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Moduli spaces of anti-self-dual connections on ALE gravitational instantons

Dedicated to Professor Akio Hattori on his sixtieth birthday

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0. Introduction

In [Kr1, Kr2] Kronheimer constructed and classified all ALE hyperkähler 4-manifolds which were originally discovered by physicists [EH, GH]. There he proved that for each finite subgroup $\Gamma \subset SU(2)$, there exists a family of ALE hyperkähler metrics on the minimal resolution of the quotient variety $\mathbb{C}^2/\Gamma$ and then showed that they exhaust all ALE hyperkähler 4-manifolds. On the other hand in the joint work with Bando and Kasue [BKN], the author pointed out that they bubble off from points at which the curvature of a sequence of Einstein metrics become concentrated (see also [An, Na]) and proved that the existence of ALE coordinate system results from the finiteness of L^2-norm of the curvature and the maximal volume growth order condition. In both works it becomes apparent that ALE hyperkähler 4-manifolds share the same properties with Yang-Mills instantons on $\mathbb{R}^4$, and so we think that they have as rich mathematical structures as Yang-Mills instantons.

In this paper we study the (framed) moduli space M of anti-self-dual connections on a principal bundle over an ALE hyperkähler 4-manifold. It has the natural Riemannian metric g_M and the hyperkähler structure (I_M, J_M, K_M) induced from those on the base manifold. The constructions of these structures are very natural, and were carried out over *compact* hyperkähler 4-manifolds by Itoh [I3] and Hitchin [H2]. The existence of these structures is strong enough so that we can almost determine the moduli space at least when it is 4-dimensional. In fact, we shall prove the following:

(0.1) **Theorem.** *Let* (X, g, I_X, J_X, K_X) *be an* ALE *hyperkähler* 4*-manifold which is diffeomorphic to the minimal resolution of* $\mathbb{C}^2/\Gamma$ *and* P *a principal bundle over*

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X with the structure group G (a compact Lie group). Let $M = M(P, k, \rho)$ denote the (framed) moduli space of anti-self-dual connections on P with an instanton number k and asymptotic to a flat connection on S^3/Γ associated with a homomorphism $\rho: \Gamma \to G$ (see § 2 for more precise definition). Suppose that M is nonempty and 4-dimensional. Then

(1) *the moduli space M is a nonsingular manifold which has a natural Riemannian metric g_M and a hyperkähler structure (I_M, J_M, K_M),*

(2) *the metric g_M is complete,*

(3) *each noncompact component of (M, g_M) is an* ALE *4-manifold.*

We remark that the existence of a hyperkähler structure on noncompact gravitational instanton is announced by Itoh [13, p. 583, Remark (v)].

It is easy to figure the shape of the moduli space. When the sequence of anti-self-dual connections goes to infinity in the moduli space, their curvature become concentrated at infinity of the base manifold. If we choose an appropriate sequence of rescalings of the metric, the connections converge to an anti-self-dual connections on $\mathbb{C}^2/\Gamma$. Conversely the Taubes' existence theorem gives a diffeomorphism Φ from the moduli space on $\mathbb{C}^2/\Gamma$ to the end of the moduli space on X. We shall prove that the moduli space M^Γ on $\mathbb{C}^2/\Gamma$ is isometric to $\mathbb{C}^2\setminus\{0\}/\Gamma'$ for some finite subgroup $\Gamma' \subset \mathrm{SU}(2)$ (possibly different from Γ) when $\dim M^\Gamma = 4$, and so that the map Φ defines a coordinate system at infinity.

Since the topology of an ALE hyperkähler 4-manifold is determined by the fundamental group of the end (see Fact 1.2), we can determine that of the noncompact component of the moduli space by studying the moduli space on $\mathbb{C}^2/\Gamma$. But to determine moreover the hyperkähler structure we must know the cohomology classes of three Kähler forms. For the examples given below we can know them by constructing a homomorphism between two cohomology groups of X and M.

(0.2) **Theorem.** *Let (X, g, I_X, J_X, K_X) be as in Theorem 0.1 and $(\omega_I^X, \omega_J^X, \omega_K^X)$ the Kähler forms associated with (I_X, J_X, K_X). We consider the root system R associated with the finite subgroup $\Gamma \subset SU(2)$ as a subset of $H_2(X; \mathbb{Z})$ (see Sect. 1 for the correspondence between binary polyhedral groups and root systems). For each root Σ of R, there exists a complex line bundle L_Σ over X such that the reducible connection D associated with the reduction $E = \underline{\mathbb{C}} \oplus L_\Sigma$ ($\underline{\mathbb{C}}$ is the trivial complex line bundle) is in a 4-dimensional moduli space $M = M(P, k, \rho)$ with*

(1) $G = U(2)$,

(2) $k = 1$,

(3) *ρ is trivial,*

(4) *$c_1(L_\Sigma)$ is the Poincaré dual of $\Sigma \in H_2(X; \mathbb{Z})$.*

Moreover M is isometric to the Eguchi-Hanson space and its Kähler forms $(\omega_I^M, \omega_J^M, \omega_K^M)$ satisfy

$$[\omega_A^M](\sigma) = [\omega_A^X](\Sigma),$$

where σ is a generator of $H_2(M; \mathbb{Z})$.

The Eguchi-Hanson space is the simplest ALE hyperkähler manifold diffeomorphic to $T'^*\mathbb{CP}^1$ the holomorphic cotangent bundle of the projective line.

(0.3) **Theorem.** *Let (X, g, I_X, J_X, K_X) be as in Theorem 0.1. There exists a noncompact component M_0 of the 4-dimensional moduli space $M = M(P, k, \rho)$ with*

(1) $G = SU(2)$,

(2) *$k = (|\Gamma| - 1)/|\Gamma|$ where $|\Gamma|$ is the order of Γ,*

(3) *ρ is equal to the inclusion map $\Gamma \hookrightarrow SU(2)$,*
(4) *M_0 is diffeomorphic to X.*

If the group Γ is a cyclic group (i.e. Γ is of type A_n), $(M_0, g_M, I_M, J_M, K_M)$ is isomorphic to the base manifold (X, g, I_X, J_X, K_X) as a hyperkähler manifold.

We owe the choices of ρ and k in Theorem 0.3 to Furuta and Hashimoto [FH] who have studied the moduli space of anti-self-dual connections on $\mathbb{C}^2/\Gamma$ in detail.

We conjecture that every component of the moduli space M in Theorem 0.1 is always noncompact. We remark our results imply that each compact component, if exists, must be a $K3$ surface or a torus since they are the only compact hyperkähler 4-manifolds.

In the explicit descriptions of moduli space of anti-self-dual connections on S^4, so called the twister method has played an important role to reduce the problem to the algebraic geometry. However in our cases our method is purely differential geometric. The keys to the proof are the existence of the hyperkähler structure on the moduli space and the study of the behavior of the metric on the end. In the spirit our method is similar to that used by Mukai who has studied the moduli space of stable sheaves on a $K3$ surface [Mu]. In fact, our results are inspired by his results. The key to his results is also the existence of the hyperkähler metric (which is essentially equivalent to the existence of holomorphic symplectic structure in his papers).

The method to determine the hyperkähler structures of the moduli spaces in Theorems 0.2, 0.3 is very similar to the calculation of the polynomial invariants defined by Donaldson for compact 4-manifolds [D4]. Recently Floer [Fl] introduced the homology groups graded by $\mathbb{Z}_8$ for homology 3-spheres. Donaldson pointed out that the polynomial invariants should take values in Floer homology groups for the compact 4-manifold with boundary which is homology 3-sphere (see [At]). Although the quotient space S^3/Γ is not homology 3-sphere except when Γ is the binary icosahedral group, our results seem to suggest that the definition of the Floer type homology groups for S^3/Γ is possible and give examples of the calculation of the polynomial invariants for manifolds with boundary S^3/Γ.

The organization of this paper is as follows. In Sect. 1, we shall review the results of [Kr1, Kr2] for the convenience of the reader. In Sect. 2, we shall study the moduli space of anti-self-dual connections on the ALE 4-manifold to prove that it is a nonsingular manifold and has a natural Riemannian metric. In Sects. 3, 4, we shall give two types of existence theorems of anti-self-dual connections. In Sect. 3, we shall use reducible connections, and in Sect. 4 we shall give Taubes' existence theorem. In Sect. 5, we study the behavior of the metric on the end of the moduli space when it is 4-dimensional using the estimates obtained in Sect. 4. In Sect. 6, we shall study the period of the moduli spaces given in Theorem 0.2, 0.3.

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After the completion of this work, in the joint work with P. Kronheimer, the author obtained a description of the moduli space by certain finite dimensional matrices when the structure group is a unitary group, which is a generalization of the ADHM construction. Theorem 0.2 and 0.3 also follow from this description.

1. ALE gravitational instantons

We say an oriented Riemannian 4-manifold (X, g) is an *asymptotically locally Euclidean* (we abbreviate it to ALE in this paper) manifold of order $\tau > 0$ (cf. [Ba, LP, BKN]) if there exist a compact set $K \subset X$, a C^∞-diffeomorphism $\mathscr{X}$: $X \setminus K \to (\mathbb{R}^4 \setminus \overline{B_R})/\Gamma$ for some $R > 0$ and a finite subgroup $\Gamma \subset SO(4)$ acting freely on $\mathbb{R}^4 \setminus \overline{B_R}$ such that the metric g is represented in the coordinates $\mathscr{X}$ as

$$g_{ij}(x) = \delta_{ij} + a_{ij}(x) \qquad \text{for } x \in \mathbb{R}^4 \setminus \overline{B_R},$$

where a_{ij} satisfies

$$(1.1) \qquad |\overbrace{\partial \ldots \partial}^{p \text{ times}} a_{ij}(x)| = O(|x|^{-p-\tau}) \qquad \text{for } p = 1, 2, 3, \ldots.$$

The definition is slightly different from the one used in [BKN]. The above definition requires the decay of all higher order derivatives of a_{ij}, though we only assume that up to $C^{1,\alpha}$ in [BKN]. But for ALE hyperkähler spaces these definitions are equivalent since we can derive the decay of higher order derivatives from the Einstein equation using Schauder estimates.

The hyperkähler structure on a Riemannian manifold (X, g) is, by definition, three parallel almost complex structures (I, J, K) which satisfy the quaternionic relation $IJ = -JI = K$. Then there exist three associated Kähler forms ω_I, ω_J, ω_K.

Let Γ be a nontrivial finite subgroup of $SU(2)$. It is well-known that these subgroups correspond to root systems and are classified as follows (see [Kr 1] and the references therein):

$$A_n\colon\ \Gamma = \left\{ \begin{bmatrix} \zeta^k & 0 \\ 0 & \zeta^{-k} \end{bmatrix} \middle| k = 0, \ldots, n \right\} \cong \mathbb{Z}_{n+1}$$

where ζ is a primitive $(n+1)$-th root of unity,

D_n: $\Gamma = \mathbf{D}^*_{n-2}$ the binary dihedral group of order $4(n-2)$,

E_6: $\Gamma = \mathbf{T}^*$ the binary tetrahedral group,

E_7: $\Gamma = \mathbf{O}^*$ the binary octahedral group,

E_8: $\Gamma = \mathbf{I}^*$ the binary icosahedral group.

The group Γ acts on $\mathbb{C}^2$ and the action is free outside the origin 0. The singularity of the quotient space $\mathbb{C}^2/\Gamma$ at the origin is called a *rational double point* and has been studied by many mathematicians. Let π: $\tilde{S} \to \mathbb{C}^2/\Gamma$ be the minimal resolution. We denote its underlying differentiable structure by X. Kronheimer [Kr 1, Kr 2] proved the following:

(1.2) **Fact.** *Let α_I, α_J, $\alpha_K \in H^2(X; \mathbb{R})$ be three cohomology classes satisfying the non-degeneracy condition*

(*) *for each $\Sigma \in H_2(X; \mathbb{Z})$ with $\Sigma \cdot \Sigma = -2$, there exists $A \in \{I, J, K\}$ with $\alpha_A(\Sigma) \neq 0$.*

Then there exist a Riemannian metric g and a hyperkähler structure (I, J, K) on X such that the cohomology classes of the Kähler forms $[\omega_A]$ are the given

α_A ($A = I, J, K$) and the metric is ALE *of order* 4. *Conversely all* ALE *hyperkähler* 4-*manifolds of order* 4 *are obtained in the above manner and their isometry classes are uniquely determined by the cohomology classes $\alpha_I, \alpha_J, \alpha_K$.*

The exceptional set $\pi^{-1}(0)$ of the minimal resolution of $\mathbb{C}^2/\Gamma$ decomposes to a union of $\mathbb{CP}^1$

$$\pi^{-1}(0) = \Sigma_1 + \Sigma_2 + \ldots + \Sigma_n,$$

and the intersection matrix $(\Sigma_i \cdot \Sigma_j)$ is the negative of the Cartan matrix associated with the root system. In this way the set $\{\Sigma_1, \ldots, \Sigma_n\}$ of irreducible components of the exceptional set can be identified with the set of simple roots. On the other hand the set of the homology classes $\{[\Sigma_1], \ldots, [\Sigma_n]\}$ gives a basis of $H_2(X; \mathbb{Z})$. Hence there is an isomorphism between $H_2(X; \mathbb{Z})$ and the root lattice L.

The same correspondence between the group Γ and the root system was discovered by McKay [Mc] in a different manner. Let $\{\rho_0, \rho_1, \ldots, \rho_n\}$ be the set of all irreducible representations of Γ with ρ_0 the trivial representation. Let ρ_Q be the canonical 2-dimensional representation defined by the inclusion $\Gamma \hookrightarrow SU(2)$. We define the matrix $A = (a_{ij})$ by the decomposition formula

$$\rho_Q \otimes \rho_i = \bigoplus_j a_{ij} \rho_j,$$

where a_{ij} denotes the multiplicity of ρ_j in $\rho_Q \otimes \rho_i$. Then the matrix $2I - A$ is the extended Cartan matrix with ρ_0 corresponds to the negative of the highest root.

Gonzalez-Sprinberg and Verdier [GV] give a geometrical explanation of the McKay correspondence as follows:

Let M_i be a reflexive $\mathcal{O}_{\mathbb{C}^2/\Gamma}$-module defined by a nontrivial irreducible representation ρ_i of Γ. We denote by $\tilde{M}_i$ the $\mathcal{O}_{\tilde{S}}$-module $\pi^* M_i$/torsion. Then $\tilde{M}_i$ is locally free, and $\{c_1(\tilde{M}_1), \ldots, c_1(\tilde{M}_n)\}$ defines the dual basis of $\{[\Sigma_1], \ldots, [\Sigma_n]\}$.

2. Local structures of moduli spaces

In this section we study local structures (e.g., manifold structure, Riemannian metric, Kähler structure, hyperkähler structure) of moduli spaces of anti-self-dual connections on general ALE 4-manifolds (not necessarily hyperkähler). The corresponding results for compact 4-manifolds have been obtained in [D1, FU, I2, I3, H2]. Our results are modifications of their results to ALE manifolds. Such modifications to non-compact manifolds were already done by Taubes [T4] for manifolds with periodic ends. To save labor, we refer to his results by changing the ALE metric to a cylindrical metric conformally, though it is also possible to prove results by using analysis on ALE manifolds directly (see e.g. [Ba]).

Let (X, g) be an oriented ALE Riemannian 4-manifold of order $\tau > 0$ with the coordinate system at infinity $\mathcal{X}\colon X \setminus K \to (\mathbb{R}^4 \setminus \overline{B_R})/\Gamma$. Let P be a principal bundle over X with a structure group G (a compact Lie group) and $\operatorname{Ad} P$ the adjoint bundle associated with P (i.e. the associated vector bundle with fiber the Lie algebra $\mathfrak{g}$ by the adjoint representation $\operatorname{Ad}\colon G \to GL(\mathfrak{g})$). We take a homomorphism $\rho: \Gamma \to G$ which will be identified with a flat connection on S^3/Γ. We assume that P has a connection A_0 such that

(2.1) $$A_0 = \rho \quad \text{on } \{t\} \times S^3/\Gamma \subset [R, \infty) \times S^3/\Gamma \cong X \setminus K \qquad \text{for } t \geqq R.$$

We define weighted Sobolev norms on the space $\Omega_0^k(\operatorname{Ad} P)$ of $\operatorname{Ad} P$-valued k-forms with compact supports. We follow the notation of [Ba, LP]. Let $r(p) = |\mathscr{X}(p)|$ on $X \setminus K$, extended to a smooth positive function on all of X. For a nonnegative integer l, $\delta \in \mathbb{R}$ and $p \geqq 1$, the Sobolev norms $\|\cdot\|_{l,p,\delta}$ on $\Omega_0^k(\operatorname{Ad} P)$ are defined by

(2.2) $$\|\alpha\|_{l,p,\delta} = \sum_{j=0}^{l} \{\int_X |\overbrace{\nabla_{A_0} \cdots \nabla_{A_0}}^{j\,\text{times}} \alpha|^p r^{-(\delta - j)p - 4} \, dV\}^{1/p},$$

where ∇_{A_0} is the covariant differentiation associated with A_0. We denote by $W_\delta^{l,p}(\Omega^k(\operatorname{Ad} P))$ the completion of $\Omega_0^k(\operatorname{Ad} P)$ by the norm $\|\cdot\|_{l,p,\delta}$.

We say a connection A is *asymptotic* to the flat connection (or to the homomorphism) ρ in $W_\delta^{l,p}$, if we can write $A = A_0 + \alpha$ with

$$\|\alpha\|_{l,p,\delta} < \infty.$$

The connection A naturally induces the exterior differential operator d_A: $\Omega_0^k(\operatorname{Ad} P) \to \Omega_0^{k+1}(\operatorname{Ad} P)$ and its formal adjoint d_A^*: $\Omega_0^{k+1}(\operatorname{Ad} P) \to \Omega_0^k(\operatorname{Ad} P)$.

Fix an integer $l > 1$. We define the space $\mathscr{A}^l$ of Sobolev connections and the group $\mathscr{G}_0^{l+1}$ of Sobolev gauge transformations by

$$\mathscr{A}^l := \{A_0 + \alpha \mid \alpha \in W_{-2}^{l,2}(\Omega^1(\operatorname{Ad} P))\},$$

$$\mathscr{G}_0^{l+1} := \{s \in W_{\text{loc}}^{l+1,2}(\Omega^0(\operatorname{End}(V)) \mid \|s - \mathrm{id}\|_{l+1,2,-1} < \infty, s \in G \text{ a.e.}\},$$

where G is considered as a linear subgroup by a faithful representation $G \to GL(V)$ and $\operatorname{End}(V)$ is the associated vector bundle. The group $\mathscr{G}_0^{l+1}$ acts smoothly on $\mathscr{A}^l$ by pullback

$$\nabla_{s^*(A)} = s^{-1} \circ \nabla_A \circ s.$$

Our group $\mathscr{G}_0^{l+1}$ is slightly different from the usual gauge group since it only contains automorphisms converging to the identity. We introduce the gauge group $\mathscr{G}^{l+1}$ which naturally acts on $\mathscr{A}^l$. Let G_ρ be a subgroup of G defined by

$$G_\rho = \{s \in G \mid s \rho s^{-1} = \rho\}.$$

We regard $s \in G_\rho$ as a section of $\operatorname{End}(V)$ by setting s on $X \setminus K$ and extending smoothly to the whole X. Then $\mathscr{G}^{l+1}$ is defined by

$$\mathscr{G}^{l+1} := \{s \in W_{\text{loc}}^{l+1,2}(\Omega^0(\operatorname{End}(V)) \mid s \in G \text{ a.e.},$$
$$\|s - s_\infty\|_{l+1,2,-1} < \infty \text{ for some } s_\infty \in G_\rho\}.$$

The quotient space $\mathscr{B}^l = \mathscr{A}^l / \mathscr{G}_0^{l+1}$ is the moduli space of connections on P asymptotic to ρ. (The moduli space is usually defined as the quotient of the gauge group $\mathscr{G}^{l+1}$ for compact manifolds, and the quotient of $\mathscr{G}_0^{l+1}$ is called a "framed" (or "based") moduli space. But since we are concerned only with $\mathscr{B}^l$, we call it the moduli space for brevity.)

To give a manifold structure on $\mathscr{B}^l$, we change the metric conformally and refer to results of [T4]. From coordinates at infinity $\mathscr{X}$: $X\backslash K \to (\mathbb{R}^4\backslash \overline{B_R})/\Gamma$, we have cylindrical coordinates

$$\mathscr{Y}\colon\ X\backslash K \to S^3/\Gamma \times (R, \infty),$$

where $\mathscr{Y}^{-1}(\theta, u) = \mathscr{X}^{-1}(e^u\theta)$. By a conformal change with factor r^{-2} the metric g is approximated by the standard metric of the cylinder:

$$(\mathscr{Y}^{-1})^*(r^{-2}g) = du^2 + d\theta^2 + O(e^{-(2+\tau)u}).$$

We denote the new metric $r^{-2}g$ by g'. Then

$$\mathscr{A}^l = \left\{ A_0 + \alpha \,\middle|\, \sum_{j=0}^{l} \int_X e^{2u} \,|\overbrace{\nabla_{A_0} \cdots \nabla_{A_0}}^{j\text{times}} \alpha|_{g'}^2 \, dV_{g'} < \infty \right\},$$

$$\mathscr{G}_0^{l+1} = \Big\{ s \in W_{\mathrm{loc}}^{l+1,2}(\Omega^0(\mathrm{End}(V))) \,\Big|\, \sum_{j=0}^{l} \int_X e^{2u} \,|\overbrace{\nabla_{A_0} \cdots \nabla_{A_0}}^{j\text{times}}(s - \mathrm{id})|_{g'}^2 \, dV_{g'} < \infty,\ s \in G \text{ a.e.} \Big\},$$

where the norm $|\ |_{g'}$ and the volume element $dV_{g'}$ are with respect to g'. Hence the spaces $\mathscr{A}^l$, $\mathscr{G}_0^{l+1}$ are isomorphic to the corresponding spaces in [T4] where the weight (δ in the notation of [T4]) is equal to 2. The Hodge star operator $*_{g'}$ with respect to g' relates to the original star operator by

$$* = e^{(4-2k)u} *_{g'} \quad \text{on } k\text{-forms}.$$

In particular, we have

$$d_A^* \alpha = 0 \qquad \text{if and only if } *_{g'} d_A *_{g'} e^{2u} \alpha = 0 \text{ for a 1-form } \alpha.$$

We then have the following "slice theorem" [T4, Lemma 7.3]:

(2.3) **Proposition.** *The quotient space* $\mathscr{B}^l = \mathscr{A}^l/\mathscr{G}_0^{l+1}$ *is a* C^∞*-Banach manifold and the projection* π: $\mathscr{A}^l \to \mathscr{B}^l$ *defines a principal* $\mathscr{G}_0^{l+1}$*-bundle. The tangent space to* $[A] \in \mathscr{B}^l$ *is isomorphic to the slice*

$$S_A := \{ \alpha \in W_{-2}^{l,2}(\Omega_0^1(\mathrm{Ad}\, P)) \mid d_A^* \alpha = 0 \}.$$

Remark. It is not explicit that results in [T4, Sect. 5, 7, 8] hold for the specific weight value 2. But if we restrict our concern to ALE manifolds, it can be shown to be true (cf. [LM, Sect. 9]).

Next we consider the moduli space of anti-self-dual connection. Let $\mathscr{A}_{\mathrm{asd}}^l$ be the set of anti-self-dual connections asymptotic to ρ, i.e.

$$\{ A \in \mathscr{A}^l \mid A \text{ is anti-self-dual} \}.$$

Adapting the argument of [T4, Proposition 8.2] as above, we have

(2.4) **Theorem.** *The quotient space* $M=\mathcal{A}^l_{\mathrm{asd}}/\mathcal{G}^{l+1}_0$ *is a nonsingular* C^∞ *manifold in a neighborhood of* $[A]\in M$ *if* A *satisfies*

$$0=H^2_{A,-1}:=\operatorname{Ker} d^*_A\colon\ W^{l,2}_{-2}(\Omega^+(\operatorname{Ad}P))\to W^{l,2}_{-1}(\Omega^1(\operatorname{Ad}P)).$$

The tangent space $T_{[A]}M$ *is isomorphic to*

$$H^1_{A,-2}:=\{\alpha\in W^{l,2}_{-2}(\Omega^1(\operatorname{Ad}P))\,|\,d^+_A\alpha=d^*_A\alpha=0\}.$$

It is worth while remarking that the moduli space is non-singular even at reducible connections. It is the main difference from the compact case.

When we want to specify the bundle P, the instanton number

$$k=\frac{1}{8\pi^2}\int_X |R_A|^2\,\mathrm{d}V,$$

and the homomorphism $\rho\colon\Gamma\to G$, we use the notation $M(P,k,\rho)$ for the moduli space M.

For a later purpose, we define the Kuranishi map following [I2]. For an anti-self-dual connection A, consider an operator defined by

$$\Delta_A=d^+_A\,d^*_A\colon\ W^{l+1,2}_{-1}(\Omega^+(\operatorname{Ad}P))\to W^{l-1,2}_{-3}(\Omega^+(\operatorname{Ad}P)).$$

If the assumption in Theorem 2.4 is satisfied, Δ_A is invertible. Hence we can define

$$\Xi_A\colon\ W^{l,2}_{-2}(\Omega^1(\operatorname{Ad}P))\to W^{l,2}_{-2}(\Omega^1(\operatorname{Ad}P))$$

$$\alpha\mapsto\alpha+\tfrac{1}{2}d^*_A\Delta_A^{-1}[\alpha\wedge\alpha]^+.$$

Then

$$\alpha\in S_A,\ A+\alpha\in\mathcal{A}^l_{\mathrm{asd}}\Leftrightarrow\Xi_A(\alpha)\in H^1_{A,-2}.$$

Since the Fréchet derivative of Ξ_A at 0 is the identity map, we have the inverse Ξ_A^{-1} defined on a neighborhood $\mathcal{O}$ of 0 in $W^{l,2}_{-2}(\Omega^1(\operatorname{Ad}P))$ by the inverse mapping theorem. This Ξ_A^{-1} defines a coordinate system of M around $[A]$.

We set $\hat{M}=\{[A]\in M\,|\,H^2_{A,-1}=0\}$. As in [I2] we define a Riemannian metric g_M on the moduli space $\hat{M}$ by

(2.5) $$g_M(\alpha,\beta)=\int_X g(\alpha,\beta)\,\mathrm{d}V\quad\text{for }\alpha,\beta\in H^1_{A,-2}=T_{[A]}\hat{M}$$

where the fiber metric on $\Omega^1(\operatorname{Ad}P)$ is induced from the Riemannian metric g and an Ad-invariant metric on $\mathfrak{g}$.

Now suppose (X,g) is a Kähler manifold. The almost complex structure I_X induces in a natural way an almost complex structure I_M on $\Omega^1_0(\operatorname{Ad}P)$. As is proved in [I2], the almost complex structure I_M preserves the space $H^1_{A,-2}$, and is *covariant constant* with respect to the Levi-Civita connection of g_M. Hence $(\hat{M},I_M,g_M)$ is a Kähler manifold. When (X,g) has a hyperkähler structure (I_X,J_X,K_X), the moduli space $(\hat{M},g_M)$ has also the hyperkähler structure (I_M,J_M,K_M) (see [I3]). (In fact, the existence of the quarternion structure on the tangent space of $\hat{M}$, when the base manifold is an ALE gravitational instanton, was already noticed in [T1]. Our assertion is that they are *covariant constant*.)

(2.6) **Theorem.** *The moduli space $\hat{M}$ has the natural Riemannian metric g_M defined by* (2.5). *When the base manifold X has the Kähler* (*resp. hyperkähler*) *structure, $\hat{M}$ also has the Kähler* (*resp. hyperkähler*) *structure.*

These properties can be proved using the moment map and the symplectic quotient method [HKLR] as in [H2, Ko, IN].

The quotient group $G_\rho/Z(G)$ acts naturally on M. In fact, the "larger" gauge group $\mathcal{G}^{l+1}$ acts on the space $\mathcal{A}^l_{\mathrm{asd}}$ and induces the residual action of $\mathcal{G}^{l+1}/\mathcal{G}^{l+1}_0 = G_\rho$ on M. The center $Z(G)$ regarded as a subgroup of $\mathcal{G}^{l+1}$ acts trivially on M. Since the fiber metric of $\operatorname{Ad} P$ is $\mathcal{G}^{l+1}$-invariant, the action of G_ρ on M is isometric. Moreover it is holomorphic (resp. triholomorphic) when X is Kähler (resp. hyperkähler). If we translate the results proved on compact manifolds, we see that $G_\rho/Z(G)$-action is free outside reducible connections (cf. [FU, Theorem 3.1]). So each orbit of an irreducible connection is diffeomorphic to $G_\rho/Z(G)$. The quotient space M/G_ρ is equal to $\mathcal{A}_{\mathrm{asd}}/\mathcal{G}$, and this coincides with the moduli space of anti-self-dual connections on the orbifold $\hat{X}$ which is usually used in the literature ([FS, La, FH]).

Now we calculate the dimension of the moduli space $\hat{M}$. We use results of [FS, La], compactifying X to an orbifold $\hat{X} = X \cup \{\infty\}$ by a conformal change of the metric g (see [Kr2, p. 687]). Uhlenbeck's removable singularities theorem [U1] implies the anti-self-dual connection A on X with finite curvature integral can be identified with an anti-self-dual connection on $\hat{X}$.

For simplicity, we assume that the structure group G is a unitary group $U(r)$. (Although the dimension formula [FS, La] was proved only for $G = SO(3)$, the adaptation is straightforward.) Let E denote the associated complex vector bundle of rank r. The flat connection ρ, to which anti-self-dual connections are asymptotic, defines an r-dimensional representation. We denote by χ_ρ its character.

(2.7) **Theorem.** *The dimension of the moduli space $\hat{M}$ at $[A]$ is given by the formula*

$$\dim M = \dim G_\rho - \int_X \operatorname{ch}(E^* \otimes E)\operatorname{ch}(S^+)\hat{A}(X)$$

$$- \frac{1}{2|\Gamma|} \sum_{\gamma \neq e} \chi_\rho(\gamma^{-1})\,\chi_\rho(\gamma)\left(1 - \cot\frac{r(\gamma)}{2}\cot\frac{s(\gamma)}{2}\right),$$

where S^+ is the positive spinor bundle, $|\Gamma|$ is the order of Γ, $r(\gamma)$ and $s(\gamma)$ are the rotation numbers corresponding to the action of $\gamma \in \Gamma$ at ∞.

At the end of this section we remark that the results of this section are applied to the case that the base manifold is the quotient space $\mathbb{R}^4/\Gamma$, though $\mathbb{R}^4/\Gamma$ has a singular point 0 if Γ is nontrivial. In fact, if we consider $\mathbb{R}^4/\Gamma$ as an orbifold and work in the equivariant setting, our results are easily adapted. We explain more precisely; let P be a bundle over $\mathbb{R}^4/\Gamma$, which is by definition Γ-equivariant bundle over $\mathbb{R}^4$. Since the action of Γ on $\mathbb{R}^4$ has a fixed point 0, the action of Γ on the fiber P_0 induces a homomorphism $\rho_0\colon \Gamma \to G$. As before we fix a connection A_0 on P satisfying (2.1), and denote by ρ_∞ the associated homomorphism of Γ. Then through the trivialization induced from the connection A_0 the bundle P is extended to a principal bundle over the compactification S^4/Γ of $\mathbb{R}^4/\Gamma$. Let $M^\Gamma = M^\Gamma(P, k, \rho_0, \rho_\infty)$ be the moduli space of anti-self-dual connections on P which is asymptotic to A_0 and with the instanton number k. Then as above we can show that M^Γ has a structure of C^∞-manifold (it is easy to see that $H^2_{A,-1}=0$ for all $[A]\in M^\Gamma$ cf. Proposition 5.1) with a natural Riemannian metric g_M defined by L^2-inner product. If $\Gamma\subset U(2)$, the complex structure $I_{\mathbb{R}^4/\Gamma}$ is invariant under the action of Γ, and gives the complex structure on M^Γ. Moreover if $\Gamma\subset Sp(1)\cong SU(2)$, M^Γ has the natural hyperkähler structure $(I_{M^\Gamma}, J_{M^\Gamma}, K_{M^\Gamma})$. We also have the dimension formula:

$$\text{(2.8)}\qquad \dim M=\dim G_{\rho_\infty}+2r\int_{\mathbb{C}^2/\Gamma} c_2(E) -\frac{1}{2|\Gamma|}\sum_{p=0,\infty}\sum_{\gamma\neq e}\chi_{\rho_p}(\gamma^{-1})\,\chi_{\rho_p}(\gamma)\left(1-\cot\frac{r_p(\gamma)}{2}\cot\frac{s_p(\gamma)}{2}\right),$$

where $r_p(\gamma)$ and $s_p(\gamma)$ are the rotation numbers corresponding to the action of γ at $p=0,\infty$ (so $r_0=r_\infty$, $s_0=-s_\infty$).

3. Reducible connections

In this section and the next section, we give two types of existence theorems of anti-self-dual connections on ALE 4-manifolds with negative definite intersection form. First one (given in this section) is the existence theorem of reducible connections and relates to the intersection form of the base manifold. The other one is Taubes' implicit function theorem and relates to the fundamental group Γ of the end of the base manifold. In Sect. 5, it is proved that the first one gives "interior" points in the moduli space M, the other one corresponds to the end of M, and they are connected by M.

In this section we only treat the case that the structure group G is $U(2)$. Let (X,g) be an oriented ALE 4-manifold with the coordinate at infinity $\mathscr{X}$: $X\setminus K\to\mathbb{R}^4/\Gamma$. Let E be a complex vector bundle of rank 2 over X. A connection A on E is said to be *reducible* if (E,A) decomposes into a sum of line bundles with connections as $(E,A)=(L_1,A_1)\oplus(L_2,A_2)$. Hereafter we shall identify the connections on L_i with that on E and use the same notation A.

Since the L^2-norm and the harmonicity on 2-forms are invariant under the conformal change of the metric, we can transcribe the results on an orbifold $\hat{X}$ to show that each element of $H^2(X;\mathbb{R})$ has the unique L^2-harmonic representative form, and there is the decomposition $H^2(X;\mathbb{R})=H^+\oplus H^-$ into L^2-self-dual and L^2-anti-self-dual harmonic 2-forms. Hence as [FS, Proposition 5.3]

(3.1) **Lemma.** *Suppose X has negative definite intersection form and satisfies $H^1(X;\mathbb{R})=0$. Then each complex line bundle L over X has a unique gauge equivalence class $[A]$ of anti-self-dual connections.*

There is 1-1 correspondence between $H^2(X;\mathbb{Z})$ and the isomorphism class of complex line bundle L over X given by the first Chern class c_1. By Lemma 3.1 the complex line bundle L has an anti-self-dual connection A, and the de Rham class $[(1/2\pi i)R_A]\in H^2(X;\mathbb{R})$ is the harmonic representative of the first Chern class $c_1(L)_\mathbb{R}$. As in Sect. 2, the anti-self-dual connection A induces a homomorphism $\rho\colon\Gamma\to U(1)$ by the action on the fiber L_∞ over ∞. If we identify the homomorphism ρ with the flat connection on S^3/Γ induced from ρ, we have the relation $i^*(c_1(L)) = c_1(\rho)$. As we remarked in Sect. 2, although reducible connections appear as singular points of the moduli space for compact manifolds, but this is not the case for ALE manifolds.

As in Sect. 2 we extend the bundle E to a bundle (also denoted by E) over the orbifold $\hat{X}$ using the trivialization induced from ρ. Then the instanton number of the anti-self-dual connection is determined by the formulas

$$c_2(E)[\hat{X}] = 2c_1(L_1) \cdot c_1(L_2).$$

We are interested in the case that the moduli space is 4-dimensional. First consider the case corresponding Theorem 0.2. We seek a complex line bundle L which satisfies

(1) the associated representation $\rho_\infty: \Gamma \to U(1)$ is trivial,

(2) $c_1(L)^2 = -2$.

The dimension formula (2.17) implies that the moduli space is 4-dimensional around the connection A on the $U(2)$-bundle $\underline{\mathbb{C}} \oplus L$, where $\underline{\mathbb{C}}$ is the trivial line bundle.

Now suppose that the space X is an ALE hyperkähler 4-manifold diffeomorphic to the minimal resolution of $\mathbb{C}^2/\Gamma$ for some nontrivial finite subgroup $\Gamma \subset SU(2)$. Let R be the corresponding root system (see Sect. 1). Via the Poincaré duality $H^2(X, \partial X; \mathbb{Z}) \cong H_2(X; \mathbb{Z})$ the intersection form on $H^2(X, \partial X; \mathbb{Z})$ is given by the negative of the Cartan matrix associated to Γ. The element $\Sigma \in H^2(X, \partial X; \mathbb{Z})$ with $\Sigma^2 = -2$ corresponds to a root. More geometrically, it is realized as a Poincaré dual of a sum of irreducible components of the exceptional set of the resolution. The complex line bundle L associated with $j^*(\Sigma) \in H^2(X; \mathbb{Z})$ satisfies the condition (2), and has an anti-self-dual connection A. Moreover we have $i^* j^*(\Sigma) = 0$, hence the connection A is asymptotic to the trivial flat connection.

(3.2) **Theorem.** *Let Γ be a nontrivial finite subgroup of $SU(2)$, R the set of roots associated with Γ, and (X, g, I, J, K) an ALE hyperkähler 4-manifold diffeomorphic to the minimal resolution of $\mathbb{C}^2/\Gamma$. For each $\Sigma \in R \subset H^2(X, \partial X; \mathbb{Z})$, there exists a reducible anti-self-dual $U(2)$-connection A associated with the decomposition $E = \mathbb{C} \oplus L$ over X such that*

(1) $\dim M = 4$,

(2) $c_1(L) = j^*(\Sigma)$.

Next consider the case corresponding Theorem 0.3. We assume that the group Γ is cyclic (namely Γ is of type A_n), and hence all irreducible nontrivial representations $\rho_1, \ldots, \rho_n$ are 1-dimensional ($n = |\Gamma| - 1$). Changing the order, we may assume

$$\rho_i = \overbrace{\rho_1 \otimes \ldots \otimes \rho_1}^{i\,\text{times}},$$

and the canonical representation $\rho: \Gamma \to SU(2)$ is given by $\rho_1 \oplus \rho_n$. As is proved in [GV] (see Sect. 1), each representation ρ_i determines a holomorphic line bundle $\tilde{M}_i$ over the minimal resolution $\tilde{S}$ which is asymptotic to the flat connection ρ_i on S^3/Γ, and the set $\{c_1(\tilde{M}_1), \ldots, c_1(\tilde{M}_n)\}$ gives a basis for $H^2(X; \mathbb{Z})$ (where X is the underlying differentiable manifold of $\tilde{S}$). Moreover it is a dual basis of the basis $\{[\Sigma_1], \ldots, [\Sigma_n]\}$ given by the irreducible components of the exceptional set. Then by direct calculation using the Cartan matrix for the A_n-type root system, we can determine the Poincaré dual of $[\Sigma_i] \in H^2(X, \partial X; \mathbb{Z})$ as

$$(3.3) \qquad j^* \text{P.D.}[\Sigma_1] = -2\alpha_1 + \alpha_2, \quad j^* \text{P.D.}[\Sigma_2] = \alpha_1 - 2\alpha_2 + \alpha_3, \ldots$$
$$j^* \text{P.D.}[\Sigma_{n-1}] = \alpha_{n-2} - 2\alpha_{n-1} + \alpha_n, \quad j^* \text{P.D.}[\Sigma_n] = \alpha_{n-1} - 2\alpha_n,$$

where $\alpha_i = c_1(\tilde{M}_i)$. We define complex line bundles $L_1, \ldots, L_{n+1}$ by

$$(3.4) \qquad L_1 = \tilde{M}_1, \quad L_2 = \tilde{M}_1^* \otimes \tilde{M}_2, \quad L_3 = \tilde{M}_2^* \otimes \tilde{M}_3, \ldots$$
$$L_n = \tilde{M}_{n-1}^* \otimes \tilde{M}_n, \quad L_{n+1} = \tilde{M}_n^*.$$

Then (3.3) shows that line bundles L_i satisfy the relation $(c_1(L_i), c_1(L_i)) = -n/(n+1)$ and are asymptotic to the flat connection ρ_1. In particular, they give reductions of the common vector bundle $E = L_i \oplus L_i^*$. By the above observation an anti-self-dual connection on L_i induces one on the vector bundle E which is asymptotic to the flat connection associated with the canonical representation ρ. Then by the index formula (2.17) we can see that the moduli space is 4-dimensional.

(3.5) **Theorem.** *Let Γ and X be as in Theorem 3.2 and assume that Γ is a cyclic group of order $|\Gamma|$. Then there exist reducible connections $[A_1], \ldots, [A_{|\Gamma|}]$ in the moduli space $M = M(P, k, \rho)$ where*
(1) *P is a principal bundle with the structure group $SU(2)$,*
(2) $k = (|\Gamma| - 1)/|\Gamma|$,
(3) *ρ is the canonical 2-dimensional representation $\Gamma \to SU(2)$,*
(4) $\dim M = 4$.

Although Theorem 0.3 holds for general Γ not only of type A_n, the results of this section cannot be applied to Γ of other types, since ρ is irreducible in these cases.

4. Taubes' existence theorem and Γ-invariant instantons on $\mathbb{R}^4$

In [T2] Taubes obtained the existence theorem of anti-self-dual connections on compact 4-manifolds with negative definite intersection form. Essentially he proved it by the implicit function theorem. Anti-self-dual connections on S^4 whose curvatures are localized at a point is grafted onto the manifold X to become "almost" anti-self-dual connections. Then by the implicit function theorem, there exist anti-self-dual connections near them, where the existence of the inverse mapping is guaranteed by the negative definiteness of intersection form. Curvatures of the constructed connections concentrate around a point in X.

An ALE manifold X is compactified as an orbifold $\hat{X}$ by a conformal change of the metric and adding a point ∞. Since the anti-self-duality of connections is invariant under the conformal change, we may construct anti-self-dual connections on $\hat{X}$. To carry out the Taubes' procedure on $\hat{X}$ with the curvature concentrating point ∞, the only difference is to use anti-self-dual connections on S^4/Γ instead of S^4. But here we give a different proof by using analysis on ALE manifolds since the estimates which we will obtain in the course of the proof are important when we study the end of the moduli space in Sect. 5. If we blow up the metric of a compact 4-manifold to get an ALE metric, our proof gives a different proof of Taubes' existence theorem. Moreover the proof becomes simpler since curvatures of almost anti-self-dual connections are uniformly bounded on the ALE manifold. This idea is the same as that of Freed-Uhlenbeck [FU] who give another different proof of Taubes' existence theorem by using a conformal change and analysis on manifolds with cylindrical ends.

In this section G is assumed to be a compact Lie group. Let P^Γ be a principal bundle with the structure group G over $\mathbb{R}^4/\Gamma$. Since Γ has a fixed point 0, we have a homomorphism ρ_0: $\Gamma \to G$ induced from the action on the fiber over 0.

(4.1) **Theorem.** *Let (X, g) be an ALE 4-manifold of order $\tau > 0$ with the asymptotic coordinate at infinity $\mathscr{X}$: $X\setminus K \to (\mathbb{R}^4 \setminus \overline{B_R})/\Gamma$, and ρ_0, ρ_∞: $\Gamma \to G$ be homomorphisms which are identified with the flat connections over S^3/Γ. Suppose that there exist principal bundles $P_0 \to X$, $P^\Gamma \to \mathbb{R}^4/\Gamma$ with the structure group G and anti-self-dual connections A_0 on P_0, A^Γ on P^Γ satisfying*

(1) $k_0 = \dfrac{1}{8\pi^2} \int_X |R_{A_0}|^2 \, dV,$

(2) $k_1 = \dfrac{1}{8\pi^2} \int_{\mathbb{R}^4/\Gamma} |R_{A^\Gamma}|^2 \, dV,$

(3) *A_0 is asymptotic to the flat connection ρ_0,*

(4) *A^Γ is asymptotic to the flat connection ρ_∞,*

(5) *P^Γ induces the homomorphism ρ_0 as the action on the fiber $(P^\Gamma)_0$ at the origin,*

(6) *L^2-kernel of $d^*_{A_0}$: $\Omega^+(\operatorname{Ad} P) \to \Omega^1(\operatorname{Ad} P)$ is trivial.*

Then there exists an irreducible anti-self-dual connection A on a principal bundle P over X such that

(7) $k_0 + k_1 = \dfrac{1}{8\pi^2} \int_X |R_A|^2 \, dV,$

(8) *A is asymptotic to the flat connection ρ_∞.*

Remark. The condition (6) preserved under a conformal change of the metric since the L^2-norm is a conformal invariant.

The notation $A = A_0 \# A^\Gamma$ which is used in [D3] illustrates the impression of the above construction in the analogy of the connected sum of manifolds.

Throughout the proof we use the constant C in the generic sense. So the symbol C may mean different constants in different equations. We take a positive function r on X which is equal to $|\mathscr{X}|$ on $X\setminus K$ and suppose $r \leqq R$ on K (recall $|\mathscr{X}| = R$ on ∂K).

For simplicity we assume that the bundle P is trivial and the anti-self-dual connection A_0 is the trivial connection. Hence the assumption (6) is equivalent to the negative-definiteness of the intersection form of X. See [I1] for the general case.

First we construct an almost anti-self-dual connection on X. We fix a cut-off function β: $[0, \infty) \to [0, 1]$ satisfying

$$\beta(t) = \begin{cases} 0 & \text{for } t \in [0, 1] \\ 1 & \text{for } t \in [2, \infty), \end{cases}$$

$$\beta'(r) \geqq 0.$$

For $\lambda \gg R^2$ we define a map $\Phi_\lambda\colon X \to \mathbb{R}^4/\Gamma$ using a coordinate at infinity $\mathscr{X}$ by

$$\Phi_\lambda(x) = \begin{cases} 0 & \text{if } r(x) < \sqrt{\lambda} \\ \beta(r(x)/\sqrt{\lambda})\,\mathscr{X}(x)/\lambda & \text{otherwise.} \end{cases}$$

Let A^Γ be an anti-self-dual connection on $\mathbb{R}^4/\Gamma$ which satisfies the condition of Theorem 4.1. The removable singularities theorem [U1] (see also [IN, Sect. 4.2]) implies

$$(4.2) \qquad |R_{A^\Gamma}(x)| \leqq \frac{C}{1+|x|^4} \quad \text{for } x \in \mathbb{R}^4/\Gamma$$

with a constant C independent of x. We consider the connection $A = \Phi_\lambda^* A^\Gamma$ on X. Then direct calculations show the followings (see [T2, FU]).

(4.3) **Lemma.** *The curvature R_A and its anti-self-dual part R_A^+ satisfy*

(1) $R_A(x) = 0$ *for* $r(x) < \sqrt{\lambda}$,

(2) $|R_A(x)| \leqq \dfrac{C\lambda^2}{\lambda^4 + r(x)^4}$ *for* $r(x) \geqq 2\sqrt{\lambda}$,

(3) $|R_A^+(x)| \leqq \dfrac{C\lambda^2}{(\lambda^4 + r(x)^4)\, r(x)^\tau} \leqq \dfrac{C}{r(x)^{\tau+2}}$ *for* $r(x) \geqq 2\sqrt{\lambda}$,

(4) $|R_A(x)| \leqq \dfrac{C\beta^2\lambda^2}{\lambda^4 + \beta^4\, r(x)^4} \leqq \dfrac{C}{r(x)^4}$ *for* $\sqrt{\lambda} < r(x) < 2\sqrt{\lambda}$

for some constant C independent of λ.

We say a connection A is λ-ASD when it satisfies the conclusions of Lemma 4.3. We denote the domain $\{x \mid r(x) \leqq n\}$ by Ω_n.

From Lemma 4.3 we have estimates for $L^p_{\delta-2}$-norm of R_A^+ for $p>1$, $-2<\delta<0$.

(4.4) **Lemma.** *For $p>1$ and $-2<\delta<0$ we have*

$$\|R_A^+\|_{0,p,\delta-2} \leqq C\lambda^{-\min(\delta/2+1,\,\tau+\delta)}$$

for some constant C.

Proof. It follows from direct calculation as follows:

$$\begin{aligned} &\int_X |R_A^+|^p r^{-(\delta-2)p-4}\,\mathrm{d}V \\ &= \int_{\Omega_{2\sqrt{\lambda}}} |R_A^+|^p r^{-(\delta-2)p-4}\,\mathrm{d}V + \int_{X\setminus\Omega_{2\sqrt{\lambda}}} |R_A^+|^p r^{-(\delta-2)p-4}\,\mathrm{d}V \\ &\leqq C\int_{2\sqrt{\lambda}}^{\infty} \left(\frac{\lambda^2}{\lambda^4+r^4}\right)^p r^{(2-\delta-\tau)p-1}\,\mathrm{d}r + C\int_{\sqrt{\lambda}}^{2\sqrt{\lambda}} r^{(-2-\delta)p-1}\,\mathrm{d}r \\ &\leqq C(\lambda^{-\delta' p} + \lambda^{-(\delta+2)p/2}), \end{aligned}$$

where

$$\delta' = \begin{cases} \delta+\tau, & \text{if } \delta+\tau<2 \\ 1+(\tau+\delta)/2-\varepsilon, & \text{if } \delta+\tau \geqq 2 \end{cases}$$

for arbitrary positive number ε. Since $1+(\tau+\delta)/2 > 1+\delta/2$, the conclusion follows directly. □

We give the scale-broken a priori estimates for $d_A^+ d_A^*$ which is the key of our proof of Taubes' theorem.

(4.5) **Lemma.** *Suppose that A be a λ-ASD connection over a bundle P on X and $\delta \in \mathbb{R}$ is nonexceptional and nonnegative. For $p>1$ and $-2<\delta<0$ the map $d_A^+ d_A^*$: $W_\delta^{2,p}(\Omega^+(\mathrm{Ad}\,P)) \to L_{\delta-2}^p(\Omega^+(\mathrm{Ad}\,P))$ is Fredholm and for any $u \in W_\delta^{2,p}(\Omega^+(\mathrm{Ad}\,P))$,*

$$\|u\|_{2,p,\delta} \leqq C(\|d_A^+ d_A^* u\|_{0,p,\delta-2} + \|u\|_{L^p(\Omega_{R_0})}), \tag{4.6}$$

where C and R_0 are constants independent of λ.

Proof. The Fredholm property follows directly from [Ba, Theorem (1.10), Proposition 1.14]. The only nontrivial assertion is that the constant C in (4.6) is independent of λ. The result of [Ba] only yields (4.6) with the constant C depending on the difference between A and the trivial connection d. If we simply estimate the difference between A and d, the required inequality (4.6) does not follow. So we shall obtain the estimate by rescaling the metric of $\mathbb{C}^2/\Gamma$ and comparing the connection A with A^Γ.

First we prove the corresponding estimate for the anti-self-dual connection A^Γ on $\mathbb{R}^4/\Gamma$ which satisfies (4.2). By taking Coulomb gauge and using a priori estimates for elliptic system, we have some gauge over $\{x \in \mathbb{R}^4/\Gamma \mid |x|>1\}$ for which $d_{A^\Gamma} = d + \alpha$ with

$$|\overbrace{\nabla \ldots \nabla}^{j\,\text{times}} A(x)| \leqq C_j\,|x|^{-j-3} \qquad \text{for all } j \geqq 0,$$

where C_j is constant depending on j and A^Γ. Then by [Ba, Theorem 1.10] for any $u \in W_\delta^{2,p}(\mathbb{R}^4/\Gamma;\, \Omega^+(\mathrm{Ad}\,P))$

$$\|u\|_{2,p,\delta} \leqq C(\|d_{A^\Gamma}^+ d_{A^\Gamma}^* u\|_{0,p,\delta-2} + \|u\|_{L^p(B_{R_0})}),$$

where C and R_0 are constants depending on C_j ($j=0, 1$), δ and p. Moreover since $\mathrm{Ker}\, d_{A^\Gamma}^+ d_{A^\Gamma}^* = H_{A^\Gamma,\delta}^2$ is zero (it easily follows from Weitzenböck formula, see Proposition 5.1), we have

$$\|u\|_{2,p,\delta} \leqq C\, \|d_{A^\Gamma}^+ d_{A^\Gamma}^* u\|_{0,p,\delta-2}.$$

Now let T_λ be a diffeomorphism $x \mapsto x/\lambda$ on $\mathbb{R}^4/\Gamma$ and consider the anti-self-dual connection $T_\lambda^* A^\Gamma$. Since we have

$$C^{-1}\, \|u\|_{k,p,\delta} \leqq \lambda^{-\delta-2}\, \|T_\lambda^* u\|_{k,p,\delta} \leqq C\, \|u\|_{k,p,\delta}$$

with a constant C independent of $\lambda \geqq 1$, the above estimate implies

$$\|u\|_{2,p,\delta} \leqq C\, \|d_{T_\lambda^* A^\Gamma}^+ d_{T_\lambda^* A^\Gamma}^* u\|_{0,p,\delta-2},$$

where C is a constant independent of $\lambda \geqq 1$. Since A is λ-ASD, the difference between the connection A and $T_\lambda^* A^\Gamma$ on Ω_{R_0} can be estimated by a constant independent of A as

$$|A - (\mathscr{X}^{-1})^*\, T_\lambda^* A^\Gamma| \leqq C r^{-3}, \qquad |\nabla(A - (\mathscr{X}^{-1})^*\, T_\lambda^* A^\Gamma)| \leqq C r^{-4}.$$

This gives (4.6) (see the proof of [Ba, Theorem (1.10)]). □

Now we use the negative-definiteness of the intersection form.

(4.7) **Proposition.** *Suppose that the space of L^2-anti-self-dual harmonic 2-forms H^+ is trivial. Then for $p>1$, $-2<\delta<0$ there exists $\lambda_0>0$ such that if A is λ-ASD, $\lambda \geqq \lambda_0$, then for $u \in W_\delta^{2,p}(\Omega^+(\operatorname{Ad} P))$*

$$\|u\|_{2,p,\delta} \leqq C \|d_A^+ d_A^* u\|_{0,p,\delta-2},$$

where C is a constant independent of λ. In particular, $d_A^+ d_A^*$ gives an isomorphism between $W_\delta^{2,p}(\Omega^+(\operatorname{Ad} P))$ and $L_{\delta-2}^p(\Omega^+(\operatorname{Ad} P))$.

Proof. Suppose the contrary. Then we can find sequences $\{\lambda_i\}$, $\{u_i\}$ and $\{A_i\}$ such that

(1) λ_i goes to infinity as $i \to \infty$,
(2) A_i is λ_i-ASD,
(3) $\|u_i\|_{2,p,\delta}=1$,
(4) $\|d_{A_i}^+ d_{A_i}^* u_i\|_{0,p,\delta-2}$ converges to 0 as $i \to \infty$.

Since the estimate (4.6) yields

$$1=\|u_i\|_{2,p,\delta} \leqq C(\|d_{A_i}^+ d_{A_i}^* u_i\|_{0,p,\delta-2}+\|u_i\|_{L^p(\Omega_{R_0})}),$$

the above condition (4) implies $\|u_i\|_{L^p(\Omega_{R_0})} \geqq \varepsilon$ for sufficiently large i with a positive constant ε independent of i. So on a compact domain Ω_{R_0} we apply the usual Rellich lemma to show that $\{u_i\}$ has a subsequence which converges strongly in $L^p(\Omega_{R_0})$. We denote this limit by u_∞, so $u_\infty \neq 0$. On the other hand A_i converges to a trivial connection uniformly on each compact subset of X, and $\{u_i\}$ has a subsequence which converges weakly in $W_\delta^{2,p}$. Hence we obtain a nonzero element $u_\infty \in W_\delta^{2,p}(\Omega^+)$ which satisfies $d^+ d^* u_\infty=0$. But this contradicts to $H^+=0$ by the following Lemma.

(4.8) **Lemma.** *Let $\delta<0$. If $u \in W_\delta^{2,p}(\Omega^+)$ satisfies $d^+ d^* u=0$, u has the following asymptotic behaviour:*

$$|u|=O(r^{-4}).$$

Proof. This is proved by a similar technique as [BKN, Appendix] (see also [IN, Chap. 4]). The basic idea is due to [SSY].

First we show $du=0$. By [Ba, Theorem 1.17] u satisfies $u=O(r^{-2})$. By the elliptic estimates and the Sobolev inequality, we have

$$|u|=O(r^{-2}), \qquad |\nabla_A u|=O(r^{-3}). \tag{4.9}$$

We use the integration by parts on a domain $X_R=\{x\colon r(x) \leqq R\}$ to get

$$0=\int_{X_R}(\Delta u, u)=\int_{X_R}|d^* u|^2+\int_{\partial X_R}\left(\frac{\partial}{\partial n} u, u\right).$$

By (4.9) the second term of right hand side converges to 0 as $R \to \infty$. This implies $d^* u=0$, and hence $du=0$.

The same argument as in [BKN, Appendix] shows

$$\tfrac{3}{2}|(\nabla u, u)|^2 \leqq |\nabla u|^2 |u|^2$$

where $(\nabla u, u)$ is a 1-form defined by $(\nabla u, u)(V) = (\nabla_V u, u)$ for a tangent vector V.

So we get

$$(4.10) \qquad |\nabla u|^2 \geqq \tfrac{3}{2} |(\nabla u, u)|^2 |u|^{-2} = \tfrac{3}{2} |d|u||^2.$$

Then we use Weitzenböck formula for Ad P-valued self-dual 2-forms

$$0 = d^+ d^* u = \nabla^* \nabla u - 2W^+(u) + \frac{S}{3} u,$$

where $W^+: \Omega^2 \to \Omega^2$ is the self-dual part of the Weyl tensor, and S is the scalar curvature. Using (4.10), we get

$$\begin{aligned} \Delta |u|^{1/2} &= \frac{1}{4} |u|^{-3/2} \Delta |u|^2 - \frac{3}{4} |u|^{-3/2} |d|u||^2 \\ &\geqq -\left(|W^+| + \frac{|S|}{6}\right) |u|^{1/2} \\ &\geqq -\frac{C}{r^{2+\tau}} |u|^{1/2}. \end{aligned}$$

In the last inequality we have use the ALE property of the metric g. Then by the argument of [Ba, Theorem 1.17], we have

$$|u|^{1/2} = O(r^{-2}).$$

(-2 is the greatest negative exceptional value in $\mathbb{R}^4$.) □

Since the trivial connection A_0 satisfies

$$0 = \operatorname{Ker} d_{A_0}^* d_{A_0}: W_\delta^{2,p}(\Omega^0(\operatorname{Ad} P)) \to W_{\delta-2}^{0,p}(\Omega^0(\operatorname{Ad} P))$$

(it follows from the maximal principle), the same argument shows the following estimates which will be necessary in Sect. 5.

(4.10) **Proposition.** *For $p > 1$, $-2 < \delta < 0$ there exists $\lambda_0 > 0$ such that if A is λ-ASD, $\lambda \geqq \lambda_0$, then for $u \in W_\delta^{2,p}(\Omega^0(\operatorname{Ad} P))$*

$$\|u\|_{2,p,\delta} \leqq C \|d_A^* d_A u\|_{0,p,\delta-2},$$

where C is a constant independent of λ. In particular, $d_A^* d_A$ gives an isomorphism between $W_\delta^{2,p}(\Omega^0(\operatorname{Ad} P))$ and $L_{\delta-2}^p(\Omega^0(\operatorname{Ad} P))$.

We remark that Propositions 4.7, 4.10 give the estimates for Green operators on $\Omega^0(\operatorname{Ad} P)$ and $\Omega^+(\operatorname{Ad} P)$ independent of $\lambda \geqq \lambda_0$.

By the same argument as [FU, Theorem 7.26] we obtain

(4.11) **Proposition.** *Under the same assumption on Proposition 4.7 consider the operator $L_B = d_A^+ d_A^* + [B \wedge d_A^*]^+$ with $B \in L_{-2}^q(\Omega^1(\operatorname{Ad} P))$ $(q = 4p/(3p-4))$. Then there exists a constant $\varepsilon > 0$ independent of λ such that if $\|B\|_{0,q,-2} \leqq \varepsilon$, then for $u \in W_\delta^{2,p}(\Omega^+(\operatorname{Ad} P))$*

$$\|u\|_{2,p,\delta} \leqq C \|L_B u\|_{0,p,\delta-2}.$$

As in [FU, Theorem 7.27] we use the continuity method to the equation

$$L_t u_t = d_A^+ d_A^* u_t + \tfrac{1}{2}[d_A^* u_t \wedge d_A^* u_t]^+ + t R_A^+ = 0,$$

we have an anti-self-dual connection $A + d_A^* u_1$ near A.

(4.12) **Theorem.** *Under the same assumption on Proposition* 4.7, *there exists* $u \in W_\delta^{2,p}(\Omega^+(\operatorname{Ad} P))$ *such that the connection* $A + d_A^* u$ *is anti-self-dual and satisfies the estimate*

$$\|u\|_{2,p,\delta} \leqq C \|R_A^+\|_{0,p,\delta-2}. \tag{4.13}$$

Moreover such u is uniquely determined by A.

In the rest of this section, we give some examples of anti-self-dual connections over S^4/Γ. We are interested in the case that $G = SU(2)$, $\Gamma \subset SU(2)$, and the moduli space is 4-dimensional (cf. Theorem 0.3). Furuta and Hashimoto [FH], and independently Austin [Au], obtained more general and detailed results when Γ is a cyclic group.

(4.14) **Theorem.** *Suppose* $\Gamma \subset SU(2)$ *is a finite subgroup of order* $|\Gamma|$. *Let* $k = |\Gamma| - 1$. *Then there exists a principal bundle* P^Γ *on* S^4/Γ *with the structure group* $SU(2)$ *such that* $c_2(P^\Gamma)[S^4/\Gamma] = k/(k+1)$, *and an irreducible anti-self-dual connection* A^Γ *on* P^Γ, *where the actions* ρ_0 *and* ρ_∞ *of* Γ *on the fibers* P_0^Γ *and* P_∞^Γ *are given by*

(1) ρ_0 *is the trivial,*

(2) ρ_∞ *is the canonical 2-dimensional representation* $\Gamma \to SU(2)$.

Moreover the moduli space M^Γ *of anti-self-dual connections on* P^Γ *is diffeomorphic to* $(\mathbb{C}^2 \setminus \{0\})/\Gamma$.

By the dimension formula (2.18) the moduli space becomes 4-dimensional in the above situation.

For the proof of (4.14), we use the result of Atiyah-Drinfeld-Hitchin-Manin [ADHM] (see also [D2]) which gives a parametrization of framed moduli spaces of anti-self-dual connections on S^4 (or of moduli spaces on $\mathbb{R}^4$). Anti-self-dual connections on S^4/Γ are Γ-equivariant anti-self-dual connections on S^4, and their parametrization is also possible. See [FH, Au] for detail.

(4.15) **Fact.** *For an* $SU(2)$ *principal bundle P over* $\mathbb{R}^4/\Gamma$, *there is a one-to-one correspondence between* $M(P, k/|\Gamma|, \rho_0, \rho_\infty)$ *and the quotient of the set of matrices* $(\alpha_1, \alpha_2, a, b)$ *satisfying:*

(1) $\alpha_1, \alpha_2 \in M(k, k; \mathbb{C})$, $a \in M(2, k; \mathbb{C})$, $b \in M(k, 2; \mathbb{C})$,

(2) $[\alpha_1, \alpha_2] + b a = 0$,

(3) $[\alpha_1, \alpha_1^*] + [\alpha_2, \alpha_2^*] + b b^* - a^* a = 0$,

(4) *for all* λ, $\mu \in \mathbb{C}$, ${}^t[\alpha_1 + \lambda, \alpha_2 + \mu, a]$ *is injective and* $[\lambda - \alpha_1, \alpha_2 - \mu, b]$ *is surjective,*

(5) $(\alpha_1, \alpha_2, a, b)$ *is* Γ*-equivariant in the following sense*

$$\begin{cases} \rho_U(\gamma^{-1})\,\alpha_1\,\rho_U(\gamma) = \gamma_1\,\alpha_1 + \gamma_2\,\alpha_2 \\ \rho_U(\gamma^{-1})\,\alpha_2\,\rho_U(\gamma) = -\overline{\gamma_2}\,\alpha_1 + \overline{\gamma_1}\,\alpha_2 \\ a = \rho_\infty(\gamma^{-1})\,a\,\rho_U(\gamma) \\ b = \rho_U(\gamma^{-1})\,b\,\rho_\infty(\gamma) \end{cases} \quad \textit{for } \gamma = \begin{bmatrix} \gamma_1 & \gamma_2 \\ -\bar{\gamma}_2 & \bar{\gamma}_1 \end{bmatrix},$$

by the action of $U_\Gamma(k)=\{u\in U(k)\,|\,\rho_U(\gamma^{-1})\,u\,\rho_U(\gamma)=u\}$:

$$\alpha_i\mapsto u^{-1}\alpha_i u,\qquad a\mapsto a\,u,\qquad b\mapsto u^{-1}b.$$

Here $\rho_U\colon \Gamma\to U(k)$ *is a representation and can be uniquely determined by* ρ_0, ρ_∞, k.

The representation ρ_0 at 0 is calculated by the following way. The fiber of the associated vector bundle over 0 is identified with the set

$$\left\{(u_1,u_2,u_3)\in\mathbb{C}^k\oplus\mathbb{C}^k\oplus\mathbb{C}^2\,\middle|\,\begin{cases}-\alpha_2 u_1+\alpha_1 u_2+b\,u_3=0\\ \alpha_1^* u_1+\alpha_2^* u_2+a^* u_3=0\end{cases}\right\}. \tag{4.16}$$

Then Γ acts on this set by

$$\begin{aligned} u_1&\mapsto\rho_U(\gamma^{-1})(\overline{\gamma_1}\,u_1-\gamma_2\,u_2)\\ u_2&\mapsto\rho_U(\gamma^{-1})(\overline{\gamma_2}\,u_1+\gamma_1\,u_2)\\ u_3&\mapsto\rho_\infty(\gamma^{-1})\,u_3.\end{aligned}\tag{4.17}$$

Matrices satisfying the above conditions are obtained from the following Lemma which was used in [Kr1] for the construction of ALE hyperkähler 4-manifolds.

Let (ρ_Q, Q) be the canonical 2-dimensional representation of Γ and (ρ_R, R) the regular representation. Define $Y=(Q\otimes\mathrm{End}(R))^\Gamma$, the space of Γ-invariant elements in $Q\otimes\mathrm{End}(R)$. Taking an orthonormal basis for Q, we represent an element of Y as a pair of endomorphisms (f,g). The group $U_\Gamma(R)$ consisting of unitary transformations of R which commutes with the action of Γ on R acts on Y by

$$(f,g)\in Y\mapsto(u^{-1}fu,\,u^{-1}g\,u)\qquad\text{for } u\in U_\Gamma(R).$$

Dividing out the group of scalar matrices, we have an effective action of $U_\Gamma(R)/S^1$.

By its definition, the regular representation has a basis $\{e_\gamma\}$ indexed by $\gamma\in\Gamma$ with the property $R(\delta)(e_\gamma)=e_{\delta\gamma}$. We define an inclusion map $i\colon\mathbb{C}^2\to Y$ by $i(x,y)=(f,g)$, where

$$f(e_\gamma)=x_\gamma\,e_\gamma,\qquad g(e_\gamma)=y_\gamma\,e_\gamma,\qquad \begin{pmatrix}x_\gamma\\ y_\gamma\end{pmatrix}=Q(\gamma)\begin{pmatrix}x\\ y\end{pmatrix}.$$

The map i is equivariant for the action of Γ on $\mathbb{C}^2$ and that of $U_\Gamma(R)$ on Y. We then have

(4.18) **Lemma** [Kr1, Lemma 3.1, Corollary 3.2]. *The quotient space*

$$\{(f,g)\in Y\,|\,[f,g]=0,\ [f,f^*]+[g,g^*]=0\}/(U_\Gamma(R)/S^1)$$

is isomorphic to $\mathbb{C}^2/\Gamma$.

Now we are in a position to start the proof of Theorem 4.14. The regular representation of Γ decomposes as

$$(\rho_R,R)=\bigoplus_i\mathbb{C}^{n_i}\otimes(\rho_i,R_i),$$

where $\{(\rho_0, R_0), (\rho_1, R_1), \ldots, (\rho_n, R_n)\}$ is the set of all irreducible representation of Γ, with (ρ_0, R_0) the trivial representation. Set

$$(\rho_{V_1}, V_1) = \bigoplus_{i>0} \mathbb{C}^{n_i} \otimes R_i, \qquad (\rho_{V_2}, V_2) = (\rho_0, R_0).$$

The quotient group $U_\Gamma(R)/S^1$ is identified with $\prod_{i>0} U(R_i) = U_\Gamma(V_1)$. Via the decomposition $(\rho_R, R) = (\rho_{V_1}, V_1) \oplus (\rho_{V_2}, V_2)$, an element $(f, g) \in Y$ is represented as

$$f = \begin{bmatrix} \alpha_1 & b_2 \\ a_1 & 0 \end{bmatrix}, \qquad g = \begin{bmatrix} \alpha_2 & -b_1 \\ a_2 & 0 \end{bmatrix}.$$

If we set

$$a = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, \qquad b = [b_1\, b_2],$$

a direct calculation shows that $(\alpha_1, \alpha_2, a, b)$ satisfies (4.14) with $\rho_U = \rho_{V_1}, \rho_\infty = \rho_Q$. Moreover (f, g) satisfies the equation $[f, g] = [f, f^*] + [g, g^*] = 0$ if and only if $(\alpha_1, \alpha_2, a, b)$ satisfies the equations (2), (3) in Fact 4.15.

Next we check the property (4). Suppose that $(f, g) \neq (0, 0)$ and v_1 is an element in the kernel of $\begin{bmatrix} \alpha_1 + \lambda \\ \alpha_2 + \mu \end{bmatrix}$. Then $v = (v_1, 0) \in V_1 \oplus V_2 = R$ satisfies

$$(f + \lambda \,\mathrm{id})\, v = 0, \qquad (g + \mu \,\mathrm{id})\, v = 0.$$

We represent v by the basis $\{e_\gamma\}$ as

$$v = \sum_\gamma v_\gamma e_\gamma.$$

For $(f, g) \in i(\mathbb{C}^2) \subset Y$,

$$(f + \lambda \,\mathrm{id})\, v = \sum v_\gamma (x_\gamma + \lambda)\, e_\gamma = 0$$

$$(g + \mu \,\mathrm{id})\, v = \sum v_\gamma (y_\gamma + \mu)\, e_\gamma = 0.$$

Hence $v_\gamma = 0$ unless $-(\lambda, \mu) = (x_\gamma, y_\gamma)$, namely $-{}^t(\lambda, \mu) = Q(\gamma)^t(x, y)$. Since the action of Γ on $\mathbb{C}^2$ is free outside 0, there exists at most only one nonzero v_γ. On the other hand, since v has no V_2-component, we have

$$(v, \sum_\gamma e_\gamma) = 0,$$

where the inner product on R is defined so that $\{e_\gamma\}$ is an orthonormal basis. This implies $\sum v_\gamma = 0$, hence $v_\gamma = 0$ for all γ. The surjectivity can be checked similarly.

Finally we calculate the action on the fiber over 0. The set defined in (4.16) is isomorphic to the set

$$\left\{ (v, w) \in R \times R \,\middle|\, \begin{cases} f v + g w \in V_2 \\ f^* w - g^* v \in V_2 \end{cases} \right\}$$

under the correspondence

$$v=\begin{bmatrix}u_2\\u_3''\end{bmatrix},\qquad w=\begin{bmatrix}-u_1\\-u_3'\end{bmatrix},$$

where

$$u_3=\begin{bmatrix}u_3'\\u_3''\end{bmatrix}.$$

If we write v and w as

$$v=\sum v_\gamma e_\gamma,\qquad w=\sum w_\gamma e_\gamma,$$

they must satisfy

$$x_\gamma v_\gamma+y_\gamma w_\gamma=x\,v_e+y\,w_e$$

$$\overline{y_\gamma}\,v_\gamma-\overline{x_\gamma}\,w_\gamma=\bar{y}\,v_e-\bar{x}\,w_e.$$

The definition of (x_γ, y_γ) implies

(4.19) $$(v_\gamma, w_\gamma)=(v_e, w_e)\,Q(\gamma^{-1}).$$

The action defined in (4.17) corresponds to the action defined by

$$v\mapsto R(\delta^{-1})(\gamma_1 v-\overline{\gamma_2}\,w)$$

$$w\mapsto R(\delta^{-1})(\gamma_2 v+\overline{\gamma_1}\,w).$$

Then (4.19) shows the action is trivial.

5. Geometry of the end

In this section we study the behavior of Riemannian metric g_M on the end of the moduli space M. We shall show that on each end the metric g_M is ALE. Our calculations are similar to those of [D1, GP] which show that an end of a moduli space on a compact definite 4-manifold X is $(0,1]\times X$ ($\{0\}\times X$ is the infinity) in both topologically and metrically in C^0-level, but our purpose requires more detail since we must study the curvature of the moduli space.

Throughout this section we assume that G is a compact Lie group and (X, g) is an ALE Riemannian manifold of order $\tau=4$ with a hyperkähler structure (I_X, J_X, K_X). Let P be a principal G-bundle over X, $\rho\colon \Gamma\to G$ a homomorphism. We consider the moduli space $M=M(P, k, \rho)$ of anti-self-dual connections on P which are asymptotic to ρ and have the instanton number k. As in the previous sections we use constant C in generic sense.

Before starting the study of the ends of M, we prove the smoothness of M.

(5.1) **Proposition.** *For each anti-self-dual connection $[A]\in M=M(P, k, \rho)$, it holds*

$$H^2_{A,-1}=0.$$

In particular, the moduli space M is a nonsingular C^∞-manifold.

Proof. We use Weitzenböck formula for $\operatorname{Ad} P$-valued self-dual 2-forms. For all $\alpha \in \Omega^+(\operatorname{Ad} P)$ we have

$$2\, d_A^+ d_A^* \alpha = \nabla_A^* \nabla_A \alpha - 2 W^+(\alpha) + \frac{S}{3} \alpha + [R_A^+, \alpha],$$

where $W^+ \colon \Omega^2 \to \Omega^2$ is the self-dual part of the Weyl curvature and S is the scalar curvature. Since A is anti-self-dual, we have $R_A^+ = 0$. We also have $W^+ = 0$ and $S = 0$ because X has a hyperkähler structure. Hence for $\alpha \in H^2_{A,-1}$ we obtain

$$\Delta\, |\alpha|^2 = 2\, |\nabla_A \alpha|^2 + 2(\nabla_A^* \nabla_A \alpha, \alpha) \geqq 0.$$

But by the Sobolev inequality [Ba, Theorem 1.2] we have $|\alpha| = O(r^{-1})$ as $r \to \infty$ and the strong maximal principle implies that $\alpha = 0$. □

This lemma implies that the space of L^2-self-dual harmonic 2-forms are trivial; $H^+ = 0$ especially. Hence the intersection form of X is negative definite.

We compactify X to an orbifold $(\hat{X}, \hat{g})$ adding the point ∞ (see § 2) and identify anti-self-dual connections on X with ones on $\hat{X}$.

(5.2) **Theorem.** *Suppose that the dimension of the moduli space M is equal to 4. Then for every sequence $\{[A_i]\}$ in the moduli space $M = M(P, k, \rho)$, there exist a subsequence (also denoted by $[A_i]$), gauge transformations $s_i \in \mathscr{G}_0$ such that one of the following* (1), (2) *holds.*

(1) *There exists an anti-self-dual connection $[A_\infty] \in M(P, k, \rho)$ such that $s_i^*(A_i)$ converges to A_∞ in the C^∞-topology on $\hat{X}$.*

(2) *There exist an anti-self-dual connection $[A_\infty] \in M(P', k', \rho')$ on a different principal bundle P' over $\hat{X}$ and an anti-self-dual connection $[A^\Gamma] \in M(P^\Gamma, k_1, \rho_0, \rho_\infty)$ on a principal bundle over S^4/Γ such that*

(2.a) *$s_i^*(A_i)$ converges to A_∞ outside ∞.*

(2.b) *There exists a divergent sequence $\{\lambda_i\}$ such that $(\Phi_{\lambda_i}^{-1})^* A_i$ converges to A^Γ (after gauge transformations) in the C^∞_{loc}-topology on $\mathbb{R}^4/\Gamma$ (Φ_λ is defined in Sect.* 4).

(2.c) *A^Γ is not flat.*

(2.d) $k = k' + k_1$.

(2.e) $\rho = \rho_\infty$, $\rho' = \rho_0$.

Proof. We use the compactness theorem of Uhlenbeck [U2] on $\hat{X}$ (see also [T3, Proposition 4.4]). Since X has a hyperkähler structure, the second cohomology group $H^2_{A,-1}$ vanishes for each anti-self-dual connection A. And the dimensions of moduli spaces are given by the dimension formulas (2.7), (2.8).

We take a subsequence of $\{[A_i]\}$ and gauge transformations s_i such that $s_i(A_i)$ converges to an anti-self-dual connection A_∞ on P' outside a finite set (possibly empty). If curvature concentration happens a point other than ∞, an anti-self-dual connection on S^4 bubbles off from there. Then the instanton number decreases so that the dimension of the moduli space decrease more than 8, so it is impossible if $\dim M = 4$.

When curvature concentration happens at ∞, anti-self-dual connection $[A_n^\Gamma] \in M(P_n^\Gamma, k_n, \rho_0^n, \rho_\infty^n)$ ($n = 1, \ldots, N$) on S^4/Γ bubble off. A connection A_i, for sufficiently large i, resembles $A_\infty \# A_1^\Gamma \# \ldots \# A_N^\Gamma$. Then we have

$$\dim M(P, k, \rho) = \dim M(P', k', \rho') + \sum_{n=1}^{N} \dim M^\Gamma(P_n^\Gamma, k_n, \rho_0^n, \rho_\infty^n).$$

Since each $M^\Gamma(P_n^\Gamma, k_n, \rho_0^n, \rho_\infty^n)$ has a hyperkähler structure, its dimension is a multiple of 4. It has a nontrivial action of $\mathbb{R}^+$, hence it has the dimension greater than 1. In particular, it is not 0. So if $\dim M=4$, we have $N=1$ and

$$\dim M(P', k', \rho')=0$$

$$\dim M(P_1^\Gamma, k_1, \rho_0^1, \rho_\infty^1)=4. \quad \square$$

Next we recall the following lemma of Uhlenbeck [U 1]:

(5.3) **Lemma.** *Let $B(p, r)$ be a geodesic ball of radius r in a 4-dimensional complete Riemannian manifold (X, g). There exists a positive constant $\varepsilon=\varepsilon(g)$ such that if A is a Yang-Mills connection on $B(p, r)$ with $\int_{B(p,r)} |R_A|^2<\varepsilon$, then*

$$\sup_{B(p,r/2)} |R_A|^2 \leqq C r^{-4} \int_{B(p,r)} |R_A|^2 \, dV$$

for some $C=C(g)$.

(5.4) **Proposition.** *Suppose that* $\dim M=4$. *Then the Riemannian manifold (M, g_M) is complete.*

Proof. First we shall show that

$$|R_A|^2 \leqq C r^{-2} \tag{5.5}$$

fore some constant C independent of $[A]\in M$. Suppose the contrary. So there exists a sequence $\{A_i\}$ such that

$$\sup_X r^2 |R_{A_i}| \to \infty \quad \text{as} \quad i\to\infty,$$

and by Theorem 5.2 we may assume

$$A_i \to A_\infty \quad \text{in } C^\infty_{\text{loc}}\text{-topology on } X, \tag{5.6}$$

$$(\Phi_{\lambda_i}^{-1})^* A_i \to A_1^\Gamma \quad \text{in } C^\infty_{\text{loc}}\text{-topology on } \mathbb{R}^4/\Gamma\setminus\{0\},$$

$$\int_X |R_{A_i}|^2 \, dV = \int_X |R_{A_\infty}|^2 \, dV + \int_{\mathbb{R}^4/\Gamma} |R_{A_1^\Gamma}|^2 \, dx.$$

We take a sufficiently large R so that

$$\int_{r>R} |R_{A_\infty}|^2 \, dV \leqq \tfrac{1}{4}\varepsilon,$$

$$\int_{|x|<R^{-1}} |R_{A_1^\Gamma}|^2 \, dx \leqq \tfrac{1}{4}\varepsilon.$$

Then for sufficiently large i, we have

$$\int_{r\leqq R} |R_{A_i}|^2 \, dV \geqq \int_{r\leqq R} |R_{A_\infty}|^2 \, dV - \tfrac{1}{4}\varepsilon$$

$$\geqq \int_X |R_{A_\infty}|^2 \, dV - \tfrac{1}{2}\varepsilon.$$

Similarly we get

$$\int_{r \geqq R^{-1}\lambda_i} |R_{A_i}|^2 \, dV \geqq \int_{\mathbb{R}^4/\Gamma} |R_{A_1^\Gamma}|^2 \, dx - \tfrac{1}{2}\varepsilon.$$

Combining the above inequalities, we have

$$\int_{R \leqq r \leqq R^{-1}\lambda_i} |R_{A_i}|^2 \, dV \leqq \varepsilon.$$

We apply Lemma 5.3 to a ball $B(x, r/2) \subset \{R \leqq r \leqq R^{-1}\lambda_i\}$ $(2R \leqq r = r(x) \leqq R^{-1}\lambda_i/2)$ by rescaling the metric with the factor r^{-1}. Then we find

$$\sup_{2R \leqq r \leqq R^{-1}\lambda_i/2} |R_{A_i}| \leqq C r^{-2}$$

for some constant C. On the other hand by (5.6) we have

$$\sup_{r \leqq 2R} |R_{A_i}| \leqq C$$

$$\sup_{r \geqq R^{-1}\lambda_i/2} |R_{A_i}| = \lambda_i^2 \sup_{|x| > R^{-1}/2} |R_{(\Phi_{\lambda_i}^{-1})^* A_i}(x)| \leqq C r^{-2}.$$

Hence we have verified (5.5).

To prove the completeness of (M, g_M) we have to show that an open curve $[A_t]$ $(t \in [0, t_0))$ in M of finite length has a limit point. We may assume that $\varphi_t = \dfrac{d}{dt} A_t$ satisfies the followings:

$$\text{(5.7)} \qquad \Delta_{A_t} \varphi_t = 0, \qquad \int_X |\varphi_t|^2 = 1$$

for all $t \in [0, t_0)$. By Weitzenböck formula for $\operatorname{Ad} P$-valued 1-forms [BL] we get

$$\begin{aligned} 0 &= \Delta_{A_t} \varphi_t \\ &= \nabla_{A_t}^* \nabla_{A_t} \varphi_t + [R_{A_t}, \varphi_t]. \end{aligned}$$

This together with (5.5) implies

$$\Delta |\varphi_t| \geqq -\frac{C}{r^2} |\varphi_t|.$$

By the L^∞-estimate for subsolutions of elliptic equation (see e.g. [GT, Theorem 8.17]) we have

$$|\varphi_t(x)|^2 \leqq C r^{-4} \int_{B(x, r/2)} |\varphi_t|^2 \, dV \leqq C r^{-4}.$$

Hence we obtain

$$|A_t - A_s| \leqq C r^{-2} |t - s|.$$

On the other hand by (5.7) we get

$$\int_X |\nabla_{A_t}\varphi_t|^2 \leqq C \int_X |\varphi_t|^2 \leqq C.$$

Combining these, we have

$$\int_X |\nabla_{A_0}\varphi_t|^2 \leqq C,$$

and hence

$$\int_X |\nabla_{A_0}(A_t - A_s)|^2 \leqq C\,|t-s|.$$

This implies that $R_{A_t} = R_{A_0} + d_{A_0}(A_t - A_0) + \frac{1}{2}[A_t - A_0 \wedge A_t - A_0]$ satisfies

$$\int_X |R_{A_t} - R_{A_s}|^2 \leqq C\,|t-s|.$$

This means that R_{A_t} is a Cauchy sequence in L^2, and hence converges strongly as $t \to t_0$. So the sequence $[A_t]$ converges to $A_{t_0} \in M$. □

(5.8) **Corollary.** *Suppose (X, g, I_X, J_X, K_X) is an* ALE *hyperkähler* 4*-manifold and $G_\rho = SO(3)$. Then each component of the moduli space $M = M(P, k, \rho)$ is isometric to the Eguchi-Hanson space if* $\dim M = 4$. *The fixed points set of G_ρ-action is written as $G_\rho[A_0]$ for a reducible connection $[A_0]$ and is isometric to S^2. In this way there is a bijective correspondence between the components of M and the reduction of the bundle P.*

Proof. Since $G_\rho = SO(3)$ acts on the complete hyperkähler 4-manifold (M, g, I_M, J_M, K_M) triholomorphically and isometrically, each component of the moduli space M is isometric to the Eguchi-Hanson space ([AH], see also [GR, Proposition 2.7]). Moreover we know that the fixed point set F of $SO(3)$-action is isometric to S^2. Hence F is the orbit of a reducible connection. □

Now we can determine the Riemannian structure of the moduli space M^Γ on $\mathbb{R}^4/\Gamma$ when Γ is a nontrivial subgroup of $SU(2)$.

(5.9) **Proposition.** *Each component of the moduli space M^Γ (with the natural Riemannian metric g_{M^Γ}) is isometric to $(\mathbb{R}^4 \setminus \{0\})/\Gamma'$ for some finite subgroup $\Gamma' \subset SU(2)$ when* $\dim M^\Gamma = 4$.

Proof. As is observed in Sect. 4, there is a natural $\mathbb{R}^+$-action on M^Γ. For $\lambda \in \mathbb{R}^+$ let T_λ be a diffeomorphism

$$T_\lambda(x) = \lambda^{-1} x \qquad \text{for } x \in \mathbb{R}^4/\Gamma.$$

We take a point $[A] \in M^\Gamma$ and a tangent vector $\alpha \in T_{[A]} M^\Gamma \cong H^1_{A,-2}$. Since $T_\lambda^* \alpha \in H^1_{T_\lambda^*(A),-2}$, the action on the tangent vector $\lambda^* \alpha$ is equal to $T_\lambda^* \alpha$. Hence we have

$$g_{M^\Gamma}(\lambda^*\alpha, \lambda^*\alpha) = \int_{\mathbb{R}^4/\Gamma} |T_\lambda^*\alpha(x)|^2\,\mathrm{d}x = \lambda^2 \int_{\mathbb{R}^4/\Gamma} |\alpha(y)|^2\,\mathrm{d}y = \lambda^2 g_{M^\Gamma}(\alpha, \alpha).$$

This shows that M^Γ is isometric to the warped product $(\mathbb{R}^+ \times S^\Gamma, d\lambda^2 + \lambda^2 g_{S^\Gamma})$ for some Riemannian manifold S^Γ (this holds even if $\dim M^\Gamma \neq 4$). Hence the sectional curvature of g_{M^Γ} vanishes for any plane tangent to $\partial/\partial\lambda$.

We have seen in Sect. 2 that the metric g_{M^Γ} is hyperkähler. We take a normal coordinate at a point $[A]$ and calculate the curvatures by using the index notation for tensors. We may assume

$$R_{1i1i} = 0 \quad \text{for all } i.$$

Since g_{M^Γ} is hyperkähler, the curvature tensor is anti-self-dual. Combining with the above, we have

$$R_{jkjk} = 0 \quad \text{for all } j, k.$$

Hence the metric is flat.

On the other hand we apply the argument of Lemma (5.3) to this situation to prove that the metric space $[1, \infty) \times S^\Gamma \subset M^\Gamma$ is complete as a metric space. So S^Γ is a space form of constant curvature $+1$. This completes the proof of the proposition. □

It is conjectural that the moduli space M^Γ itself is connected. When $G = SU(2)$ and Γ is a cyclic group, it is proved by [FH] by the ad hoc method.

In examples of Theorem 4.14 we already observed that M^Γ is diffeomorphic to $(\mathbb{R}^4 \setminus \{0\})/\Gamma$. Combining this with Proposition 5.9, we get that this is in fact isometric. Since M is "asymptotic" to M^Γ at the end in a certain sense, this gives an evidence that M is ALE.

We recall some facts on Riemannian geometry of the moduli space (M, g_M) ([12]). In Sect. 2 we defined the Kuranishi map Ξ_A: $W^{l,2}_{-2}(\Omega^1(\operatorname{Ad} P)) \to W^{l,2}_{-2}(\Omega^1(\operatorname{Ad} P))$ for $[A] \in M$ and it was shown that if $H^2_{A,-1} = 0$, then Ξ_A: $S^{\text{asd}}_A \to H^1_{A,-2}$ defines a coordinate system of M around $[A]$. Itoh observed that this gives in fact a normal coordinate system ([12, Proposition 3.4]) with respect to the Riemannian metric g_M. Let $\{V_i\}$ be an orthonormal basis of $H^1_{A,-2}$ with respect to g_M. We denote by $(x^1, \ldots, x^n)$ the normal coordinate system associated with $\{V_i\}$. In this coordinate system, the second derivatives of the metric tensor g_{ij} are written as ([12, (5.18)])

$$(5.10)\ \frac{\partial^2 g_{lj}}{\partial x^i \partial x^k}(0) = -g_M(\{V_i, V_l\}, G_A\{V_k, V_j\}) + g_M([V_i \wedge V_l]^+, G_A([V_k \wedge V_j]^+)) - g_M(\{V_k, V_l\}, G_A\{V_i, V_j\}) + g_M([V_k \wedge V_l]^+, G_A([V_i \wedge V_j]^+)),$$

where G_A is the Green operator and $\{\cdot,\cdot\}$: $\Omega^a(\operatorname{Ad} P) \times \Omega^b(\operatorname{Ad} P) \to \Omega^{(a+b-2)}(\operatorname{Ad} P)$ is defined by contraction

$$\{V, W\}(x) := \sum_{i=1}^{4} [i_{E_\mu} V \wedge i_{E_\mu} W],$$

where $i.$ is the interior product and $\{E_\mu\}_{\mu=1,2,3,4}$ is an orthonormal basis of $T_x X$ with respect to the Riemannian metric g of X. In particular, the Riemannian curvature tensor R is represented as ([12, (5.19)])

$$\begin{aligned} R_{lkij}(0) &= g_M(R(V_i, V_j) V_k, V_l) \\ &= -g_M(\{V_i, V_l\}, G_A\{V_k, V_j\}) - 2g_M(\{V_k, V_l\}, G_A\{V_i, V_j\}) \\ &\quad + g_M(\{V_j, V_l\}, G_A\{V_k, V_i\}) + g_M([V_i \wedge V_l]^+, G_A([V_k \wedge V_j]^+)) \\ &\quad - g_M([V_j \wedge V_l]^+, G_A([V_k \wedge V_i]^+)). \end{aligned}$$

When the curvature tensor R vanishes identically, we have $\partial^2 g_{lj}/\partial x^i \partial x^k = 0$. Then (5.10) implies

$$\{V_i, V_l\} = 0, \quad [V_i \wedge V_l]^+ = 0 \tag{5.11}$$

for all i, l (remark that $\{\cdot,\cdot\}$ is skew-symmetric and $[\cdot \wedge \cdot]^+$ is symmetric on 1-forms).

Now we start the study of the end of the moduli space M. For simplicity we only treat the case that the limiting connection A_∞ (see Theorem 5.2) is the trivial connection. It is easy to adapt the proof to the general case (see [I1]). Let λ: $M^\Gamma \to (0, \infty)$ be the projection to the first factor of $M^\Gamma = (0, \infty) \times S^\Gamma$. We shall identify S^Γ with a submanifold $\{[A^\Gamma] \in M^\Gamma \mid \lambda([A^\Gamma]) = 1\}$. By Theorem 4.12 we can define a smooth map Ψ: $(\lambda_0, \infty) \times S^\Gamma \to M$ for sufficiently large λ_0 into M by

$$\Psi(\lambda, [A^\Gamma]) := [A + d_A^* u_A] \quad \text{for } [A^\Gamma] \in S^\Gamma,$$

where $A = \Phi_\lambda^* A^\Gamma$ is the almost anti-self-dual connection constructed in Lemma 4.3, and $u = u_A$ is the solution of the equation

$$d_A^+ d_A^* u + \tfrac{1}{2}[d_A^* u \wedge d_A^* u]^+ + R_A^+ = 0$$

with the condition

$$\|u\|_{2,p,\delta} \leqq C \|R_A^+\|_{0,p,\delta-2}, \quad (-2 < \delta < 0)$$

(see Theorem 4.12). Moreover this construction is gauge equivariant, and the map Ψ is well-defined as the map from the moduli space M^Γ.

To study the behavior of the map Ψ we need estimates of tangent vectors of the moduli space M^Γ.

(5.12) **Proposition.** *There exists a positive constant C such that for $V^\Gamma \in T_{[A^\Gamma]} M^\Gamma$ we have*

$$|V^\Gamma(x)| \leqq \frac{C}{1+|x|^3}, \quad |\nabla_{A^\Gamma}(x)| \leqq \frac{C}{1+|x|^4},$$

where we suppose $\lambda([A^\Gamma]) = 1$, and $g_{M^\Gamma}(V^\Gamma, V^\Gamma) = 1$.

The proof is the same as that of Lemma 4.8.

Using (5.12), we have some estimates on $\Phi_\lambda^* V^\Gamma$.

(5.13) **Lemma.** *Let A^Γ be an anti-self-dual connection with $\lambda([A^\Gamma]) = 1$. We take unit length tangent vectors $V^\Gamma, W^\Gamma \in T_{[A^\Gamma]} M^\Gamma$ Then $A = \Phi_\lambda^* A^\Gamma$, $V = \lambda^{-1} \Phi_\lambda^* V^\Gamma$ and $W = \lambda^{-1} \Phi_\lambda^* W^\Gamma$ satisfy the followings:*

$$1 - C\lambda^{-2} \leqq g_M(V, V) \leqq 1 + C\lambda^{-2}, \tag{1}$$

$$\|V\|_{1,p,\delta-1} \leqq C\lambda^{-(\delta+1)}, \tag{2}$$

$$\|d_A^+ V\|_{0,p,\delta-2} \leqq C\lambda^{-(\delta/2+2)}, \tag{3}$$

$$\|d_A^* V\|_{0,p,\delta-2} \leqq C\lambda^{-(\delta/2+2)}, \tag{4}$$

$$\|[V \wedge W]^+\|_{0,p,-3} \leqq C\lambda^{-5/2}, \tag{5}$$

$$\|\{V, W\}\|_{0,p,-3} \leqq C\lambda^{-5/2}, \tag{6}$$

where $-2 < \delta < 0$, and C is a constant independent of λ.

Proof. Since these inequalities follow from direct calculation, we only show (1). Other inequalities is proved in a similar way (see also the proof of Lemma (4.3)). For (5) (6) see (5.11).

Since $|(\Phi_\lambda)_*| \leqq C\lambda^{-1}$, Lemma 5.12 implies estimates for V:

$$|V| \leqq \frac{C\lambda}{\lambda^3 + (\beta r)^3}.$$

This gives

$$\left| \int_{X \setminus \Omega_{2\sqrt{\lambda}}} |V|^2 \, dV - \lambda^{-2} \int_{|\chi| > 2\sqrt{\lambda}} |T_\lambda^* V^\Gamma|^2 d\chi \right| \leqq C \int_{2\sqrt{\lambda}}^{\infty} r^{-1} \frac{\lambda^2}{(\lambda^3 + r^3)^2} dr \leqq C\lambda^{\varepsilon - 4}$$

with arbitrary positive number ε. On the other hand we have

$$\int_{\Omega_{2\sqrt{\lambda}}} |V|^2 dV \leqq C\lambda^{-4} \int_{\sqrt{\lambda}}^{2\sqrt{\lambda}} r^3 dr \leqq C\lambda^{-2}.$$

Since

$$\left| \lambda^{-2} \int_{|\chi| > 2\sqrt{\lambda}} |T_\lambda^* V^\Gamma|^2 d\chi - 1 \right| = \left| \int_{|y| > 2/\sqrt{\lambda}} |V^\Gamma|^2 dy - 1 \right| \leqq C\lambda^{-2},$$

we have got (1). □

(5.14) **Proposition.** *The map Ψ gives an into diffeomorphism for sufficiently large λ_0. Moreover there exists a positive constant C such that*

(1) $$\left| \frac{1}{\lambda} d_M(\Psi([A^\Gamma]), o) - 1 \right| \leqq C\lambda^{-3/4},$$

(2) $$|g_M(\Psi_* V^\Gamma, \Psi_* W^\Gamma) - g_{M^\Gamma}(V^\Gamma, W^\Gamma)| \leqq C\lambda^{-3/2},$$

(3) $$\|[\Psi_* V^\Gamma \wedge \Psi_* W^\Gamma]^+\|_{0,p,-3} \leqq C\lambda^{-9/4},$$

(4) $$\|\{\Psi_* V^\Gamma, \Psi_* W^\Gamma\}\|_{0,p,-3} \leqq C\lambda^{-9/4}$$

where $o \in M$ is a fixed point, $[A^\Gamma]$ is a point in M^Γ with $\lambda = \lambda([A^\gamma])$ and V^Γ, $W^\Gamma \in T_{[A^\Gamma]} M^\Gamma$ are tangent vectors satisfying $g_M(V^\Gamma, V^\Gamma) = g_M(W^\Gamma, W^\Gamma) = 1$.

Proof. We treat two types of tangent vectors i) $V^\Gamma = \lambda^{-1} T_\lambda^* V_0^\Gamma$ with $V_0^\Gamma \in TS^\Gamma$, and ii) $V^\Gamma = \partial/\partial\lambda$ separately.

First let $V^\Gamma = \lambda^{-1} T_\lambda^* V_0^\Gamma$ with $V_0^\Gamma \in TS^\Gamma$. Let $\{A_t^\Gamma\}$ ($|t| < \varepsilon$) be a smooth family of anti-self-dual connections in M^Γ with $\lambda([A_t^\Gamma]) = 1$ and $\dot{A}_0^\Gamma = V_0^\Gamma$. Here the symbol "$\cdot$" means the differentiation with respect *to t*. Then we have a family of λ-ASD connections $\{\tilde{A}_t = \Phi_\lambda^* A_t^\Gamma\}$. The tangent vector $\dot{\tilde{A}}_0$ is given by

$$\dot{\tilde{A}}_0 = \lambda^{-1} \Phi_\lambda^* V_0^\Gamma.$$

By Theorem 4.12 we have a family of the solutions of

$$d_{\tilde{A}_t}^+ d_{\tilde{A}_t}^* u_t + \tfrac{1}{2}[d_{\tilde{A}_t}^* u_t \wedge d_{\tilde{A}_t}^* u_t]^+ + R_{\tilde{A}_t}^+ = 0$$

with

$$\|u_t\|_{2,p,\delta} \leqq C\,\|R^+_{\tilde{A}_t}\|_{0,p,\delta-2} \leqq C\,\lambda^{-(\delta/2+1)} \tag{5.15}$$

for $-2<\delta<0$. Differentiating the above equality by t and evaluating at $t=0$, we obtain

$$\begin{aligned} &d^+_{\tilde{A}_0} d^*_{\tilde{A}_0} \dot{u}_0 - d^+_{\tilde{A}_0}\{\dot{\tilde{A}}_0, u_0\} + [\dot{\tilde{A}}_0 \wedge d^*_{\tilde{A}_0} u_0]^+ \\ &\quad + [d^*_{\tilde{A}_0} u_0 \wedge d^*_{\tilde{A}_0} \dot{u}_0]^+ - [d^*_{\tilde{A}_0} u_0 \wedge \{\dot{\tilde{A}}_0, u_0\}]^+ + d^+_{\tilde{A}_0} \dot{\tilde{A}}_0 = 0. \end{aligned}$$

Substituting (5.13) and (5.15) into this equality, we get

$$\|d^+_{\tilde{A}_0} d^*_{\tilde{A}_0} \dot{u}_0 + [d^*_{\tilde{A}_0} u_0 \wedge d^*_{\tilde{A}_0} \dot{u}_0]^+\|_{0,p,\delta-2} \leqq C\,\lambda^{-(\delta/2+2)}.$$

By Proposition 4.11

$$\|\dot{u}_0\|_{2,p,\delta} \leqq C\,\lambda^{-(\delta/2+2)}. \tag{5.16}$$

Now we estimate $\Psi_*(V^\Gamma)$. Let $A_t = \tilde{A}_t + d^*_{\tilde{A}_t} u_t$. Then we have

$$\dot{A}_0 = \dot{\tilde{A}}_0 + d^*_{\tilde{A}_0} \dot{u}_0 - \{\dot{\tilde{A}}_0, u_0\}.$$

Since $d^+_{A_0}(\dot{A}_0)=0$, the harmonic part $\mathbb{H}_{A_0}(\dot{A}_0)$ is given by

$$\mathbb{H}_{A_0}(\dot{A}_0) = \dot{A}_0 - d_{A_0} \Delta^{-1}_{A_0}(d^*_{A_0} \dot{A}_0).$$

The invertibility of Δ_{A_0} is guaranteed in Proposition 4.7. More precisely using (5.15), (5.16) we have

$$\|\Delta^{-1}_{A_0}(d^*_{A_0} \dot{A}_0)\|_{2,p,-1} \leqq C\,\|d^*_{A_0} \dot{A}_0\|_{0,p,-3} \leqq C\,\lambda^{-3/2}$$

$$\|\Delta^{-1}_{A_0}(d^*_{A_0} \dot{A}_0)\|_{2,p,-1/2} \leqq C\,\|d^*_{A_0} \dot{A}_0\|_{0,p,-5/2} \leqq C\,\lambda^{-7/4}.$$

So finally we obtain

$$|g_M(\Psi_* V^\Gamma, \Psi_* V^\Gamma) - 1| = |\int_X |\mathbb{H}_{A_0}(\dot{A})|^2\, dV - 1| \leqq C\,\lambda^{-3/2}.$$

If tangent vectors V^Γ, W^Γ of unit length are written respectively as $V^\Gamma = \lambda^{-1} T^*_\lambda V^\Gamma_0$, $W^\Gamma = \lambda^{-1} T^*_\lambda W^\Gamma_0$ with $V^\Gamma_0, W^\Gamma_0 \in TS^\Gamma$, then we have

$$\begin{aligned} &|g_M(\Psi_* V^\Gamma, \Psi_* W^\Gamma) - g_{M^\Gamma}(V^\Gamma, W^\Gamma)| \leqq C\,\lambda^{-3/2} \\ &\|[\Psi_* V^\Gamma \wedge \Psi_* W^\Gamma]^+\|_{0,2,-3} \leqq C\,\lambda^{-9/4} \\ &\|\{\Psi_* V^\Gamma, \Psi_* W^\Gamma\}\|_{0,2,-3} \leqq C\,\lambda^{-9/4}, \end{aligned} \tag{5.17}$$

where in the second and third inequalities we viewed $\Psi_* V^\Gamma$, $\Psi_* W^\Gamma$ as element of $H^1_{\Psi[A^\Gamma],-2}$. Hence we have verified the assertion (2), (3), (4) in this case.

Next we consider the case $V^\Gamma = \partial/\partial\lambda$. We take a family of diffeomorphisms $S_{\lambda,t}\colon X \to X$ ($|t|<\varepsilon$) which depend smoothly on t and satisfy

$$\Phi_{\lambda+t} = \Phi_\lambda \circ S_{\lambda,t},$$

$$S_{\lambda,0} = \mathrm{id}.$$

More precisely we take the radial coordinate system $(r, \psi) \in [R, \infty) \times S^3/\Gamma$ on $X \setminus K$, and then $S_{\lambda,t}$ is represented as

$$S_{\lambda,t}(r, \theta) = (f_{\lambda,t}(r), \theta),$$

where $f_{\lambda,t}$ is a function satisfying

$$\beta\left(\frac{f_{\lambda,t}(r)}{\sqrt{\lambda}}\right)\frac{f_{\lambda,t}(r)}{\lambda} = \beta\left(\frac{r}{\sqrt{\lambda+t}}\right)\frac{r}{\lambda+t}.$$

The function $f_{\lambda,t}$ is uniquely determined by the above equation on $r \geqq \sqrt{\lambda+t}$, and extended smoothly on $\mathbb{R}^+$ so that $f_{\lambda,t}(r) = r$ if $r \leqq \sqrt{\lambda}/2$. In particular we have

$$f_{\lambda,t}(r) = \frac{\lambda r}{\lambda+t} \qquad \text{if } r \geqq \max(2\sqrt{\lambda+t}, 2(\lambda+t)/\sqrt{\lambda}).$$

Moreover $f_{\lambda,t}$ satisfies

$$\left|\frac{d}{dt}\bigg|_{t=0} f_{\lambda,t}(r)\right| \leqq \frac{Cr}{\lambda}, \qquad \left|\frac{d}{dt}\bigg|_{t=0} df_{\lambda,t}(r)\right| \leqq \frac{C}{\lambda}.$$

We lift the diffeomorphism $S_{\lambda,t}$ to a bundle map $\tilde{S}_{\lambda,t}: P \to P$ by setting

$$\tilde{S}_{\lambda,t}(u) := \tau_t^0(u) \qquad \text{for } u \in P_x,$$

where τ_t^0 is the parallel translation along the curve $x(s) = S_{\lambda,s}(x)\,(0 \leqq s \leqq t)$ from x to $S_{\lambda,t}(x)$. Then we have a family of λ-ASD connections $\{\tilde{A}_t = \tilde{S}^*_{\lambda,t}\Phi^*_\lambda A^\Gamma\}$ for fixed $A^\Gamma \in M^\Gamma$ with $\lambda([A^\Gamma]) = 1$. Remark that $\tilde{A}_t$ and $\Phi^*_{\lambda+t} A^\Gamma$ are gauge equivalent. Differentiating by t, we have

$$\dot{\tilde{A}}_0 = i_{X_\lambda} R_{\tilde{A}_0}$$

where X_λ is the generating vector field of $S_{\lambda,t}$ and i_{X_λ} is the interior product (see [BL]). Then direct calculation shows (cf. Lemma 5.13)

$$\|i_{X_\lambda} R_{\tilde{A}_0}\|_{1,p,\delta-1} \leqq C\lambda^{-(\delta+1)}$$

$$\|d^*_{\tilde{A}}(i_{X_\lambda} R_{\tilde{A}_0})\|_{0,p,\delta-2} \leqq C\lambda^{-(\delta/2+2)}$$

$$\|d^+_{\tilde{A}}(i_{X_\lambda} R_{\tilde{A}_0})\|_{0,p,\delta-2} \leqq C\lambda^{-(\delta/2+2)}.$$

Moreover remarking that $\dot{\tilde{A}}_0 = \lambda^{-1}\Phi^*_\lambda V_0^\Gamma$ on $X \setminus \Omega_{2\sqrt{\lambda}}$, we see

$$\left|\int_X |i_{X_\lambda} R_{\tilde{A}_0}|^2 \, dV - 1\right| \leqq C\lambda^{-2}$$

$$\left|\int_X (i_{X_\lambda} R_{\tilde{A}_0}, \lambda^{-1}\Phi^*_\lambda W_0^\Gamma)\, dV\right| \leqq C\lambda^{-2}$$

$$\|[\dot{\tilde{A}}_0 \wedge \dot{\tilde{A}}_0]^+\|_{0,p,-3} \leqq C\lambda^{-9/4}$$

$$\|[\dot{\tilde{A}}_0 \wedge \lambda^{-1}\Phi^*_\lambda W_0^\Gamma]^+\|_{0,p,-3} \leqq C\lambda^{-9/4}$$

$$\|\{\dot{\tilde{A}}_0, \lambda^{-1}\Phi^*_\lambda W_0^\Gamma\}\|_{0,p,-3} \leqq C\lambda^{-9/4},$$

where W_0^Γ is a unit length tangent vector of TS^Γ. Then as above we obtain the inequalities of the statement (2), (3), (4).

The above calculation shows that the map Ψ gives a finite covering map from (λ_0, ∞) to M for sufficiently large λ_0. We now prove that Ψ is, in fact, a diffeomorphism onto its image (if we replace the constant λ_0 by a larger one). Suppose the contrary, and assume that there exist sequences $\{(\lambda_{i,1}, [A_{i,1}^\Gamma])\}$ and $\{(\lambda_{i,2}, [A_{i,2}^\Gamma])\}$ in $(\lambda_0, \infty) \times S^\Gamma$ such that

$$\lambda_{i,1}, \lambda_{i,2} \to \infty \quad \text{as} \quad i \to \infty,$$

$$\Psi(\lambda_{i,1}, [A_{i,1}^\Gamma]) = \Psi(\lambda_{i,2}, [A_{i,2}^\Gamma]) \quad \text{for all } i.$$

Since S^Γ is compact, we may assume that $A_{i,k}^\Gamma$ converges to an anti-self-dual connection A_k^Γ on each compact subset of $\mathbb{R}^4/\Gamma \setminus \{0\}$ $(k=1, 2)$. We shall prove that $\lambda_{i,1} \lambda_{i,2}^{-1}$ converges to 1 as $i \to \infty$ and $[A_1^\Gamma] = [A_2^\Gamma]$.

We denote by $A_{i,k}$ the anti-self-dual connection constructed from $(\lambda_{i,k}, A_{i,k}^\Gamma)$ by Taubes existence theorem $(k=1, 2)$. From the assumption there exists a sequence of gauge transformations $\{s_i\} \in \mathcal{G}_0$ such that $A_{i,2} = s_i^* A_{i,1}$. From the definition of the map Ψ and Theorem (4.10) the connection $T_{\lambda_{i,k}^{-1}}^* (\mathcal{X}^{-1})^* A_{i,k}$ converges to A_k^Γ on each compact subset of $\mathbb{R}^4/\Gamma \setminus \{0\}$. This, in particular, implies that there exists a positive constant C such that

$$C^{-1} \leqq \lambda_{i,1} \lambda_{i,2}^{-1} \leqq C.$$

We may assume that $\lambda_{i,1} \lambda_{i,2}^{-1}$ converges to a positive number λ as $i \to \infty$. Hence the connection $T_{\lambda_{i,1}^{-1}}^* (\mathcal{X}^{-1})^* A_{i,2}$ converges to $T_{\lambda^{-1}}^* A_2$. If we pull back the gauge transformation s_i by the map $\mathcal{X}^{-1} \circ T_{\lambda_{i,1}}^{-1}$, it converges to a gauge transformation s on $\mathbb{R}^4/\Gamma \setminus \{0\}$ such that

$$s(x) \to \mathrm{id} \quad \text{as} \quad x \to \infty,$$

$$T_{\lambda^{-1}}^* A_2^\Gamma = s^* A_1^\Gamma.$$

Thus we have $[T_{\lambda^{-1}}^* A_2^\Gamma] = [A_1^\Gamma]$. But since $\lambda(A_1^\Gamma) = \lambda(A_2^\Gamma) = 1$, we must have $\lambda = 1$ and $[A_1^\Gamma] = [A_2^\Gamma]$. Hence Ψ is an into diffeomorphism. The assertion (1) follows directly combining this with (2). □

(5.18) **Corollary.** *Each noncompact component of the moduli space M is an* ALE *hyperkähler* 4-*manifold.*

Proof. Proposition 5.14 (1)(2) implies that Ψ gives coordinates at infinity at least in the level of C^0-norm. The curvature of g_M is written ([I2, Theorem 6.1])

$$g_M(R(V, W)W, V) = 3g_M(\{V, W\}, G_A\{V, W\}) - g_M([V \wedge W]^+, G_A[V \wedge W]^+)$$
$$+ g_M([V \wedge V]^+, G_A[W \wedge W]^+).$$

Since the operator norm of the Green operator G_A is bounded (Propositions 4.7, 4.8), Proposition 5.14, 3, 4 implies

$$|R(p)| = O(d(o, p)^{-9/2}).$$

Hence by [BKN, Theorem 1.1] we have coordinates at infinity of order 5/2 on this end.

Since (M, g) is hyperkähler, it is Ricci-flat. Then each component of M has at most one end by Cheeger-Gromoll splitting theorem [CG]. So the end corresponding to Taubes' existence theorem is the only end of a component of M. □

6. Periods of the moduli spaces

The results in previous sections determine the differentiable structures of the moduli spaces in principle. So the remained problem is to determine the hyperkähler structures. These are determined by cohomology classes of three Kähler forms (see Fact 1.2). In this section we relate the homology group $H_2(M; \mathbb{Z})$ of the moduli space to that of the base space and compute the values of Kähler forms evaluated on the homology group for examples given in Theorems 0.2, 0.3.

Throughout this section we assume that the group G is a unitary group $U(r)$ and (X, g, I_X, J_X, K_X) is an ALE hyperkähler 4-manifold diffeomorphic to the minimal resolution $\tilde{S}$ of $\mathbb{C}^2/\Gamma$. Let ω_I^X, ω_J^X, ω_K^X denote the associated Kähler forms. The irreducible components $\Sigma_1, \ldots, \Sigma_n$ of the exceptional set give a basis of the homology group $H_2(X; \mathbb{Z})$ and the intersection matrix (Σ_i, Σ_j) is the negative of the Cartan matrix.

Let P be a principal bundle over X with the structure group G which can be extended to the orbifold $\hat{X} = X \cup \{\infty\}$, $\rho\colon \Gamma \to G$ the homomorphism induced by the action on the fiber P_∞, and $M = M(P, k, \rho)$ the moduli space of anti-self-dual connections on P asymptotic to ρ. We denote by E the associated complex vector bundle.

Now we assume $\dim M = 4$. In the previous sections we have observed that M has a natural complete metric g_M with the hyperkähler structure (I_M, J_M, K_M). It may have several components, but we already know that each noncompact component is ALE, and each compact component must be a $K3$ surface or a torus. We denote by ω_I^M, ω_J^M, ω_K^M the Kähler forms associated with the hyperkähler structure on M.

Since the reduced gauge group $\mathscr{G}_0$ acts freely on $\mathscr{A}_{\mathrm{asd}}$, there exists a universal bundle $\mathbb{P} = \mathscr{A}_{\mathrm{asd}} \times_{\mathscr{G}_0} P$ over $M \times X$. We take the associated vector bundle $\mathbb{E}$ over $M \times X$. The bundle $\mathbb{P}$ admits a natural universal connection $\mathbb{A}$ which is equivalent to A when restricted to $\{[A]\} \times X$ (cf. [AS 2]).

Following [Mu, D3], we define homomorphisms $f\colon H^2(X; \mathbb{R}) \to H^2(M; \mathbb{R})$, $f'\colon H_c^2(M; \mathbb{R}) \to H^2(X; \mathbb{R})$ (where H_c^2 denotes the cohomology group with compact supports) by

$$(6.1) \qquad f(\alpha) = -\Big(\int_X \alpha\, \mathrm{ch}(\mathbb{E})\Big)^{(2)}, \quad f'(\beta) = -\Big(\int_M \beta\, \mathrm{ch}(\mathbb{E})\Big)^{(2)},$$

where $(\cdot)^{(2)}$ means a 2-form component.

(6.2) **Theorem.** *If the* Ad-*invariant inner product on* $\mathfrak{g}$ *is suitable chosen, then*
(1) $(f(\alpha), \beta) = (\alpha, f'(\beta))$ *for* $\alpha \in H^2(X; \mathbb{R})$ *and* $\beta \in H_c^2(M; \mathbb{R})$,
(2) $f([\omega_I^X]) = [\omega_I^M]$, $f([\omega_J^X]) = [\omega_J^M]$, $f([\omega_K^X]) = [\omega_K^M]$
where $[\cdot]$ *denotes the cohomology class.*

Proof. The statement (1) follows easily. In fact,

$$(f(\alpha), \beta) = (\alpha \cdot \mathrm{ch}(\mathbb{E}) \cdot \beta)[X \times M] = (\alpha, f'(\beta)).$$

The statement (2) was proved in [D4, (5.10)]. Using the universal connection $\mathbb{A}$, we take the representation $-(1/8\pi^2)\,\mathrm{tr}(R_{\mathbb{A}}^2)$ of $\mathrm{ch}^{(4)}(\mathbb{E})$ via the Chern-Weil theory. For $(a, v) \in TM \times TX$ the curvature form $R_{\mathbb{A}}$ satisfies

$$R_{\mathbb{A}}(a, v) = a(v)$$

where in the right-hand side the tangent vector a is viewed as $\mathrm{Ad}\,P$-valued 1-form on X [AS 2]. Hence for $a, b \in TM$

$$-(\int_X \mathrm{ch}(\mathbb{E}) \wedge \omega_I^X)(a, b) = \frac{-1}{4\pi^2} \int_X \mathrm{tr}(a \wedge b) \wedge \omega_I^X.$$

The right-hand side is the Kähler form ω_I^M. The correspondence for J and K can be proved similarly. This shows (2). □

The meaning of the map f is explained by using the determinant line bundle (see [D3, Sect. 2], [BF]). Suppose that $\alpha \in H^2(X; \mathbb{R})$ is the Poincaré dual of the homology class $[\Sigma_i]$. We couple the Dirac operator on Σ_i with the family of connections on $\mathbb{E}$ to get the family of twisted Dirac operators on Σ_i. Then we have the determinant line bundle $\mathscr{L}_i = \mathscr{L}_{\Sigma_i}$ whose fiber over $[A] \in M$ is defined by

$$(\mathscr{L}_{\Sigma_i})_A = (\bigwedge^{\max} \ker D_A)^* \otimes (\bigwedge^{\max} \operatorname{coker} D_A).$$

Then the class $f(\mathrm{P.D.}[\Sigma_i])$ can be calculated by the families index theorem [AS 1]

$$f(\mathrm{P.D.}[\Sigma_i]) = -\int_{\Sigma_i} \mathrm{ch}(\mathbb{E}) = -c_1(\mathrm{ind}\, D_i) = c_1(\mathscr{L}_i).$$

There is a canonical section $\det D_i$ of $\mathscr{L}_i$ which is nonzero exactly where D_A is invertible. Moreover when Σ_i is a complex submanifold of (X, g), we can define a natural holomorphic structure on $\mathscr{L}_i$ since $\mathbb{E}$ with the connection $\mathbb{A}$ is a holomorphic vector bundle over $X \times M$ (see e.g., [I4]). Then there is a correspondence (cf. [H1])

$$\ker D_A \cong H^0(\Sigma_i; \mathcal{O}(K^{1/2} \otimes E)) \tag{6.3}$$
$$\operatorname{coker} D_A \cong H^1(\Sigma_i; \mathcal{O}(K^{1/2} \otimes E)),$$

where $K^{1/2}$ is the square root of the canonical bundle (i.e. a holomorphic line bundle with $K^{1/2} \otimes K^{1/2} \cong K$). In this case the canonical section $\det D_i$ is holomorphic.

First we study the case considered in Theorem 0.2. We take a complex line bunle L over X which is asymptotic to the trivial connection and satisfy $c_1(L)^2[\hat{X}] = -2$. Let $\Sigma \in H_2(X; \mathbb{Z})$ be the Poincaré dual of $c_1(L)$. Since $c_1(E) = c_1(\mathbb{C} \oplus L) = c_1(L)$, the line bundle L is determined from a topological datum of E. We already know that each component of the moduli space M is isometric to the Eguchi-Hanson space up to a constant factor (Theorem 5.11) and contain a reducible connection corresponding S^2 (the exceptional set of the resolution). This shows that M itself is connected. In particular, it is diffeomorphic to the minimal resolution of $\mathbb{C}^2/\mathbb{Z}_2$, and has a generator σ of $H_2(M; \mathbb{Z})$ determined by the exceptional set of the resolution.

(6.4) **Theorem.**

$$f(\alpha)[\sigma]=(c_1(L),\alpha)=\alpha(\Sigma) \quad \textit{for} \ \alpha\in H^2(X;\mathbb{R}).$$

(6.5) **Corollary.**

$$[\omega_A^M](\sigma)=[\omega_A^X](\Sigma) \quad \textit{for all} \ A=I, J, K.$$

Proof. Theorem 6.4 implies

$$(f(\alpha), \text{P.D.}\,\Sigma)=(\alpha, c_1(L)) \quad \text{for } \alpha\in H^2(X;\mathbb{R}).$$

Together with Theorem 6.2, 1, this implies $f'(\text{P.D.}\,\Sigma)=c_1(L)$. Then using Theorem 6.2, 2, we have

$$[\omega_A^M](\sigma)=f([\omega_A^X])(\sigma)=([\omega_A^X], f'(\text{P.D.}\,\Sigma))=([\omega_A^X], c_1(L)). \quad \square$$

Proof of Theorem 6.4. We fix a point $[A_0]\in M$ which corresponds to the reduction $E=\mathbb{C}\oplus L$. The group $G_\rho=SO(3)$ acts on the set $\mathscr{R}\subset M$ of reducible connections transitively, and the isotropy subgroup at $[A_0]$ is isomorphic to $S^1\subset\mathbb{C}^*$ which acts the complex line bundle L as the scalar multiplication. In particular, $\mathscr{R}$ is isometric to S^2, in fact coincides with the exceptional set of the minimal resolution of $\mathbb{C}^2/\mathbb{Z}_2$.

We compute the values of the map f on $\text{P.D.}[\Sigma_1], \ldots, \text{P.D.}[\Sigma_n]\in H^2(X;\mathbb{R})$ by studying the action of the isotropy subgroup S^1 on the fiber of the determinant line bundle $\mathscr{L}_i$ at $[A_0]$ (cf. [D3, Lemma 2.28]). Via the decomposition $E=\underline{\mathbb{C}}\oplus L$, the determinant line at $[A_0]$ is written as

$$\mathscr{L}_i=\det D_{\Sigma_i,\underline{\mathbb{C}}}\det D_{\Sigma_i,L}.$$

Since S^1 acts on L with weight 1, the index theorem implies the action on $\mathscr{L}_i$ is with weight

$$-c_1(L)[\Sigma_i].$$

This shows

$$c_1(\mathscr{L}_i)[\mathscr{R}]=2c_1(L)[\Sigma_i].$$

Since $H_2(M;\mathbb{Z})$ is generated by $[\mathscr{R}]$

$$c_1(\mathscr{L}_i)=-c_1(L)[\Sigma_i]\,\text{P.D.}[\mathscr{R}].$$

The conclusion follows directly from the above. $\square$

Next we turn to the case considered in Theorem 0.3. The method is almost the same as the above and we use the action of $G_\rho=S^1$ essentially. The fixed points $\text{Fix}(S^1)$ of S^1-action are $n+1$ reducible connections $[A_1], \ldots, [A_{n+1}]$ (Theorem 3.6). In particular, the Euler number of the moduli space M is equal to $n+1$. Since M has a component M_0 diffeomorphic to X whose Euler number is equal to $n+1$, the Euler number of the other components must be zero. So the other components are tori. The component M_0 is diffeomorphic to the

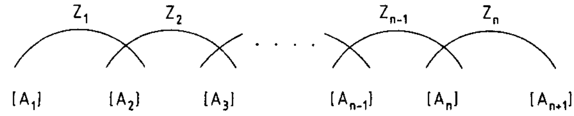

Fig. 1

minimal resolution of $\mathbb{C}^2/\Gamma$, and has a basis $\sigma_1, \ldots, \sigma_n$ of $H_2(M_0; \mathbb{Z})$ determined by the irreducible components of the exceptional set. The intersection matrix (σ_i, σ_j) is the negative of the Cartan matrix.

(6.6) **Theorem.** *The classes* $f(\text{P.D.}[\Sigma_i])$ $(i=1, \ldots, n)$ *belong to* $H^2(M_0, \partial M_0; \mathbb{Z})$ *and the intersection matrix* $(f(\text{P.D.}[\Sigma_i]), \text{f}(\text{P.D.}[\Sigma_j]))$ *is equal to the negative of the Cartan matrix.*

(6.7) **Corollary.** *The component* $(M_0, g_M, I_M, J_M, K_M)$ *of the moduli space* M_0 *is isomorphic to* (X, g, I_X, J_X, K_X) *as a hyperkähler manifold.*

Proof. By Theorem 6.6, the map f gives an isomorphism between $H_2(X; \mathbb{Z})$ and $H_2(M_0; \mathbb{Z})$ which preserves the intersection product. Then we have

$$[\omega_A^M](\sigma_i) = [\omega_A^X](\Sigma_i),$$

where $\sigma_i = f(\Sigma_i)$. This implies (M_0, g_M) and (X, g) are isomorphic. □

Proof of Theorem 6.6. Since the bilinear form $(f(\cdot), f(\cdot))$ on $H^2(X; \mathbb{R})$ is independent of the metric g on X (see [D4]), we may assume that (X, g) is biholomorphic to the minimal resolution of $\mathbb{C}^2/\Gamma$. So $\Sigma_1, \ldots, \Sigma_n$ are complex submanifolds biholomorphic to $\mathbb{CP}^1$ and the determinant line bundle $\mathscr{L}_i$ have holomorphic structures under which the canonical section $\det D_i$ is holomorphic.

When the connection $[A] \in M_0$ goes to the infinity, it converges to the trivial connection on a compact subset of X. So there exists a compact subset C of M_0 such that the restriction of a connection A in $M_0 \backslash C$ to Σ_i is trivial as a holomorphic bundle, and hence the Dirac operator D_i is invertible (see (6.3)). Hence the zero set Z_i of $\det D_i$ is a compact complex submanifold of M_0. It is invariant under the action of $G_\rho = S^1$, and the intersection $\text{Fix}\, S^1 \cap Z_i$ is two points $[A_i]$ and $[A_{i+1}]$. Thus Z_i must be biholomorphic to $\mathbb{CP}^1$. Moreover for $i<j$ the zero sets Z_i and Z_j intersect if and only if $i+1=j$ and the intersection point is $[A_{i+1}]$ (Fig. 1). Hence the set $\{[Z_1], \ldots, [Z_n]\}$ gives a basis for $H_2(M_0; \mathbb{Z})$ and its intersection matrix is the same as that of $\{\Sigma_1, \ldots, \Sigma_n\}$.

The class $c_1(\mathscr{L}_i)$ is written as $a_i \text{P.D.}[Z_i]$ with a positive integer a_i. To calculate a_i we study the S^1-action on the fiber of $\mathscr{L}_i$ at fixed points. At $[A_i]$ the bundle E splits into $L_i \oplus L_i^{-1}$ (see (3.5)), and S^1 acts with weight 1 on L_i and with weight (-1) on L_i^{-1}. So S^1 acts on $\mathscr{L}_i$ with weight

$$c_1(L_i)[\Sigma_i] - c_1(L_i^{-1})[\Sigma_i] = 2.$$

On the other hand (-1) acts trivially on Z_i, and in fact $S^1/\{\pm 1\}$ is acts effectively. Thus we have $c_1(\mathscr{L}_i)[Z_i] = -2$ and hence $c_1(\mathscr{L}_i) = \text{P.D.}[Z_i]$. □

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