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Determining representations from invariant dimensions

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This paper is motivated by the following "Tannakian" question: to what extent is a complex Lie group, G, and a finite dimensional representation, (ρ, V) of G, determined by the dimensions of the various invariant spaces W^G, where the W are obtained from V by linear algebra? That is, given $\dim((\mathrm{Sym}^2(V)^G)$, $\dim((\Lambda^3 V)^G)$, etc., can one determine (G, ρ)? This problem arises, for instance, in the cohomological study of exponential sums; we intend to apply the below results to the problem of "l-independence of monodromy" for compatible systems of exponential sums in a subsequent paper.

Let us first fix ideas concerning dimensions of spaces of invariants.

Definition. *We call* dimension data *for* (G, ρ) *the data associating*

$$\dim W^G \tag{1}$$

to every Lie group homomorphism $GL(V) \to GL(W)$.

Note that this definition makes sense only if $\dim(V)$ is given. If $\det(\rho)=1$, we can define dimension data to consist of

$$\dim(\mathrm{Homs}_{S_k}(U, V^{\otimes k})^G)$$

for every $k \in \mathbb{N}$ and every irreducible representation of the symmetric group S_k (which acts on $V^{\otimes k}$ by permuting the factors). This makes sense even when $\dim(V)$ is unknown; and it determines $\dim(V)$ as the largest integer k such that $\Lambda^k V$ has a non-trivial G-invariant. The classical invariant theory of $SL(V)$ ([6] 4.4.D, 7.5.C) tells us that the two definitions are equivalent.

From now on, we assume that G is connected and semisimple except in § 1, where we allow it to be connected reductive. For obvious reasons we assume that ρ is faithful. Our main results are the following:

Theorem 1. *For any faithful finite dimensional representation ρ of a connected semisimple Lie group G, dimension data uniquely determines G up to isomorphism.*

Theorem 2. *Under the hypotheses of Theorem 1, if ρ is irreducible, dimension data uniquely determines ρ up to isomorphism.*

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Theorem 3. *In the full generality of Theorem 1, ρ is not determined up to isomorphism by dimension data.*

A few notes may be in order here. We observe that dimension data determines $\dim(G)$ quite easily. Indeed, let nV be the direct sum of $n=\dim(V)$ copies of V. The set of bases of V is an open dense subset of nV. Thus the stabilizer in G of the generic point of nV is the kernel of ρ, which is trivial by assumption. It follows that

$$\dim(\mathbf{C}[nV]^G)=n^2-\dim(G).$$

Dimension data determines the Hilbert polynomial of the graded algebra $\mathbf{C}[nV]^G$ and hence its Krull dimension. Thus $\dim(G)$ is determined.

Note also that Theorem 1 may be generalized as follows: if H is any linear algebraic group and $\rho\colon G\to H$ is an injective homomorphism, the dimension data obtained by restricting all representations of H to G determine G up to isomorphism. This follows by embedding H in $GL(V)$ and taking representations of $GL(V)$.

In Sect. 1, we show that dimension data is equivalent to certain data involving only a maximal torus of G. In Sects. 2 and 3, we study to what extent the root system of G is determined by this data, and we prove Theorems 1 and 3. We give an effective procedure to construct a counter-example in Theorem 3. It can occur only if a certain root system naturally associated to dimension data (see § 2) is not reduced. This does not happen, for instance, for irreducible ρ, and this fact can be used to prove Theorem 2. In Sect. 4, however, we follow a different approach, using only the weights of the (irreducible) representation ρ. Theorem 4 completely determines which non-isomorphic pairs (G, ρ), with ρ irreducible, have the same weight configuration. These ideas lead to the proof of Theorem 2 that we present.

This problem was suggested to us by N. Katz in his 1985–1986 Princeton University course on exponential sums. We enjoyed many useful conversations with him during the course of the work and numerous helpful comments on the exposition of this paper. It gives us both great pleasure to acknowledge our various debts to him.

1. Sato-Tate measure

Let G be a connected complex reductive Lie group and $\rho\colon G\to GL(V)$ a faithful representation, of dimension n. Let K be a maximal compact subgroup of G, and T a maximal torus of K. We choose a basis of V so that

$$\begin{array}{ccccc}\rho(T) & \subset & \rho(K) & \subset & \rho(G)\\ \cap & & \cap & & \cap\\ U(1)^n & \subset & U(n) & \subset & GL(n).\end{array}$$

Let the superscript $^\natural$ denote the set of conjugacy classes. Then we have the commutative diagram

$$\begin{array}{ccc} K & \xhookrightarrow{\rho} & U(n) \\ p_K \big\downarrow & & \big\downarrow p_U \\ K^\natural & \xrightarrow{\rho^\natural} & U(n)^\natural \\ \pi_T \big\uparrow & & \big\uparrow \pi_U \\ T & \xhookrightarrow{\rho_T} & U(1)^n \end{array}$$

where $K^\natural \cong T/W$, $U(n)^\natural \cong U(1)^n/S_n$, and the maps π_T, π_U are the quotient maps. Let $\mathrm{d}k$ denote Haar measure on K, normalized to total volume 1. Given a measurable function $f\colon X \to Y$ and a measure μ on X, we write $f_*\mu$ for the measure on Y such that

$$\int_X g(f(x))\,\mu = \int_Y g(y) f_*\,\mu.$$

For any representation $\sigma\colon GL(V) \to GL(W)$,

$$\dim(W^G) = \dim(W^K) = \int_K \operatorname{tr}(\sigma\rho(x))\,\mathrm{d}k = \int_{U(n)} \operatorname{tr}(\sigma)\,\rho_*\,\mathrm{d}k.$$

By the Peter-Weyl theorem ([5] § 6, p. 754), the values of these integrals determine the measure

$$p_{U*}\,\rho_*\,\mathrm{d}k = \rho^\natural_*\,p_{K*}\,\mathrm{d}k$$

on $U(n)^\natural$

This measure is determined by dimension data. We write $\operatorname{supp}(\mu)$ for the support of a measure μ, that is, the smallest closed set such that the restriction of μ to the complement is 0. Since $\mathrm{d}k$ is Haar measure on K, $\operatorname{supp}(p_{K*}\,\mathrm{d}k) = K^\natural = T/W$, $\operatorname{supp}(\rho^\natural_*\,p_{K*}\,\mathrm{d}k) = \rho^\natural(T/W)$, and therefore

$$Y := \pi_U^{-1}\operatorname{supp}(\rho^\natural_*\,p_{K*}\,\mathrm{d}k) = \pi_U^{-1}(\rho^\natural(T/W)) = \bigcup_{\sigma\in S_n} \rho_T(T)^\sigma.$$

The irreducible components of Y are the $\rho_T(T)^\sigma$. These components differ only by the numbering of the coordinates of $U(1)^n$. We choose one such component and label it $\rho_T(T)$. As ρ is faithful, we can identify T with $\rho_T(T)$.

What we would like to know is the *Sato-Tate measure* $p_{K*}\,\mathrm{d}k$ on $K^\natural$. Let

$$F_\Phi(t) = \prod_{\alpha\in\Phi} (1-\alpha(t)).$$

The Weyl integration formula ([1] IX § 6 Th. 1 Cor. 2(b)) says

$$\frac{1}{|W|}\,\pi_{T*}\,F_\Phi(t)\,\mathrm{d}t = p_{K*}\,\mathrm{d}k, \tag{2}$$

where $\mathrm{d}t$ denotes normalized Haar measure on T and Φ denotes the set of roots of G. If we knew the measure (2), we would know Φ by unique factorization (up to units) of Laurent polynomials. Unfortunately, $\rho^\natural$ is not always injective, even on the complement of a set of measure zero. The configuration of weights of V may have symmetries outside the Weyl group $W=W(G,T)$, or equivalently, an element of $S_n=W(U(n), U(1)^n)$ which stabilizes T may not act on T by any element of W. In this case $\rho^\natural$ is generically many-to-one. This reflects the fact that symmetries are preserved by the operations of linear algebra, and that consequently, our dimension data bears only on a subcategory of Rep(K). This is the central difficulty in proving theorems 1 and 2, and it makes possible the counter-examples of theorem 3.

What we know is the measure

$$\frac{1}{|W|}\rho^\natural_* \pi_{T*} F_\Phi(t)\,\mathrm{d}t=\frac{1}{|W|}\pi_{U*}\rho_{T*} F_\Phi(t)\,\mathrm{d}t$$

on $U(n)^\natural$. The unique S_n-invariant measure μ on $U(1)^n$ such that

$$\frac{1}{n!}\pi_{U*}\mu=\frac{1}{|W|}\pi_{U*}\rho_{T*} F_\Phi(t)\,\mathrm{d}t$$

is

$$\frac{1}{|W|}\sum_{\sigma\in S_n}\sigma_*\rho_{T*} F_\Phi(t)\,\mathrm{d}t.$$

The restriction of this measure to $T=\rho_T(T)$ is

$$\frac{1}{|W|}\sum_{\sigma\in \mathrm{Stab}_{S_n} T}\sigma_* F_\Phi(t)\,\mathrm{d}t,$$

where $\mathrm{Stab}_{S_n} T$ is the group of automorphisms of S_n which preserve the set T. Let Γ° be the image of $\mathrm{Stab}_{S_n} T$ in $\mathrm{Aut}(T)$ or equivalently the group of automorphisms of T induced by permutation automorphisms of $U(1)^n$. We have seen that dimension data determines the measure

$$\frac{1}{|W|}\sum_{\gamma\in \mathrm{Stab}_{S_n} T}\gamma_*(F_\Phi(t)\,\mathrm{d}t)=\frac{1}{|W|}\sum_{\gamma\in \mathrm{Stab}_{S_n} T}(\gamma(F_\Phi))(t)\,\mathrm{d}t$$

on T, so it determines the group algebra element

$$F^\circ=\sum_{\gamma\in\Gamma^\circ}\gamma(F_\Phi)\in\mathbf{Q}[X(T)] \tag{3}$$

up to a non-zero rational scalar multiple. (This scalar depends on $|W|$ which at this point is unknown; *a posteriori*, *i.e.*, as a consequence of Theorem 1, $|W|$ and hence F° is determined exactly.) We summarize:

Proposition 1. *Let G be a connected reductive group and $\rho: G\to GL(V)$ a faithful representation. Let T, ρ_T, and F° be as above. Then $(T, \rho_T, \mathbf{Q}F^\circ)$ is determined up to isomorphism by dimension data for (G,ρ). That is, given (G,ρ) and (G',ρ')*

with the same dimension data, there exist maximal tori T and T' of G and G' respectively and an isomorphism of triples $(T, \rho_T, \mathbf{Q}F^\circ) \cong (T', \rho'_T \rho'_{T'}, \mathbf{Q}F'^\circ)$.

The converse is left as an exercise to the reader:

Proposition 2. *If G is a connected reductive group, ρ: $G \to GL(V)$ is a faithful representation, and $GL(V) \to GL(V')$ is a Lie group homomorphism, then* $(T, \rho_T, \mathbf{Q}F^\circ)$ *determines* $\dim(V'^G)$.

We fix some notation to be used in the following paragraphs. Let $\Lambda = X(T)$ and $X = \Lambda \otimes \mathbf{R}$. The weights of ρ_T form a finite set S of elements $s \in X$ taken with multiplicity $m(s)$. We define an inner product on X^* by setting

$$(x_1^*, x_2^*) = \sum_{s \in S} m(s)\, x_1^*(s)\, x_2^*(s).$$

We denote the dual inner product on X by $\langle , \rangle$. Since ρ is faithful, S spans X so $(,)$ and hence $\langle , \rangle$, is positive definite. As S is W-invariant, so is $\langle , \rangle$. Note that this invariance alone determines $\langle , \rangle$ up to a positive scalar factor on each simple root system, by the W-irreducibility of simple root systems ([1] VI § 1 Prop. 5 Cor. (i)). Let $\mathbf{Z}\Phi \subset X$ denote the root lattice, and

$$\Lambda_\Phi = \left\{ \lambda \in X \,\middle|\, \frac{2\langle \lambda, \alpha \rangle}{\langle a, a \rangle} \in \mathbf{Z}\ \forall \alpha \in \Phi \right\}$$

the weight lattice. Since $\langle , \rangle$ is determined by T and ρ_T, by the above proposition dimension data corresponds to the data $(T, \rho_T, X, \Lambda, \langle , \rangle, \Gamma^\circ, \mathbf{Q}F^\circ)$ up to isomorphism. In the rest of the paper we study this 7-tuple in pieces: Sects. 2 and 3 concentrate on $(\Lambda, \langle , \rangle, \mathbf{Q}F^\circ)$, and Sect. 4 on (T, ρ_T).

2. A root argument

Definition 1. *By a* root system *we mean a finite set Φ spanning a real vector space, and an inner product $\langle , \rangle$ on $\mathbf{R}\Phi$ such that*

i) $0 \notin \Phi$, and $x \in \Phi$ if and only if $-x \in \Phi$.

i) For all $\alpha \in \Phi$, S_α, the reflection through the hyperplane perpendicular to α, preserves Φ.

iii) $\dfrac{2\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle} \in \mathbf{Z}$ *for all α, $\beta \in \Phi$.*

If, in addition, α, $K\alpha \in \Phi$ implies $K = \pm 1$, we say Φ is reduced.

Every reduced root system is associated to some semimsimple Lie algebra. A simple Lie algebra determines the corresponding reduced root system up to scalar multiplication; more generally, a semisimple Lie algebra $\mathbf{g}$ determines the corresponding root system up to scalar multiplication on each simple factor. In what follows we will often speak loosely of "the" root system of type $\mathbf{g}$. Whenever the normalization of $\langle , \rangle$ matters, however, we will take care to specify it more precisely.

Definition 2. *A* root subsystem *$\Phi \subset \Psi$ of a root system Ψ is a subset of Ψ such that Φ is root system and the inner products $\langle , \rangle_\Phi$ and $\langle , \rangle_\Psi$ coincide on* $\mathbf{R}\Phi \subset \mathbf{R}\Psi$.

The Weyl group of a root system is generated by reflections S_α; these depend on α and the inner product on X. In particular, with our definitions, the Weyl group of a root subsystem of Φ is a subgroup of the Weyl group $W(\Phi)$.

Definition 3. *A sum of root systems* $\Phi_1+\Phi_2$ *is a root system in* $\mathbf{R}\Phi_1\oplus\mathbf{R}\Phi_2$ *obtained by taking the union of the (orthogonal) sets* $\Phi_i\subset\mathbf{R}\Phi_i\subset\mathbf{R}\Phi_1\oplus\mathbf{R}\Phi_2$. *The* Φ_i *are* factors *of* Φ. *A root system which is not of the form* $\Phi_1+\Phi_2$ *is* simple *or* irreducible.

Theorem 1 is a consequence of the following theorem:

Theorem 1′. *Fix a finite dimensional* $\mathbf{R}$*-vector space* X *with a positive definite inner product* $\langle\,,\rangle$ *and a lattice* $\Lambda\subset X$, *and let* Γ *be the (necessarily finite) group of all isometries of* Λ. *Then every reduced root system* Φ *in* X *of rank* $\dim(X)$ *such that* $\mathbf{Z}\Phi\subset\Lambda\subset\Lambda_\Phi$ *is determined up to conjugation by* Γ *by the 1-dimensional subspace* $\mathbf{Q}F\subset\mathbf{Q}[X]$, *where*

$$F=\sum_{\gamma\in\Gamma}\gamma(F_\Phi)\quad\textit{and}\quad F_\Phi=\prod_{\alpha\in\Phi}(1-[\alpha])$$

as in (2) *and* (3).

We observe that this implies in particular that the triple $(\Lambda, \langle\,,\rangle, \Phi)$ is determined up to isomorphism by $(X, \Lambda, \langle\,,\rangle, \mathbf{Q}F)$. Since Γ contains the group Γ° defined in § 1, $\mathbf{Q}F$ is determined by $\mathbf{Q}F^\circ$. Theorem 1 follows: $Lie(G)$ is determined by dimension data (because Φ is), and the center of G is determined by dimension data (because Λ is). If Theorem 1′ remained true with Γ replaced by an arbitrary group of isometries containing $W(\Phi)$, it would imply Theorem 2 without the irreducibility hypothesis. Unfortunately, as Theorem 3 shows, this is too much to hope for. The rest of this section gives a proof of Theorem 1′, modulo some technical results proved in § 3. Note that throughout we employ additive notation for the group law on characters.

Definition 4. *A* short root *in a root system* Φ *is any root which is short in its irreducible component of* Φ. *We denote the set of short roots* Φ°.

Lemma 1. *For every root system* Φ, *the short roots generate the root lattice* $\mathbf{Z}\Phi$.

Proof. The root lattice of a sum of root systems is the direct sum of the root lattices of the components, so it suffices to check irreducible root systems. If Φ is reduced, every pair α, β of roots of different length generates a root subsystem of rank 2, either $B_2=C_2$, G_2, or A_1+A_1. In the first two cases the long root is a sum of short roots. Thus every long root which is not a sum of short roots is orthogonal to all short roots. The Weyl group of a simple root system Φ acts irreducibly on $\mathbf{R}\Phi$ ([1] VI § 1 Prop. 5 Cor. (i)) and stabilizes the set of short roots, so this is impossible. If Φ is BC_n, the root lattice is $\mathbf{Z}^n$, which is generated by the short roots e_i. □

Lemma 2. *The set* Φ° *is a root system.*

Proof. Every reflection $\sigma\in W(\Phi)$ fixes all but one component, Ψ of Φ. If $\alpha\notin\Psi$, $\sigma(\alpha)=\alpha$. If $\alpha\in\Psi$, $\|\sigma(\alpha)\|^2=\|\alpha\|^2$, so $\sigma(\alpha)$ is short in Ψ and hence in Φ. Finally, the short roots span $\mathbf{R}\Phi$ by Lemma 1. □

Lemma 3. *If Φ is a root system, Λ a lattice such that $\mathbf{Z}\Phi \subset \Lambda \subset \Lambda_\Phi$, and Γ a group of isometries such that $W(\Phi) \subset \Gamma \subset \mathrm{Aut}(\Lambda)$, then $\Gamma\Phi$ is a (not necessarily reduced) root system.*

Proof. As Γ is contained in a compact (orthogonal) group and fixes a lattice, it is finite. Hence $\Gamma\Phi$ is finite. It obviously spans $\mathbf{R}\Phi$. As Φ satisfies Definition 1 (i), so does $\Gamma\Phi$. If $\gamma\alpha \in \Gamma\Phi$, the reflection in $(\gamma\alpha)^\perp$ is

$$S_{\gamma\alpha} = \gamma S_\alpha \gamma^{-1}.$$

As $\Gamma \supset W(\Phi)$, γ and S_α belong to Γ, so $S_{\gamma\alpha} \in \Gamma$ fixes $\Gamma\Phi$. Finally, if $\gamma_1 \alpha_1, \gamma_2 \alpha_2 \in \Gamma\Phi$, then

$$\frac{2\langle \gamma_1 \alpha_1, \gamma_2 \alpha_2 \rangle}{\|\gamma_1 \alpha_1\|^2} = \frac{2\langle \alpha_1, \gamma_1^{-1} \gamma_2 \alpha_2 \rangle}{\|\alpha_1\|^2} \in \mathbf{Z},$$

since $\gamma_1^{-1} \gamma_2 \alpha_2 \in \Gamma\Phi \subset \Lambda \subset \Lambda_\Phi$. $\square$

Lemma 4. *Under the hypotheses of Theorem* 1′, $\mathbf{Q}F$ *determines* $(\Gamma\Phi)^\circ$.

Proof. Let δ denote one half the sum of the roots of Φ^+. As

$$\begin{aligned} F_\Phi &= \Big[\prod_{\alpha\in\Phi^+}\Big(\Big[-\frac{\alpha}{2}\Big]-\Big[\frac{\alpha}{2}\Big]\Big)\Big]\Big[\prod_{\alpha\in\Phi^+}\Big(\Big[\frac{\alpha}{2}\Big]-\Big[-\frac{\alpha}{2}\Big]\Big)\Big] \\ &= \Big[\sum_{w\in W} \mathrm{sgn}(w)\,[-w\delta]\Big]\Big[\sum_{w\in W} \mathrm{sgn}(w)\,[wd]\Big] \\ &= \sum_{w'\in W} w'\Big(\sum_{w\in W} \mathrm{sgn}(w)\,[\delta - w\delta]\Big), \end{aligned}$$

we have

$$F = \sum_{\gamma\in\Gamma} \gamma(F_\Phi) = |W| \sum_{\gamma\in\Gamma} \gamma\Big(\sum_{w\in W} \mathrm{sgn}(w)\,[\delta - w\delta]\Big).$$

Now,

$$\|\delta - w\delta\|^2 = 2\|\delta\|^2 - 2\langle \delta, w\delta\rangle = \langle 2\delta, \delta - w\delta\rangle,$$

and

$$\delta - w\delta = \frac{1}{2}\sum_{\alpha\in\Phi^+} \alpha - \frac{1}{2}\sum_{\alpha\in\Phi^+} w\alpha = \sum_{\alpha\in\Phi^+\cap w\Phi^-} \alpha,$$

so

$$\|\delta - w\delta\|^2 = \sum_{\Phi^+\cap w\Phi^-} 2\langle \delta, \alpha\rangle.$$

If $\alpha = \Sigma r_i \alpha_i$, where the α_i are simple roots,

$$2\langle \delta, \alpha\rangle = \sum_i r_i \|\alpha_i\|^2,$$

since δ is the sum of the fundamental weights ([1] VI § 1 Prop. 29). Thus, if $w \neq 1$,

$$\|\delta - w\delta\|^2 \geqq \min_{\alpha\in\Phi} \|\alpha\|^2,$$

with equality if and only if $\Phi^+ \cap w(\Phi^-) = \{\alpha\}$ and α is a simple root of shortest possible length. By [1] VI § 1 Prop. 17, this occurs if and only if the Weyl chambers associated to Φ^+ and $w(\Phi^+)$ are joined by the wall $\alpha^\perp$, i.e., if and only if $w = S_\alpha$. As all occurrences of terms of length $\min_{\alpha\in\Phi}\|\alpha\|^2$ in F_Φ (and therefore in F) have $\mathrm{sgn}(S_\alpha) = -1$, there can be no cancellation. Therefore the terms of minimal non-zero length in F are precisely the $[\gamma\alpha]$, where $\gamma\in\Gamma$, and α is a root of minimal length in Φ. Let X' be the vector space generated by these $\gamma\alpha$, and let X'' be its orthogonal complement. By construction X' is Γ-stable, hence W-stable, and hence the span of a (root system) factor Φ' of Φ. If Φ'' is the complementary factor,

$$F_{\Phi'} \otimes F_{\Phi''} = F_\Phi \in \mathbf{Q}[\mathbf{Z}\Phi] = \mathbf{Q}[\mathbf{Z}\Phi'] \otimes \mathbf{Q}[\mathbf{Z}\Phi''].$$

The constant term of $F_{\Phi'}$ is non-zero, so the terms in F_Φ which belong to $\mathbf{Q}[X'']$ are, up to a constant non-zero multiple, just the terms of $F_{\Phi''}$. By induction on rank, the lemma follows. □

Lemma 5. *Consider a root system $\Omega = \Omega^\circ$ and a lattice Λ with $\mathbf{Z}\Omega \subset \Lambda \subset \Lambda_\Omega$. Then the collection of all (possibly non-reduced) root systems $\Psi \supset \Omega$ such that $\Psi^\circ = \Omega$ and $\Lambda_\Psi \supset \Lambda$ possesses a unique maximal element.*

Proof. Denote the collection of all such Ψ by S. Since Ω is an element, S is non-empty. By Lemma 1, the short roots generate the root lattice. In any simple root system, $\langle\alpha, \alpha\rangle \leqq 4\langle\beta, \beta\rangle$ for all roots α and β ([1] VI § 1 Prop. 12–14), so every element of every $\Psi \in S$ lies in a bounded subset of $\mathbf{Z}\Omega = \mathbf{Z}\Psi^\circ = \mathbf{Z}\Psi$. It follows that S is finite, so it suffices to prove that for any $\Psi_1, \Psi_2 \in S$ there exists a root system $\Psi_3 \in S$ containing both. Elements of $W(\Psi_i)$ preserve length and therefore preserve $\Psi_i^\circ = \Omega$. We let W_{12} denote the subgroup of $\mathrm{Aut}(\Omega)$ generated by $W(\Psi_1)$ and $W(\Psi_2)$. We set $\Psi_3 = W_{12}(\Psi_1 \cup \Psi_2)$. Given $w \in W_{12}$, $\alpha \in \Psi_1 \cup \Psi_2$, we have

$$S_{w\alpha} = w S_\alpha w^{-1} \in W_{12},$$

so Ψ_3 is stable by $S_{w\alpha}$. The root lattices generated by Ω, Ψ_1, and Ψ_2 all coincide, and W_{12} preserves this lattice. Therefore, Ψ_3 also satisfies the integrality condition for root systems. The inclusions $\mathbf{Z}\Psi_i \subset \Lambda \subset \Lambda_{\Psi_i}$ show that Λ is stable by $W(\Psi_i)$. It is therefore preserved by W_{12}, and hence

$$\Lambda \subset \bigcap_{w\in W_{12}} w(\Lambda_{\Psi_1} \cap \Lambda_{\Psi_2}) = \Lambda_{\Psi_3},$$

as desired. Finally, $\Psi_3^\circ = \Omega$. Indeed, W_{12} stabilizes Ω, and each of the systems Ψ_i already contains Ω. On the other hand, every short root of Ψ_3 is an $\mathrm{Aut}(\Omega)$-translate of an element α of some Ψ_i and can be short in Ψ_3 only if α is short in Ψ_i, i.e., only if $\alpha \in \Omega$. □

Let Ψ be the unique maximal root system given by Lemma 5, such that $\Psi^\circ = (\Gamma\Phi)^\circ$ and $\Lambda \subset \Lambda_\Psi$. We have just seen that $(X, \Lambda, \langle\,,\rangle, \mathbf{Q}F)$ determines Ψ. Moreover, Ψ contains Φ and is stable by Γ. We are therefore led to the following slightly more general question: Fix a possibly non-reduced root system Ψ and a lattice $\mathbf{Z}\Psi \subset \Lambda \subset \Lambda_\Psi$. Let $\Gamma = \mathrm{Aut}(\Psi, \Lambda)$ be the group of all isometries of Λ that preserve Ψ. Let $\Phi \subset \Psi$ be a root subsystem of equal rank; we do not require that it be a closed subsystem, i.e., that it satisfy the condition $\alpha + \beta \in \Psi$

implies $\alpha+\beta\in\Phi$ for all $\alpha, \beta\in\Phi$. To what extent is Φ determined by the 1-dimensional subspace $\mathbf{Q}F$, defined as in Theorem 1′? The following two propositions address this question.

Proposition 1. *Let Ψ be a reduced root system, and $\{\Phi_i\}$ a set of representatives for the $W(\Psi)$-conjugacy classes of subsystems $\Phi\subset\Psi$ such that $rk(\Phi)=rk(\Psi)$. Then the sums*

$$\sum_{\gamma\in W(\Psi)} \gamma(F_{\Phi_i})$$

are linearly independent.

Note that if $\Psi=\Psi'+\Psi''$, then Ψ' and Ψ'' are both reduced, and every equal rank subsystem Φ of Ψ is of the form $\Phi'+\Phi''$, where Φ' and Φ'' are equal rank subsystems of Ψ' and Ψ'' respectively. Moreover, $W(\Psi)=W(\Psi')\times W(\Psi'')$, so two subsystems Φ_1 and Φ_2 are Weyl group conjugate if and only if their corresponding components are Weyl group conjugate. As

$$F_{\Phi'}\otimes F_{\Phi''}=F_{\Phi}\in\mathbf{Q}[\mathbf{Z}\Psi]=\mathbf{Q}[\mathbf{Z}\Psi']\otimes\mathbf{Q}[\mathbf{Z}\Psi''],$$

the $W(\Psi)$-trace of F_Φ is just the tensor product of the $W(\Psi')$- and $W(\Psi'')$-traces of $F_{\Phi'}$ and $F_{\Phi''}$ respectively. If $\{u_i\}$ and $\{v_i\}$ are linearly independent vectors in U and V respectively, then $\{u_i\otimes v_i\}$ is a linearly independent set of vectors in $U\otimes V$. Applying this fact to the group algebras above, it suffices to prove Proposition 1 for simple reduced root systems.

Proposition 2. *Let $\Psi=BC_n$. Let $\{\Phi_i\}$ be a set of representatives for the $W(\Psi)$-conjugacy classes of subsystems $\Phi\subset\Psi$ of equal rank. Then the*

$$\sum_{\gamma\in W(\Psi)} \gamma(F_{\Phi_i})$$

are pairwise *linearly independent.*

Proof of Theorem 1′. Decompose $\Psi=\Psi_0+\Psi_{nr}$, where Ψ_0 is the sum of all reduced simple factors of Ψ, and Ψ_{nr} the sum of all factors of type BC. Since for every root system of type BC the root lattice coincides with the weight lattice ([1] VI Ex. § 1, no. 3), the inclusions $\mathbf{Z}\Psi\subset\Lambda\subset\Lambda_\Psi$ show that we have an orthogonal decomposition $\Lambda=\Lambda_0\oplus\mathbf{Z}\Psi_{nr}$, with $\mathbf{Z}\Psi_0\subset\Lambda_0\subset\Lambda_{\Psi_0}$. By definition Ψ is stabilized by Γ, so in particular this decomposition is stabilized. Thus, if Γ_0 is the group of all isometries of Λ_0, and Γ_{nr} the group of all isometries of $\mathbf{Z}\Psi_{nr}$, the inclusion $\Gamma_0\times\Gamma_{nr}\subset\Gamma$ is in fact an equality. Next let $\Psi_{nr}=\sum_{i=1}^s\Psi_i$ be the decomposition into simple $\Psi_i\cong BC_{n_i}$. If an element of Γ_{nr} maps a short root of Ψ_i to one of Ψ_j, with $i\neq j$, then the short roots of both have the same length, so $\Psi_i+\Psi_j$ can be embedded into some $BC_{n_i+n_j}$. As $BC_{n_i+n_j}$ has the same set of short roots and the same weight lattice as $\Psi_i+\Psi_j$, by our definition of Ψ this cannot happen. Thus Γ_{nr} stabilizes each Ψ_i. Letting Γ_i be the group of all isometries of $\mathbf{Z}\Phi_i$, it follows as above that $\Gamma_{nr}=\prod_{i=1}^s\Gamma_i$. In particular the decomposition $\Psi=\sum_{i=0}^s\Psi_i$ corresponds to a decomposition $\Gamma=\prod_{i=0}^s\Gamma_i$. With $\Phi_i=\Phi\cap\Psi_i$ this implies

$$\sum_{\gamma\in\Gamma}\gamma(F_\Phi)=\prod_{i=0}^{s}\Big(\sum_{\gamma\in\Gamma_i}\gamma(F_{\Phi_i})\Big),$$

so all our data decomposes. Reasoning as in Proposition 1, we reduce to the following two cases: Ψ is reduced, or $\Psi = BC_n$.

In the latter case $\Gamma = \mathrm{Aut}(\Psi) = W(\Psi)$, so the assertion is an immediate consequence of Proposition 2. In the former case observe that if Γ is any group of automorphisms of Ψ that contains $W(\Psi)$, the linear independence of Γ-traces is weaker than the linear independence of $W(\Psi)$-traces. Applying this to our given Ψ and Γ, we obtain Theorem 1′.

To prove Propositions 1 and 2 we first make a list of all possible equal rank subsystems of simple root systems. In order to do this, we introduce some notation. When there are exactly two different simple root subsystems Φ of Ψ which differ by a non-trivial scalar factor in $\langle\, ,\rangle$, we distinguish them by writing Φ^s and Φ^l for the embeddings corresponding to the smaller and larger inner product respectively. For instance, every pair of roots $\pm\alpha$ in a root system Ψ corresponds to a root subsystem $A_1 \subset \Phi$. If Ψ has roots of two different lengths, A_1^s and A_1^l correspond to short and long embeddings respectively. The problem of how to denote embeddings of A_1 in BC_n is solved by the following conventions: $C_1 = A_1$ consists of a pair of long roots of BC_n, $B_1 = A_1$ consists of a pair of short roots, and $D_2 = A_1 + A_1$ consists of two orthogonal pairs of roots of intermediate length. We also identify D_3 and A_3.

Proposition 3. *The following list gives for each simple Ψ a complete list of reduced root subsystems Φ of equal rank. For each type there is just one conjugacy class under $W(\Psi)$.*

A_r : A_r.

BC_r: $\sum B_{b_i} + \sum C_{c_i} + \sum D_{d_i}$
$(\sum b_i + \sum c_i + \sum d_i = r$, *all* $b_i, c_i \geqq 1, d_i \geqq 2.)$

B_r : $\sum B_{b_i} + \sum D_{d_i}$
$(\sum b_i + \sum d_i = r$, *all* $b_i \geqq 1, d_i \geqq 2.)$

C_r : $\sum C_{c_i} + \sum D_{d_i}$
$(\sum c_i + \sum d_i = r$, *all* $c_i \geqq 1, d_i \geqq 2.)$

D_r : $\sum D_{d_i}$
$(\sum d_i = r$, *all* $d_i \geqq 2.)$

E_6 : $E_6, A_5 + A_1, 3A_2$.

E_7 : $E_7, D_6 + A_1, A_5 + A_2, 2A_3 + A_1, A_7, D_4 + 3A_1, 7A_1$.

E_8 : $E_8, A_8, D_8, A_7 + A_1, A_5 + A_2 + A_1, 2A_4, 4A_2, A_2 + E_6, A_1 + E_7,$
$D_6 + 2A_1, D_5 + A_3, 2D_4, D_4 + 4A_1, 2A_3 + 2A_1, 8A_1$.

F_4 : $F_4, B_4, D_4^l, B_3 + A_1^s, A_3^l + A_1^s, C_4, D_4^s, C_3 + A_1^l, A_3^s + A_1^l, 2B_2,$
$B_2 + 2A_1^l, B_2 + 2A_1^s, 4A_1^l, 2A_1^l + 2A_1^s, 4A_1^s, A_2^l + A_2^s$.

G_2 : $G_2, A_2^l, A_2^s, A_1^l + A_1^s$.

Proof. Suppose first that all the roots of Ψ have equal length. Then $\alpha, \beta, \alpha + \beta \in \Psi$ implies

$$\|\alpha\|^2 = \|\beta\|^2 = \|\alpha + \beta\|^2; \qquad \langle \alpha, \beta \rangle = \frac{-\|\alpha\|^2}{2}.$$

Therefore if $\alpha, \beta \in \Phi$,

$$S_\alpha(\beta) = \beta - \frac{2\langle \alpha, \beta \rangle \alpha}{\|\alpha\|^2} = \alpha + \beta \in \Phi.$$

Hence, root subsystems of Ψ correspond to Lie subalgebras. In these cases, that is when $\Psi = A_r$, D_r, or E_r, the assertions follow directly from [2], Table 10, and the remarks after Table 10.

If $\Psi = BC_r$, we define

$$i \sim j \Leftrightarrow i = j \text{ or } e_i - e_j \in \Phi \text{ or } e_i + e_j \in \Phi,$$

where $e_1, \ldots, e_r$ are the standard basis vectors of $\mathbf{Z}^r = \mathbf{Z}\Psi$. The invariance of Φ under reflections at any of its elements implies that this is an equivalence relation. Let S be an equivalence class for this relation. Then the roots

$$\Phi \cap \{e_i, 2e_i, \pm e_i \pm e_j \mid i, j \in S\}$$

form a root system Φ_S which is a factor of Φ. Since $\operatorname{rank}(\Phi) = r$, $\operatorname{rank}(\Phi_S) = |S|$. Changing the signs and indices of e_i if necessary, we may assume that $e_1 - e_2$, $\ldots, e_{|S|-1} - e_{|S|} \in \Phi_S$. Therefore Φ_S contains $A_{|S|-1}$. As $\operatorname{rank}(\Phi_S) = |S|$, some e_i, $2e_i$ or $e_i + e_j$ must belong to Φ_S. In the first case $\Phi_S = B_{|S|}$, in the second, $\Phi_S = C_{|S|}$, and otherwise $\Phi_S = D_{|S|}$. The conjugacy modulo $W(\Psi)$ is clear, and this shows the assertions for $\Psi \in \{B_r, C_r\}$ as well.

Next let $\Psi = F_4$. Observe that the orthogonal complement of A_1^s in D_4^s is of type $3A_1^s$. Therefore if $\Phi \cap D_4^s$ contains a simple factor of type A_1^s, or is empty, then Φ is contained in a copy of B_4 inside F_4. By the above calculation this gives rise to the following types, each of which represents one conjugacy class under $W(B_4)$:

$$B_4, B_3 + A_1^s, 2B_2, B_2 + 2A_1^s, 4A_1^s, B_2 + 2A_1^l, 2A_1^s + 2A_1^l, 4A_1^l, A_1^s + A_3^l, D_4^l.$$

By the remark after [2], Table 10, any two subsystems of F_4 of type B_4 are conjugate under $W(F_4)$. Thus each of these types occurs in precisely one conjugacy class under $W(F_4)$. Next by dualizing we obtain the complete list

$$C_4, C_3 + A_1^l, A_1^l + A_3^s, D_4^s$$

of all subsystems that are contained in a copy of C_4, but not in any copy of B_4. Again each such type corresponds to precisely one conjugacy class under $W(F_4)$. If Φ is any other subsystem, then by the above remark and by dualizing it must contain roots of both lengths, and neither $\Phi \cap D_4^s$ nor $\Phi \cap D_4^l$ may have a simple factor of type A_1. If Φ is simple, this leaves only the case $\Phi = F_4$. Otherwise Φ must decompose into two simple factors of rank 2. Clearly such a factor must be of type A_2, so only the case $\Phi = A_2^l + A_2^s$ remains. Since none of the other possible cases has a direct factor of type A_2^l, the factor A_2^s must be the orthogonal complement of A_2^l in F_4. Since by the above argument any A_2^l corresponds to a Lie subalgebra of D_4^l, and hence of F_4, it follows that any $\Phi = A_2^l + A_2^s$ corresponds to a Lie subalgebra of F_4. By [2], Table 10, there exists a unique conjugacy class of this type. This confirms the list given above.

The case $\Psi = G_2$ is trivial. □

Lemma 6. *Proposition* 1 *holds for every* $\Psi \in \{A_n, E_n, F_4, G_2\}$.

Proof. These case $\Psi = A_n$ is trivial. For the other cases observe that reasoning as in Lemma 4, the square length $\|\gamma(\delta - w\delta\|^2$ attains its greatest value exactly when $w\delta = -\delta$. Since there is a unique such $w \in W(\Phi)$, all these terms occur with the same sign in F. In particular the terms $\operatorname{sgn}(w)\,\gamma[\delta - w\delta]$ do not cancel in F, so $\|2\delta\|^2$ is the greatest square length of a vector occurring in F. We enumerate these values for $\Psi \in \{E_n, F_4, G_2\}$. The lengths are normalized so that $\|\alpha\|^2 = 1$ for every short root α. This table can easily be checked using the formulas in [1] ch. VI (planches) for the sum of all roots:

Ψ	Φ	$\|\lambda\|^2$	Ψ	Φ	$\|\lambda\|^2$
E_6	E_6	156	E_7	E_7	399
E_6	A_5+A_1	36	E_7	D_6+A_1	111
E_6	$3A_2$	12	E_7	A_7	84
			E_7	A_5+A_2	39
E_8	E_8	1240	E_7	D_4+3A_1	31
E_8	E_7+A_1	400	E_7	$2A_3+A_1$	21
E_8	D_8	280	E_7	$7A_1$	7
E_8	E_6+A_2	160			
E_8	A_8	120	F_4	F_4	156
E_8	D_6+2A_1	112	F_4	B_4	84
E_8	A_7+A_1	85	F_4	C_4	60
E_8	D_5+A_3	70	F_4	D_4^l	56
E_8	$2D_4$	56	F_4	$B_3+A_1^s$	36
E_8	$A_5+A_2+A_1$	40	F_4	$C_3+A_1^l$	30
E_8	$2A_4$	40	F_4	D_4^s	28
E_8	D_4+4A_1	32	F_4	$A_3^l+A_1^s$	21
E_8	$2A_3+2A_1$	22	F_4	$2B_2$	20
E_8	$4A_2$	16	F_4	$B_2+2A_1^l$	14
E_8	$8A_1$	8	F_4	$A_3^s+A_1^l$	12
			F_4	$A_2^l+A_2^s$	12
G_2	G_2	28	F_4	$B_2+2A_1^s$	12
G_2	A_2^l	12	F_4	$4A_1^l$	8
G_2	A_2^s	4	F_4	$2A_1^l+2A_1^s$	6
G_2	$A_1^l+A_1^s$	4	F_4	$4A_1^s$	4

For E_6 and E_7 this directly implies the desired linear independence, since every length occurs just once. In the remaining cases, we must modify the above lists, replacing the offending vectors by suitable linear combinations so that no length occurs twice. To find such linear combinations write $m(\Phi, l)$ for the sum of the coefficients of all the terms with $\|\ \|^2 = l$ in $\sum_{\gamma \in W(\Psi)} \gamma(F_\Phi)$. For $\Psi = E_8$ one checks that

$$\frac{m(\Phi, 38)}{m(\Phi, 40)} = \begin{cases} 23 & \text{if} \quad \Phi = A_5 + A_2 + A_1, \\ 22 & \text{if} \quad \Phi = 2A_4, \end{cases}$$

and for $\Psi = G_2$ that

$$\frac{m(\Phi, 3)}{m(\Phi, 4)} = \begin{cases} -1 & \text{if} \quad \Phi = A_1^l + A_1^s, \\ -2 & \text{if} \quad \Phi = A_2^s. \end{cases}$$

For $\Psi = F_4$ let $\Phi_1 = A_3^s + A_1^l$, $\Phi_2 = A_2^l + A_2^s$, and $\Phi_3 = B_2 + 2A_1^s$. One verifies the matrix equation

$$\left(1, \frac{m(\Phi_i, 11)}{m(\Phi_i, 12)}, \frac{m(\Phi_i, 10)}{m(\Phi_i, 12)}\right)_{1 \leq i \leq 3} = \begin{pmatrix} 1 & -3 & 0 \\ 1 & -2 & -2 \\ 1 & -3 & 2 \end{pmatrix},$$

and observes that this matrix is invertible. Thus in each of the three cases the length inequalities hold after the troublesome terms are replaced by suitable linear combinations. Now the above argument applies. □

The proof of Propositions 1 and 2 in the cases B_n, C_n, D_n, BC_n will be carried out in the next paragraph, where we develop the appropriate machinery.

3. Root subsystems of BC_n, and counter-examples

In this section we study root subsystems of BC_n, and construct the counter-example promised in the introduction. The idea is that if all BC_n are considered together, the functions F in theorem 1′ appear naturally as elements of a certain commutative algebra, such that direct sums of subsystems correspond to products in this algebra. The linear independence in § 2 Proposition 1 is then equivalent to algebraic independence in this algebra, and the assertion of § 2 Proposition 2 translates into a question about unique factorization. The counter-example can be constructed precisely because the analogue of Proposition 1 is false for BC_n.

Let $Z_n = \mathbf{Q}[\mathbf{Z}^n]$, $W_n = \{\pm 1\}^n \rtimes S_n$. For $m \leq n$, the injection

$$\mathbf{Z}^m \hookrightarrow \mathbf{Z}^n\colon (a_1, \ldots, a_m) \mapsto (a_1, \ldots, a_m, 0, \ldots, 0)$$

extends to an injection $i_{m,n}\colon Z_m \hookrightarrow Z_n$. We define $\phi_{m,n}\colon Z_m \to Z_n$:

$$\phi_{m,n}(z) = \frac{1}{|W_n|} \sum_{w \in W_n} w(i_{m,n}(z)).$$

Evidently $\phi_{m,n}\phi_{k,m} = \phi_{k,n}$ for $k \leq m \leq n$. The image of $\phi_{m,n}$ lies in $Y_n = Z_n^{W_n}$, so we can form the direct limit under $\phi_{m,n}$:

$$Y = \varinjlim_n Y_n.$$

We define maps $j_n\colon Z_n \to Y$ by composing $\phi_{n,p}$ with the injection $Y_p \hookrightarrow Y$ for any $p \geq n$. The maps $\phi_{m,n}$ are not ring homomorphisms, so *a priori*, Y is only a vector space. It can be endowed with an algebra structure as follows: The obvious isomorphism $\mathbf{Z}^m \oplus \mathbf{Z}^n \xrightarrow{\sim} \mathbf{Z}^{m+n}$ gives a canonical isomorphism M: $Z_m \otimes Z_n \xrightarrow{\sim} Z_{m+n}$. Given two elements of Y represented by $y \in Y_m$ and $y' \in Y_n$, we define

$$y y' = j_{m+n}(M(y \otimes y')).$$

This product is independent of the choice of m and n and is commutative and associative.

Let $\{e_i\}$ denote the standard basis for $\mathbf{Z}^n$. The monomials $[e_1]^{a_1}\dots[e_n]^{a_n}$ form a $\mathbf{Q}$-basis of Z_n. Therefore, Y has a $\mathbf{Q}$-basis

$$e(a_1, a_2, \dots, a_n) = j_n([e_1]^{a_1}\dots[e_n]^{a_n}),$$

indexed by $n \geqq 0$ and integers $a_1 \geqq a_2 \geqq \dots \geqq a_n > 0$. Mapping each such element to $x_{a_1} x_{a_2} \dots x_{a_n}$, we obtain a $\mathbf{Q}$-linear map from Y to the polynomial algebra in countably many variables $\mathbf{Q}[x_1, x_2, \dots]$.

Lemma. *This map is an algebra isomorphism.*

Proof. Since bijectivity is clear, it remains to prove that

$$e(a_1, \dots, a_m)\, e(b_1, \dots, b_n) = e(a_1, \dots, a_m, b_1, \dots, b_n).$$

This easy calculation is left to the reader. □

From now on we identify Y with $\mathbf{Q}[x_1, x_2, \dots]$ through the above isomorphism. The resemblance of this construction to that of the algebra of symmetric functions in infinitely many variables is perhaps somewhat misleading. Note, in particular, the very different way in which the multiplicative structure arises.

Let Φ be a root subsystem of equal rank of some BC_n. As usual we let F_Φ denote the Weyl product $\prod_{\alpha\in\Phi}(1-[\alpha])$. Setting $F(\Phi) = j_n(F_\Phi) \in Y$, we have $F(\Phi_1 + \Phi_2) = F(\Phi_1)\, F(\Phi_2)$ by construction. Let $b_n = F(B_n)$, $c_n = F(C_n)$ for $n \geqq 1$, and $d_n = F(D_n)$ for $n \geqq 2$. By Proposition 3 the $F(\Phi)$ for all $W(BC_n)$-conjugacy classes of root subsystems $\Phi \subset BC_n$ are precisely the monomials $\prod b_{l_i} \prod c_{m_j} \prod d_{n_k}$, where all $l_i, m_j \geqq 1$, $n_k \geqq 2$, and $\sum l_i + \sum m_j + \sum n_k = n$.

Proposition.
(a) *For every n, the elements $b_1, \dots, b_n$, $c_1, \dots, c_n$, and $d_2, \dots, d_{n+1}$ all lie in* $\mathbf{Q}[x_1, \dots, x_{2n}]$.
(b) *These elements are pairwise inequivalent primes in this ring.*
(c) *Each of the subset $\{b_1, \dots, b_n, c_1, \dots, c_n\}$, $\{b_1, \dots, b_n, d_2, \dots, d_{n+1}\}$, and $\{c_1, \dots, c_n, d_2, \dots, d_{n+1}\}$ is algebraically independent.*

Proof. We recall that

$$F_\Phi = \sum_{w_1\in W(\Phi)} w_1\Big(\sum_{w_2\in W(\Phi)} \operatorname{sgn}(w_2)[\delta - w_2 d]\Big).$$

For $\Phi \in \{B_n, C_n, D_n\}$ we have, in standard notation, $\delta = (n-\varepsilon, n-1-\varepsilon, \dots, 1-\varepsilon)$, where ε is $\frac{1}{2}$ for B_n, 0 for C_n, and 1 for D_n. By the known structure of the Weyl group we get the formula

$$\frac{1}{|W(\Phi)|} F(\Phi) = \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma) \sum_{e\in\{\pm 1\}^n} \prod_{i=1}^{n} e_i\, x_{|(i-\varepsilon) - e_i(\sigma(i)-\varepsilon)|}, \tag{4}$$

where in the case $\Phi = D_n$ the second sum is extended only over those $e \in \{\pm 1\}^n$ with $\prod_{i=1}^n e_n = 1$, and where we write $x_0 = 1$. It suffices to prove the assertions for $b'_n = \dfrac{b_n}{2^n n!}$, $c'_n = \dfrac{c_n}{2^n n!}$, and $d'_n = \dfrac{d_n}{2^{n-1} n!}$. We immediately find that $b'_1 = -x_1 + 1$, $c'_1 = -x_2 + 1$, and $d'_2 = x_2 - 2x_1^2 + 1$, so all three assertions are obvious for $n = 1$. Let $R_n = \mathbf{Q}[x_1, \dots, x_n]$ for every $n \geqq 0$, and for simplicity set $b_0 = c_0 = d_1 = 1$. In the above sum let us separate the terms into (i) those with $\sigma n = n$ and $e_n = -1$,

(ii) those with $\sigma n = n-1$, $\sigma(n-1)=n$, and $e_n = e_{n-1} = -1$, and (iii) the remaining terms. Then for $n \geqq 2$ (4) implies

$$\begin{aligned} b'_n &\in -x_{2n-1}\, b'_{n-1} - x_{2n-2}^2\, b'_{n-2} + x_{2n-2}\, R_{2n-3} + R_{2n-3}, \\ c'_n &\in -x_{2n}\, c'_{n-1} - x_{2n-1}^2\, c'_{n-2} + x_{2n-1}\, R_{2n-2} + R_{2n-2}, \\ d'_{n+1} &\in -x_{2n}\, d'_n - x_{2n-1}^2\, d'_{n-1} + x_{2n-1}\, R_{2n-2} + R_{2n-2}. \end{aligned} \tag{5}$$

In particular we have $b'_n \in R_{2n-1}$, and c'_n, $d'_{n+1} \in R_{2n}$, which by induction proves (a).

Let us do (c) next. By induction it suffices to prove that each of the sets $\{b'_n, c'_n\}$, $\{b'_n, d'_{n+1}\}$, and $\{c'_n, d'_{n+1}\}$ is algebraically independent over R_{2n-2}, for $n \geqq 2$. Since (5) implies that $b'_n \in R_{2n-1} \setminus R_{2n-2}$ and $c'_n \in R_{2n} \setminus R_{2n-1}$, this is so for $\{b'_n, c'_n\}$. The same argument applies to $\{b'_n, d'_{n+1})$. Suppose that c'_n and d'_{n+1} are algebraically dependent over R_{2n-2}. Then by (5) we must have

$$d'_n\, c'_n - c'_{n-1}\, d'_{n+1} \in x_{2n-1}^2 (c'_{n-1}\, d'_{n-1} - d'_n\, c'_{n-2}) + x_{2n-1}\, R_{2n-2} + R_{2n-2}.$$

This implies $c'_{n-1}\, d'_{n-1} = d'_n\, c'_{n-2}$. But by the induction hypothesis, c'_{n-1} and d'_n are algebraically independent over R_{2n-4}. Since c'_{n-2}, d'_{n-1} are non-zero elements of R_{2n-4}, we get a contradiction.

Returning to (b), observe that by (c) any two b_l, c_m, d_{n+1} are algebraically independent. Thus it suffices to prove that each of these is a prime. Again we proceed by induction over n, noting that any prime of R_k stays prime in R_{k+m}, since the latter is a polynomial ring over the former. Thus it remains to prove the primality of b'_n in R_{2n-1} and of c'_n, d'_{n+1} in R_{2n}. By the inductive assumption, b'_{n-1} is a prime in the ring R_{2n-3}, and by the above formula it is not contained in any smaller R_k. Since, again by the inductive assumption, b'_{n-2} is nonzero, and lies in R_{2n-4}, the prime b'_{n-1} cannot divide b'_{n-2}. Therefore, by the above formula it does not divide b'_n. This in turn implies that b'_n is prime. The proofs for c'_n and d'_{n+1} are analogous. □

It is now easy to prove § 2 Proposition 2 and finish off § 2 Proposition 1. In fact, for $\Psi = B_n$ and $\Psi = C_n$ the assertion of Proposition 1 is equivalent to the linear independence of all monomials in $\{b_1, b_2, \ldots, d_2, d_3, \ldots\}$, respectively in $\{c_1, c_2, \ldots, d_2, d_3, \ldots\}$. This is just the algebraic independence in (c) above. For $\Psi = D_n$ this gives the same assertion, but with $W(D_n)$ replaced by $W(C_n)$. But by § 2 Proposition 3, the root subsystems of D_n form the same conjugacy classes under either of these groups, so Proposition 1 also follows for D_n. The assertion of Proposition 2 is equivalent to the linear independence of any two monomials in $\{b_1, b_2, \ldots, c_1, c_2, \ldots, d_2, d_3, \ldots\}$. This follows from (b) above, and unique factorization in Y.

Also it is easy to see that the analogue of Proposition 1 is false for every sufficiently large BC_n. In fact, consider

$$\frac{b_k}{b_1^k}, \quad \frac{c_k}{b_1^k}, \quad \frac{d_{k+1}}{b_1^{k+1}} \in \mathbf{Q}(x_1, \ldots, x_{2m})$$

for $1 \leqq k \leqq m$. Leaving out $1 = b_1/b_1$, they form $3m-1$ distinct nonscalar elements in a field of transcendence degree $2m$ over $\mathbf{Q}$. Therefore, if $m > 1$, they satisfy non-trivial polynomial equations with rational coefficients, i.e. there exist non-trivial linear relations between monomials in these elements. After multiplying through by a sufficiently large power of b_1, we get non-trivial linear relations between monomials of equal weight, where each of b_k, c_k, d_k has weight k. If the total weight of such a relation is n, then this shows that there exist $W(BC_n)$-inequivalent subsystems $\Phi_1, \ldots, \Phi_r \subset BC_n$ of rank n, such that the $F(\Phi_1), \ldots, F(\Phi_r)$ are linearly dependent.

We can now prove Theorem 3. Fix such $\Phi_i \subset BC_n$. For each i let G_i be a semisimple group and $T_i \subset G_i$ a maximal torus with $\Phi_i \subset \mathbf{Z}^n = X(T_i)$ the associated root system. We shall construct elements $v_1, \ldots v_r$ in Y_n, such that for all i, j there exists a faithful representation V_{ij} of G_i with formal character v_j. If we require that the image of G_i in $GL(V_{ij})$ contains no scalar matrix except the identity, then

$$V = \bigoplus_{\sigma \in A_r} V_{1\sigma(1)} \otimes \ldots \otimes V_{r\sigma(r)}$$

and

$$V' = \bigoplus_{\sigma \in S_r \setminus A_r} V_{1\sigma(1)} \otimes \ldots \otimes V_{r\sigma(r)}$$

are faithful representations of $G = \prod_{i=1}^{r} G_i$.

Lemma. *We can choose v_i and V_{ij} as above so that the stabilizer in $GL(\mathbf{Z}^{rn})$ of the formal character of V is $W_n^r \rtimes A_r$. (Recall that $W_n = \{\pm 1\}^n \rtimes S_n$.)*

Proof. Given any semi-simple Lie algebra $\mathbf{g}$ with weight lattice $X = \Lambda_{\Phi(\mathbf{g})}$ and Weyl group W, every element $\rho \in \mathbf{Z}[X]^W$ corresponds to a virtual representation of $\mathbf{g}$. The condition that ρ correspond to an effective representation can be expressed by saying that the coefficient of every vector $x \in X$ must be larger than some linear combination of the coefficients of vectors of greater length than x. In particular, if we start with some value of x and declare that its coefficient is 1 and that no longer vectors has non-zero coefficient, we can then proceed inward, making each coefficient sufficiently large as we go. Of course, in choosing the v_i we have to satisfy effectivity conditions for many Lie algebras simultaneously, but we can always satisfy a finite number of conditions of the form (coefficient of x)$> C_i$. If we choose v_1 with longest vector $(1, 2, \ldots, n)$, v_2 with longest vector $(n+1, n+2, \ldots, 2n)$, and so on, we see that the orbit of $v_1 \otimes \ldots \otimes v_r$ under W_{rn}/W_n^r consists of linearly independent elements of $\mathbf{Z}[\mathbf{Z}^{rn}]$. Indeed, each $w(v_1 \otimes \ldots \otimes v_r)$ possesses a $w([1, 2, \ldots, rn])$ term, which no other $\tau(v_1 \otimes \ldots \otimes v_r)$ can have. We conclude that the trace of $v_1 \otimes \ldots \otimes v_r$ under A_r is invariant by $W_n^r \rtimes A_r$ and no more. Finally the v_i can be chosen such that every V_{ij} contains at least one copy of the trivial representation. This guarantees that the condition about scalars is also satisfied. □

We have constructed faithful representations (V, ρ) and (V', ρ') of G. Letting $T = \prod_{i=1}^r T_i$, by the lemma we have $\Gamma = \mathrm{Aut}(T, \rho_T) = W_n^r \rtimes A_r$. Thus by construction $\rho'_T \cong w(\rho_T)$ for every $w \in W_n^r \rtimes S_r - W_n^r \rtimes A_r$, but for no element $w \in W_n^r \rtimes A_r = \Gamma$. In particular the pairs (G, V) and (G, V') are not isomorphic, and $\mathrm{Aut}(T, \rho'_T)$ is also equal to Γ. Finally the linear dependence of the $F(\Phi_i)$ means that

$$F(\Phi_1) \wedge \ldots \wedge F(\Phi_r) = 0$$

in $A^r\mathbf{Q}[\mathbf{Z}^n]$. This implies that

$$F=\sum_{\gamma\in\Gamma}\gamma(F_\Phi)=\sum_{\gamma\in W_n^r\rtimes A_r}\gamma(F_{\Phi_1}\otimes\ldots\otimes F_{\Phi_r})$$

$$=\sum_{\sigma\in A_r}\sigma\left(\bigotimes_{i=1}^r\sum_{w\in W_n}w(F_{\Phi_i})\right)=\sum_{\sigma\in A_r}\sigma\left(\bigotimes_{i=1}^r F(\Phi_i)\right)$$

is invariant under the whole of $W_n^r\rtimes S_r$. Thus the triples $(T,\rho_T,\mathbf{Q}F)$ and $(T,\rho'_T,\mathbf{Q}F)$ are isomorphic, so by § 1 Proposition 2, the dimension data for (G, V) and (G, V') coincide. □

One of the simplest examples that can be constructed by the above method is the following. We have seen that there is a non-trivial linear relation between monomials in b_1, b_2, c_1, c_2, d_2, d_3 of equal weight. It is easy to check that there exists no such relation for a proper subset of these elements. Explicit calculation shows that there is a non-trivial relation between 13 distinct monomials of total weight 6. The above construction yields a semisimple group of rank $13\times 6=78$. It can be checked that it consists of 60 almost simple factors of abstract type A_1, B_2, and A_3. Clearly the smallest representation in the above lemma must be rather large.

4. A weight argument

In this section we consider a faithful irreducible representation (V,ρ) of a connected semisimple group G. Let T be a maximal torus of G, and ρ_T the restriction of ρ to T. We want to study to what extent the isomorphism class of (G, V) is determined by the isomorphism class of (T,ρ_T). Under the given assumptions, it turns out that the ambiguities can be described completely.

If (G_i, V_i) and (G'_i, V'_i) are pairs of irreducible representations of semisimple groups with the same formal character, then $(G_1\times\ldots\times G_k, V_1\otimes\ldots\otimes V_k)$ and $(G'_1\times\ldots\times G'_k, V'_1\otimes\ldots\otimes V'_k)$ have the same formal character. So in order to describe all ambiguities, it is enough to give a generating set of ambiguous pairs. This set is described in the following theorem. Using it, at the end of this section we prove theorem 2.

Theorem 4. *Let T be a torus, and ρ_T a faithful representation of T. Suppose that there exists a connected semisimple group G, a faithful irreducible representation ρ of G, and an isomorphism between T and a maximal torus of G, such that ρ_T is the pull-back of ρ. Then the pair (G,ρ) is uniquely determined up to isomorphism by (T,ρ_T), except for the equivalences generated by the following exceptions:*
(a) *The spin representation of B_n restricts to the tensor product of the spin representations of $\sum B_{n_i}\subset B_n$, $\sum n_i=n$.*
(b) *For every $n\geqq 2$ the representations of C_n and D_n with the highest weight $(k, k-1, \ldots, 1, 0, \ldots, 0)$ for $1<k\leqq n-1$ have the same formal character.*
(c) *There are unique irreducible representations of A_2 and G_2, of dimension 27, which have the same formal character.*
(d) *There are unique irreducible represenations of D_4, C_4, and F_4, of dimension $4096=2^{12}$, which have the same formal character.*

Here as usual the conventions $B_1 = A_1^s$, $D_2 = 2A_1^s \subset C_2$, *and* $D_3 = A_3$ *are in force. Moreover the relation for* $D_2 \subset C_2$ *in* (b) *is the same as that for* $2B_1 \subset B_2$ *in* (a), *and the relation for* $D_4 \subset C_4$ *in* (b) *is the same as that in* (d).

Proof. Since ρ is faithful, it suffices to prove that the root system of G is uniquely determined in $X = X(T) \otimes \mathbf{R}$, with the exceptions mentioned above. Let $\Gamma = \mathrm{Aut}(T, \rho_T)$, and $X = \bigoplus X_i$ be a decomposition into Γ-irreducible subspaces. Letting Φ denote the root system of G, the representation of $W(\Phi)$ on X contains every irreducible representation at most once. Since $\Gamma \supset W(\Phi)$, the same follows for Γ, so the realization of each X_i as a subspace of X is unique. Now with $\Phi_i = X_i \cap \Phi$ we have $\Phi = \sum \Phi_i$. Moreover $\rho_T \in \mathbf{Z}[X] \cong \bigotimes Z[X_i]$ splits canonically as $\bigotimes \rho_i$, where ρ_i is the formal character of a faithful irreducible representation of the Lie algebra with root system Φ_i. This decomposition cannot depend on G because an element of $\mathbf{C}[x_1^{\pm}, \ldots, x_r^{\pm}, y_1^{\pm}, \ldots, y_s^{\pm}]$ can be written as a product of a Laurent polynomial in x and a Laurent polynomial in y in at most one way (up to scalars). Indeed, $f_1(x)\, g_1(y) = f_2(x)\, g_2(y)$ implies $f_1(x)/f_2(x) = g_2(y)/g_1(y)$, which means both sides must be constant. The ρ_i are normalized by the condition that they are formal characters of irreducible representations and therefore have non-negative integral coefficients with g.c.d. 1. This allows us to reduce the problem to the case that Γ acts irreducibly on X.

Consider the subgroup of X generated by all differences of weights in ρ_T. Since ρ is an irreducible representation of G, any two such weights differ by a linear combination of roots, so this subgroup is contained in the root lattice. But ρ is faithful, so in fact we have equality. This shows that the root lattice $\mathbf{Z}\Phi$ is determined by (T, ρ_T). As in § 1, ρ_T determines a canonical positive definite inner product $\langle\,,\rangle$ on X. Our next aim is to extract as much information as possible from the data $(\mathbf{Z}\Phi, \langle\,,\rangle)$.

We say that a lattice Λ in an inner product space *factors* as $\Lambda_1 \times \Lambda_2$ if $\Lambda = \Lambda_1 \oplus \Lambda_2$ and $\Lambda_1 \perp \Lambda_2$. A lattice is *irreducible* if it does not have a non-trivial factorization. It is well-known, and easy to check, that with respect to a positive definite inner product, factorization into irreducible lattices is unique.

Lemma. *Let* Φ *be a simple root system in a euclidean space* X.
(a) *The shortest non-zero vectors in* $\mathbf{Z}\Phi$ *are just the short roots.*
(b) *If all roots in* Φ *have the same length, then the root lattice of* Φ *is irreducible.*

Proof. A consideration of the rank-2 root systems shows that every pair of roots which form an obtuse angle have a sum which is also a root. This shows that every non-zero sum of roots λ can be written as a sum of roots, every one of which makes an angle $\leqq \pi/2$ with every other. Fix one such root, α. Then $\langle\lambda, \alpha\rangle \geqq \langle\alpha, \alpha\rangle$, so $\langle\lambda, \lambda\rangle \geqq \langle\alpha, \alpha\rangle$, with equality if and only if $\lambda = \alpha$. This gives (a). For (b), suppose that $\Lambda = \mathbf{Z}\Phi$ splits non-trivially as $\Lambda_1 \oplus \Lambda_2$. Then the shortest non-zero vectors in Λ all lie in $\Lambda_1 \cup \Lambda_2$. By (a) this forces Φ to split, contrary to the assumption. □

Proposition. *Let* Φ *be an arbitrary root system in a euclidean space* $(X, \langle\,,\rangle)$. *Then* Φ° *is determined by* $(\mathbf{Z}\Phi, \langle\,,\rangle)$.

Proof. The short roots generate the root lattice, i.e. $\mathbf{Z}\Phi = \mathbf{Z}\Phi^\circ$. Let $\Phi^\circ = \sum \Omega_i$ be the decomposition into simple factors. Then by part (b) of the lemma, $\mathbf{Z}\Phi = \bigoplus \mathbf{Z}\Omega_i$ is the (unique) decomposition into irreducible lattices. By part (a) of the lemma, every Ω_i is determined by its lattice. □

Coming back to our situation, this shows that Φ° is determined by the given data. Moreover by the definition of $\langle\, ,\rangle$, Γ consists of isometries of $\mathbf{Z}\Phi$. Since Γ acts irreducibly on X, the lattice $\mathbf{Z}\Phi$ is an orthogonal direct sum of isometric irreducible lattices. In particular Φ° is isotypic, say of simple type Ω. Since $(B_n)^\circ \cong A_1^n$, $(C_n)^\circ \cong D_n$, $(F_4)^\circ \cong D_4$, and $(G_2)^\circ \cong A_2$, by classification Ψ° is simple for every simple root system Ψ not of type B_n.

Assume that $\Omega \not\cong A_1$, then $\Phi = \sum \Phi_i$ with Φ_i simple and $(\Phi_i)^\circ \cong \Omega$. Then $\rho_T = \bigotimes \rho_i$, where each ρ_i is the formal character of an irreducible representation of a Lie algebra of type Φ_i. Since all ρ_i are determined by our data, we are reduced to the case where Φ is simple. Then the only ambiguities are between different simple root systems with the same short roots. By classification we only have to consider the following pairs of types: $D_n \subset C_n$ $(n \geqq 3)$, $D_4 \subset F_4$, $C_4 \subset F_4$, and $A_2 \subset G_2$.

We introduce the following notation. For any reduced root system Φ and any $f \in \mathbf{Z}[\Lambda_\Phi]$ we let

$$\mathrm{alt}_\Phi(f) = \sum_{w \in W(\Phi)} \mathrm{sgn}(w) \cdot wf.$$

If $\lambda \in X$ is regular, the Weyl orbit of $[\lambda]$ consists of $|W|$ linearly independent elements of $\mathbf{Z}[\Lambda_\Phi]$, so $\mathrm{alt}_\Phi([\lambda]) \neq 0$. Moreover, for any collection of regular $\{\lambda_i\}$, no two of which are Weyl-conjugate, the $\mathrm{alt}_\Phi([\lambda_i])$ are linearly independent. On the other hand, if λ is singular, it lies on a wall of some Weyl chamber and is therefore invariant under some reflection in W; it follows that $\mathrm{alt}_\Phi([\lambda]) = 0$. If $\lambda \in \Lambda_\Phi$ is a dominant weight, we let

$$q_\Phi(\lambda) = \mathrm{alt}_\Phi([\lambda + \delta_\Phi)]),$$

where δ_Φ is one half the sum of all positive roots in Φ. Weyl's theorem ([3] § 24.3) says that $q_\Phi(0)\,\rho_T = q_\Phi(\lambda)$ if ρ_T is the formal character of the irreducible representation with highest weight λ. Moreover

$$q_\Phi(0) = \prod_{\alpha \in \Phi^+} \left(\left[\frac{\alpha}{2}\right] - \left[-\frac{\alpha}{2}\right]\right).$$

Let $\Phi \subset \Psi$ be as above, then any order on Ψ induces one on Φ. By the last formula we get

$$q_\Psi(0) = q_\Phi(0) \prod_{\alpha \in \Psi^+ \setminus \Phi^+} \left(\left[\frac{\alpha}{2}\right] - \left[-\frac{\alpha}{2}\right]\right).$$

If λ is a dominant weight for Ψ, and the irreducible representations of both Lie algebras with this highest weight have the same formal character ρ_T, then it follows that

$$q_\Psi(\lambda) = \frac{q_\Psi(0)}{q_\Phi(0)}\, q_\Phi(0)\, \rho_T = q_\Phi(\lambda) \prod_{\alpha \in \Psi^+ \setminus \Phi^+} \left(\left[\frac{\alpha}{2}\right] - \left[-\frac{\alpha}{2}\right]\right).$$

Let us apply this in the case $\Phi = D_n \subset C_n = \Psi$, for $n \geqq 2$. If $\lambda = (\lambda_1, \ldots, \lambda_n)$ is the highest weight of our given representation, then $\lambda_1 \geqq \ldots \geqq \lambda_n \geqq 0$. Since $(\lambda_1, \ldots, \lambda_{n-1}, -\lambda_n)$ occurs in the irreducible representation of D_n of highest weight λ, and is itself dominant for D_n, we must have $\lambda_n=0$. Writing $\{\varepsilon_i | 1 \leqq i \leqq n\}$ for the standard basis of $\mathbf{Z}^n$, we calculate

$$q_{D_n}(\lambda) \prod_{\alpha \in C_n^+ \setminus D_n^+} \left(\left[\frac{a}{2}\right]-\left[-\frac{a}{2}\right]\right)=\operatorname{alt}_{D_n}\left([\lambda+\delta_{D_n}] \prod_{i=1}^{n}([\varepsilon_i]-[-\varepsilon_i])\right)$$

$$=\operatorname{alt}_{C_n}\left([\lambda+\delta_{C_n}] \prod_{i=1}^{n-1}(1-[-2\varepsilon_i])\right).$$

This is equal to $q_{C_n}(\lambda)$ if and only if

$$\sum_S (-1)^{|S|} \operatorname{alt}_{C_n}([\lambda+\delta_{C_n}-\sum_{i\in S} 2\varepsilon_i])=0, \tag{6}$$

where the sum is extended over all nonempty subsets $S \subset \{1, \ldots, n-1\}$. Since $\delta_{C_n}=(n, \ldots, 2, 1)$, $\mu_S=\lambda+\delta_{C_n}-\sum_{i\in S} 2\varepsilon_i$ is regular unless at least one of the following conditions holds:

i) $\lambda_i=\lambda_{i+1}=\lambda_{i+2}$, $i\in S$, $i+2\notin S$ (for some $1\leqq i\leqq n-2$).
ii) $\lambda_i=\lambda_{i+1}+1$, $i\in S$, $i+1\notin S$ (for some $1\leqq i\leqq n-1$).
iii) $\lambda_{n-1}=0$ and $n-1\in S$.

If, for $S\neq S'$, μ_S and $\mu_{S'}$ are both regular, they cannot be Weyl-conjugate. Indeed, the coordinates of both are positive, so they could only differ by a permutation. But we have coordinate inequalities

$$(\mu_S)_i \geqq (\mu_S)_j-1, \quad (\mu_{S'})_i \geqq (\mu_{S'})_j-1,$$

for $j\geqq i$ and the parity condition $\mu_S-\mu_{S'}\in 2\mathbf{Z}^n$, so this is impossible. We conclude that (6) is equivalent to the condition that μ_S is singular for every $\emptyset\neq S\subset \{1, \ldots, n-1\}$. When $S=\{n-1\}$, conditions (ii) and (iii) imply $\lambda_{n-1}\in\{0, 1\}$. When $S=\{i\}$, $1\leqq i\leqq n-2$, conditions (i) and (ii) imply $\lambda_i\in\{\lambda_{i+1}+1, \lambda_{i+2}\}$. Taking into account the inequalities $\lambda_1\geqq \ldots \geqq\lambda_n=0$ one easily proves that this leaves only those λ specified in the theorem. Conversely, for such λ, it is easy to verify that the above condition holds for all S. This finishes the case $D_n\subset C_n$.

The cases $D_4\subset F_4$, $C_4\subset F_4$, and $A_2\subset G_2$ can be treated along the same lines, with no new idea needed. Note that these cases can also be treated by finite computations since the rate of growth in Weyl's dimension formula guarantees that

$$\dim(V_{F_4}(\lambda))>\dim(V_{C_4}(\lambda))>\dim(V_{D_4}(\lambda)) \quad \text{and} \quad \dim(V_{G_2}(\mu))>\dim(V_{A_2}(\mu))$$

for all but finitely many dominant weights λ, μ. The remaining cases can be checked using tables, e.g. [4].

We come back to the case $\Phi^\circ=A_1^n$. Identifying the root lattice with $\mathbf{Z}^n$ in the standard way, Φ becomes a root subsystem of B_n, and Γ becomes a subgroup of $W(B_n)=\{\pm 1\}^n \rtimes S_n$ that contains $\{\pm 1\}^n$. The weight lattice of Φ is contained in that of Φ°, which is $\frac{1}{2}\mathbf{Z}^n$ under our identification, so we can consider ρ_T as an element of $\mathbf{Z}[\frac{1}{2}\mathbf{Z}^n]$.

Define an equivalence relation on the set $\{1, \ldots, n\}$ by $i\sim j$ if and only if the permutation $(i\,j)$ is in Γ. Since Γ acts irreducibly on X, it acts transitively on this set, so all equivalence classes have the same size, say m. These equivalence classes define a decomposition $\mathbf{Z}^n=(\mathbf{Z}^m)^k$. Since Γ contains the Weyl group of Φ, every root in Φ must already lie in one of the factors. In other words, we have a corresponding decomposition $\Phi=\sum_{i=1}^{k}\Phi_k$. Now ρ_T splits accordingly as $\bigotimes_{i=1}^{k}\rho_i$. Up to scalars, the ρ_i are determined by ρ_T. Since they correspond to irreducible representations, the coefficients of the highest weights are 1, so the requirement that all coefficients be non-negative integers determines them completely. Having decomposed Φ and ρ_T canonically into products, we can check the assertion factor by factor. We may now assume that $\Gamma=\{\pm 1\}^n \rtimes S_n$.

By unique factorization in the ring $\mathbf{Z}[\frac{1}{2}\mathbf{Z}^n]$, there exists a unique finest decomposition $\mathbf{Z}^n=\bigoplus_{i=1}^{k}\mathbf{Z}^{n_i}$ so that $\rho_T=\bigotimes_{i=1}^{k}\rho_i$ with $\rho_i\in\mathbf{Z}[\frac{1}{2}\mathbf{Z}^{n_i}]$. Again the ρ_i are uniquely determined if we require each coefficient to be a non-negative integer. Since Γ fixes ρ_T and acts irreducibly on X, it transitively permutes the factors $\mathbf{Z}^{n_i}$ and the ρ_i. But clearly the only S_n-invariant decompositions are the trivial decomposition ($k=1$) and the complete decomposition ($k=n$). In the first case ρ_T cannot decompose at all, hence Φ is simple. Then we must have $\Phi=B_n$, and there is no ambiguity.

In the second case we have $\rho_T=\rho_1^{\otimes n}$ with $\rho_1\in\mathbf{Z}[\frac{1}{2}\mathbf{Z}]$, and we may assume that $n\geqq 2$. The highest weight of ρ_T is then of the form $\lambda=\left(\frac{m}{2},\ldots,\frac{m}{2}\right)$ for some integer $m\geqq 1$. Since ρ_T is the formal character of an irreducible representation, it is well-known (e.g. [3] § 22 ex. 1) that the multiplicity of $\left(\frac{m}{2},\ldots,\frac{m}{2},\frac{m}{2},\frac{m-i}{2}\right)$ is 1 for any even integer $0\leqq i\leqq 2m$, and 0 for all other $i\in\mathbf{Z}$. This implies that all non-zero coefficients of ρ_1 are 1, and the same follows for ρ_T. But with Freudenthal's formula ([3] § 22.3) one easily checks that the multiplicity of $\left(\frac{m}{2},\ldots,\frac{m}{2},\frac{m}{2}-1,\frac{m}{2}-1\right)$ is 2 if $m\geqq 2$. Thus we must have $m=1$, and $\rho_1=[\frac{1}{2}]+[-\frac{1}{2}]\in\mathbf{Z}[\frac{1}{2}\mathbf{Z}]$. This is the formal character of the standard representation of SL_2, viewed as the spin representation of B_1. Clearly $\rho_1^{\otimes k}$ is the formal character of the spin representation of B_k, for every k. This shows that, with the given choice of ρ_1, $\rho_T=\rho_1^{\otimes n}$ comes from an irreducible representation of every root system $\Phi=\sum B_{n_i}$: this is the exception (a) in theorem 4. □

Proof of Theorem 2. By § 1 Proposition 1, (T,ρ_T) is determined by dimension data. Let $\Gamma=\mathrm{Aut}(T,\rho_T)$, and as in the proof of Theorem 4 consider the unique decomposition $X=X(T)\otimes\mathbf{R}=\bigoplus X_i$ into Γ-irreducible subspaces. Since Γ and the root system Φ of G respect this decomposition, so does

$$F=\sum_{\gamma\in\Gamma}\gamma(F_\Phi)\in\mathbf{Q}[X(T)].$$

(Cf. § 2 Proposition 1.) We are therefore reduced to the case where X is Γ-irreducible.

In this case, Theorem 4 shows that either Φ is isotypic and is uniquely determined by ρ_T, or ρ_T is an ambiguous representation, and Φ is one of cC_n+dD_n ($n\geqq 3$), $cC_4+dD_4+fF_4$, aA_2+gG_2, or $\sum B_{m_j}$. By Theorem 1, the latter cases are distinguished by F. Moreover, Theorem 4 implies that $\Gamma=\mathrm{Aut}(\Phi^\circ)$, so the isomorphism class of Φ determines that of (Φ,ρ_T), as desired. (Remark: After the reduction to the Γ-irreducible case, as above, an estimate of the longest vector occurring in F, as in § 2, leads to an independent proof of Theorem 2.)

References

1. Bourbaki, N.: Groupes et algèbres de Lie. Paris: Masson 1981
2. Dynkin, E.B.: Semisimple subalgebras of semisimple Lie algebras. Am. Math. Soc. Transl. Ser. 2, **6**, 111–244 (1957)
3. Humphreys, J.E.: Introduction to Lie algebras and representation theory. Berlin-Heidelberg-New York: Springer 1972
4. McKay, W.G., Patera, J.: Tables of dimensions, indices, and branching rules for representations of simple Lie algebras (Lecture Notes in Pure and Applied Math.). New York: Marcel Dekker 1981
5. Peter, F., Weyl, H.: Vollständigkeit der Darstellungen kontinuierlicher Gruppen. Math. Ann. **97**, 737–755 (1927)
6. Weyl, H.: The classical groups. Princeton, NJ: Princeton University Press 1939