A NEW REGULARIZATION OF THE RESTRICTED THREE-BODY PROBLEM AND AN APPLICATION

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Abstract. A new regularizing transformation for the three-dimensional restricted three-body problem is constructed. It is explicitly derived and is equivalent to a simple rational map. Geometrically it is equivalent to a rotation of the 3-sphere. Unlike the KS map it is dimension preserving and is valid in n dimensions. This regularizing map is applied to the restricted problem in order to prove the existence of a family of periodic orbits which continue from a family of collision orbits.

1. Introduction

A. THE EQUATIONS OF MOTION

We write the equations of motion for the circular three-dimensional restricted three-body problem in a rotating coordinate system $q = (q_1, q_2, q_3)$ of rotational frequency $\omega = 1$ in the following Hamiltonian form:

$$\dot{q} = H_p, \qquad \dot{p} = -H_q; \qquad \dot{} \equiv \mathrm{d}/\mathrm{d}t, \tag{1}$$

where $p = (p_1, p_2, p_3)$, and where

$$H = \tfrac{1}{2}|p|^2 - |q|^{-1} + \omega(q_2 p_1 - p_2 q_1) + \mu G(p, q), \tag{2}$$

and

$$G(p, q) = |q|^{-1} - \Delta^{-1} - \omega p_1, \qquad \Delta = |q - e_2|,$$

$|q|^2 = \sum_{k=1}^{3} q_k^2$, $e_2 = (0, 1, 0)$. In these coordinates the larger of the two primaries, m_1, of mass $1 - \mu$, $0 < \mu \ll 1$, is at the origin while the smaller primary, m_2, of mass μ is at the position e_2 (see Figure 1).

We carry the parameter ω along, but keep in mind that it is normalized to 1 for our problem. It will be assumed that p, q belong to $\mathbb{R}^3$ throughout, unless otherwise stated.

B. EXISTENCE RESULTS

One of our aims will be to prove the existence of a family of periodic orbits for (1) for $\mu \neq 0$ sufficiently small which are obtained by continuation from a periodic collision orbit $\phi^*(t) = (p^*(t), q^*(t))$ for $\mu = 0$. The collision orbit $\phi^*(t)$ for $\mu = 0$ moves on the positive q_3-axis (i.e. $q_1(t) = q_2(t) = 0$) and, of course, corresponds to negative energy $H|_{\mu=0} = -h < 0$. After appropriate regularization this solution can be

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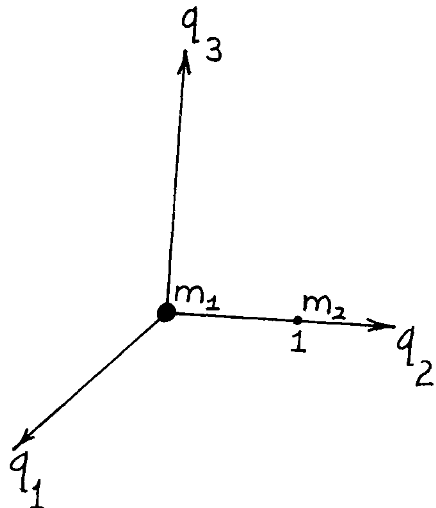

Fig. 1. The rotating coordinate system.

continued through the collision and then gives rise to a periodic orbit with period $2\pi(2h)^{-(3/2)}$. For each negative value of the energy there are exactly two such orbits, namely $\phi^*(t)$ and the orbit $-\phi^*(t)$ obtained by reflection in the q_1q_2-plane. These two collision orbits share with the circular orbits in the q_1q_2-plane about the origin the property that they are carried into themselves under rotation about the q_3-axis which is essential for their continuation for small $\mu \neq 0$.

We will prove:

THEOREM A: On each fixed energy surface $H = -h < 0$ there exists a unique† periodic orbit $\phi(t, \mu)$ for sufficiently small μ for which $\phi(t, 0) = \phi^*(t)$ and whose period $T(\mu)$ tends continuously to the period $T(0) = 2\pi/\omega^*$ of $\phi^*(t)$, provided

$$\frac{1}{\omega^*} \neq j,$$

where $j = 1, 2, 3, \ldots$.††

Moreover $\phi(t, \mu)$ is symmetric with respect to reflection in the q_2q_3-plane in q-space.

Theorem A was proven already by Guillaume (see reference [9]). He used the Kustaanheimo-Stiefel transformation which requires increasing the dimension of the phase space; this leads to additional unpleasant degeneration. We will instead derive a new regularization which avoids such degenerations and consequently is a simpler approach. Moreover, our approach is more advantageous for numerical

† Of course $\phi(t + c, \mu)$ is identified with $\phi(t, \mu)$, $c \in \mathbb{R}$

†† If we ignore the normalizations made, this condition reads

$$\frac{\omega}{\omega^*} \neq \pm j.$$

work since we remain in 6-dimensional phase space and avoid the 8-dimensional extension of the KS transformation. In fact, these orbits are numerically studied in reference [2] and exhibit an unexpected behavior. Also, see reference [3]. In Section 5 we will compare our map to the KS transformation.

C. REGULARIZATION

It is clear from Equation (2) that the system of differential equations given by (1) is singular at $q = 0$. Also, the energy manifold given by

$$\Sigma = \{(p, q) | H(p, q) = -h < 0\}$$

has a singularity for $q = 0$. Indeed from (2) one derives that $|q| \to 0$ implies $|p| \to \infty$. This is easily seen if one writes (2) in the form

$$H = \tfrac{1}{2}(p_1 + (q_2 - \mu))^2 + \tfrac{1}{2}(p_2 - q_1)^2 + \tfrac{1}{2}[p_3^2 + q_1^2 - (q_2 - \mu)^2] - \\ - (1 - \mu)|q|^{-1} - \mu\Delta^{-1}.$$

Thus, for $(p, q) \in \Sigma$, letting $|q| \to 0$ forces $|p| \to \infty$.

Since $\phi^*(t)$ passes through this singularity on Σ our first goal will be to remove this singularity by an appropriate regularization. We will construct such a regularization in Section 2. It will be derived by making use of a result of Moser (see Section 2 or reference [1]) which says that on the energy manifold for the Kepler problem† for negative energy the Kepler flow is equivalent to the geodesic flow of $S^3 : \sum_{k=0}^{3} \xi_k^2 = 1$, with the north pole, $\xi_0 = 1$, corresponding to the collision, after a change of the independent variable. The equivalence is established by making use of the stereographic projection from S^3 to the momentum space $p = (p_1, p_2, p_3) \in \mathbb{R}^3$. Under the stereographic projection, points with $|p| \to \infty$ are mapped into points approaching the north pole, and the regularization is accomplished by extending the geodesic flow on $\mathring{S}^3 = \{\xi \in S^3, \xi_0 \neq 1\}$ to the north pole. Since this flow is obviously regular at the north pole regularization of the Kepler problem is achieved by simply restoring the north pole and extending the flow across it. Moreover, we see that on the 5-dimensional energy surface $\Sigma^0 = \{H^0(p, q) = -h < 0\}$ the singular locus corresponds to the two sphere given by the tangent vectors at the north pole of fixed length.

Now, the above procedure refers to regularization of the Kepler problem on the energy surface Σ^0. To obtain a regularization of the restricted problem we will proceed somewhat differently, however, and construct a canonical transformation of the whole phase space (p, q) of the Kepler problem into itself which maps the singular locus into a compact manifold which will also turn out to be the 2-sphere. This will be accomplished by the following consideration: Since the geodesic flow on S^3 is invariant under the rotation group $O(4)$ we choose a fixed rotation R taking the north pole to some different point ξ. Subjecting this map R to the stereographic projection gives rise to a rational transformation of the p-space taking the point at infinity to a

† The Kepler problem in inertial coordinates is defined by (1), (2), where $\omega = \mu = 0$. We call the corresponding Hamiltonian H^0.

finite one. Subsequently we will extend the mapping so obtained to a canonical one of the phase space (p, q) which will serve for our regularization of the restricted problem as will be seen in Sections 2, 3, and 4. In particular, see Propositions 1, 2 in Section 2, and Summary A in Section 3, and also Theorem B in Section 4.

We illustrate this consideration with the reflection

$$\begin{aligned} &\xi_0 \to -\xi_0 \\ &\xi_k \to \xi_k, \qquad k = 1, 2, 3, \end{aligned} \tag{3}$$

in $S^3: \sum_{k=0}^{3} \xi_k^2 = 1$ so that the north pole gets mapped into the south pole and conversely. If we denote by ρ the canonical extension to (3) and by ψ the extension of the stereographic projection mapping the tangent bundle $T\mathring{S}^3 = \{(\xi, \eta) \mid \xi \in \mathring{S}^3, \sum_{k=0}^{3} \xi_k \eta_k = 0\}$ to the phase space $(p, q) \in \mathbb{R}^6$, satisfying $\sum_{k=1}^{3} q_k \mathrm{d}p_k = -\sum_{k=0}^{3} \eta_k \mathrm{d}\xi_k$, then the desired transformation of $\mathbb{R}^6 \to \mathbb{R}^6$ is given by $\psi \circ \rho \circ \psi^{-1}$, or more explicitly by

$$p \to \frac{p}{|p|^2}, \qquad q \to |p|^2 q - 2(p, q)p, \tag{4}$$

where $p = (p_1, p_2, p_3)$, $q = (q_1, q_2, q_3)$ and $(p, q) = \sum_{k=1}^{3} p_k q_k$ (see reference [1]). The canonical map given by (4) gives rise to a regularization of the Kepler problem which was used by Sundman: It is simply a canonical extension of the inversion with respect to the sphere. Clearly, (4) takes $p = \infty \to p = 0$, however, since the south pole, corresponding to $p = 0$, is interchanged with the north pole, the previously regular state $p = 0$ becomes singular. In our case this is not desirable because on $\phi^*(t)$ the value $p = 0$ is taken on when m_3 reaches its maximum point. A way to remedy this situation is, e.g. to rotate S^3 by 90° so that instead of (3) we have (see Figure 2)

$$\begin{aligned} &\xi_0 \to -\xi_1 = \tilde{\xi}_0 \\ &\xi_1 \to \xi_0 = \tilde{\xi}_1 \\ &\xi_k \to \xi_k = \tilde{\xi}_k, \qquad k = 2, 3. \end{aligned} \tag{5}$$

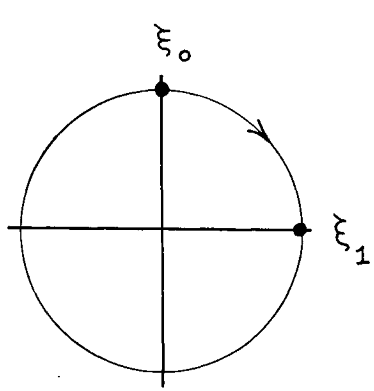

Fig. 2.

We calculate the desired canonical transformation as above by $\psi \circ \rho \circ \psi^{-1} : \mathbb{R}^6 \to \mathbb{R}^6$ where ρ now represents the canonical extension of (5). This is done in detail in Section 2, and the relevant map in momentum space turns out to be

$$p_k \to \tilde{p}_k = \begin{cases} \dfrac{|p|^2 - 1}{|p + e_1|^2} & k = 1 \\ \\ \dfrac{2p_k}{|p + e_1|^2} & k = 2, 3 \end{cases}, \tag{6}$$

where $e_1 = (1, 0, 0)$. Thus, unlike the Sundman map, (6) maps $p = \infty \to \tilde{p} = -e_1$ and $p = 0 \to \tilde{p} = e_1$. Since the map of the momentum in (4) and Map (6) correspond to an inversion and rotation of S^3 respectively, then they are Möbius maps preserving the family of lines and circles. The map of momentum in (4) and Map (6) become respectively, in the two-dimensional complex notation,

$$p \to \frac{1}{p}, \qquad p \to \frac{p - 1}{p + 1}.$$

It will be proven in Section 3 that these maps can be described by the same expressions $1/p, (p - 1)/(p + 1)$ in the n-dimensional case by using a Jordan algebra which will offer a generalization of the complex notation. In particular, see Lemma 1.

It is important to note that (6), together with the corresponding map $q \to \tilde{q}$ as given by (14), represents a *local* regularization. This is easily seen as follows:

ϕ^*, in momentum space, is the p_3-axis. Our map (6) takes this line into the unit circle $|\tilde{p}| = 1$, and therefore ϕ^* in the new coordinates $(\tilde{p}, \tilde{q})$ will be regular. However, according to (6), the previously regular value $p = -e_1$ gets mapped into $\tilde{p} = \infty$; but this presents no difficulty since $p = -e_1$ is never taken on by ϕ^*. Thus this singular value is avoided on ϕ^* and evidently $p = -e_1$ will not be taken on for those orbits sufficiently close to ϕ^*. Therefore (6) represents a local regularization. In fact. (6), (14) regularizes everywhere except at those points where $p = -e_1$. Hence any collision orbit where $p = -e_1$ is not taken on can be regularized by (6), (14). If $p = -e_1$ is taken on for a collision orbit then a suitable regularizing map can be constructed by using a different rotation of S^3.

We finally remark that our map applied to the nonsingular circular orbit given in momentum coordinates by $|p| = 1$ gets mapped into the $\tilde{p}_3$-axis which is singular. In this context it is emphasized that our map is of main interest in its application to singular or nearly singular orbits.

2. The Regularizing Map

In this section we derive the regularizing map which we will use. It is given by formula (11) below.

To derive this map we recall from Section 1C that it is given by the map

$\psi \circ \rho \circ \psi^{-1}: \mathbb{R}^6 \rightarrow \mathbb{R}^6$, where ψ is the extension of the stereographic projection defined in Section 4C, mapping $T\mathring{S}^3$ to the phase space $(p, q) \in \mathbb{R}^6$, and where ρ is the canonical extension to (5). More explicitly the stereographic projection gives rise to the following map, ψ,

$$p_k = \frac{\xi_k}{1 - \xi_0}, \quad q_k = \eta_k(\xi_0 - 1) - \xi_k \eta_0 \tag{7}$$

of the tangent bundle $T\mathring{S}^3 = \{\xi, \eta \,||\, \xi| = 1, (\xi, \eta) = 0, \xi_0 \neq 1\}$ onto $\mathbb{R}^6$ such that

$$\sum_{k=0}^{3} \eta_k \mathrm{d}\xi_k = -\sum_{k=1}^{3} q_k \mathrm{d}p_k,$$

which we write more compactly as

$$\eta \mathrm{d}\xi = -q \mathrm{d}p \tag{8}$$

(see reference [1]). The map ρ is constructed by canonically extending (5) with the map

$$\begin{aligned} \eta_0 &\rightarrow -\eta_1 = \tilde{\eta}_0 \\ \eta_1 &\rightarrow \eta_0 = \tilde{\eta}_1 \\ \eta_k &\rightarrow \eta_k = \tilde{\eta}_k, \qquad k = 2, 3. \end{aligned} \tag{9}$$

Clearly,

$$\tilde{\eta}\, \mathrm{d}\tilde{\xi} = \eta\, \mathrm{d}\xi. \tag{10}$$

One notes that with the matrix

$$E = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

we can write (5) and (9), i.e. ρ, as

$$\tilde{\xi} = E\xi, \qquad \tilde{\eta} = E\eta$$

respectively.

PROPOSITION 1. *The map* $\tilde{\xi} = E\eta, \tilde{\eta} = E\eta$ *gives rise to the map* $\tilde{p} = f(p), \tilde{q} = q(p, q)$ *on* $\mathbb{R}^6$ *given by*

$$\tilde{p}_k = f_k(p) = \begin{cases} \dfrac{|p|^2 - 1}{|p + c_1|^2} & k = 1 \\[2ex] \dfrac{2p_k}{|p + e_1|^2} & k = 2, 3 \end{cases} \tag{11}$$

$$\tilde{q}_k = q_k(p, q) = \begin{cases} \dfrac{1 - |p|^2}{2} q_1 + (p, q)(p_1 + 1) & k = 1 \\[2ex] \dfrac{|p|^2 + 1}{2} q_k + p_1 q_k - p_k q_1 - (p, q) p_k, & k = 2, 3. \end{cases}$$

Proof. Clearly we must compute $\psi \circ \rho \circ \psi^{-1}$, where ψ^{-1} is given by

$$\xi_0 = \frac{|p|^2 - 1}{|p|^2 + 1}, \quad \xi_k = \frac{2p_k}{|p|^2 + 1} \quad k = 1, 2, 3 \tag{12}$$

and

$$\eta_0 = -(p, q), \quad \eta_k = \frac{|p|^2 + 1}{2} q_k + (p, q) p_k, \quad k = 1, 2, 3 \tag{13}$$

(see reference [1]). We derive $\tilde{p}_k = f_k(p)$ first: Using (5) and (12) yields

$$p_1 = \frac{\xi_1}{1 - \xi_0} \to \tilde{p}_1 = \frac{\tilde{\xi}_1}{1 - \tilde{\xi}_0} = \frac{\xi_0}{1 + \xi_1} = \frac{\dfrac{|p|^2 - 1}{|p|^2 + 1}}{1 + \dfrac{2p_1}{|p|^2 + 1}}$$

$$= \frac{|p|^2 - 1}{|p|^2 + 2p_1 + 1} = \frac{|p|^2 - 1}{|p + e_1|^2}.$$

Similarly, for $k = 2, 3$

$$p_k = \frac{\xi_k}{1 - \xi_0} \to \tilde{p}_k = \frac{\tilde{\xi}_k}{1 - \tilde{\xi}_0} = \frac{\xi_k}{1 + \xi_1} = \frac{2p_k}{|p + e_1|^2}.$$

In the same exact way we can also derive $\tilde{q}_k = q_k(p, q)$ where Equations (5), (6), (12), and (13) are used.

The map given by (11) is our desired regularizing map as will be seen.

PROPOSITION 2. *The map given by* (11) *is canonical.*

Proof. Clearly under ψ^{-1}, i.e. (12), (13), we have

$$q\,\mathrm{d}p = -\eta\,\mathrm{d}\xi$$

as follows from (8). Now, we recall the relation given by (10). Finally, mapping $T(\mathring{S}^3)$ to $\mathbb{R}^6$ by ψ implies

$$\tilde{\eta}\,\mathrm{d}\tilde{\xi} = -\tilde{q}\,\mathrm{d}\tilde{p}.$$

Thus

$$q\,\mathrm{d}p = -\eta\,\mathrm{d}\xi = -\tilde{\eta}\,\mathrm{d}\tilde{\xi} = -(-\tilde{q}\,\mathrm{d}\tilde{p})$$

or

$$q\,\mathrm{d}p = \tilde{q}\,\mathrm{d}\tilde{p},$$

and hence $\mathrm{d}q \wedge \mathrm{d}p = \mathrm{d}\tilde{q} \wedge \mathrm{d}\tilde{p}$.

We could have constructed (11) by canonically extending $\tilde{p} = f(p)$ to the map $\tilde{q} = q(p, q)$ using a generating function $w = r(\tilde{p}, q)$ where by $p = r_q, \tilde{q} = r_{\tilde{p}}$. $\tilde{q} = r_{\tilde{p}}$ defines our extension if we choose

$$r = \sum_{k=1}^{3} q_k f_k^{-1}(\tilde{p}).$$

Of course, the most general extension is obtained by adding an arbitrary function $f_0(\tilde{p})$ to r.

Before proceeding to the next section, we calculate the inverse to (11). This can be done by observing that one must reverse the direction of rotation of (5), (6). For example, to compute the inverse to the momentum map in (11), $p = f^{-1}(\tilde{p})$, (5) takes the form (see Figure 3)

$$\begin{aligned} \tilde{\xi}_0 &\to \quad \tilde{\xi}_1 = \xi_0 \\ \tilde{\xi}_1 &\to -\tilde{\xi}_0 = \xi_1 \\ \tilde{\xi}_k &\to \quad \tilde{\xi}_k = \xi_k, \quad k = 2, 3, \end{aligned}$$

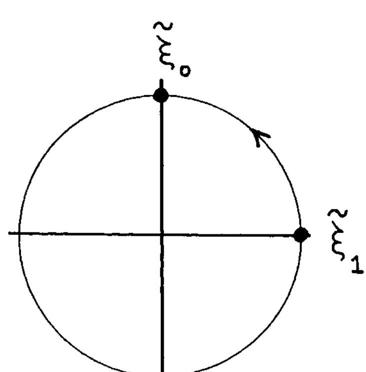

Fig. 3.

and similarly for (6). Proceeding exactly as in the proof of Proposition 1, we obtain

$$p_k = f_k^{-1}(\tilde{p}) = \begin{cases} \dfrac{1 - |\tilde{p}|^2}{|\tilde{p} - e_1|^2} & k = 1, \\[2ex] \dfrac{2\tilde{p}_k}{|\tilde{p} - e_1|^2} & k = 2, 3. \end{cases} \tag{14}$$

$$q_k = g_k^{-1}(\tilde{p}, \tilde{q}) = \begin{cases} \dfrac{1-|\tilde{p}|^2}{2}\tilde{q}_1 + (\tilde{p}, \tilde{q})(\tilde{p}_1 - 1) & k = 1, \\[2ex] \dfrac{|\tilde{p}|^2+1}{2}\tilde{q}_k + \tilde{q}_1\tilde{p}_k - \tilde{p}_1\tilde{q}_k - (\tilde{p}, \tilde{q})\tilde{p}_k & k = 2, 3. \end{cases}$$

It is interesting to note that the maps $\tilde{q} = g(p, q)$ and $q = g^{-1}(\tilde{p}, \tilde{q})$ have no denominators. This is also the case with Sundman's map–see formula (4).

3. A Specialized Jordan Algebra

The above transformation (11), which we refer to as Φ, can be conveniently presented in terms of a non-associative algebra which we now describe. Its elements z are of the form

$$z = z_0 + i_1 z_1 + i_2 z_2 + \dots i_n z_n,$$

where $z_\nu \in \mathbb{R}$, $\nu = 0, 1, 2, \dots, n$, and where

$$i_\alpha i_\beta = -\delta_{\alpha\beta}.$$

The set of all such elements forms a non-associative algebra which is called a 'special' Jordan algebra (see references [4] and [5]). This algebra is clearly commutative. If we now define the operation of 'conjugation'

$$\bar{z} = z_0 - i_1 z_1 - i_2 z_2 - \dots - i_n z_n,$$

we obtain a set of elements with algebraic properties similar to the complex numbers, e.g.

$$z\bar{z} = |z|^2 \equiv \sum_{k=0}^{n} z_k^2,$$

$$\operatorname{Re} z \equiv z_0 = \tfrac{1}{2}(z + \bar{z}),$$

$$\operatorname{Im}_{i_\alpha} z \equiv z = \frac{1}{2i_\alpha}(z - \bar{z}), \quad \alpha = 1, \dots, n,$$

$$\operatorname{Re} z\bar{w} = (z, w) \equiv \sum_{k=0}^{n} z_k w_k.$$

Division is defined via

$$\frac{1}{z} = \frac{\bar{z}}{|z|^2}.$$

We call this non-associative algebra A_n. Clearly, $A_1 = \mathbb{C}$, $A_0 = \mathbb{R}$.

It is now seen how much A_n misses being associative by computing the 'associator' $a = a(x, y, z) = x(yz) - (xy)z$ where $x, y, z \in A_n$. In particular, for the case $n = 2$, which concerns us,

$$a = (x_2 z_1 - x_1 z_2)(iy_2 - jy_1),$$

where we have set $i_1 = i$, $i_2 = j$. One must be careful of the nonassociativity when doing calculations. For example, $\alpha\beta = \gamma$ does not imply $\alpha = \gamma\beta^{-1}$, where $\alpha, \beta, \gamma \in A_n$, since $(\alpha\beta)\beta^{-1}$ need not equal $\alpha(\beta\beta^{-1}) \equiv \alpha$.

We now write Φ in terms of A_2. For the map $\tilde{p} = f(p)$ of (11) we have

LEMMA 1.

For $p, \tilde{p} \in A_2$, $\tilde{p} = f(p)$ becomes

$$\tilde{p} = \frac{p-1}{p+1}. \tag{15}$$

Proof. Set $\tilde{p} = \tilde{p}_1 + i\tilde{p}_2 + j\tilde{p}_3$ and similarly for p, then

$$\tilde{p} = \frac{p-1}{p+1} = \frac{(p-1)(\bar{p}+1)}{|p+1|^2} = \frac{|p|^2 - 1 + p - \bar{p}}{|p+1|^2},$$

but $p - \bar{p} = 2ip_2 + 2jp_3$; therefore

$$p = \frac{|p|^2 - 1 + 2ip_2 + 2jp_3}{|p+1|^2}.$$

Equating 'real' and 'imaginary' parts proves the Lemma.

We see that (15) agrees with the complex version in Section 1C. The fractional linear character of $\tilde{p} = f(p)$ is now evident. One verifies that the other half of the map $\tilde{q} = g(p, q)$ of (11) can be written as

$$\tilde{q} = \frac{q}{2}(1 + \bar{p})^2 - (qp)\bar{p} + q(p\bar{p}), \tag{16}$$

where $p, q \in A_2$. Thus Φ is also represented by (15), (16). Formula (15), (16) can also be immediately generalized to n-dimensions, i.e. $p, q \in A_{n-1}$.

Before proceeding further we make a remark on notation. In what follows we will do computations and state formula in both vector notation and in A_n. As we saw above, Φ is represented by (11) in vector notation and by (15), (16) in A_2. Also, a term which often arises is $(p_1 - 1)^2 + p_2^2 + p_3^2$. In vector notation it is written as $|p - e_1|^2$ while in A_2 we write it as $|p - 1|^2$. We will always specify what notation is being used.

Now, one would like to perform 'usual' algebraic operations on (15) without having to be concerned with nonassociativity, e.g. see the comment before Lemma 1. It turns out that we can in fact freely manipulate (15) without the concern of nonassociativity. For the proof of this fact see reference [2]. The inverse to (15) is therefore given by

$$p = \frac{1 + \tilde{p}}{1 - \tilde{p}} \tag{17}$$

which agrees with the map $p_k = f_k^{-1}(\tilde{p})$ in (14) when it is written in component form.
Several useful identities can be obtained from (15).

PROPOSITION 3. *Let* $p, \tilde{p} \in A_2$, *then*

$$|\tilde{p} - 1||p + 1| = 2, \tag{18a}$$

$$|p| = |\tilde{p} + 1|/|\tilde{p} - 1|, \tag{18b}$$

$$|p|^2 = [2(|\tilde{p}|^2 + 1)/|\tilde{p} - 1|^2] - 1. \tag{18c}$$

Proof. By (15), $\tilde{p} - 1 = (p - 1)/(p + 1) - 1 = -(2)/(p + 1)$, and taking the absolute value of both sides of $\tilde{p} - 1 = -(2)/(p + 1)$ proves (18a). Taking the absolute value of both sides of (17) yields (18b). To prove (18c) we use (17) and form

$$|p|^2 + 1 = \frac{|\tilde{p} + 1|^2}{|\tilde{p} - 1|^2} + 1 = \frac{|\tilde{p} + 1|^2 + |\tilde{p} - 1|^2}{|\tilde{p} - 1|^2},$$

but $|\tilde{p} + 1|^2 + |\tilde{p} - 1|^2 = 2(|\tilde{p}|^2 + 1)$ which yields (18c).
With a little more work we have for the canonical extension Φ, see reference [2],

PROPOSITION 4. *Let* $p = (p_1, p_2, p_3)$, $q = (q_1, q_2, q_3)$, *etc., then in the usual vector notation*

$$q_2 p_1 - q_1 p_2 = \tilde{q}_2 \frac{1 - |\tilde{p}|^2}{2} + \tilde{p}_2(\tilde{p}, \tilde{q}), \tag{19a}$$

$$|q| = \frac{|\tilde{p} - e_1|^2}{2} |\tilde{q}|. \tag{19b}$$

We mention that the second part of the inverse $q = g^{-1}(\tilde{p}, \tilde{q})$ in (14) can be written in A_2 as

$$q = \frac{\tilde{q}}{2}(1 - \bar{\tilde{p}})^2 + \tilde{q}(\tilde{p}\bar{\tilde{p}}) - (\tilde{q}\tilde{p})\bar{\tilde{p}}.$$

The results obtained above for our canonical map Φ, and Φ^{-1}, are now summarized where we allow right away n-dimensions:

SUMMARY A. Φ and Φ^{-1} are given in the usual vector notation as

$$\tilde{p}_k = \begin{cases} \dfrac{|p|^2 - 1}{|p + e_1|^2} & k = 1, \\[2ex] \dfrac{2p_k}{|p + e_1|^2} & k = 2, \ldots, n \end{cases} \tag{20a}$$

$$\tilde{q}_k = \begin{cases} \dfrac{1 - |p|^2}{2} q_1 + (p, q)(p_1 + 1) & k = 1, \\[2ex] \dfrac{|p|^2 + 1}{2} q_k + p_1 q_k - p_k q_1 - (p, q) p_k & k = 2, \ldots, n \end{cases}$$

$$p_k = \begin{cases} \dfrac{1-|\tilde{p}|^2}{|\tilde{p}-e_1|^2} & k=1 \\ \\ \dfrac{2\tilde{p}_k}{|\tilde{p}-e_1|^2} & k=2,\ldots,n \end{cases}$$

$$q_k = \begin{cases} \dfrac{1-|\tilde{p}|^2}{2}\tilde{q}_1 + (\tilde{p},\tilde{q})(\tilde{p}_1 - 1) & k=1, \\ \\ \dfrac{|\tilde{p}|^2+1}{2}\tilde{q}_k + \tilde{q}_1\tilde{p}_k - \tilde{p}_1\tilde{q}_k - (\tilde{p},\tilde{q})\tilde{p}_k & k=2,\ldots,n \end{cases} \tag{20b}$$

or in A_{n-1} by

$$\tilde{p} = \frac{p-1}{p+1}, \quad \tilde{q} = \frac{q}{2}(1+p)^2 - (qp)\bar{p} + q(p\bar{p}), \tag{20c}$$

$$p = \frac{1+\tilde{p}}{1-\tilde{p}}, \quad q = \frac{\tilde{q}}{2}(1-\bar{\tilde{p}})^2 - (\tilde{q}\tilde{p})\bar{\tilde{p}} + \tilde{q}(\tilde{p}\bar{\tilde{p}}). \tag{20d}$$

In addition, we have the following useful identities which are written in vector notation:

$$|\tilde{p}-e_1||p+e_1| = 2, \quad |p| = |\tilde{p}+e_1|/|\tilde{p}-e_1|, \tag{20e}$$

$$|p|^2 = [2(|\tilde{p}|^2+1)/|\tilde{p}-e_1|^2] - 1$$

$$q_2p_1 - q_1p_2 = \tilde{q}_2\frac{1-|\tilde{p}|^2}{2} + \tilde{p}_2(\tilde{p},\tilde{q}), \, |q| = \frac{|\tilde{p}-e_1|^2}{2}|\tilde{q}|. \tag{20f}$$

4. Existence and Properties of a Family of Periodic Orbits

A. REGULARIZATION

We use the transformation Φ to regularize our system near ϕ^*. The standard iso-energetic transformation will be used (see reference [10]): The system defined by (1) in Section 1A on $H = -h < 0$ can be transformed under a canonical transformation Φ,

$$P = f(p, q)$$

$$Q = q(p, q),$$

and by a time transformation

$$s = \int^t r(P, Q)\, \mathrm{d}t,$$

or $\mathrm{d}s/\mathrm{d}t = r(P, Q) > 0$, into

$$P' = \Gamma_Q, \qquad Q' = -\Gamma_P \quad \text{on} \quad \Gamma = 0, \tag{21}$$

$' \equiv \mathrm{d}/\mathrm{d}s$, and where

$$\Gamma = r^{-1} H \circ \Phi^{-1}.$$

We take for Φ our transformation as defined in Summary A with $\tilde{p}$ and $\tilde{q}$ replaced replaced by P, Q respectively for notation. We also take

$$r^{-1} = |q| = \tfrac{1}{2}|P - e_1|^2 |Q|.$$

THEOREM B: *System* (21) *obtained from* (1) *on* $H = -h$ *by the above isoenergetic transformation is regular near* $\Phi \circ \phi^*$.

Proof: By our map Φ, $H(p, q)$ as defined by (2) becomes

$$\tilde{H} = H \circ \Phi^{-1} = \frac{1}{2} \frac{|P + e_1|^2}{|P - e_1|^2} - \frac{2}{|P - e_1|^2 |Q|} + \omega \tilde{\alpha} + \mu \tilde{G}$$

on $\tilde{H} = -h < 0$, where by Propositions 3, 4 in Section 3

$$\tilde{\alpha} = \alpha \circ \Phi^{-1} = Q_2 \frac{1 - |P|^2}{2} + P(P, Q),$$

$$\tilde{G} = G \circ \Phi^{-1} = \frac{2}{|P - e_1|^2 |Q|} - \tilde{\Delta}^{-1} - \omega \frac{1 - |P|^2}{|P - e_1|^2},$$

with $\omega = 1$, and

$$\tilde{\Delta} = \Delta \circ \Phi^{-1} = \left[\frac{1}{4} |P - e_1|^4 |Q|^2 - \right.$$
$$\left. - 2\left(\frac{|P|^2 + 1}{2} Q_2 + Q_1 P_2 - P_1 Q_2 - (P, Q) P_2 \right) + 1 \right]^{1/2}.$$

We can cancel out the denominator $|P - e_1|^2$ in $\tilde{H}$ which has a zero on $\Phi \circ \phi^*$ by an appropriate time transformation. To make this time transformation we consider the Hamiltonian $\tilde{H} + h (= 0)$ and then apply the above time transformation, written as

$$t = \int^s |q| \, \mathrm{d}\sigma = \frac{1}{2} \int^s |P - e_1|^2 |Q| \, \mathrm{d}\sigma,$$

which has the effect of multiplying $H + h$ by $\frac{1}{2}|P - e_1|^2 |Q|$. We remark that the time transformation $t = \int^s |P - e_1|^2 \mathrm{d}\sigma$ could have been used, but we prefer the independent variable s defined by $t = \int^s |q| \mathrm{d}\sigma$ because for $\mu = 0$ this transformation carries the Kepler problem into itself and therefore the solutions are explicitly available.

If we call the Hamiltonian $\frac{1}{2}|P-e_1|^2|Q|(H+h)$, Γ, then

$$\Gamma=\frac{|Q|}{4}\{|P+e_1|^2+2(h+\omega\tilde{\alpha})|P-e_1|^2\}-1+\mu F(P,Q) \quad (=0), \tag{22}$$

where

$$F(P,Q)=\tfrac{1}{2}|P-e_1|^2|Q|\tilde{G}=1-\tfrac{1}{2}|P-e_1|^2|Q|\bar{\Delta}^{-1}-\frac{\omega}{2}|Q|(1-|P|^2),$$

with $\omega=1$.

Now, we must prove that Γ is regular on our reference orbit $\phi^*=(p^*,q^*)$ in the coordinates P,Q which we call $X^*=(P^*,Q^*)$. In particular, we will show that $Q^*(s)\neq 0$, $\tilde{\Delta}(X^*(s))\neq 0$. We now list $\phi^*(s)$ and $X^*(s)$:

$$\phi^*(s)=(p^*(s),q^*(s))=\left(0,0,\sqrt{2h}\frac{C}{S},0,0,h^{-1}S^2\right), \tag{23}$$

$$X^*(s)=(P^*(s),Q^*(s))=$$

$$\left(-\frac{S^2-2hC^2}{S^2+2hC^2},0,\frac{4\sqrt{\frac{h}{2}}CS}{S^2+2hC^2},\sqrt{\frac{h}{2}}CS,0,\frac{1}{2h}S^2-C^2\right), \tag{24}$$

where $S=\sin\sqrt{h/2}s$, $C=\cos\sqrt{h/2}s$. Clearly, $Q^*(s)\neq 0$ for $Q^*(s)=0$ only when $\cos(\sqrt{h/2}s)=\sin(\sqrt{h/2}s)=0$ which is impossible. In addition, $\tilde{\Delta}(X^*(s))=$ $=(\frac{1}{4}|P^*(s)-e_1|^4|Q^*(s)|^2+1)^{1/2}$ which clearly does not vanish: This concludes the proof of Theorem B. The analyticity of Γ on X^* is now evident.

We remark that the collision states on the energy surface $\Gamma(P,Q)=0$ correspond to the 2-dimensional sphere

$$P=e_1, \quad |Q|=1-\mu.$$

For $\mu=0$ the orbit (24) intersects this collision manifold at $P=e_1, Q=-e_3$ for $s=0$.

B. PROOF OF THEOREM A

We prove Theorem A by the Poincaré continuation method for orbits with symmetry (For an alternate argument see Section 4C). We will look at solutions which are symmetric with respect to the q_2q_3-plane. In other words, we search for solutions which are invariant under the reflection

$$t\to -t,(q_1,q_2,q_3,p_1,p_2,p_3)\to(-q_1,q_2,q_3,p_1,-p_2,-p_3)$$

If we write this reflection in the algebra $\dot{A}_2$, it becomes $t \to -t$, $q \to -\bar{q}$, $p \to \bar{p}$. Applying this to Φ written in A_2, see (22c), and to our time transformation $t \to s = \int |q|^{-1} \, dt$ yields $s \to -s$, $Q \to -\bar{Q}$, $P \to \bar{P}$ or,

$$s \to -s, (Q_1, Q_2, Q_3, P_1, P_2, P_3) \to (-Q_1, Q_2, Q_3, P_1, -P_2, -P_3).$$

Thus the reflection ρ defined by this formula commutes with Φ:

$$\rho \circ \Phi = \Phi \circ \rho.$$

We note that Γ is invariant under ρ, i.e.

$$\Gamma \circ \rho = \Gamma.$$

which implies: If $X(s) = (P(s), Q(s))$ is a solution of (23) so is $\rho X(-s)$ a solution, and more generally, so is $\rho X(s_0 - s)$ where s_0 is a fixed value of s.

We call a solution symmetric with respect to ρ if

$$\rho X(-s) = X(s).$$

A symmetric solution is characterized by its initial conditions $X(0)$ satisfying $\rho X(0) = X(0)$, or

$$Q_1(0) = P_2(0) = P_3(0) = 0$$

so that symmetric solutions are characterized by only 3 initial values $Q_2(0)$, $Q_3(0)$, $P_1(0)$.

In addition, we restrict the initial values to the energy surface $\Gamma(P(0), Q(0)) = 0$. Since, at $X^*(0) = (1, 0, 0, 0, 0, -1)$,

$$\frac{\partial \Gamma}{\partial Q_3} = -1 \neq 0$$

we can use $Q_2(0)$, $P_1(0)$ as independent variables near $X^*(0)$ and express $Q_3(0)$, via the implicit function theorem, in terms of $Q_2(0)$, $P_1(0)$. One can easily prove the following result (see reference [2]).

LEMMA 2: If for a symmetric solution $X(s)$ one has for some value of $S > 0$

$$Q_1(S) = P_2(S) = P_3(S) = 0, \tag{25}$$

then $X(s) = X(s + 2S)$, i.e. the symmetric orbit has period $2S$.

We note that $X^*(s)$ is a symmetric solution satisfying (25) with $S = S^* = (\pi/2)(2/h)$. This follows since $Q^*(0) = (0, 0, -1)$, $P^*(0) = (1, 0, 0)$ and for $S = S^*$ one has $Q^*(S) = (0, 0, 1/2h)$, $P^*(S) = (-1, 0, 0)$ which again lies on $Q_1 = P_2 = P_3 = 0$, and $2S^* = \pi \, 2/h$ is the corresponding period. Thus, the two points of intersections with the symmetry manifold $Q_1 = P_2 = P_3 = 0$ correspond to the collision and the maximum point of the orbit.

In order to find symmetric solutions of period $2S$ on $\Gamma = 0$ for μ different from zero we merely have to determine $S = S(\mu)$, $Q_2(0)$, $P_1(0)$ near S^*, $Q_2^*(0)$, $P_1^*(0)$ such that (25) holds by applying the implicit function theorem, which requires that the functional determinant

$$D = \det \frac{\partial(Q_1, P_2, P_3)}{\partial(S, Q_2^0, P_1^0)}$$

does not vanish for $\mu = 0$, $S = S^*$, $X^0 \equiv X(0) = X^*(0)$.

To calculate the determinant D we need only consider the unperturbed problem for $\mu = 0$: For $\mu = 0$ we have the Kepler problem in rotating coordinates where all the solutions are known explicitly, and the calculation of D is in principal simple. One finds that up to a constant factor, see reference [2],

$$D = \det \begin{pmatrix} 1 & -\sin\dfrac{\pi}{\omega^*} & 0 \\ 0 & 0 & \sin\dfrac{\pi}{\omega^*} \\ -1 & 0 & 0 \end{pmatrix} = \sin^2 \frac{\pi}{\omega^*}.$$

Hence for $1/\omega^* \neq j, j = 1, 2, 3, \ldots,$ we can determine $S = S(\mu)$, $Q_2^0(\mu)$, $P_1^0(\mu)$ near S^*, $Q_2^*(0) = 0$, $P_1^*(0) = 1$ such that the initial values $X(0, \mu)$ give rise to a symmetric solution $X(s, \mu)$ of period $2S$ near $2S^*$ which proves Theorem A.

It is important to note that it is not clear whether the continuation $\phi(t, \mu) = (q(t, \mu), \dot{q}(t, \mu))$ is a noncolliding periodic orbit or a collision orbit. Towards this end the following result can be proven:

COROLLARY A: If $\phi(t, \mu)$ is a collision orbit then $\dot{q} \in \mathscr{S}$ at collision for $|\mu| \neq 0$ sufficiently small, where $\mathscr{S}$ is the symmetry plane, $\mathscr{S} = \{q \mid q_1 = 0\}$. The proof of this result is accomplished by assuming the contrary, namely that $\dot{q} \notin \mathscr{S}$ at collision. A contradiction is obtained by making use of the uniqueness and the symmetry properties of ϕ. For the full proof of Corollary A see reference [2]. Numerical results indicate that $\dot{q} \in \mathscr{S}$ at collision for a discrete set of ω^* only. These numerical results will be presented in a forthcoming paper (see reference [3]).

D. THE ORBIT MANIFOLD

Geometrically we can obtain an overview of the orbit manifold for μ sufficiently small by noting that for each fixed μ Theorem A yields a one-parameter family of periodic orbits parameterized by the energy E, or by the period T since T is functionally related to E by Kepler's equation. This one-parameter family of periodic orbits ϕ_E sweeps out a 2-dimensional manifold M_2 in the 6-dimensional phase space $\mathbb{R}^6$. For each

E, ϕ_E lies on the 5-dimensional energy surface Σ_5 $(E) = \{(p, q) | H(p, q) = E < 0\}$. Thus

$$\phi_E = M_2 \cap \Sigma_5(E).$$

If we let $d = \max\{|\phi_E(s_1) - \phi_E(s_2)|\}$ then from the behavior of $\phi_E^*(s)$, see (23), for $s \in I$ one finds that as $E \to -\infty$; $d \to \infty$ and as $E \to {}^-0$; $d \to \infty$, at least for $\mu = 0$. The above is summed up in the following picture (Figure 4). For $\mu = 0$ we must eliminate those orbits of M_2 s.t. $1/\omega^* = j$, $j = 1, 2, 3, \ldots$.

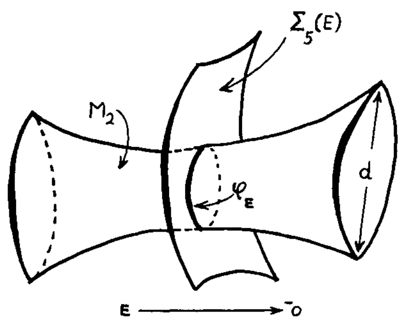

Fig. 4. The orbit manifold.

E. CALCULATION OF THE FLOQUET MULTIPLIERS

It is very easy to compute the Floquet multipliers of ϕ^* for $\mu = 0$ and in this way get an alternate existence proof, which, however, does not yield the symmetry of the orbits. This fully carried out in reference 2 where the corresponding Poincaré map is constructed. The Floquet multipliers are calculated to be $\lambda = e^{\pm 2\pi i(\omega/\omega^*)}$, 1, each taken double. Clearly, the pair of ones is due to the existence of the energy integral. They can be eliminated by the restriction of the Poincaré map to an appropriate four dimensional transversal section to ϕ^*. Thus, to insure that the remaining multipliers, which are the eigenvalues of the restricted Poincaré map, do not equal one means that we must require $1/\omega^* \neq j$, $j = 1, 2, 3, \ldots$, where we have normalized $\omega = 1$. This is precisely our previous condition in Theorem A.

5. Comparison with the KS Map

The KS map is a 3-dimensional regularization of the restricted problem. We now compare it to our regularizing map Φ as given in Summary A, Section 3.

The KS map is a generalization of the 2-dimensional Levi–Civita map. The Levi–Civita map in complex notation is given by

$$q = u^2, \quad p = \frac{1}{2|u|^2} uv,$$

where $u = u_1 + iu_2$, $v = v_1 + iv_2$, and is thus a map of $\mathbb{R}^4 \to \mathbb{R}^4$. This map provides a regularization for the 2-dimensional restricted problem, i.e. where m_3 is restricted to lie in the plane determined by m_1, m_2. The Levi–Civita map was developed at the turn of the century. There was the desire to generalize it to 3 or more dimensions. This was accomplished in 1964 by Kustaanheimo and Stiefel (see reference [6]) and is based on the Hopf map (see references [7], [8]) when restricted to the position space $q \in \mathbb{R}^3$. The resulting map on the full phase space $(p, q) \in \mathbb{R}^6$ is called the KS map. Its restriction to the q-space is given by

$$\begin{aligned} &q_1 = u_1^2 - u_2^2 - u_3^2 + u_4^2, \qquad q_2 = 2(u_1 u_2 - u_3 u_4), \\ &q_3 = 2(u_1 u_3 + u_2 u_4). \end{aligned} \tag{26}$$

This is a map from $\mathbb{R}^4 \to \mathbb{R}^3$. One finds that, similar to the Levi–Civita map, $|q| = |u|^2$. Thus, when $|u| = 1$, i.e. $u \in S^3 \subset \mathbb{R}^4$, this restriction is a map of $S^3 \to S^2$. One sees that this restriction is the Hopf map.

A drawback to the KS map is that it introduces another integral when one requires it to regularize the perturbed Kepler problem (see reference [6]). In particular, if the variables conjugate to u are called v, this integral is given by

$$I = v_1 u_4 - v_2 u_3 + v_3 u_2 - v_4 u_1,$$

where one has canonically extended (26) to a map of the p-space:

$$p = \frac{1}{2|u|^2} \Lambda v, \tag{27}$$

where $p = (p_1, p_2, p_3) \in \mathbb{R}^3$, $v = (v_1, v_2, v_3, v_4) \in \mathbb{R}^4$,

$$\Lambda = \begin{pmatrix} u_1 & -u_2 & -u_3 & u_4 \\ u_2 & u_1 & -u_4 & -u_3 \\ u_3 & u_4 & u_1 & u_2 \end{pmatrix}.$$

It is required that $I = 0$ along solutions (see references [6] and [11]). The introduction of another integral complicates the perturbation theory – since the system for the flow is autonomous, another pair of ones as Floquet multipliers will be introduced. To make the Poincaré continuation argument go through, to prove the existence of a continuation to ϕ^*, the dimension of the Poincaré map has to be reduced by two so that the ones can be eliminated. This adds complications. This problem is avoided with our map because no new integrals are introduced.

The increase of the dimension of the phase space from 6 to 8 is a disadvantage of the KS map since all computational work, whether it be on the computer or analytical, is significantly increased. Φ does not increase dimension.

Another advantage of Φ is that it is automatically valid for n-dimensions whereas the KS map is valid for only 3.

An advantage, however, of the KS map is that it not only regularizes the singularity at $q = 0$ but introduces global variables for the energy surface, while Φ introduces new singularities. However, this disadvantage of Φ does no harm in our case since these new singularities lie outside of our periodic orbit as was seen in the Introduction.

It is finally mentioned that other regularizations can be found in reference [12].

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