STABILITY AND BIFURCATIONS OF SITNIKOV MOTIONS

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Abstract. A family of straight line periodic motions, known as the Sitnikov motions and existing in the case of equal primaries of the three body problem, is studied with respect to stability and bifurcations. Continuation of the bifurcations into the case of unequal primaries is also discussed and some of the bifurcating families of three-dimensional periodic motions are computed.

1. Introduction

In the restricted three-body problem there exists a unique family of rectilinear motions of the infinitesimal particle perpendicular to the plane of motion of the two primaries. This family exists throughout the range of the mass parameter μ but consists of rectilinear orbits only for $\mu = 1/2$. These orbits are segments of the z-axis and are simply called 'the Sitnikov motions' as they were originally studied by Sitnikov (1961); see also Moser (1973). They cross the plane of motion of the primaries at the inner collinear equilibrium point L_1. The family of periodic Sitnikov motions can be recognised as a special case ($\mu = 1/2$) of the family L_1^3 of three-dimensional orbits emanating at L_1 for all admissible values of μ and will be denoted here by L_1^1.

The existence of such a family offers the opportunity to study the stability of bifurcations of one-dimensional motions under perturbations in two additional dimensions which tend to take the moving particle directly into the full three-dimensional space. In this sense the present work can be considered as a special case of the investigation of bifurcations in systems of three degrees of freedom (see, e.g., Davoust and Broucke, 1982; Contopoulos, 1985, 1986). In relation to the increase of orbit dimensionality involved in the bifurcation mechanism in this special case we may mention here especially the work of DeVogelaere (1950).

The treatment presented here is similar to the one applied by Bennett (1965) in the study of the stability of the equilibrium points of the elliptic restricted three-body problem, an apparently different stability problem.

The resulting bifurcations of the family of periodic Sitnikov motions L_1^1 and their continuation into the general case $\mu \neq 1/2$, where the family L_1^3 no longer consists of rectilinear motions, are studied in some detail. Numerical data are given for some of the bifurcating branch-families.

It is evident from the results that the family L_1^3 has a very complex evolution as the z-axis amplitude of its member orbits is increased, a fact which explains the difficulty involved in its numerical determination.

2. Equations of Motion and Variation

The equation governing the Sitnikov motions $z(t)$ is easily obtained from the usual restricted three-body problem formulation for $\mu = 1/2$, $x(t) = y(t) = 0$, and can be written in the form

$$\ddot{z} = \frac{z}{(z^2 + \frac{1}{4})^{3/2}}. \tag{1}$$

The usual expression of the Jacobi integral reduces in this case to

$$C = \frac{2}{(z^2 + \frac{1}{4})^{1/2}} - \dot{z}^2. \tag{2}$$

(see e.g. Szebehely, 1967).

It is probably easier to study equation (1) numerically. One starts the integration with initial conditions $z_0 = 0$, $\dot{z}_0$ arbitrary, and integrates until the velocity $\dot{z}$ becomes zero for the first time at time, say, T. The solution is periodic with period $T = 4T$ and the whole family of periodic solutions can be traced by repeating the numerical procedure for a sufficient number of discrete values of the family parameter ($\dot{z}_0$ in this case). In the present paper we adopt the above numerical treatment which we use for the purpose of studying numerically the stability properties of the family of periodic Sitnikov motions.

Consider small perturbations $x = \xi$, $y = \eta$ to the 'horizontal' components of the rectilinear motion $x(t) = 0$, $y(t) = 0$. It can be shown that small (first order) perturbations ζ to the 'vertical' component $z(t) \neq 0$ do not change the equations of motion and we therefore focus our attention on a perturbed motion $\xi(t)$, $\eta(t)$, $z(t)$, where only ξ and η are small.

The linearized equations of the perturbed almost rectilinear motion are

$$\begin{aligned} \ddot{\xi} - 2\dot{\eta} &= [F_1(z) + F_2(z)]\xi + F_3(z) \\ \ddot{\eta} + 2\dot{\xi} &= F_1(z)\eta \\ \ddot{z} &= [F_1(z) - 1]z + F_4(z)\xi z \end{aligned} \tag{3}$$

where we have abbreviated:

$$\begin{aligned} F_1(z) &= 1 - [(1-\mu)\Phi_1^{-3/2} + \mu\Phi_2^{-3/2}] \\ F_2(z) &= 3\mu(1-\mu)\,[\mu\Phi_1^{-5/2} + (1-\mu)\Phi_2^{-5/2}] \\ F_3(z) &= -\mu(1-\mu)(\Phi_1^{-3/2} - \Phi_2^{-3/2}) \\ F_4(z) &= 3\mu(1-\mu)(\Phi_1^{-5/2} - \Phi_2^{-5/2}) \\ \Phi_1 &= \mu^2 + z^2, \quad \Phi_2 = (\mu - 1)^2 + z^2 \end{aligned} \tag{4}$$

The first two Equations (3) are written in the form:

$$\dot{\boldsymbol{\xi}} = A(z(t))\boldsymbol{\xi}, \tag{5}$$

where $\boldsymbol{\xi} = (\xi_1, \xi_2, \xi_3, \xi_4)^T$, $\xi_1 = \xi$, $\xi_2 = \eta$, $\xi_3 = \dot{\xi}$, $\xi_4 = \dot{\eta}$,

$$A(z(t)) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ F_{10}(z) + F_{20}(z) & 0 & 0 & 2 \\ 0 & F_{10}(z) & -2 & 0 \end{bmatrix} \tag{6}$$

and $F_{10}(z)$, $F_{20}(z)$ are the above functions $F_1(z)$, $F_2(z)$ for $\mu = 1/2$.

Equations (5) are independent of the third of Equations (3) and can be treated separately. They describe the evolution of the perturbations ξ, η of the Sitnikov motion $z(t)$ and can be called the 'variational' equations of this rectilinear motion. A study of Equations (5) will reveal the stability properties of the rectilinear motions $z(t)$ under perturbations ξ, η which are 'normal' to it. This kind of stability of a one-dimensional motion will be called *normal stability*.

3. Stability Properties of Periodic Sitnikov Motions

Consider $z^*(t)$, a particular solution of (1), i.e. a member of the Sitnikov family L_1^1 of rectilinear motions $z(t)$, and let T^* be the period of this member-solution. Then, the linear system of variational Equations (5) of this solution is periodic with period T^* and the Floquet theory applies.

The general solution of Equations (5) will determine the normal stability of the basic solution $z^*(t)$ and this general solution depends on the characteristic roots s_k, $k = 1, 2, 3, 4$ i.e. the roots of the characteristic equation

$$\det(\tilde{C} - sI) = 0, \tag{7}$$

where I is the 4×4 unit matrix and

$$\tilde{C} = X^{-1}(t)\, X(t + T^*), \tag{8}$$

with $X(t)$ a fundamental solution of system (5). With no loss of generality we can set $X(0) = I$ and Equation (8) gives

$$\tilde{C} = X(T^*). \tag{9}$$

If the roots s_k of (6) are distinct, then there exist four linearly independent solutions $\boldsymbol{x}_k$ satisfying the property

$$\boldsymbol{x}_k(t + T^*) = s_k \boldsymbol{x}_k(t), \qquad k = 1, 2, 3, 4. \tag{10}$$

It follows that unit roots correspond to periodic solutions of (5), with the same period T^* of the coefficient matrix $A(z(t))$.

For non-unit roots the solutions are bounded (correspondingly unbounded) if the roots satisfy the condition $|s_k| < 1$ (correspondingly $|s_k| > 1$).

The fundamental solution $X(t)$ of (5) is a linear combination of the canonical solutions $\boldsymbol{x}_k$, $k = 1, 2, 3, 4$. It follows that the stability of the Sitnikov motion (1) under 'normal' perturbations ξ, η is determined from the characteristic roots s_k, i.e. from Equations (7) and (9). This requires the determination of the fundamental solution at $t = T^*$.

In the restricted three-body problem, if s is a root of (7) then so is s^{-1}, the product of the four roots being 1 (Wintner, 1947). The characteristic equation has the form

$$s^4 + ps^3 + qs^2 + ps + 1 = 0 \tag{11}$$

where the left-hand side of the equation can be written as the product of two quadratic factors giving

$$s^2 + a_1^* s + 1 = 0, \qquad s^2 + a_2^* s + 1 = 0 \tag{12}$$

with

$$a_1^* = \tfrac{1}{2}(p + \sqrt{\Delta}), \qquad a_2^* = \tfrac{1}{2}(p - \sqrt{\Delta}) \tag{13}$$

and

$$\begin{aligned} &\Delta = p^2 - 4(q - 2) \\ &p = -\sum_{i=1}^{4} C_{ii} = -\mathrm{Tr}\,\tilde{C} \\ &q = \sum_{j=i+1}^{4} \sum_{i=1}^{4} (C_{ii}C_{jj} - C_{ij}C_{ji}). \end{aligned} \tag{14}$$

As is well known, in such cases the stability conditions are

$$\Delta > 0, \quad |a_1^*| \leqslant 2, \quad |a_2^*| \leqslant 2. \tag{15}$$

In this paper we determine the matrix $\tilde{C}$ from (9) by integrating numerically the fundamental solution $X(t)$ from $t = 0$ to $t = T^*/4$, and applying the relation

$$X(T^*) = [MX^{-1}(T^*/4)MX(T^*/4)]^2, \tag{16}$$

which is due to the special symmetry of the Sitnikov motions and can be easily established. The matrix M involved in (16) is the constant symmetry matrix

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{17}$$

The economy in computing time involved in using (16) instead of a direct computation of $X(t)$ from $t = 0$ to $t = T^*$ is obvious. For better accuracy we have applied the Newton identities and verified the form (11) of the characteristic equation. We observed that when the elements of $\tilde{C}$ were of the order of 10^3 to 10^4 the accuracy of the constant term in (11) was approximately two decimal digits. When the elements of $\tilde{C}$ were smaller (of the order of 10^2 and 10^3) then the accuracy of the constant term was much better (four to six decimal digits). For even smaller elements of $\tilde{C}$ (10^1 to 10^2) the accuracy of the constant term was even better (8 to 10 decimal digits).

All members of the family were found to be unstable. The stability conditions (15) are never satisfied along the family $z(t)$ of rectilinear orbits. This is due to the fact that one of the two stability parameters (a_2^*) is always absolutely larger than 2. However, the other stability parameter a_1^* does obtain values in the interval $[-2, 2]$ and this fact is responsible for the occurrence of bifurcations as described in Section 4 below. The behaviour of the stability parameter a_1^* along the basic family is illustrated in Figure 1.

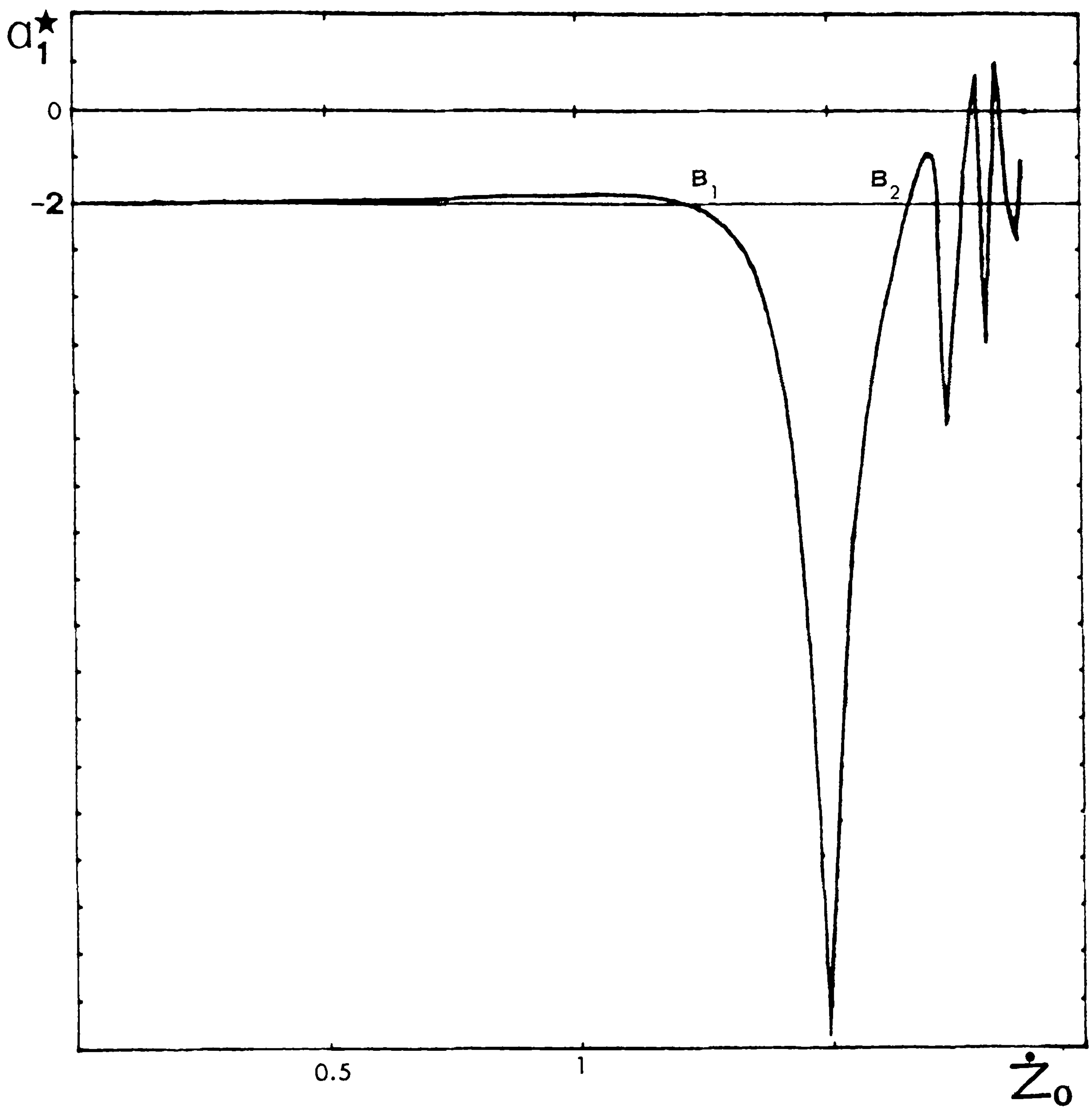

Fig. 1. Stability diagram for the basic family L_1^1.

3.1. THE LIMITING CASE $\dot{z}_0 \to 0$

The rectilinear motions $z(t)$ given in Table I correspond to finite values of the family parameter $\dot{z}_0$. It is interesting and useful, for verification, to examine the situation for $\dot{z}_0$ tending to 0. In that case the (infinitesimal) rectilinear motion is governed by the linearized version of (1),

$$\ddot{\zeta} + 8\zeta = 0. \tag{18}$$

This is a simple harmonic motion and the stability analysis can easily be performed without resort to computer calculations. The two stability parameters a_1^* and a_2^* are decoupled and can be determined from separate stability considerations. One of the two parameters describes the usual 'horizontal stability' (i.e. under 'horizontal' perturbations ξ, η) of the infinitesimal rectilinear motion $\zeta(t)$ and should be given by

$$a_i^* = -2\cosh \lambda T^*, \qquad i = 1 \text{ or } 2, \tag{19}$$

where $\lambda = 3.7833462$ (root of the characteristic equation of the equilibrium point for $\mu = 1/2$; se Szebehely *loc. cit.*, p. 311). This calculation gives the following limiting value of the second stability parameter a_2^*, which is thus identified as the one corresponding to the usual 'horizontal' stability parameter: $a_2^* = -4467.0441$. The other stability parameter can also be determined, in the limiting case. It is provided by

$$a_1^* = -\cos(2\pi s/s_z) = -1.9851325 \tag{20}$$

where s and s_z are the horizontal and vertical angular velocities (Szebehely, *loc.cit.*, pp. 571–574 and p. 311). The above limiting values of a_1^* and a_2^* agree perfectly with our numerical results. For $\dot{z}_0 = 0.001$, for example, the numerical procedure gives the values $a_1^* = -1.9851322$, $a_2^* = -4467.0394$.

4. Critical Solutions and Bifurcations

In relation to property (10) above, we note that the solutions satisfying

$$|a_i^*| = 2, \qquad i = 1 \text{ or } 2, \tag{21}$$

are called critical and are of special interest because they mark the occurence of bifurcations of new families of periodic orbits. In general this does not signify a change of stability along the basic family since such a change would occur only at a point where the full set of stability conditions (15) either start or cease to be valid. On the other hand bifurcations occur every time (21) is satisfied for one of the two stability parameters (see, e.g., Contopoulos, 1986).

All quantities associated with a particular member of the family depend entirely on 'the family parameter' $\dot{z}_0$. Here we seek those orbits for which

$$|a_1^*(\dot{z}_0)| = 2. \tag{22}$$

Applying a simple differential-correction procedure we obtain the required correction:

$$\delta\dot{z}_0 = -[2 + a_1^*(\dot{z}_0)]/[\partial a_1^*(\dot{z}_0)/\partial \dot{z}_0], \tag{23}$$

TABLE I
Bifurcation orbits of the family L_1^1.

	$\dot{z}_0$	$z(T^*/4)$	a_2^*	$T^*/4$	C
$B1$	1.2058314	0.6058850	−462.53	0.9508536	2.5459707
$B2$	1.6466218	1.4692823	−91.284	2.3420700	1.2886366
$B3$	1.7043778	1.7565471	−25.533	2.9389476	1.0950962
$B4$	1.7522080	2.0921592	−42.385	3.7045148	0.9297670
$B5$	1.7942219	2.5123094	−9.6904	4.7566414	0.7807677
$B6$	1.8067357	2.6721000	−21.310	5.1821617	0.7357059
$B7$	1.8394755	3.2062645	−11.840	6.6986793	0.6163297
$B8$	1.8614253	3.7040543	−6.9449	8.2335006	0.5350957

where the derivative in the denominator can be calculated (approximately) with an extra integration. Using this simple procedure we have determined numerically the first eight bifurcation points of the basic Sitnikov family L_1^1. The numerical data for these bifurcations are given in Table I.

4.1. EXAMPLES OF BRANCHES

In order to verify and illustrate the occurrence of branches of the basic family at the bifurcation points, we have computed two such branches, which we call $B3$ and $B6$, occuring at the bifurcations $B3$ and $B6$ of Table I, respectively. These branches are families of three-dimensional (no longer rectilinear) periodic solutions of double symmetry (with respect to the x-axis and the x–z plane). The numerical determination of these branch-families and the stability parameters of their member solutions was carried out using known procedures.

The initial conditions and other numerical data (including the stability parameters

TABLE II
Numerical data for the family of periodic solutions branching from the basic Sitnikov family at the bifurcation point $B6$.

$\dot{z}_0$	x_0	$\dot{y}_0$	$z(T/4)$	$T/4$	a_1	a_2	C
1.8067357	0	0	2.6721000	5.1821617	−2	−21.310	0.7357059
1.8000000	0.0347889	−0.2248319	2.6579033	5.1826805	−3.1173	−18.603	0.7301193
*1.6845256	0.1377585	−0.9293338	2.4259411	5.1904655	−2.0000	−62.403	0.6462703
1.6000000	0.1708902	−1.1840226	2.2677930	5.1951005	0.3181	−148.94	0.5963508
1.4000000	0.2174734	−1.5861314	1.9229769	5.2036236	1.8888	−415.71	0.5047515
1.2000000	0.2445190	−1.8542726	1.6084103	5.2097859	1.8904	−689.74	0.4387977
1.0000000	0.2621261	−2.0480479	1.3149807	5.2143414	1.2694	−930.36	0.3902363
0.8000000	0.2741035	−2.1907298	1.0366379	5.2177103	0.3840	−1125.7	0.3544596
0.6000000	0.2822701	−2.2939349	0.7690377	5.2201412	−0.5320	−1274.2	0.3287174
0.4000000	0.2875949	−2.3641204	0.5088379	5.2217876	−1.3072	−1377.8	0.3113195
0.2000000	0.2906107	−2.4049629	0.2532937	5.2227423	−1.8207	−1438.8	0.3012451
0.1000000	0.2913450	−2.4150336	0.1265079	5.2229773	−1.9547	−1453.9	0.2987672
0.0050000	0.2915876	−2.4183702	0.0063231	5.2230553	−1.9999	−1458.9	0.2979468

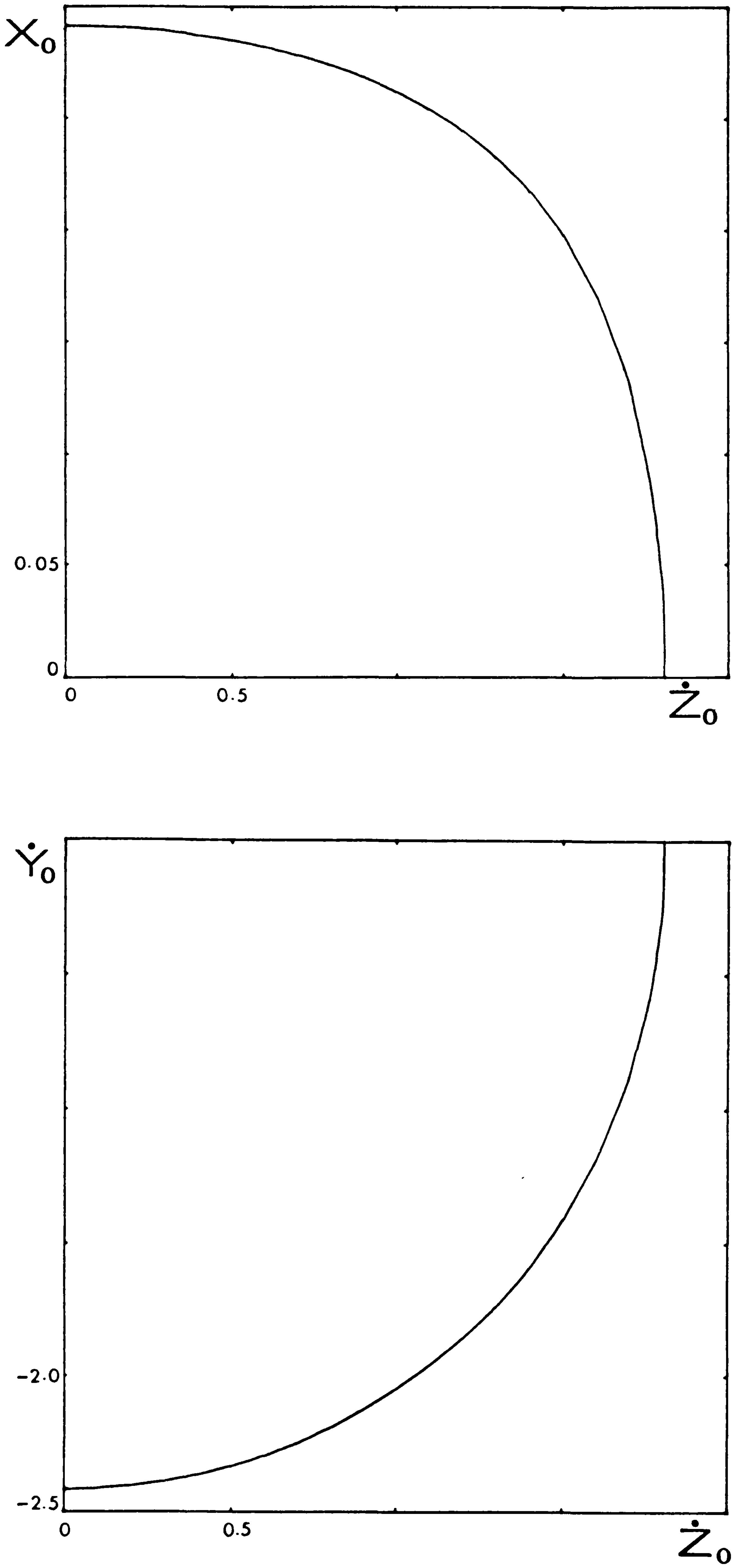

Fig. 2. Family characteristic of the branch-family $B6$. Projections in the $(\dot{z}_0, x_0)$ and $(\dot{z}_0, \dot{y}_0)$ planes.

a_1 and a_2) of the typical member solutions of the branch $B6$ are given in Table II. All members of the two computed branches are unstable. Both branches begin with their respective bifurcations from the basic Sitnikov family L_1^1 and terminate with planar periodic solutions. The terminating members of the branches $B3$ and $B6$ were identified as the 'vertical-critical orbits' h_2v and h_23v, respectively, given in Hénon (1973). The numerical data for the termination of $B6$ are approximated well with the data in the last entry of Table III. The entry marked with an asterisk corresponds to another bifurcation along the branch $B6$, i.e. the beginning of a 'branch of the brach'. The same phenomenon occurs along the other branch $B3$.

The 'family characteristic' of the branch $B6$ is given in Figure 2, and some typical member solutions of this branch are presented graphically in Figure 3. The members of the other branch $B3$ have similar, but simpler, shapes.

5. Continuation of the Bifurcations for Non-Equal Primaries

It is easy to see that the bifurcations of the basic family continue to exist when $\mu \neq 1/2$, i.e. when the basic family ceases to consist of rectilinear solutions, its members becoming three-dimensional periodic solutions of double symmetry (as given for $\mu = 0.4$ by Bray and Goudas, 1967). Let us consider the linearized Equations (3) with $\mu = 1/2 + \varepsilon$. Linearizing with respect to ε we obtain, for the expressions (4),

$$\begin{aligned} F_1(z) &= 1 - \Phi_{10}^{-3/2} \\ F_2(z) &= \tfrac{3}{4}\Phi_{10}^{-5/2} \\ F_3(z) &= \tfrac{3}{4}\Phi_{10}^{-5/2}\varepsilon \\ F_4(z) &= -\tfrac{15}{4}\Phi_{10}^{-7/2}\varepsilon \end{aligned} \tag{24}$$

with

$$\Phi_{10} = \tfrac{1}{4} + z^2, \tag{25}$$

and Equations (3) become

$$\ddot{\xi} - 2\dot{\eta} = (1 - \Phi_{10}^{-3/2} + \tfrac{3}{4}\Phi_{10}^{-5/2})\xi + \tfrac{3}{4}\Phi_{10}^{-5/2}\varepsilon, \tag{26a}$$

$$\ddot{\eta} + 2\dot{\xi} = (1 - \Phi_{10}^{-3/2})\eta, \tag{26b}$$

$$\ddot{z} = -\Phi_{10}^{-3/2} z. \tag{26c}$$

Equations (26) are now the equations of motion for the basic family (small ξ, η) for μ slightly different from 1/2 (small ε). We see that the equation for z does not change, therefore the period of a member of the basic family remains the same in this linear theory for $\mu = 1/2 + \varepsilon$.

It can be shown numerically that if the first two of Equations (26) have a periodic solution (for $\varepsilon \neq 0$) with the period T^* of one of the critical solutions of the Sitnikov family of rectilinear motions ($\varepsilon = 0$), then this periodic solution is the continuation of the critical Sitnikov solution to the case $\mu = 1/2 + \varepsilon$.

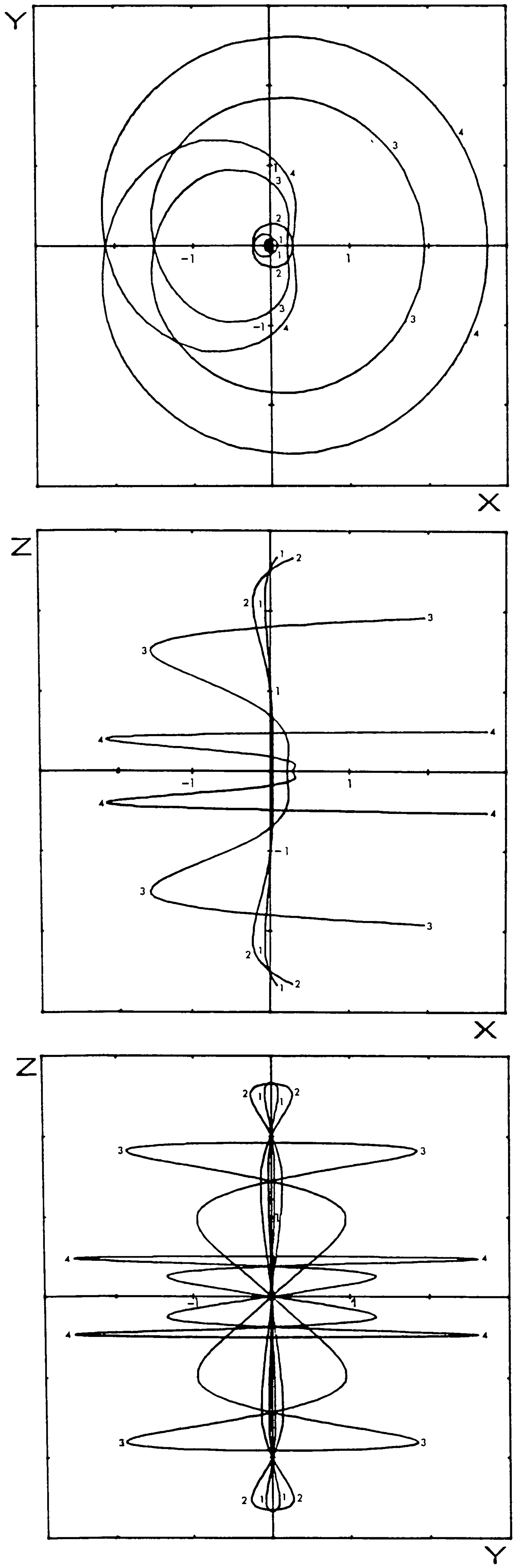

Fig. 3. Typical member-solutions of the branch-family $B6$. Projections in the (x, y), (x, z) and (y, z) planes.

To illustrate that a bifurcation of the Sitnikov family continues to exist for $\mu = 1/2 + \varepsilon$ and sufficiently small ε, one needs to show that the first two of (26) admit a periodic solution with the period T^* of the bifurcation. The periodicity conditions are

$$\begin{aligned} \eta(\xi_0, \dot{\eta}_0) &= 0 \\ \dot{\xi}(\xi_0, \dot{\eta}_0) &= 0, \end{aligned} \tag{27}$$

where the left-hand sides are assumed to be evaluated at $t = T^*/4$ (we seek periodic solutions of double symmetry). A simple differential-corrections procedure for the satisfaction of (27) is to start with $\xi_0 = \dot{\eta}_0 = 0$ and obtain the corrections $\Delta\xi_0$, $\Delta\dot{\eta}_0$ from

$$\begin{bmatrix} \partial\eta/\partial\xi_0 & \partial\eta/\partial\dot{\eta}_0 \\ \partial\dot{\xi}/\partial\xi_0 & \partial\dot{\xi}/\partial\dot{\eta}_0 \end{bmatrix} \begin{bmatrix} \Delta\xi_0 \\ \Delta\dot{\eta}_0 \end{bmatrix} = - \begin{bmatrix} \eta(0,0) \\ \dot{\xi}(0,0) \end{bmatrix} \tag{28}$$

Since for $\mu = 1/2 + \varepsilon$ ($\varepsilon \neq 0$) the quantities $\eta(0,0)$ and $\dot{\xi}(0,0)$ will not be simultaneously zero at $t = T^*/4$, it suffices to show that for the bifurcation solution ($\mu = 1/2$) the determinant of the coefficient matrix is non-zero. This has been established numerically for the four bifurcations $B1$, $B4$, $B5$, $B7$ which we have continued for $\mu \neq 1/2$ (see below).

We note that the corrector (28) only needs to be applied once. The values $\xi_{01} = \xi_0 + \Delta\xi_0 = \Delta\xi_0$, $\dot{\eta}_{01} = \dot{\eta}_0 + \Delta\dot{\eta}_0 = \Delta\dot{\eta}_0$ obtained after one application of the corrector are sufficiently accurate to lead to exact periodicity (within the accuracy of the numerical integration and of the machine itself).

For large deviations of μ from the value 1/2 the above linear approximation is no longer helpful and we have used the full equations of the restricted three-body problem to determine the bifurcations, i.e. to determine the initial conditions $x_0, \dot{y}_0, \dot{z}_0$ satisfying the following periodicity and criticality conditions

$$\begin{aligned} \dot{x}(x_0, \dot{y}_0, \dot{z}_0; \mu) &= 0 \\ \dot{z}(x_0, \dot{y}_0, \dot{z}_0; \mu) &= 0 \\ a_1(x_0, \dot{y}_0, \dot{z}_0; \mu) &= -2, \end{aligned} \tag{29}$$

where the left-hand sides are computed at the appropriate crossing of the x–z plane. We shall not describe this differential-corrections procedure in detail. We merely note that as implied from (29) the bifurcations are isolated for fixed μ but if μ is allowed to vary they comprise monoparametric 'series'. Four of these (bifurcations $B1$, $B4$, $B5$, and $B7$) we have computed in the range $\mu = 0.5$ to $\mu = 0.4$ and present here in Tables III to VI. Typical shapes of these bifurcation solutions are presented in Figure 4.

It must be stressed that, due to the rectilinear shape and high instability of the bifurcation solutions for $\mu = 1/2$, their numerical continuation to other values of μ would be extremely difficult without the use of the linear approximation, presented in the beginning of this section, which provides the knowledge of the appropriate in each cace intersection of the x–z plane and the accurate starting values necessary for the above continuation.

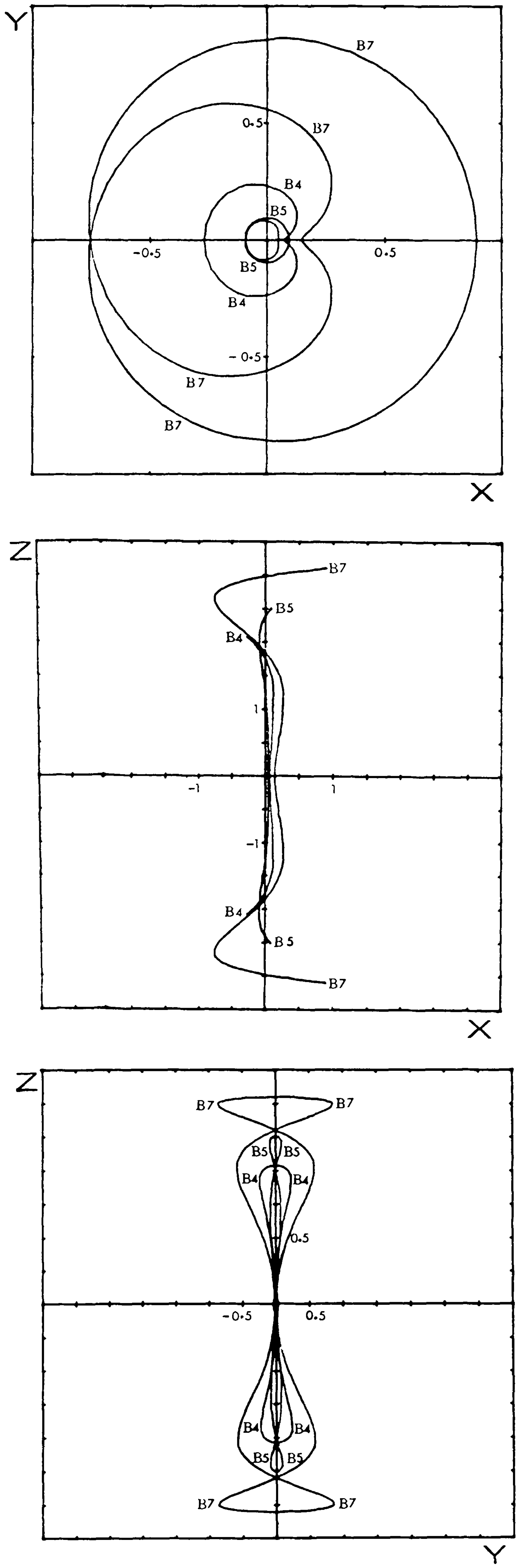

Fig. 4. Typical shapes of the bifurcation solutions $B4$, $B5$, $B7$. Projections in the (x, y), (x, z) and (y, z) planes.

TABLE III
The bifurcation $B1$ for different values of μ.

μ	x_0	$\dot{y}_0$	$\dot{z}_0$	$z(T/4)$	$T/4$	a_2	C
0.50	0	0	1.2058314	0.6058850	0.9508536	−462.52	2.5459707
0.48	0.0298848	0.0051117	1.2052117	0.6056123	0.9508992	−462.68	2.5467313
0.46	0.0597898	0.0102254	1.2033487	0.6047924	0.9510369	−463.13	2.5490127
0.44	0.0897356	0.0153428	1.2002295	0.6034187	0.9512677	−463.89	2.5528163
0.42	0.1197432	0.0204658	1.1958319	0.6014802	0.9515934	−464.95	2.5581438
0.40	0.1498349	0.0255958	1.1901247	0.5989610	0.9520168	−466.32	2.5649976

TABLE IV
The bifurcation $B4$ for different values of μ.

μ	x_0	$\dot{y}_0$	$\dot{z}_0$	$z(T/4)$	$T/4$	a_2	C
0.50	0	0	1.7522080	2.0921592	3.7045148	−42.385	0.9297670
0.48	0.0762509	−0.0680192	1.7589751	2.0816796	3.7128352	−56.007	0.9402387
0.46	0.1465625	−0.1264225	1.7764571	2.0541703	3.7356879	−102.62	0.9685877
0.44	0.2083896	−0.1708594	1.7988606	2.0178108	3.7688621	−193.30	1.0084609
0.42	0.2620263	−0.2015093	1.8211013	1.9799681	3.8089296	−336.04	1.0541840
0.40	0.3087579	−0.2203394	1.8398066	1.9457685	3.8540193	−533.82	1.1018528

TABLE V
The bifurcation $B5$ for different values of μ.

μ	x_0	$\dot{y}_0$	$\dot{z}_0$	$z(T/4)$	$T/4$	a_2	C
0.50	0	0	1.7942219	2.5123094	4.7566414	−9.6904	0.7807677
0.48	0.0495133	−0.1383391	1.7919226	2.5079365	4.7509766	−13.674	0.7768351
0.46	0.0969675	−0.2587052	1.7862877	2.4970151	4.7354886	−27.594	0.7673196
0.44	0.1415898	−0.3524838	1.7797891	2.4836867	4.7132205	−55.631	0.7571086
0.42	0.1836932	−0.4191705	1.7743114	2.4707339	4.6866887	−102.27	0.7509342
0.40	0.2239843	−0.4613504	1.7707145	2.4587892	4.6572772	−172.74	0.7525913

TABLE VI
The bifurcation $B7$ for different values of μ.

μ	x_0	$\dot{y}_0$	$\dot{z}_0$	$z(T/4)$	$T/4$	a_2	C
0.50	0	0	1.8394755	3.2062645	6.6986793	−11.840	0.6163297
0.48	0.1496513	−0.1827156	1.8889273	3.1080677	6.7453021	−231.21	0.6648238
0.46	0.2338265	−0.2594361	1.9528468	2.9818016	6.8125440	−1003.3	0.7351749
0.44	0.2922441	−0.2927061	2.0035197	2.8805544	6.8775355	−2216.1	0.8017587
0.42	0.3381886	−0.3042306	2.0401672	2.8041504	6.9407029	−3772.2	0.8632777
0.40	0.3766009	−0.3029268	2.0631786	2.7510309	7.0035629	−5622.8	0.9195886

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