a general derivation of Hill's equations and we show that, contrary to a widespread belief, Hill's problem is not a particular case of the restricted problem. Specifically, the ratio of the masses of the two satellites can have any value; it need not be small. Also the mean orbit described by M_2 and M_3 does not have to be circular; it can be elliptical, provided that the radial excursion remains finite in Hill's coordinates. All this does not affect the final equations, which have the same form in all cases.

In the remainder of the paper, we restrict our attention to encounter-type orbits, in which the distance between the two satellites becomes infinite (in Hill's coordinates) both for $t \to -\infty$ and for $t \to +\infty$. In Section 3, we derive expansions describing the asymptotic motion of the two satellites for large mutual distance. This is done first in the comparatively simple case where the asymptotic motion corresponds to circular orbits in the physical frame of reference, and then in the general case. These expansions can be matched to a numerical integration over a finite time to give an accurate description of the whole orbit from $t = -\infty$ to $t = +\infty$ (Petit and Hénon, 1985).

Section 4 examines the case where the impact parameter is small, i.e. the two satellites are almost exactly on the same orbit. The two satellites then "repel" each other azimuthally, and describe what is known as a "horseshoe motion" (Brown, 1911; Dermott and Murray, 1981; Yoder et al., 1983). The satellites never come in close vicinity; therefore their mutual attraction remains small, and the whole orbit can be obtained from a perturbation theory. Using Kruskal's formalism (1962), we show that the horseshoe motion has an adiabatic invariant, which we compute explicitly. Series describing the orbit with high accuracy are also obtained. These results explain completely the numerically observed properties of horseshoe orbits.

At the other extreme, Section 5 examines the case where the impact parameter is large. Again the mutual attraction remains small during the whole encounter; each particle's orbit is only slightly deflected by the other, and a perturbation theory can again be worked out (Goldreich and Tremaine, 1979, 1980, 1982). Here we rederive these results in the context of Hill's problem.

As this paper was being prepared, we were informed of the existence of recent work on the same subject by Spirig and Waldvogel (1985).

2. Basic equations

2.1. Reduction to Hill's equations

The derivation of Hill's equations usually found in the literature assumes a hierarchy of masses for the three bodies:

$$m_1 \gg m_2 \gg m_3, \tag{1}$$

and proceeds in two steps: first the limit $m_3 \to 0$ is taken, which gives the restricted three-body problem; then the limit $m_2 \to 0$ is taken. Hill's problem is thus presented as a sub-case of the restricted problem.

Here we shall consider a more general situation: the ratio of the two masses m_2 and m_3 can be arbitrary; the only condition is that both masses should be small compared to m_1:

$$m_1 \gg m_2, \qquad m_1 \gg m_3. \tag{2}$$

Our procedure will be to fix the ratio m_2/m_3 and to let both m_2 and m_3 tend to zero simultaneously. Remarkably, the equations obtained in this limit are identical to the classical Hill's equations. Thus, these equations have a greater generality than is generally thought; and *Hill's problem is not a subset of the restricted problem.* The true state of affairs is as follows:

The restricted problem is applicable to situations where one mass is much smaller than the two others; Hill's problem is applicable to situations where one mass is much larger than the two others.

The two problems have thus in a sense the same degree of generality. Neither contains the other; but they have in common the "hierarchical case" (1). Perhaps the misconception arose from the fact that the first and most famous application of Hill's equations was to the Sun-Earth-Moon problem, which belongs to the hierarchical case, so that the greater generality of these equations was not perceived.

We start from the equations of the plane problem of three bodies in an inertial reference system (X,Y):

$$\begin{aligned} \ddot{X}_1 &= \frac{Gm_2(X_2 - X_1)}{R_{12}^3} + \frac{Gm_3(X_3 - X_1)}{R_{13}^3}, \\ \ddot{Y}_1 &= \frac{Gm_2(Y_2 - Y_1)}{R_{12}^3} + \frac{Gm_3(Y_3 - Y_1)}{R_{13}^3}, \end{aligned} \tag{3}$$

and similar equations for $\ddot{X}_2$, $\ddot{Y}_2$, $\ddot{X}_3$, $\ddot{Y}_3$. G is the gravitational constant; X_i, Y_i are the coordinates of body M_i; m_i is its mass, and R_{ij} is the distance between bodies i and j:

$$R_{ij} = \sqrt{(X_i - X_j)^2 + (Y_i - Y_j)^2}. \tag{4}$$

If the three distances R_{ij} are of the same order, the problem becomes trivial in the limit $m_2 \to 0$, $m_3 \to 0$: it reduces to the superposition of two two-body problems. A non-trivial problem arises, however, if the distance R_{23} is small. We assume this to be the case. In a first approximation, the two satellites can then be considered as describing the same orbit around the planet; we shall call it the *mean orbit* (the precise definition of this mean orbit is not important). We shall restrict our attention here to the case where the mean orbit is circular; in other words, we consider the *circular Hill's problem.* (The general case is called the elliptic Hill's problem; see Ichtiaroglou, 1980).

Let a_0 be the radius of the mean orbit, and

$$m = m_1 + m_2 + m_3 \tag{5}$$

the total mass of the system; the angular velocity of m_2 and m_3 on the mean orbit is of the order of

$$\omega = \sqrt{Gma_0^{-3}}. \tag{6}$$

We introduce dimensionless variables:

$$X_i' = \frac{X_i}{a_0}, \qquad Y_i' = \frac{Y_i}{a_0}, \qquad R_{ij}' = \frac{R_{ij}}{a_0}, \qquad m_i' = \frac{m_i}{m}, \qquad t' = \omega t. \tag{7}$$

In these new variables, the gravitational constant, the total mass of the system, the radius of the mean orbit, and the mean angular velocity are all equal to 1. The equations of motion in these variables are similar to (3), with G replaced by 1, and the time derivative being taken now with respect to t'. Only the dimensionless variables (7) will be used in the following, so we drop the primes from now on for simplicity.

By analogy with the restricted problem, we write

$$m_1 = 1 - \mu, \qquad m_2 + m_3 = \mu; \tag{8}$$

μ is a small number. We introduce also

$$\frac{m_3}{m_2 + m_3} = \nu. \tag{9}$$

We introduce a new reference system (x, y), centered on the planet M_1 and rotating with angular velocity 1:

$$\begin{aligned} x_2 &= (X_2 - X_1)\cos(t - t_0) + (Y_2 - Y_1)\sin(t - t_0), \\ y_2 &= -(X_2 - X_1)\sin(t - t_0) + (Y_2 - Y_1)\cos(t - t_0), \end{aligned} \tag{10}$$

and similar equations for x_3, y_3. The parameter t_0 is chosen in such a way that the two satellites are close to the positive x axis. The equations become

$$\begin{aligned} \ddot{x}_2 &= 2\dot{y}_2 + x_2 - \frac{(m_1 + m_2)x_2}{R_{12}^3} + \frac{m_3(x_3 - x_2)}{R_{23}^3} - \frac{m_3 x_3}{R_{13}^3}, \\ \ddot{y}_2 &= -2\dot{x}_2 + y_2 - \frac{(m_1 + m_2)y_2}{R_{12}^3} + \frac{m_3(y_3 - y_2)}{R_{23}^3} - \frac{m_3 y_3}{R_{13}^3}, \end{aligned} \tag{11}$$

and similar equations for $\ddot{x}_3$, $\ddot{y}_3$, obtained by permuting indices 2 and 3, with

$$R_{12} = \sqrt{x_2^2 + y_2^2}, \qquad R_{13} = \sqrt{x_3^2 + y_3^2}, \qquad R_{23} = \sqrt{(x_3 - x_2)^2 + (y_3 - y_2)^2}. \tag{12}$$

In this system, the satellites M_2 and M_3 are in the vicinity of the point (1,0). The problem is non-trivial when the distance R_{23} is such that the gravitational force between the satellites is of the same order as the differential force from the planet. As is well known, this corresponds to a distance R_{23} of the order of $\mu^{1/3}$. Therefore we make yet another change of coordinates:

$$\begin{aligned} x_2 &= 1 + \mu^{1/3}\xi_2, \qquad & y_2 &= \mu^{1/3}\eta_2, \\ x_3 &= 1 + \mu^{1/3}\xi_3, \qquad & y_3 &= \mu^{1/3}\eta_3, \end{aligned} \tag{13}$$

$$R_{23} = \mu^{1/3}\rho. \tag{14}$$

ξ_i, η_i will be called *Hill's coordinates* of the satellite M_i. We have then

$$R_{12} = 1 + \mu^{1/3}\xi_2 + O(\mu^{2/3}), \qquad R_{13} = 1 + \mu^{1/3}\xi_3 + O(\mu^{2/3}). \tag{15}$$

Substituting in (11), we obtain

$$\begin{aligned} \ddot{\xi}_2 &= 2\dot{\eta}_2 + 3\xi_2 + \frac{\nu(\xi_3 - \xi_2)}{\rho^3} + O(\mu^{1/3}), \\ \ddot{\eta}_2 &= -2\dot{\xi}_2 + \frac{\nu(\eta_3 - \eta_2)}{\rho^3} + O(\mu^{1/3}), \end{aligned} \tag{16}$$

and

$$\begin{aligned} \ddot{\xi}_3 &= 2\dot{\eta}_3 + 3\xi_3 + \frac{(1-\nu)(\xi_2 - \xi_3)}{\rho^3} + O(\mu^{1/3}), \\ \ddot{\eta}_3 &= -2\dot{\xi}_3 + \frac{(1-\nu)(\eta_2 - \eta_3)}{\rho^3} + O(\mu^{1/3}), \end{aligned} \tag{17}$$

with

$$\rho = \sqrt{(\xi_3 - \xi_2)^2 + (\eta_3 - \eta_2)^2}. \tag{18}$$

We go over to new coordinates ξ^*, η^*, ξ , η, which describe respectively the center of mass and the relative position of the two satellites:

$$\xi^* = \frac{m_2\xi_2 + m_3\xi_3}{m_2 + m_3} = (1-\nu)\xi_2 + \nu\xi_3, \qquad \eta^* = \frac{m_2\eta_2 + m_3\eta_3}{m_2 + m_3} = (1-\nu)\eta_2 + \nu\eta_3,$$

$$\xi = \xi_3 - \xi_2, \qquad \eta = \eta_3 - \eta_2. \tag{19}$$

ξ and η will be called *relative Hill's coordinates*. The equations of motion are

$$\ddot{\xi}^* = 2\dot{\eta}^* + 3\xi^* + O(\mu^{1/3}), \qquad \ddot{\eta}^* = -2\dot{\xi}^* + O(\mu^{1/3}), \tag{20}$$

and

$$\ddot{\xi} = 2\dot{\eta} + 3\xi - \frac{\xi}{\rho^3} + O(\mu^{1/3}), \qquad \ddot{\eta} = -2\dot{\xi} - \frac{\eta}{\rho^3} + O(\mu^{1/3}), \tag{21}$$

with

$$\rho = \sqrt{\xi^2 + \eta^2}. \tag{22}$$

Note that no approximation has been made so far; all we have done is to change coordinates. Now we assume that μ , the combined relative mass of the two satellites, is small. The last terms in (20) and (21) can be neglected. The two sets of equations are now separated. The equations (20) for the center of mass are linear and easily solved; their general solution is

$$\xi^* = D_1^* \cos t + D_2^* \sin t + D_3^*, \qquad \eta^* = -2D_1^* \sin t + 2D_2^* \cos t - \frac{3}{2}D_3^* t + D_4^*, \tag{23}$$

where D_1^*, D_2^*, D_3^*, D_4^* are constants of integration.

The equations (21) for the relative motion are the familiar *Hill's equations*. It will be frequently useful to write them as a system of four first-order differential equations, by introducing the velocity coordinates u, v:

$$\dot{\xi} = u, \qquad \dot{\eta} = v, \qquad \dot{u} = 2v + 3\xi - \frac{\xi}{\rho^3}, \qquad \dot{v} = -2u - \frac{\eta}{\rho^3}. \tag{24}$$

Hill's equations admit the integral

$$\Gamma = 3\xi^2 + \frac{2}{\rho} - u^2 - v^2 \tag{25}$$

which will be called the *Jacobi integral* by analogy with the restricted problem; in the hierarchical case (1) which is common to both problems, Γ is related to the classical integral C by

$$C = 3 + \mu^{2/3}\Gamma. \tag{26}$$

Hill's equations contain no parameter; perhaps this has contributed to the false impression that Hill's problem is a particular case compared to the restricted problem, which contains a parameter μ . Actually, our present problem contains also a parameter ν , which describes the mass ratio $m_3/(m_2 + m_3)$ just as μ in the restricted problem describes the mass ratio $m_2/(m_1 + m_2)$. This parameter has gone into hiding when we did the change of coordinates (19), and reappears when we go back to the coordinates of the two bodies:

$$\xi_2 = \xi^* - \nu\xi, \qquad \eta_2 = \eta^* - \nu\eta, \qquad \xi_3 = \xi^* + (1-\nu)\xi, \qquad \eta_3 = \eta^* + (1-\nu)\eta. \tag{27}$$

2.2. Unperturbed solutions

When the two satellites are sufficiently far apart, the attractive terms ξ/ρ^3 and η/ρ^3 can be neglected. Then (21) has exactly the same form as (20), and therefore its general solution has the same form as (23):

$$\xi = D_1 \cos t + D_2 \sin t + D_3, \qquad \eta = -2D_1 \sin t + 2D_2 \cos t - \frac{3}{2} D_3 t + D_4, \quad (28)$$

where D_1, D_2, D_3, D_4 are constants. We shall frequently need also the corresponding velocities:

$$u = -D_1 \sin t + D_2 \cos t, \qquad v = -2D_1 \cos t - 2D_2 \sin t - \frac{3}{2} D_3. \quad (29)$$

In the Jacobi constant (25), the term $2/\rho$ can be neglected, and we find that Γ is related to the D_i by:

$$\Gamma = \frac{3}{4} D_3^2 - D_1^2 - D_2^2. \quad (30)$$

Substituting (23) and (28) into (27), we find that the motion of the two particles taken individually has again the same form:

$$\begin{aligned} \xi_i &= D_{i1} \cos t + D_{i2} \sin t + D_{i3}, \\ \eta_i &= -2D_{i1} \sin t + 2D_{i2} \cos t - \frac{3}{2} D_{i3} t + D_{i4}, \end{aligned} \qquad (i = 2, 3) \quad (31)$$

with

$$D_{2j} = D_j^* - \nu D_j, \qquad D_{3j} = D_j^* + (1 - \nu) D_j \qquad (j = 1, 2, 3, 4). \quad (32)$$

Equations (31) represent the familiar epicyclic motion in rotating axes, corresponding to keplerian elliptic motion in fixed axes. Going back to the physical variables with (13), (10) and (7), we find that the semi-major axes and eccentricities of the orbits are related to the constants D_{ij} by

$$a_i = a_0(1 + \mu^{1/3} D_{i3}), \qquad e_i = \mu^{1/3} \sqrt{D_{i1}^2 + D_{i2}^2} \qquad (i = 2, 3). \quad (33)$$

3. Series for approach and departure

We consider now the full equations (24). We shall develop analytic approximations, in the form of asymptotic series, for the solution in the limit $t \to -\infty$ or $t \to +\infty$, i.e. when the two satellites are far from each other; these approximations, in conjunction with numerical integrations over a finite time interval, allow a full determination of a solution from $t = -\infty$ to $t = +\infty$ (see Petit and Hénon, 1985).

3.1. Asymptotically circular orbits

We consider first the particular case where the orbits of the two satellites are circular when they are far from each other (either before or after the encounter). This case is the easiest analytically; and the case where orbits are circular before the encounter is of interest in many applications. A more fundamental reason is that the solution of this particular case is necessary as a first step in the solution of the general case, as will be seen in Section 3.2. From (33), we have then for the limiting motion of satellite i for $|t| \to \infty$:

$$D_{i1} = 0, \qquad D_{i2} = 0, \qquad D_{i3} = \mu^{-1/3}\,\frac{a_i - a_0}{a_0}, \tag{34}$$

where a_i is the radius of the circular orbit. In Hill's coordinates, the motion is given by (31):

$$\xi_i = D_{i3}, \qquad \eta_i = -\frac{3}{2}D_{i3}t + D_{i4}. \tag{35}$$

Thus, a circular orbit appears in Hill's coordinates as a straight line, parallel to the η axis. We obtain the relative motion

$$\xi = h, \qquad \eta = -\frac{3}{2}h(t-\tau), \qquad u = 0, \qquad v = -\frac{3}{2}h, \tag{36}$$

where we have written

$$D_3 = h, \qquad D_4 = \frac{3}{2}h\tau. \tag{37}$$

h represents the radial separation of the two circular orbits in Hill's coordinates; it should properly be called the *reduced impact parameter*, but we shall abbreviate this into *impact parameter*.

Four cases can be distinguished, depending on the signs of t and h (Figure 1):

1) $t < 0$, $h > 0$: approach in first quadrant;
2) $t > 0$, $h < 0$: departure in second quadrant;
3) $t < 0$, $h < 0$: approach in third quadrant;
4) $t > 0$, $h > 0$: departure in fourth quadrant.

Because of the symmetries of the problem, it is possible to treat all four cases by a single formalism, which will be described below.

We seek now the solution which has the asymptotic behaviour (36) for $|t| \to \infty$. The first idea which comes to mind is to look for an expansion in powers of t^{-1}, of the form

$$\xi = h + a_1 t^{-1} + a_2 t^{-2} + \cdots, \qquad \eta = -\frac{3}{2}ht + b_0 + b_1 t^{-1} + \cdots. \tag{38}$$

Substituting this in the equations of motion (24), however, one soon finds that the equations for the coefficients cannot be satisfied. Thus, the solution cannot be of the simple form (38). The reason for this will be understood below.

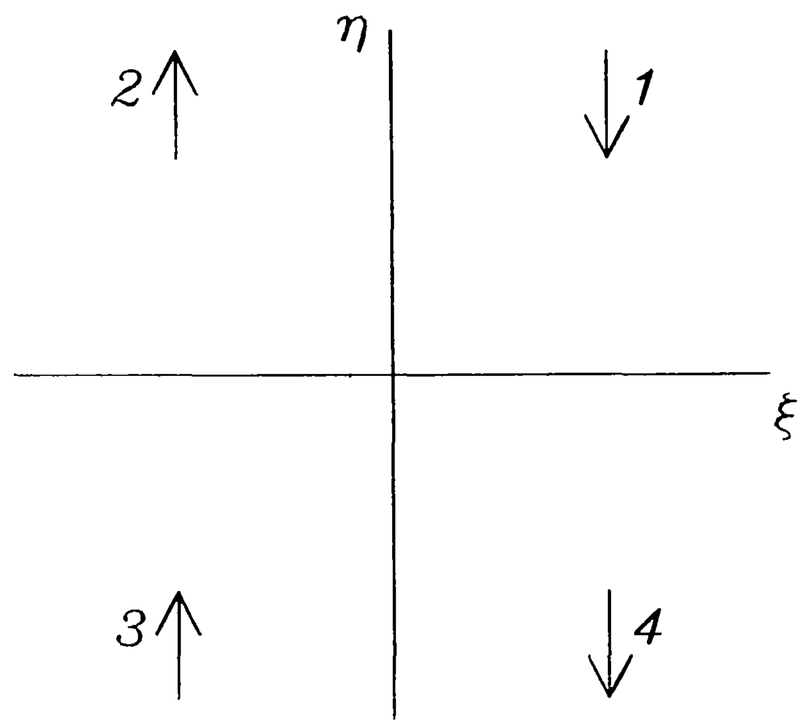

Fig. 1. The four cases of asymptotic motion for $t \to \pm\infty$.

The correct approach consists in ignoring the time altogether, and looking for an expansion in powers of η^{-1} instead. First, we eliminate the time by substituting $dt = d\eta/v$ in (24), which becomes

$$v\frac{d\xi}{d\eta} = u, \qquad v\frac{du}{d\eta} = 2v + 3\xi - \frac{\xi}{\rho^3}, \qquad v\frac{dv}{d\eta} = -2u - \frac{\eta}{\rho^3}. \tag{39}$$

We seek an expansion of the form

$$\xi = a_0 + a_1\eta^{-1} + \cdots, \qquad u = c_0 + c_1\eta^{-1} + \cdots, \qquad v = d_0 + d_1\eta^{-1} + \cdots. \tag{40}$$

From the required behaviour of the solution for $|t| \to \infty$, we have immediately

$$a_0 = h, \qquad c_0 = 0, \qquad d_0 = -\frac{3}{2}h. \tag{41}$$

The factor $1/\rho^3$ is expanded as

$$\frac{1}{\rho^3} = s\eta^{-3}\left(1 + \frac{\xi^2}{\eta^2}\right)^{-3/2} = s\eta^{-3}\left(1 - \frac{3}{2}\frac{\xi^2}{\eta^2} + \cdots\right), \tag{42}$$

where

$$s = \mathrm{sign}(\eta). \tag{43}$$

This expansion is valid provided that $|\eta| \gg |\xi|$, or, since ξ is nearly equal to h :

$$|\eta| \gg |h|. \tag{44}$$

Substituting (40) into (39), we compute the coefficients recursively as follows: assume that the coefficients are known up to a_{i-1}, c_i, d_{i-1} ; then the coefficients of $\eta^{-(i+1)}$ in (39a) and (39c) and of η^{-i} in (39b) give three equations for a_i, c_{i+1}, d_i, which can be solved.

Expansions up to order 12 have been obtained by computer algebra, using an 8-bit microcomputer. They were also verified by hand up to order 4. We quote them to order 8 only to save space:

$$\begin{aligned}\xi = h &- \frac{4}{3}h^{-1}s\eta^{-1} - \frac{8}{9}h^{-3}\eta^{-2} + \left(\frac{17}{3}h - \frac{32}{27}h^{-5}\right)s\eta^{-3} \\ &+ \left(-\frac{44}{9}h^{-1} - \frac{160}{81}h^{-7}\right)\eta^{-4} + s\left(-152h^3 - \frac{40}{27}h^{-3} - \frac{896}{243}h^{-9}\right)\eta^{-5} \\ &+ \left(\frac{2338}{3}h + \frac{32}{81}h^{-5} - \frac{1792}{243}h^{-11}\right)\eta^{-6} \\ &+ s\left(\frac{246205}{24}h^5 - 592h^{-1} + \frac{1120}{243}h^{-7} - \frac{11264}{729}h^{-13}\right)\eta^{-7} \\ &+ \left(-\frac{1969253}{18}h^3 - \frac{7568}{27}h^{-3} + \frac{11648}{729}h^{-9} - \frac{73216}{2187}h^{-15}\right)\eta^{-8} + O(\eta^{-9}),\end{aligned} \tag{45a}$$

$$\begin{aligned}u = &-2s\eta^{-2} + \frac{51}{2}sh^2\eta^{-4} - 78\eta^{-5} - 1140sh^4\eta^{-6} + 9126h^2\eta^{-7} \\ &+ s\left(\frac{1723435}{16}h^6 - \frac{49552}{3}\right)\eta^{-8} + O(\eta^{-9}),\end{aligned} \tag{45b}$$

$$\begin{aligned}v = &-\frac{3}{2}h + 2h^{-1}s\eta^{-1} + \frac{4}{3}h^{-3}\eta^{-2} + \left(-11h + \frac{16}{9}h^{-5}\right)s\eta^{-3} \\ &+ \left(\frac{32}{3}h^{-1} + \frac{80}{27}h^{-7}\right)\eta^{-4} + s\left(\frac{1215}{4}h^3 + \frac{40}{9}h^{-3} + \frac{448}{81}h^{-9}\right)\eta^{-5} \\ &+ \left(-\frac{4739}{3}h + \frac{64}{27}h^{-5} + \frac{896}{81}h^{-11}\right)\eta^{-6} \\ &+ s\left(-\frac{164135}{8}h^5 + \frac{11042}{9}h^{-1} - \frac{160}{81}h^{-7} + \frac{5632}{243}h^{-13}\right)\eta^{-7} \\ &+ \left(\frac{2639153}{12}h^3 + \frac{16120}{27}h^{-3} - \frac{3584}{243}h^{-9} + \frac{36608}{729}h^{-15}\right)\eta^{-8} + O(\eta^{-9}).\end{aligned} \tag{45c}$$

The time t itself can now be computed as an expansion in η, by a simple integration of $dt/d\eta = 1/v$. The result is

$$
\begin{aligned}
t-\tau = &-\frac{2}{3}h^{-1}\eta - \frac{8}{9}sh^{-3}\ln s\eta + \frac{16}{9}h^{-5}\eta^{-1} + \left(-\frac{22}{9}h^{-1} + \frac{160}{81}h^{-7}\right)s\eta^{-2} \\
&+ \left(-\frac{224}{81}h^{-3} + \frac{2240}{729}h^{-9}\right)\eta^{-3} + \left(\frac{135}{4}h - \frac{136}{27}h^{-5} + \frac{448}{81}h^{-11}\right)s\eta^{-4} \\
&+ \left(-\frac{2756}{45}h^{-1} - \frac{2560}{243}h^{-7} + \frac{39424}{3645}h^{-13}\right)\eta^{-5} \\
&+ \left(-\frac{164135}{108}h^{3} - \frac{3956}{81}h^{-3} - \frac{51520}{2187}h^{-9} + \frac{146432}{6561}h^{-15}\right)s\eta^{-6} \\
&+ \left(\frac{643051}{63}h - \frac{4576}{63}h^{-5} - \frac{13312}{243}h^{-11} + \frac{732160}{15309}h^{-17}\right)\eta^{-7} \\
&+ \left(\frac{165449515}{1152}h^{5} - \frac{1638293}{162}h^{-1} - \frac{31300}{243}h^{-7} - \frac{285824}{2187}h^{-13}\right. \\
&\quad \left. + \frac{6223360}{59049}h^{-19}\right)s\eta^{-8} + O(\eta^{-9}).
\end{aligned} \tag{46}
$$

This expansion contains a logarithmic term, while the expansions (45) were purely polynomial. This term is essentially due to the fact that at large distances, the attraction between the satellites is proportional to t^{-2} ; this, integrated twice with respect to time, produces perturbations proportional to $\ln|t|$. The presence of this logarithmic term explains why no formal solution with the form (38) can exist.

Inverting (46), we obtain

$$
\begin{aligned}
s\eta = &T - \frac{4}{3}h^{-2}\ln T + \frac{16}{9}h^{-4}T^{-1}\ln T + \frac{8}{3}h^{-4}T^{-1} + \frac{32}{27}h^{-6}T^{-2}\ln^2 T \\
&+ \frac{32}{27}h^{-6}T^{-2}\ln T + \left(-\frac{11}{3} - \frac{16}{27}h^{-6}\right)T^{-2} + O(T^{-3}\ln^3 T)
\end{aligned} \tag{47}
$$

where we have used the abbreviation

$$T = -\frac{3}{2}sh(t-\tau). \tag{48}$$

(47) shows that the asymptotic motion for $|t| \to \infty$ is in fact not exactly of the expected form (36): a logarithmic term is superimposed on the linear motion. Our problem has a singular perturbation at $|t| \to \infty$: the solution (36) of the unperturbed problem does not coincide with the limit of the general solution for a vanishing perturbation.

More terms could be computed in (47), and this expression of η as a function of time could be substituted in the equations (45); in this way one would obtain also ξ, u, v as explicit functions of t. However, these functions would be double series in T and $\ln T$, much more cumbersome than (45) and (46). It seems therefore preferable to use the implicit form of the solution given by (45) and (46). The logarithmic singularity is then neatly isolated in a single term in (46). In fact, most applications require only a knowledge of the shape of the orbit; the time is not needed. One can then conveniently work with the polynomial equations (45) alone.

The series (45) have been derived in a formal way, and we do not know whether they are convergent. In fact it seems likely that they are only asymptotic series (see Sec. 4). In practice this is not of much importance: the series (45), truncated at some finite order, are found to agree with the exact solution with high accuracy, provided that $|\eta|$ is large enough (Petit and Hénon, 1985).

3.2. General case

The relative motion, in the limit $|t| \to \infty$, has essentially the form (28), with D_1, ..., D_4 arbitrary. However, in the case of asymptotic circular motion, we found that the time dependence of η must be corrected according to (47). We shall assume that the same correction applies in the general case. With an appropriate choice of the origin of time, the asymptotic motion can then be rewritten as

$$\xi = h + k cos(t-\varphi), \qquad \eta = -\frac{3}{2}h(t-\tau) - \frac{4}{3}sh^{-2}\ln\left[-\frac{3}{2}sh(t-\tau)\right] - 2k\sin(t-\varphi). \tag{49}$$

It depends on four arbitrary constants h, k, φ, τ. h will be called the impact parameter, as before. k measures the radial amplitude of the relative motion; we shall call it *reduced eccentricity*. (Note that it is "reduced" in two ways: first because it involves a combination of the actual eccentricities of the two satellites and of the orientations of their semi-major axes; and second, because it is measured in Hill's coordinates rather than in physical coordinates). φ is the *phase* of the epicyclic motion. As is easily seen, the Jacobi constant (25) is related to these constants by

$$\Gamma = \frac{3}{4}h^2 - k^2. \tag{50}$$

We seek the solution which has the asymptotic behaviour (49) for $|t| \to \infty$. Again we will try to express this solution as a series, and we must guess at the appropriate form of that series.

Clearly, a simple expansion in powers of η^{-1} will not work anymore for $k \neq 0$, since the solution has an oscillatory component which does not vanish in the limit. Instead of a single solution, consider however the set of solutions obtained by keeping h, k and τ constant in (49), while giving to φ all possible values from 0 to 2π. At any given time t, the points corresponding to these solutions lie on a closed curve C. In the limit $|t| \to \infty$, this curve is an ellipse centered in $[h, -\frac{3}{2}h(t-\tau)]$ and with semi-axes $(k, 2k)$. As t changes, and as long as the asymptotic form (49) is valid, this ellipse glides smoothly along a straight line, without changing its shape: no oscillations are present. As the attraction of M_2 begins to be felt, the curve C is progressively distorted; it seems reasonable to conjecture, however, that this distortion will be "smooth", in the sense that it will not contain oscillatory components. We look therefore for an oscillation-free representation of C as a function of time.

A natural parameterization of C, at a given time, is offered by the phase φ. The coordinates ξ and η of a point of C are periodic functions of φ; thus, the curve at a given time can be represented by Fourier series for ξ and η, with φ as variable. With this simple choice, however, the coefficients of the Fourier series will oscillate; this is already obvious for the asymptotic form (49). We remedy this by redefining locally the origin of the angles: we substitute to φ the new angular variable

$$\theta = t - \varphi, \tag{51}$$

and we write the Fourier series with θ as variable. Now the coefficients of the series do not oscillate any more in the limit (49). We conjecture that this is true in general; more specifically, we conjecture that the coefficients of the Fourier series can be expressed as polynomial series.

What should the argument of these series be ? We cannot use η^{-1} itself as in the previous section, since η has now an oscillating component (it may even be a non-monotonic function of time if k is large enough). We need an "average" η, describing the position of the curve C and smoothly changing with it. This can be obtained as follows. Consider the larger family of solutions obtained by varying k. At any given time, instead of a single curve we have now a family of closed curves, with k as parameter. In the limit $|t| \to \infty$, this is a family of concentric ellipses. At the center of this family lies the curve corresponding to $k = 0$, which in fact degenerates into a single point. This point can be called the *guiding center* of the whole family. We call ξ_c, η_c its coordinates. The motion of the guiding center is known from the previous section; simply substitute ξ_c, η_c, u_c, v_c for ξ, η, u, v in (45) and (46). This motion is smooth. So we shall use η_c as the argument of the polynomial series.

It will also be convenient to use the guiding center as origin for the description of the curve C. We arrive thus at the following representation. The general solution is written as

$$\xi = \xi_c + x, \qquad \eta = \eta_c + y. \tag{52}$$

x and y are Fourier series in θ:

$$x = \sum_{j=0}^{\infty}(a_j \cos j\theta + b_j \sin j\theta), \qquad y = \sum_{j=0}^{\infty}(c_j \cos j\theta + d_j \sin j\theta). \tag{53}$$

The coefficients a_j, b_j, c_j, d_j are themselves polynomial series in η_c^{-1}:

$$a_j = \sum_{i=0}^{\infty} \alpha_{ji}\eta_c^{-i}, \qquad b_j = \sum_{i=0}^{\infty} \beta_{ji}\eta_c^{-i}, \qquad c_j = \sum_{i=0}^{\infty} \gamma_{ji}\eta_c^{-i}, \qquad d_j = \sum_{i=0}^{\infty} \delta_{ji}\eta_c^{-i}. \tag{54}$$

The coefficients $\alpha_{ji}, \ldots, \delta_{ji}$ are fixed for a given orbit. So they should be definite functions of the parameters h, k, φ, τ which define the orbit. Actually they will be seen to be functions of h and k only. The dependence on φ has been absorbed by the change of variables (51); in other words, φ appears in the expressions of ξ and η only through θ. Similarly, the dependence on τ has been absorbed by the change of variables (52); τ appears in the expressions of ξ and η only through η_c. All coefficients vanish for $k = 0$.

From the asymptotic form (49) we derive the coefficients at order 0 in η_c^{-1}:

$$\begin{aligned} &\alpha_{10} = k; \qquad \alpha_{j0} = 0 \quad \text{for} \quad j \neq 1; \\ &\beta_{j0} = 0 \quad \text{for all } j; \\ &\gamma_{j0} = 0 \quad \text{for all } j; \\ &\delta_{10} = -2k; \qquad \delta_{j0} = 0 \quad \text{for} \quad j \neq 1. \end{aligned} \tag{55}$$

The other coefficients will be obtained recursively. The procedure is different from that of the previous section. The expressions of ξ and η defined by (52), (53), (54) are substituted directly in the equations of motion (21), written in the form

$$\ddot{\xi} - 2\dot{\eta} - 3\xi = R_\xi, \qquad \ddot{\eta} + 2\dot{\xi} = R_\eta, \tag{56}$$

with

$$R = \frac{1}{\rho} = (\xi^2 + \eta^2)^{-1/2}. \tag{57}$$

ξ and η subscripts represent partial derivation. The right-hand sides in (56) are expanded around the guiding center, using (52):

$$R_\xi(\xi, \eta) = R_\xi(\xi_c, \eta_c) + R_{\xi\xi}(\xi_c, \eta_c)x + R_{\xi\eta}(\xi_c, \eta_c)y + \cdots \tag{58}$$

and a similar expression for R_η.

The derivatives $R_{\xi\xi}(\xi_c, \eta_c)$, etc. are expressed as series in η_c^{-1}, using expansions similar to (42).

When computing time derivatives, one must remember that both arguments θ and η_c are functions of time. From (51) we have simply:

$$\frac{d\theta}{dt} = 1, \tag{59}$$

while the time derivative of η_c is

$$\frac{d\eta_c}{dt} = v_c, \tag{60}$$

where v_c is given by (45c) with η_c substituted for η. Finally, ξ_c is given by (45a).

With these substitutions, both sides of equations (56) are completely expressed as double series in θ and η_c^{-1}. This provides an infinite set of equations, which can be solved recursively.

For a given order i in η_c^{-1}, it is found that the coefficients $\alpha_{ji}, \ldots, \delta_{ji}$ vanish beyond a certain order j. Specifically, non-zero terms exist only for the following combinations: $i = 0$, $j = 1$; $i = 2$, $j = 0$; $i = 2$, $j = 1$; and $i > 2$, $j < i - 1$. Therefore the correct procedure is to compute first all terms with $i = 1$, then all terms with $i = 2$, etc. (the terms $i = 0$ are already known from (55)): at each step only a finite number of new coefficients have to be computed.

The computation shows that all coefficients are thus uniquely determined. A curious feature, however, is that some coefficients "lag behind". The coefficients α_{0i}, α_{1i}, β_{1i}, γ_{1i}, δ_{1i} remain indeterminate at order i, and become determined only when terms of order $i+1$ in the equations (56) are considered. The coefficient γ_{0i} lags even more: it is determined only when terms of order $i+2$ are considered.

The computations are straightforward but lengthy. Results to order 7 in η_c^{-1} were obtained by computer algebra. They were verified independently by each of us up to order 4. The expressions are cumbersome and we give them to order 4 only:

$$\begin{aligned}\xi = {} & h + k\cos\theta - \frac{4}{3}sh^{-1}\eta_c^{-1} + \left(-\frac{8}{9}h^{-3} + \frac{7}{6}sh^{-1}k\sin\theta\right)\eta_c^{-2} \\ & + \left(\frac{17}{3}sh - \frac{32}{27}sh^{-5} - \frac{7}{3}sh^{-1}k^2 + \frac{23}{4}sk\cos\theta + \frac{28}{27}h^{-3}k\sin\theta\right)\eta_c^{-3} \\ & + \left[-\frac{44}{9}h^{-1} - \frac{160}{81}h^{-7} - \frac{14}{9}h^{-3}k^2 - \frac{49}{72}h^{-2}k\cos\theta \right. \\ & \left. + s\left(-\frac{473}{16}hk + \frac{14}{9}h^{-5}k + \frac{99}{32}h^{-1}k^3\right)\sin\theta - \frac{5}{4}sk^2\sin 2\theta\right]\eta_c^{-4} + O(\eta_c^{-5}),\end{aligned} \tag{61a}$$

$$\begin{aligned}\eta = {} & \eta_c - 2k\sin\theta + \left(\frac{7}{6}sh^{-2}k^2 + \frac{7}{3}sh^{-1}k\cos\theta\right)\eta_c^{-2} \\ & + \left(\frac{14}{9}h^{-4}k^2 + \frac{56}{27}h^{-3}k\cos\theta - \frac{29}{2}sk\sin\theta\right)\eta_c^{-3} + O(\eta_c^{-4}),\end{aligned} \tag{61b}$$

$$\begin{aligned}u = {} & -k\sin\theta + \left(-2s + \frac{7}{6}sh^{-1}k\cos\theta\right)\eta_c^{-2} \\ & + \left(\frac{28}{27}h^{-3}k\cos\theta - \frac{9}{4}sk\sin\theta\right)\eta_c^{-3} \\ & + \left[\frac{51}{2}sh^2 - \frac{21}{2}sk^2 + s\left(-\frac{59}{16}hk + \frac{14}{9}h^{-5}k + \frac{99}{32}h^{-1}k^3\right)\cos\theta \right. \\ & \left. + \frac{49}{72}h^{-2}k\sin\theta - \frac{5}{2}sk^2\cos 2\theta\right]\eta_c^{-4} + O(\eta_c^{-5}),\end{aligned} \tag{61c}$$

$$v = -\frac{3}{2}h - 2k\cos\theta + 2sh^{-1}\eta_c^{-1} + \left(\frac{4}{3}h^{-3} - \frac{7}{3}sh^{-1}k\sin\theta\right)\eta_c^{-2}$$
$$+ \left(-11sh + \frac{16}{9}sh^{-5} + \frac{7}{2}sh^{-1}k^2 - \frac{15}{2}sk\cos\theta - \frac{56}{27}h^{-3}k\sin\theta\right)\eta_c^{-3}$$
$$+ \left[\frac{32}{3}h^{-1} + \frac{80}{27}h^{-7} + \frac{7}{3}h^{-3}k^2 + \frac{49}{36}h^{-2}k\cos\theta\right.$$
$$\left. + s\left(\frac{353}{8}hk - \frac{28}{9}h^{-5}k - \frac{99}{16}h^{-1}k^3\right)\sin\theta + \frac{47}{8}sk^2\sin 2\theta\right]\eta_c^{-4} + O(\eta_c^{-5}). \quad (61d)$$

For $k = 0$, we recover of course the developments (45) for ξ, u, v, while η reduces to η_c.

These equations, together with (51) and (46) (with η_c substituted for η) define implicitly ξ, η, u, v as functions of the time and of the four parameters h, k, φ, τ.

4. Small impact parameter

4.1. THE HORSESHOE MOTION: FIRST-ORDER DESCRIPTION

In this section, we shall consider the case where the impact parameter h is small. For initially circular orbits, the satellites describe then a *horseshoe motion*, as shown by Brown (1911). A typical example is shown by Figure 2a. This peculiar motion can be explained in physical terms. Assume for the simplicity of the description that $m_3 \ll m_2$ and $\eta > 0$ (however the argument is also valid in the general case). Then M_2 stays fixed at the origin, and M_3 comes down from $\eta = +\infty$ in the first quadrant (case 1 in Figure 1). This is the situation for Figure 2a. In the physical frame of reference, M_3 continually loses energy because it is attracted from behind by M_2. Therefore its semi-major axis decreases. In Hill's coordinates: ξ constantly decreases. This process is very slow because η is large, so that the attraction is small. Therefore we can assume that the motion of M_3 remains very nearly circular at all times. The differential velocity of M_3, represented by $\dot\eta = v$ in Hill's coordinates, is then proportional to $-\xi$. Thus, η first decreases because ξ is positive; it reaches a minimal value η_{cr} when ξ crosses the value 0; and then it increases again as ξ becomes negative.

This argument is easily cast into equations as follows. In the equation of motion (24c), the radial acceleration $\dot u$ is extremely small and can be neglected; also ξ/ρ^3 is negligible compared to ξ. This equation reduces therefore to

$$v = -\frac{3}{2}\xi. \quad (62)$$

ρ is very nearly equal to η; therefore, eliminating $\dot v$ between (62) and (24d), we obtain

$$u = -\frac{2}{\eta^2}. \quad (63)$$

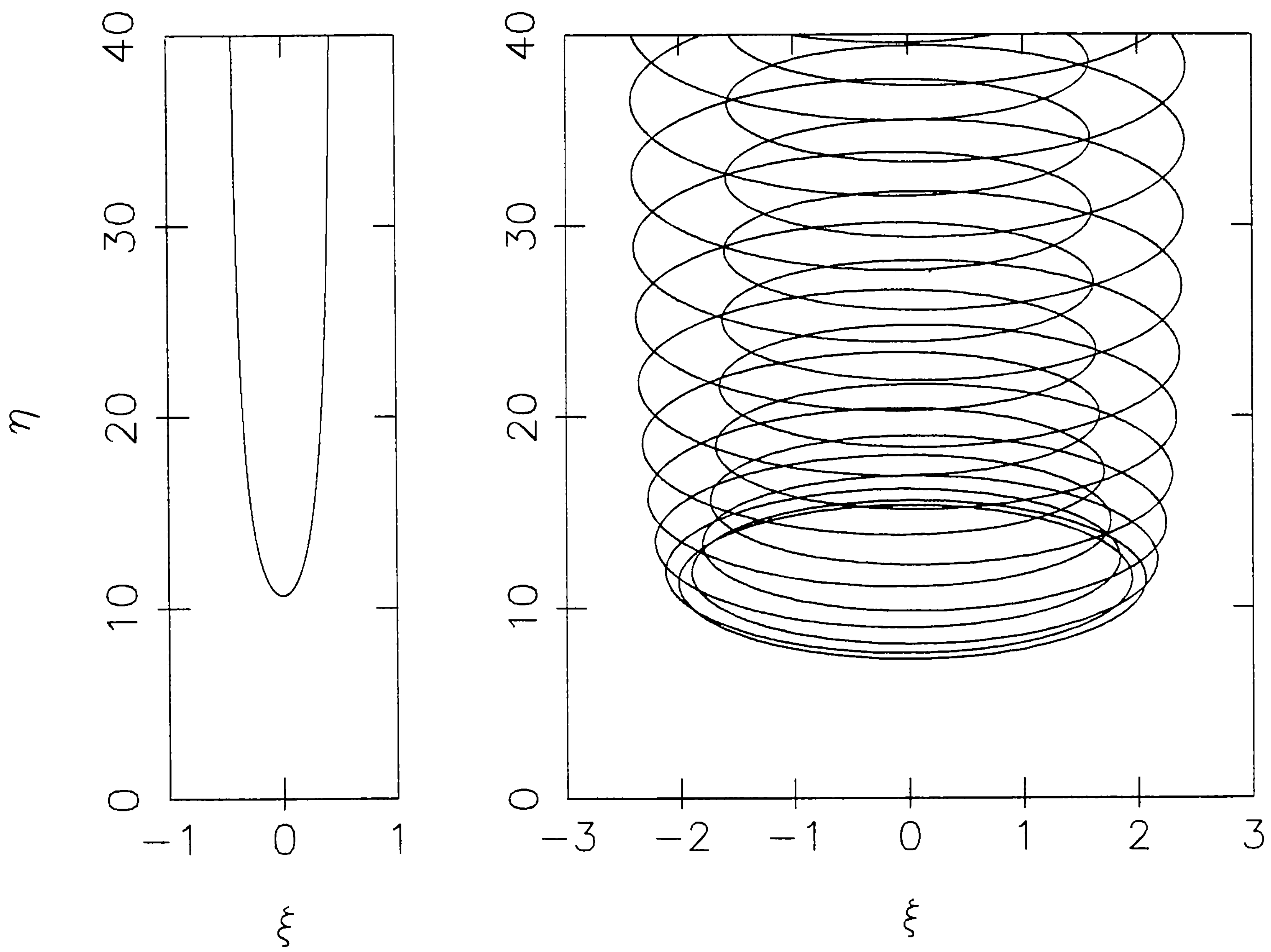

Fig. 2. (a) A horseshoe orbit. Initial values are $h = 0.5$, $k = 0$. (b) An orbit with the same impact parameter and non-zero initial eccentricity. Initial values are $h = 0.5$, $k = 2$, $\varphi = \tau = 0$. Note that the scales for ξ and η are different.

Since $u = \dot{\xi}$ and $v = \dot{\eta}$, (62) and (63) represent a simple system of two first-order equations. The solution of this system cannot be expressed in a closed form $\xi(t)$, $\eta(t)$; however, it admits a parametric representation

$$\xi = -h \tanh \lambda, \qquad \eta = \frac{8}{3h^2} \cosh^2 \lambda, \qquad t - t_{cr} = \frac{16}{9h^3}(\lambda + \frac{1}{2} \sinh 2\lambda) \tag{64}$$

where the boundary condition $\xi \to h$ for $t \to -\infty$ has been taken into account. The parameter λ increases from $-\infty$ to $+\infty$. t_{cr} is an integration constant; $t = t_{cr}$ corresponds to the point where the orbit crosses the η axis. The corresponding minimal value of η is

$$\eta_{cr} = \frac{8}{3h^2}. \tag{65}$$

(64) shows also that the orbit is symmetrical with respect to the η axis. In particular, for $t \to +\infty$, we have $\xi \to -h$ and $\eta \to +\infty$. Eliminating λ between (64a) and (64b), we obtain the equation of the orbit:

$$\eta = \frac{8}{3(h^2 - \xi^2)}. \tag{66}$$

From the solution (64), one can verify a posteriori that the approximations made are valid for h small.

In the more general case of initially eccentric orbits, numerical computations (Dermott et al., 1980; Petit, 1985) show that the orbit consists of the slow horseshoe motion described above, on which is superimposed a rapid gyration of nearly constant amplitude. A typical example is shown by Figure 2b.

4.2. ADIABATIC INVARIANT THEORY: BASIC EQUATIONS

We try now to refine the description of the horseshoe motion. The character of this motion (a rapid gyration superimposed on a slow drift), and the fact that at $t \to +\infty$ one obtains in the above approximation an impact parameter simply reversed in sign (and the same reduced eccentricity) suggest that there exists an adiabatic invariant. This was already suggested by Dermott and Murray (1981). We will show that this is indeed the case, and we will exhibit the adiabatic invariant in the form of a series.

We shall use the method described in a classical paper by Kruskal (1962). The notations of that paper will be followed, except that the independent variable s of Kruskal will be the time t in our case. In order for the method to be applicable, the system of differential equations must be such that all solutions are nearly periodic; more precisely, it must be of the form

$$\frac{d\mathbf{x}}{dt} = \mathbf{F}(\mathbf{x}, \epsilon), \tag{67}$$

where $\mathbf{x}$ is an N-component vector ($N = 4$ in our case); ϵ is a small parameter; and $\mathbf{F}$ is such that for $\epsilon = 0$, all solutions are periodic. For ϵ small but not zero, a solution is nearly periodic in the short run; in the long run, it drifts slowly along the family of periodic orbits of the $\epsilon = 0$ case.

Equations (24) of Hill's problem do not satisfy this condition: they contain no small parameter, and the approximate solution (28) contains a linear drift term $-\frac{3}{2}D_3 t$ which is not small in general, so that the solutions are far from being periodic. We remark, however, that in the present Section only large values of η are considered. This suggests the change of variables

$$\hat{\eta} = \epsilon\eta, \tag{68}$$

where ϵ is a small positive parameter. Since ξ is small, we can again use the expansion (42), which becomes

$$\frac{1}{\rho^3} = \frac{\epsilon^3 s Q}{\hat{\eta}^3} \tag{69}$$

with

$$Q = \left(1 + \frac{\epsilon^2\xi^2}{\hat{\eta}^2}\right)^{-3/2} = 1 - 3\frac{\epsilon^2\xi^2}{\hat{\eta}^2} + \frac{15}{8}\frac{\epsilon^4\xi^4}{\hat{\eta}^4} - \cdots . \tag{70}$$

The equations of motion are now

$$\dot{\xi} = u, \qquad \dot{\hat{\eta}} = \epsilon v, \qquad \dot{u} = 2v + 3\xi - \frac{\epsilon^3 s\xi Q}{\hat{\eta}^3}, \qquad \dot{v} = -2u - \frac{\epsilon^2 s Q}{\hat{\eta}^2}. \tag{71}$$

For $\epsilon = 0$, they reduce to

$$\dot{\xi} = u, \qquad \dot{\hat{\eta}} = 0, \qquad \dot{u} = 2v + 3\xi, \qquad \dot{v} = -2u, \tag{72}$$

which is linear and trivially integrable, with the general solution

$$\xi = D_1 \cos t + D_2 \sin t + D_3, \qquad \hat{\eta} = \hat{D}_4,$$
$$u = -D_1 \sin t + D_2 \cos t, \qquad v = -2D_1 \cos t - 2D_2 \sin t - \frac{3}{2} D_3. \tag{73}$$

This is periodic. The system (71) is therefore in the required form for the application of Kruskal's method. It is also in the "standardized form" (Kruskal 1962, Equs. (B4) and (B5)): the right-hand members are series in ϵ.

It will be convenient to introduce another new variable

$$\hat{v} = 2v + 3\xi. \tag{74}$$

Using this in place of v, we have the new equations

$$\dot{\xi} = u, \qquad \dot{\hat{\eta}} = \frac{\epsilon}{2}(\hat{v} - 3\xi), \qquad \dot{u} = \hat{v} - \frac{\epsilon^3 s \xi Q}{\hat{\eta}^3}, \qquad \dot{\hat{v}} = -u - \frac{2\epsilon^2 s Q}{\hat{\eta}^2}. \tag{75}$$

For $\epsilon = 0$, they reduce to

$$\dot{\xi} = u, \qquad \dot{\hat{\eta}} = 0, \qquad \dot{u} = \hat{v}, \qquad \dot{\hat{v}} = -u, \tag{76}$$

with the general solution

$$\xi = D_1 \cos t + D_2 \sin t + D_3, \qquad \hat{\eta} = \hat{D}_4,$$
$$u = -D_1 \sin t + D_2 \cos t, \qquad \hat{v} = -D_1 \cos t - D_2 \sin t. \tag{77}$$

4.3. More appropriate variables

The next step in Kruskal's method (p. 812 of his paper) consists in replacing the variables ξ, $\hat{\eta}$, u, $\hat{v}$ by "more appropriate variables" y_1, y_2, y_3, ν. These variables are such that in the limiting case $\epsilon = 0$, the three variables y_i become constants, characterizing the particular periodic solution, while ν increases monotonically with time, and increases by exactly 2π after one revolution (our ν corresponds to 2π times Kruskal's ν). The change of variables should be defined by Kruskal's equations (B11) :

$$\mathbf{y} = \mathbf{Y}(\xi, \hat{\eta}, u, \hat{v}), \qquad \nu = \Upsilon(\xi, \hat{\eta}, u, \hat{v}), \tag{78}$$

i.e. it should be time-independent.

In view of the form of (77), a natural choice would be to define the new variables by

$$\xi = y_1 + y_2 \cos \nu, \qquad \hat{\eta} = y_3, \qquad u = -y_2 \sin \nu, \qquad \hat{v} = -y_2 \cos \nu. \tag{79}$$

y_1, y_2, y_3, ν thus defined are easily seen to have the required properties. Unfortunately, this change of variables has a singularity at $u = \hat{v} = 0$. This is seen when one computes the time derivative of ν from the last two equations (79):

$$\dot{\nu} = \frac{\dot{\hat{v}} \sin \nu - \dot{u} \cos \nu}{y_2}. \tag{80}$$

For $\epsilon \neq 0$, $\dot{u}$ and $\dot{\hat{v}}$ do not vanish at $u = \hat{v} = 0$; therefore $\dot{\nu}$ is infinite at that point. This singularity propagates in later series developments, where higher and higher powers of $1/y_2$ appear. As a consequence, these developments are useless in a sizable region around $u = \hat{v} = 0$; and unfortunately this is the region of greatest interest, because it corresponds to orbits with zero or small reduced eccentricity. So the change of variables (79) is unacceptable.

There is a remedy. We remark that for $\epsilon \neq 0$, equations (75c) and (75d) still represent essentially a rotation in the $(u, \hat{v})$ plane, but with the rotation center slightly shifted from the origin. This suggests that we should correspondingly shift the origin in the change from cartesian to polar coordinates represented by (79c) and (79d). Therefore we replace (79) by

$$\xi = y_1 + y_2 \cos \nu, \qquad \hat{\eta} = y_3,$$

$$u = -y_2 \sin \nu + U(y_1, y_2, y_3), \qquad \hat{v} = -y_2 \cos \nu + V(y_1, y_2, y_3). \tag{81}$$

U and V are two as yet unspecified functions, of order ϵ; note that they do not depend on ν . We try now to choose these functions so as to eliminate the singularity; it will be seen that U and V are in fact uniquely determined by this condition. Substituting (81) into (75), we obtain the new system of differential equations for y_1, y_2, y_3, ν:

$$\begin{aligned}
\dot{y}_1 &= -\frac{2\epsilon^2 sQ}{y_3^2} - \dot{V}, \\
\dot{y}_2 &= \left[\dot{V} + U + \frac{2\epsilon^2 sQ}{y_3^2}\right] \cos \nu + \left[\dot{U} - V + \frac{\epsilon^3 sQ}{y_3^3}(y_1 + y_2 \cos \nu)\right] \sin \nu, \\
\dot{y}_3 &= -\frac{3}{2}\epsilon y_1 - 2\epsilon y_2 \cos \nu + \frac{1}{2}\epsilon V, \\
\dot{\nu} &= 1 + \frac{1}{y_2}\left[\dot{U} - V + \frac{\epsilon^3 sQ}{y_3^3}(y_1 + y_2 \cos \nu)\right] \cos \nu - \frac{1}{y_2}\left[\dot{V} + U + \frac{2\epsilon^2 sQ}{y_3^2}\right] \sin \nu,
\end{aligned} \tag{82}$$

with

$$Q = \left[1 + \frac{\epsilon^2}{y_3^2}(y_1 + y_2 \cos \nu)^2\right]^{-3/2} = 1 - \frac{3\epsilon^2}{2y_3^2}(y_1 + y_2 \cos \nu)^2 + \cdots. \tag{83}$$

$\dot{\nu}$ will have no singularity if the two brackets in (82d) vanish identically for $y_2 = 0$; we assume this to be the case. Then $\dot{y}_2$ given by (82b) vanishes. Thus, solutions which have $y_2 = 0$ at one point have $y_2 = 0$ everywhere. We temporarily restrict our attention to this subset of solutions, in order to determine U and V. Equations (82a) and (82c) reduce to

$$\dot{y}_1 = -\frac{2\epsilon^2 sQ}{y_3^2} - \dot{V}, \qquad \dot{y}_3 = -\frac{3}{2}\epsilon y_1 + \frac{1}{2}\epsilon V, \tag{84}$$

with

$$Q = 1 - \frac{3}{2}\frac{\epsilon^2 y_1^2}{y_3^2} + \cdots \tag{85}$$

The condition that the brackets should vanish in (82) gives the two equations

$$U = -\dot{V} - \frac{2\epsilon^2 sQ}{y_3^2}, \qquad V = \dot{U} + \frac{\epsilon^3 s y_1 Q}{y_3^3}. \tag{86}$$

U and V will be functions of y_1 and y_3 only since $y_2 = 0$, and their derivatives are

$$\dot{U} = \frac{\partial U}{\partial y_1}\dot{y}_1 + \frac{\partial U}{\partial y_3}\dot{y}_3, \qquad \dot{V} = \frac{\partial V}{\partial y_1}\dot{y}_1 + \frac{\partial V}{\partial y_3}\dot{y}_3. \tag{87}$$

U and V can now be found recursively, order by order in ϵ: assume that U, V, $\dot{y}_1$, $\dot{y}_3$ are known to order $i-1$; then from (87) we obtain $\dot{U}$ and $\dot{V}$ to order i (we gain one order because $\dot{y}_1$, $\dot{y}_3$, U, V are all of order ϵ). From (86) we compute U and V to order i, and from (84) we obtain $\dot{y}_1$ and $\dot{y}_3$ to order i. The recursion is started with $i = 1$, and $\dot{y}_1 = \dot{y}_3 = U = V = 0$. The results are, to order 8 in ϵ:

$$\begin{aligned} U &= -2s\epsilon^2 y_3^{-2} + \frac{51}{2} y_1^2 s\epsilon^4 y_3^{-4} - 10\epsilon^5 y_3^{-5} - 1140 y_1^4 s\epsilon^6 y_3^{-6} + 2757 y_1^2 \epsilon^7 y_3^{-7} \\ &\quad + \left(-424 + \frac{1723435}{16} y_1^6\right) s\epsilon^8 y_3^{-8} + O(\epsilon^9), \\ V &= -5 y_1 s\epsilon^3 y_3^{-3} + \frac{303}{2} y_1^3 s\epsilon^5 y_3^{-5} - 187 y_1 \epsilon^6 y_3^{-6} - \frac{82065}{8} y_1^5 s\epsilon^7 y_3^{-7} \\ &\quad + 39927 y_1^3 \epsilon^8 y_3^{-8} + O(\epsilon^9). \end{aligned} \tag{88}$$

We remark that for the particular subset of solutions which we are now considering, (81) reduces to

$$\xi = y_1, \qquad \hat{\eta} = y_3, \qquad u = U, \qquad \hat{v} = V. \tag{89}$$

At large distance, $|y_3| \to \infty$, so that $u \to 0$. Therefore these solutions are the asymptotically circular orbits considered in Section 3.1. It is remarkable that we find here again that these particular solutions must be considered first, as a necessary preamble to the study of the general solutions.

4.4. Nice variables

The change of variables (81) is now fully defined. The next step in Kruskal's method (pages 813–814) consists in determining another change of variables, from y_1, y_2, y_3, ν to the "nice variables" z_1, z_2, z_3, ϕ. The nice variables are such that the differential equations take the form

$$\dot{z}_i = \epsilon h_i(z_1, z_2, z_3), \qquad \dot{\phi} = \omega(z_1, z_2, z_3), \tag{90}$$

i.e. their right-hand members do not depend on the angle-like variable ϕ any more. The equations (90a), considered separately, represent then the slow drift along the family of periodic orbits; the fast epicyclic motion has been completely eliminated from these equations.

The nice variables are obtained as functions

$$z_i = Z_i(y_1, y_2, y_3, \nu), \qquad \phi = \Phi(y_1, y_2, y_3, \nu). \tag{91}$$

These functions are series in ϵ; they are obtained recursively by the algorithm described in Kruskal's paper (page 814), to which we refer for details. The computations were done by hand up to order 4, and by computer algebra up to order 7 . The functions defining the change of variables (91) are rather cumbersome and we reproduce them to order 6 only:

$$\begin{aligned} z_1 = y_1 &+ \left(-8sy_2\cos\nu\right)\epsilon^3 y_3^{-3} + \left(\frac{69}{4}sy_2^2\sin 2\nu\right)\epsilon^4 y_3^{-4} \\ &+ \left(204sy_1^2y_2\cos\nu + \frac{207}{4}sy_1y_2^2\cos 2\nu - 63sy_2^3\cos\nu + \frac{85}{3}sy_2^3\cos 3\nu\right)\epsilon^5 y_3^{-5} \\ &+ \left(-204y_2\cos\nu - \frac{15255}{16}sy_1^2y_2^2\sin 2\nu + 225sy_1y_2^3\sin\nu - \frac{935}{6}sy_1y_2^3\sin 3\nu \right. \\ &\left. + \frac{11165}{48}sy_2^4\sin 2\nu - \frac{15475}{384}sy_2^4\sin 4\nu\right)\epsilon^6 y_3^{-6} + O(\epsilon^7), \end{aligned} \tag{92a}$$

$$
\begin{aligned}
z_2 = y_2 &+ \left(-\frac{7}{4} s y_2 \cos 2\nu\right)\epsilon^3 y_3^{-3} \\
&+ \left(\frac{207}{16} s y_1 y_2 \sin 2\nu + \frac{15}{2} s y_2^2 \sin \nu + 2 s y_2^2 \sin 3\nu\right)\epsilon^4 y_3^{-4} \\
&+ \left(\frac{1419}{16} s y_1^2 y_2 \cos 2\nu - \frac{387}{8} s y_1 y_2^2 \cos \nu + \frac{167}{8} s y_1 y_2^2 \cos 3\nu \right. \\
&\qquad \left. + \frac{301}{16} s y_2^3 \cos 2\nu + \frac{125}{64} s y_2^3 \cos 4\nu\right)\epsilon^5 y_3^{-5} \\
&+ \left(-\frac{705}{16} y_2 \cos 2\nu - \frac{49}{64} y_2 \cos 4\nu - \frac{45765}{64} s y_1^3 y_2 \sin 2\nu - \frac{195}{2} s y_1^2 y_2^2 \sin \nu \right. \\
&\qquad - \frac{1615}{8} s y_1^2 y_2^2 \sin 3\nu + \frac{11165}{64} s y_1 y_2^3 \sin 2\nu - \frac{15475}{512} s y_1 y_2^3 \sin 4\nu \\
&\qquad \left. - \frac{3085}{32} s y_2^4 \sin \nu - 35 s y_2^4 \sin 3\nu - 2 s y_2^4 \sin 5\nu\right)\epsilon^6 y_3^{-6} + O(\epsilon^7),
\end{aligned}
\tag{92b}
$$

$$
\begin{aligned}
z_3 = y_3 &+ \left(2 y_2 \sin \nu\right)\epsilon + \left(26 s y_2 \sin \nu\right)\epsilon^4 y_3^{-3} \\
&+ \left(102 s y_1 y_2 \cos \nu + \frac{831}{16} s y_2^2 \cos 2\nu\right)\epsilon^5 y_3^{-4} \\
&+ \left(-1317 s y_1^2 y_2 \sin \nu - \frac{3051}{8} s y_1 y_2^2 \sin 2\nu - 111 s y_2^3 \sin \nu \right. \\
&\qquad \left. - \frac{499}{6} s y_2^3 \sin 3\nu\right)\epsilon^6 y_3^{-5} + O(\epsilon^7),
\end{aligned}
\tag{92c}
$$

$$
\begin{aligned}
\phi = \nu &+ \left(\frac{7}{4} s \sin 2\nu\right)\epsilon^3 y_3^{-3} + \left(\frac{207}{16} s y_1 \cos 2\nu - 15 s y_2 \cos \nu + 2 s y_2 \cos 3\nu\right)\epsilon^4 y_3^{-4} \\
&+ \left(-\frac{1419}{16} s y_1^2 \sin 2\nu + \frac{333}{8} s y_1 y_2 \sin \nu - \frac{167}{8} s y_1 y_2 \sin 3\nu \right. \\
&\qquad \left. + \frac{379}{8} s y_2^2 \sin 2\nu - \frac{125}{64} s y_2^2 \sin 4\nu\right)\epsilon^5 y_3^{-5} \\
&+ \left(\frac{705}{16} \sin 2\nu + \frac{49}{32} \sin 4\nu - \frac{45765}{64} s y_1^3 \cos 2\nu + \frac{6885}{8} s y_1^2 y_2 \cos \nu \right. \\
&\qquad - \frac{1615}{8} s y_1^2 y_2 \cos 3\nu + \frac{7285}{32} s y_1 y_2^2 \cos 2\nu - \frac{15475}{512} s y_1 y_2^2 \cos 4\nu \\
&\qquad \left. - \frac{5095}{32} s y_2^3 \cos \nu + \frac{9485}{96} s y_2^3 \cos 3\nu - 2 s y_2^3 \cos 5\nu\right)\epsilon^6 y_3^{-6} + O(\epsilon^7).
\end{aligned}
\tag{92d}
$$

The differential equations (90) are much simpler. To order 7, they are:

$$\dot{z}_1 = -2s\epsilon^2 z_3^{-2} + \left(\frac{51}{2}z_1^2 - \frac{21}{2}z_2^2\right)s\epsilon^4 z_3^{-4} - 10\epsilon^5 z_3^{-5}$$
$$+ \left(-1140z_1^4 + \frac{1995}{4}z_1^2 z_2^2 - \frac{1485}{32}z_2^4\right)s\epsilon^6 z_3^{-6}$$
$$+ \left(2757z_1^2 - \frac{2853}{4}z_2^2\right)\epsilon^7 z_3^{-7} + O(\epsilon^8),$$
$$\dot{z}_2 = -18z_1 z_2 s\epsilon^4 z_3^{-4} + \left(765z_1^3 z_2 + \frac{495}{4}z_1 z_2^3\right)s\epsilon^6 z_3^{-6} - \frac{78891}{64}z_1 z_2 \epsilon^7 z_3^{-7} + O(\epsilon^8),$$
$$\dot{z}_3 = -\frac{3}{2}z_1\epsilon - \frac{5}{2}z_1 s\epsilon^4 z_3^{-3} + \left(\frac{303}{4}z_1^3 + \frac{867}{2}z_1 z_2^2\right)s\epsilon^6 z_3^{-5} - \frac{187}{2}z_1\epsilon^7 z_3^{-6} + O(\epsilon^8),$$
$$\dot{\phi} = 1 - \frac{7}{2}s\epsilon^3 z_3^{-3} + \left(\frac{399}{4}z_1^2 - \frac{297}{16}z_2^2\right)s\epsilon^5 z_3^{-5} - \frac{449}{8}\epsilon^6 z_3^{-6}$$
$$+ \left(-\frac{109365}{16}z_1^4 + \frac{69705}{32}z_1^2 z_2^2 - \frac{10605}{128}z_2^4\right)s\epsilon^7 z_3^{-7} + O(\epsilon^8). \tag{93}$$

A further simplification of these equations can be achieved by introducing yet another set of variables w_1, w_2, w, defined by

$$w_1 = z_1^2, \qquad w_2 = z_2^2, \qquad w = \frac{s\epsilon}{z_3} \tag{94}$$

and by using w as the independent variable instead of t. Note that w is always positive since $\mathrm{sign}(z_3) = \mathrm{sign}(y_3) = \mathrm{sign}(\eta) = s$. Substituting into (93) and dividing (93a), (93b) and (93d) by (93c), we obtain

$$\frac{dw_1}{dw} = -\frac{8}{3} + \left(34w_1 - 14w_2\right)w^2 - \frac{80}{9}w^3$$
$$+ \left(-1520w_1^2 + 665w_1 w_2 - \frac{495}{8}w_2^2\right)w^4$$
$$+ \left(\frac{10454}{3}w_1 - \frac{5095}{3}w_2\right)w^5 + O(w^6),$$
$$\frac{dw_2}{dw} = -24w_2 w^2 + \left(1020w_1 w_2 + 165w_2^2\right)w^4 - \frac{25657}{16}w_2 w^5 + O(w^6), \tag{95}$$
$$\frac{d(\phi - t)}{dw} = \frac{s}{z_1}\left[-\frac{7}{3}w + \left(\frac{133}{2}w_1 - \frac{99}{8}w_2\right)w^3 - \frac{1207}{36}w^4\right.$$
$$\left. + \left(-\frac{36455}{8}w_1^2 + \frac{23235}{16}w_1 w_2 - \frac{3535}{64}w_2^2\right)w^5\right] + O(w^6).$$

From (93c) we deduce also

$$\frac{dt}{dw} = \frac{s}{z_1}\left[\frac{2}{3}w^{-2} - \frac{10}{9}w + \left(\frac{101}{3}w_1 + \frac{578}{3}w_2\right)w^3 - \frac{1072}{27}w^4\right] + O(w^5). \tag{96}$$

ϵ has disappeared from these equations; the small parameter is now the variable w. From (65) it can be deduced that

$$w = O(h^2). \tag{97}$$

4.5. Series solution

We determine now the boundary conditions for the new variables in the limit $t \to \pm\infty$. In this limit, we have $|y_3| \to \infty$, and therefore $U \to 0$, $V \to 0$, as shown by (88). Comparing (81) with the expression (49) of the asymptotic motion, we find that

$$y_1 \to h, \qquad y_2 \to k,$$

$$y_3 \simeq \epsilon\left\{-\frac{3}{2}h(t-\tau) - \frac{4}{3}sh^{-2}\ln\left[-\frac{3}{2}sh(t-\tau)\right]\right\} - 2k\sin(t-\varphi), \qquad \nu \simeq t-\varphi. \tag{98}$$

From (91) and (92) we have

$$z_1 \to h, \qquad z_2 \to k,$$

$$z_3 \simeq \epsilon\left\{-\frac{3}{2}h(t-\tau) - \frac{4}{3}sh^{-2}\ln\left[-\frac{3}{2}sh(t-\tau)\right]\right\}, \qquad \phi \simeq t-\varphi. \tag{99}$$

From (94):

$$w_1 \to h^2, \qquad w_2 \to k^2, \qquad w \to 0. \tag{100}$$

Finally, by inverting (99c) and using (94c) we obtain

$$t \simeq \tau - \frac{2}{3}h^{-1}sw^{-1} + \frac{8}{9}h^{-3}s\ln w. \tag{101}$$

The solution of the system of equations (95) and (96) can be written as series expansions in w; successive terms of these series are found recursively, starting with the lowest order expressions (100a), (100b), (99d) and (101). In fact the two equations (95a) and (95b), which describe the slow motion, can be solved separately since the right-hand members are functions of w_1, w_2, w only. The result is

$$\begin{aligned} w_1 &= h^2 - \frac{8}{3}w + \left(\frac{34}{3}h^2 - \frac{14}{3}k^2\right)w^3 - \frac{224}{9}w^4 \\ &\quad + \left(-304h^4 + 133h^2k^2 - \frac{99}{8}k^4\right)w^5 + \left(\frac{17965}{9}h^2 - \frac{10555}{18}k^2\right)w^6 + O(w^7), \\ w_2 &= k^2 - 8k^2w^3 + \left(204h^2k^2 + 33k^4\right)w^5 - \frac{22035}{32}k^2w^6 + O(w^7). \end{aligned} \tag{102}$$

We substitute then (102) into (95c) and (96) and integrate in w, taking into account the boundary conditions (99d) and (101) for $w \to 0$. We obtain

$$\phi = t-\varphi-\frac{7}{6}h^{-1}sw^2-\frac{28}{27}h^{-3}sw^3+\left(\frac{133}{8}h-\frac{14}{9}h^{-5}-\frac{99}{32}k^2h^{-1}\right)sw^4+O(w^5) \quad (103)$$

and

$$\begin{aligned} s(t-\tau) = &-\frac{2}{3}h^{-1}w^{-1}+\frac{8}{9}h^{-3}\ln w+\frac{16}{9}h^{-5}w \\ &+\left(-\frac{22}{9}h^{-1}+\frac{160}{81}h^{-7}+\frac{7}{9}k^2h^{-3}\right)w^2 \\ &+\left(-\frac{224}{81}h^{-3}+\frac{2240}{729}h^{-9}+\frac{56}{27}k^2h^{-5}\right)w^3+O(w^4). \end{aligned} \quad (104)$$

The four equations (102), (103) and (104) give the full solution, with w as the independent variable and w_1, w_2, ϕ, t as the dependent variables. This solution depends on the four integration constants h, k, φ, τ.

We have thus obtained the general solution as a formal series. Kruskal (1962) has proved that the series derived by his method are asymptotically correct solutions of the original problem.

We can now go back to the original variables; this involves some heavy but straigthforward algebra, which we will not describe in detail. First, z_1 and z_2 are computed as expansions in w from (94) and (102). Next, the equations (92) are inverted so as to give y_1, y_2, y_3, ν as functions of z_1, z_2, z_3, ϕ; this can be done recursively, using the fact that ϵ is a small parameter. Finally, using (88), (81), (74), and (68), we obtain expressions of ξ, η, u, v as functions of w. The time t is itself given as a function of w by (104). We do not quote these expressions for reasons which will become immediately apparent.

These expressions can be compared with those given in Sec. 3.2. The relation between the variable w used here and the variable η_c used in Sec. 3 is found by comparing (104) with (46):

$$w = s\eta_c^{-1}-\frac{7}{6}k^2h^{-2}\eta_c^{-4}-\frac{14}{9}k^2h^{-4}s\eta_c^{-5}+O(\eta_c^{-6}). \quad (105)$$

Substituting this, and using also (51) to substitute θ for $t-\varphi$ in (103), one finds that the present expressions are identical with the expressions (61) of Sec. 3.2. The same developments have thus been obtained by two quite different routes. This provides a welcome check of the correctness of the computations. The fact that the same developments are found in the two different situations considered in Section 3 and in the present Section can be explained: the condition for the validity of these developments is simply that $|\eta|$ be large. Thus, for the orbits with an arbitrary impact parameter h considered in Section 3, the expansions (61) describe the asymptotic behaviour of the orbit; for the orbits with a small impact parameter considered in the present Section, they describe the whole orbit since $|\eta|$ remains large at all times.

4.6. Properties

The equations (102) which describe the slow motion have some interesting properties:

1. These equations can be solved for h^2 and k^2, again as series in w:

$$h^2 = w_1 + \frac{8}{3}w + \left(-\frac{34}{3}w_1 + \frac{14}{3}w_2\right)w^3 - \frac{16}{3}w^4 + \left(304w_1^2 - 133w_1w_2 + \frac{99}{8}w_2^2\right)w^5 + \left(-\frac{739}{3}w_1 + \frac{1297}{6}w_2\right)w^6 + O(w^7),$$
$$k^2 = w_2 + 8w_2w^3 - \left(204w_1w_2 + 33w_2^2\right)w^5 + \frac{6675}{32}w_2w^6 + O(w^7). \tag{106}$$

These expressions are therefore two constants of the motion. However, a particular combination of h^2 and k^2 corresponds to an already known constant, namely the Jacobi integral, as shown by (50). Therefore we have uncovered essentially one new constant, k^2 . This quantity is the *adiabatic invariant* which characterizes the motion for small impact parameters.

2. An immediate consequence of the existence of this invariant is that the values of h^2 and k^2 characterizing the asymptotic motion must be the same for $t \to -\infty$ and for $t \to +\infty$. This explains the near-perfect reversal of $\Delta a = a_3 - a_2$ found numerically by Dermott et al. (1980) for small Δa.

3. If we keep only the first term in the right-hand side of (102b) and use (97) and (92b), we obtain

$$y_2 = k\left[1 + O(h^6)\right]. \tag{107}$$

Thus y_2 is constant to a very good approximation. This accounts for the observed fact (Petit, 1985) that the fast epicyclic motion has a nearly constant amplitude along the orbit.

4. The case of asymptotically circular orbits is of particular interest. Then $k = 0$, and (102b) shows that w_2 vanishes identically. Therefore $y_2 = 0$, and Eqs. (92a) to (92c) reduce to

$$z_1 = y_1, \qquad z_2 = 0, \qquad z_3 = y_3. \tag{108}$$

Going back to the original variables, we obtain

$$\begin{aligned}
\xi^2 = y_1^2 = w_1 &= h^2 - \frac{8}{3}w + \frac{34}{3}h^2 w^3 - \frac{224}{9}w^4 - 304h^4 w^5 + \frac{17965}{9}h^2 w^6 + O(w^7),\\
\eta &= \frac{s}{w},\\
su &= -2w^2 + \frac{51}{2}h^2 w^4 - 78w^5 - 1140h^4 w^6 + 9126h^2 w^7\\
&\quad + \left(\frac{1723435}{16}h^6 - \frac{49552}{3}\right)w^8 + O(w^9),\\
v &= \frac{V - 3\xi}{2}\\
&= \xi\left[-\frac{3}{2} - \frac{5}{2}w^3 + \frac{303}{4}h^2 w^5 - \frac{591}{2}w^6 - \frac{82065}{16}h^4 w^7 + 48177h^2 w^8 + O(w^9)\right].
\end{aligned} \tag{109}$$

Here again, it can be verified that the equations (109) are identical with the solution (45) derived in Section 3.

5. If we keep only the first two terms in (109a), the first term in (109c), and the first term in (109d), we obtain

$$\xi^2 = \left(h^2 - \frac{8s}{3\eta}\right)\left[1 + O(h^6)\right], \qquad su = -\frac{2}{\eta^2}\left[1 + O(h^6)\right], \qquad v = -\frac{3}{2}\xi\left[1 + O(h^6)\right]. \tag{110}$$

We recover the simple horseshoe approximation expressed by (62), (63) and (66) (for the case $s = 1$). Here again, as in (107), the relative error is of order h^6. Thus the horseshoe approximation is in fact very good for small impact parameters.

6. (109a) shows that ξ^2 decreases from its asymptotic value h^2 as w increases from 0. It vanishes for a particular value $w = w_{cr}$; from the first two terms of (109a), we obtain $w_{cr} \simeq 3h^2/8$, in agreement with (65). A refined value is obtained by solving (109a) for $\xi = 0$; the result is a remarkably simple series:

$$w_{cr} = \left(\frac{3}{8}h^2\right) + 2\left(\frac{3}{8}h^2\right)^4 + 36\left(\frac{3}{8}h^2\right)^7 + 1514\left(\frac{3}{8}h^2\right)^{10} + O(h^{26}). \tag{111}$$

(Note: this was derived from an extension of the series (109a) to an order higher than shown here). This value corresponds to the crossing point of the horseshoe orbit with the η axis. The corresponding value η_{cr} of η is immediately obtained from (109b). At that point, $\xi = 0$, and also $v = 0$ as shown by (109d). It follows then from the classical symmetries of Hill's problem that the whole orbit is symmetrical with respect to the η axis. Thus, the existence of an adiabatic invariant explains completely the near-perfect symmetry of the orbits for small h observed numerically by Dermott and Murray (1981).

The symmetry of the orbit is also directly apparent in (109): for a given value of η, i.e. a given value of w, (109a) gives two opposite values of ξ, corresponding to two symmetrical points of the orbit. Moreover, (109c) and (109d) show that these points have the same value of u and opposite values of v.

The expressions (109) are much simpler than the expressions (45) obtained in Section 3. The series (45) for ξ and v involve powers of h^{-1}; more precisely, they contain terms of the form $h^{1-2i}\eta^{-i}$ for all positive values of i. Therefore they are meaningful only for $|\eta| \gg h^{-2}$, i.e. for $|\eta|$ large compared to its minimal value $|\eta_{cr}|$. On the contrary, the series (109) contain only positive powers of h; thus, for h small, these series are applicable to the whole orbit.

This difference is easily explained. Because of the symmetry, ξ and v are odd functions of $(t-t_{cr})$, while η and u are even functions of $(t-t_{cr})$. Therefore, near the crossing point, we have $\xi \propto \sqrt{\eta - \eta_{cr}}$: $\xi(\eta)$ is a non-analytic function, and $d\xi/d\eta$ becomes infinite at the crossing point. Clearly, therefore, the series (45a) must fail before this point is reached. The same is true for the series (45c) describing v. On the other hand, ξ^2 and v/ξ are even functions of $t - t_{cr}$, and are analytical functions of η even in the vicinity of the crossing point. The corresponding series are therefore meaningful for the whole orbit.

4.7. Breakdown of the Adiabatic Invariant

The existence of an adiabatic invariant implies that for small h, the final eccentricity k' is very nearly equal to the initial eccentricity k. Here we verify this by obtaining the asymptotic form of k' for $h \to 0$ in the case of initially circular orbits, i.e. $k = 0$. This computation is due in part to Balbus and Tremaine (1985).

We consider a horseshoe orbit, described in first approximation by (64). We choose the origin of time so that $t_{cr} = 0$. The motion can also be represented by (28) and (29), with the D_i no more constant but slowly varying. Their variation is found by substituting (28) and (29) in (24):

$$\dot{D}_1 = \frac{\xi \sin t + 2\eta \cos t}{\rho^3}, \qquad \dot{D}_2 = \frac{-\xi \cos t + 2\eta \sin t}{\rho^3},$$

$$\dot{D}_3 = -\frac{2\eta}{\rho^3}, \qquad \dot{D}_4 = \frac{2\xi - 3\eta t}{\rho^3}. \tag{112}$$

Only the first two equations will be of interest here. Since η is positive and much larger than $|\xi|$, we can take $\rho \simeq \eta$. We integrate over time to obtain the variations of D_1 and D_2:

$$\Delta D_1 = \int_{-\infty}^{+\infty} \frac{\xi \sin t + 2\eta \cos t}{\eta^3}\, dt, \qquad \Delta D_2 = \int_{-\infty}^{+\infty} \frac{-\xi \cos t + 2\eta \sin t}{\eta^3}\, dt. \tag{113}$$

We substitute the horseshoe solution (64) in the right-hand sides of (113). ΔD_2 vanishes because of the symmetry of the orbit. The first term in (113a) can be integrated by parts after substitution of $\xi = -2\dot{\eta}/3$ from (62), giving

$$\Delta D_1 = \frac{5}{3} \int_{-\infty}^{+\infty} \frac{\cos t}{\eta^2}\, dt. \tag{114}$$

At $t = -\infty$, D_1 and D_2 are zero since we consider an initially circular orbit. Therefore at $t = +\infty$, we have $D_1 = \Delta D_1$, $D_2 = 0$, and the final eccentricity is

$$k' = \Delta D_1 = \frac{5}{6}h \int_{-\infty}^{+\infty} \frac{d\lambda}{\cosh^2 \lambda} \cos\left[p(\lambda + \frac{1}{2}\sinh 2\lambda)\right], \tag{115}$$

with

$$p = \frac{16}{9h^3}. \tag{116}$$

This can be written

$$k' = \frac{5}{6}h \int_{-\infty}^{+\infty} \frac{d\lambda}{\cosh^2 \lambda} \exp\left[ip(\lambda + \frac{1}{2}\sinh 2\lambda)\right]. \tag{117}$$

To compute this integral we look for a contour in the complex plane $\lambda = \lambda_r + i\lambda_i$ on which the argument of the exponential is real. Such a contour BD is shown on Figure 3; its equation is

$$\lambda_i = \frac{1}{2}\arccos\left(-\frac{2\lambda_r}{\sinh 2\lambda_r}\right). \tag{118}$$

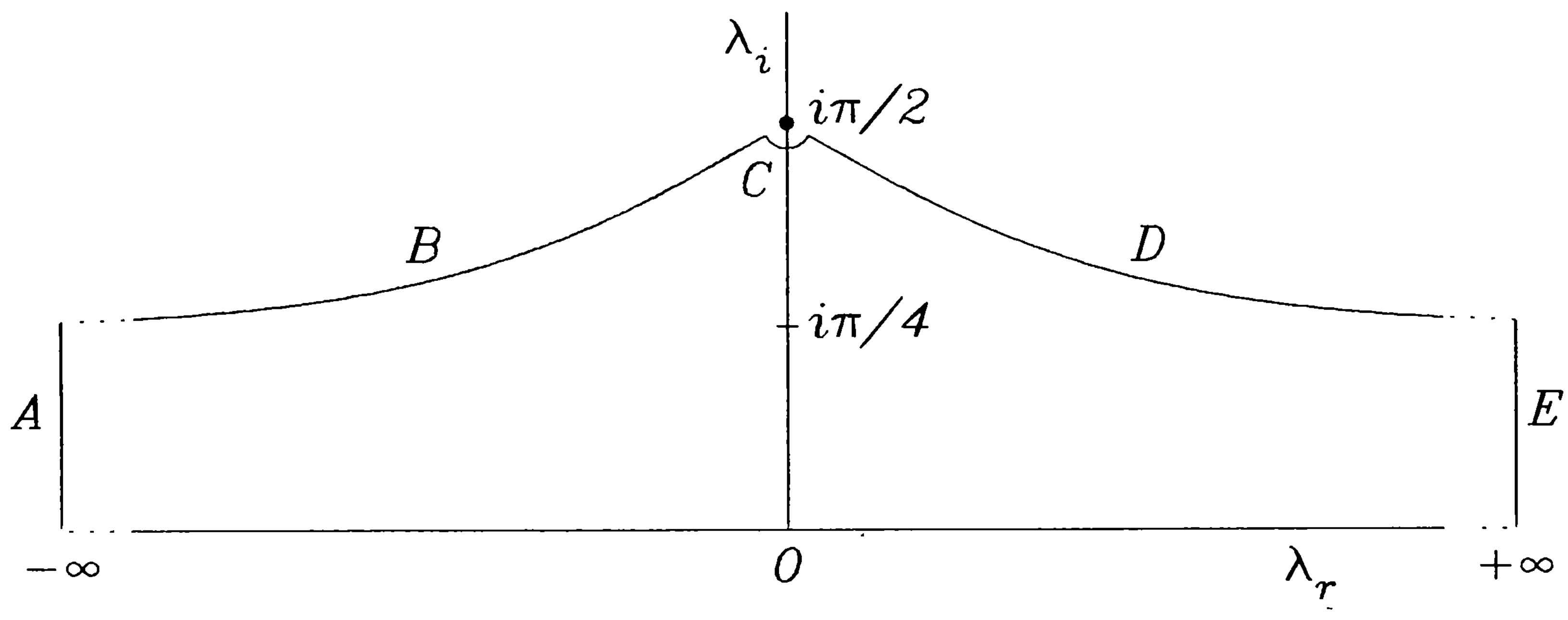

Fig. 3. Contour of integration for (117).

$\cosh\lambda$ vanishes at $\lambda = i\pi/2$; we avoid this singularity by a small detour C. We replace the integration along the real axis by an integration along $ABCDE$ (Figure 3). The contributions from A and E are zero because the first factor in (117) is vanishingly small. For BCD we use an integration by parts:

$$\begin{aligned} k' = {} & \frac{5}{6}h \tanh\lambda \exp\left[ip(\lambda + \frac{1}{2}\sinh 2\lambda)\right]\Big|_{BCD} \\ & - \frac{5}{6}iph \int_{BCD} \sinh 2\lambda \exp\left[ip(\lambda + \frac{1}{2}\sinh 2\lambda)\right] d\lambda. \end{aligned} \tag{119}$$

The first term vanishes because the argument of the exponential tends to $-\infty$ at the two ends of BCD. The second term has no singularity any more at $\lambda = i\pi/2$, so we can eliminate the detour C. Writing

$$\lambda = \frac{i\pi}{2} + z \tag{120}$$

we obtain

$$k' = \frac{5}{6} iph \int_{BD} \sinh 2z \exp\Big[p(-\frac{\pi}{2} + iz - \frac{i}{2}\sinh 2z)\Big]\, dz. \tag{121}$$

For p large, we expand around $z = 0$ and keep only the leading terms:

$$k' \simeq \frac{5}{6} iph \int_{BD} 2z \exp\Big[p(-\frac{\pi}{2} - \frac{2i}{3} z^3)\Big]\, dz. \tag{122}$$

The equation of the contour (118) near $z = 0$ is

$$z \simeq \lambda_r (1 \pm \frac{i}{\sqrt{3}}), \tag{123}$$

with a sign + on B and − on D. The integral (122) is then easily evaluated, and we obtain

$$k' = 2^{2/3}\, 3^{-3/2}\, 5\, \Gamma(\frac{2}{3}) \exp\left(-\frac{8\pi}{9h^3}\right). \tag{124}$$

This expression agrees well with numerical results (Petit and Hénon, 1985). It shows that k' decreases exponentially fast for $h \to 0$; this behaviour is typical of adiabatic invariants.

5. Large impact parameter

We consider the case where the coordinate ξ remains large at all times. .Then the orbit of each particle is only slightly deflected by the other, and a perturbation theory can be used.

A simple "impulse approximation", which treats the problem as a two-body encounter, has sometimes been used in the literature. Unfortunately it gives incorrect results (Hénon, 1984). The correct theory was first given by Goldreich and Tremaine (1980), for the case where one of the satellites has a negligible mass compared to the other (ν small). A simpler derivation is sketched in Goldreich and Tremaine (1982, p. 276). Here we rederive the result in Hill's coordinates; this gives an even simpler treatment, and also it automatically generalizes the result to the case of an arbitrary mass ratio.

The terms ξ/ρ^3, η/ρ^3 in (24) will be considered as small perturbations. In the zero-order approximation, i.e. when these terms are neglected altogether, the solution has the form (28), (29). Since ξ remains large for all t, D_3 must be large. We assume that the reduced eccentricity is moderate, so that D_1 and D_2 are small compared to D_3. We can then retain only the D_3 terms in the present approximation. (The D_4 term can be eliminated by a change in the origin of time). The relative motion has the form (36), with $\tau = 0$.

Now we insert this solution into the perturbative terms ξ/ρ^3, η/ρ^3 and we solve (24) to obtain the first-order approximation. To do this it is convenient to make a change of variables and to replace ξ, η, u, v by D_1, D_2, D_3, D_4, the correspondence being defined by (28) and (29). The D_i are no more constants; but they will vary slowly. Their variation is here again given by the equations (112), where the zero-order solution (36) should be substituted in the right-hand side members. We compute the net effect of the encounter by integrating from $-\infty$ to $+\infty$:

$$\begin{aligned}
\Delta D_1 &= D_1(+\infty) - D_1(-\infty) = h^{-2} \int_{-\infty}^{+\infty} (\sin t - 3t\cos t)\left(1 + \frac{9}{4}t^2\right)^{-3/2} dt = 0, \\
\Delta D_2 &= h^{-2} \int_{-\infty}^{+\infty} (-\cos t - 3t\sin t)\left(1 + \frac{9}{4}t^2\right)^{-3/2} dt \\
&= -\frac{8}{9}h^{-2}\left[2K_0\left(\frac{2}{3}\right) + K_1\left(\frac{2}{3}\right)\right], \\
\Delta D_3 &= h^{-2} \int_{-\infty}^{+\infty} 3t\left(1 + \frac{9}{4}t^2\right)^{-3/2} dt = 0, \\
\Delta D_4 &= h^{-2} \int_{-\infty}^{+\infty} 2\left(1 + \frac{9}{4}t^2\right)^{-1/2} dt = \infty.
\end{aligned} \tag{125}$$

The divergence of ΔD_4 is again a consequence of the logarithmic singularity mentioned in Section 3; fortunately this quantity will not be needed. The computation of ΔD_2 results from an integration by parts, and the use of Abramowitz and Stegun (1965 , formula 9.6.25); K_0 and K_1 are modified Bessel functions.

Equations (125) give a simple result in the special case where the orbit before the encounter is exactly circular. We have then

$$D_1(-\infty) = 0, \qquad D_2(-\infty) = 0, \qquad D_3(-\infty) = h; \tag{126}$$

therefore:

$$D_1(+\infty) = 0, \qquad D_2(+\infty) = -\frac{8}{9}h^{-2}\left[2K_0\left(\frac{2}{3}\right) + K_1\left(\frac{2}{3}\right)\right], \qquad D_3(+\infty) = h. \tag{127}$$

This gives the reduced eccentricity of the orbit after the encounter. An equation which contains (127b) as a particular case was derived by Julian and Toomre (1966) in the context of galactic dynamics.

In the present first-order approximation, the impact parameter D_3 does not change, so that it seems at first view necessary to go to second order to find its variation (Goldreich and Tremaine, 1980). However, the result can be obtained much more simply, using only the first-order solution, thanks to the conservation of the Jacobi constant (Goldreich and Tremaine, 1982). Using the form (30) of this constant, we have

$$\frac{3}{2}D_3\Delta D_3 = \Delta(D_1^2 + D_2^2), \tag{128}$$

and therefore, using (126) and (127):

$$\Delta D_3 = \frac{128}{243}h^{-5}\left[2K_0\left(\frac{2}{3}\right) + K_1\left(\frac{2}{3}\right)\right]^2 = 3.34379\ldots h^{-5}. \tag{129}$$

Numerical computations by Petit and Hénon (1985) give excellent agreement with this formula: at $h = 10$, the error on ΔD_3 is already less than 1 percent.

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