NORMAL FORMS FOR SYMPLECTIC MAPS OF $\mathbf{R}^{2n}$

A.Bazzani
Dipartimento di Fisica - Universitá di Padova
INFN - Sezione di Padova - ITALY

(Received: 21 August, 1987; accepted: 26 May, 1988)

Abstract

The problem of the existence of normal forms for symplectic maps in $\mathbf{R}^{2n}$ is analyzed using a technique based on the generating functions. We discuss extensively the case of a map with an elliptic fixed point: both the resonant and non resonant cases are presented.

Explicit algorithms to calculate the analytic expansions of the normal forms and the associated canonical transformation are also provided.

Introduction

The aim of this paper is to generalize the well known results for Birkhoff's normal forms of a Hamiltonian system to the symplectic maps in $\mathbf{R}^{2n}$. Studying the behaviour of the orbits near an equilibrium solution, Birkhoff [1] showed the existence of a formal canonical transformation which reduces the Hamiltonian to a very simple form when the frequencies of the linearized equations are rationally independent. Later Gustavson [2] introduced a more general normal form which also admits study of the resonant case. More recently the theory of the normal forms for Hamiltonian systems has been developed using the Lie transformations method [3] .

In spite of the connection between the Hamiltonian systems and the symplectic maps, the previous results do not include a trivial generalization to the symplectic maps in more than two dimensions. Indeed the conditions for the symplecticity of a map in $\mathbf{R}^{2n}$ take a very complicated form and it is difficult to construct a normal forms' theory explicitly satisfying these conditions. This theory has been framed for the area-preserving maps [4] while a computer code capable of calculating the normal form already exists [5] .

In order to avoid these difficulties, we use extensively the possibility of representing a

Celestial Mechanics 42 (1988), 107–128.

symplectic map by a generating function. In this way we can construct explicit algorithms to compute the normal form and the associated canonical transformation. Clearly the recurrent equations we must solve are more complicated than the corrisponding ones in the case of the differential equations, but the use of the mappings has obvious advantages from a computational point of view. Moreover the symplectic maps are useful in the study of periodic differential equations of the form:

$$\frac{d^2\mathbf{x}}{dt^2} + \sum_{k\geq 1} \lambda_k(\mathbf{x}, t) = 0 \tag{1}$$

when the homogeneous polynomials $\lambda_k(\mathbf{x}, t)$ are discontinuous functions of t. We recall for example that the motion of a particle in a hadronic accelerator is well described by equation (1) with $\lambda_k(\mathbf{x}, t)$ stepwise functions [6] . Other physical phenomena which involve instantaneous interaction, like the beam-beam interaction in modern colliders [7,12] , are easily described by means of symplectic maps.

The difficulties of visualizing the phase space of a map in $\mathbf{R}^{2n}$ with $n \geq 2$, make the study of the geometry of the orbits a very hard task. Although the transformation which brings a map into normal form is usually divergent [8] , we can extract relevant information about the behaviour of the orbits using the related first integrals of motion and the method of the interpolating Hamiltonian [9] . Anyway, a rigorous analysis with the normal forms would involve the estimates of the remainders of the perturbative expansions. This has been done in the case of Hamiltonian systems [10] obtaining an exponential estimate of Nekhoroshev's type and there is strong evidence for the area-preserving maps [11] . Yet up to now a rigorous result in this sense for the symplectic maps in $\mathbf{R}^{2n}$ is not available.

The present work is organized in the following way:

In section 1 we define the normal form with respect to a subspace $\mathbf{L} \subseteq \mathbf{Z}^n$ for the symplectic maps in $\mathbf{R}^{2n}$ with an elliptic fixed point.We analyze the properties of such normal forms by means of Noether's theorem.

In section 2 we discuss in a general context the problem of the existence of a formal canonical trasformation of variables which conjugates two given symplectic maps.

In section 3 we prove the existence of a canonical transformation of variables which brings a symplectic map in normal form up to any finite perturbative order. We treat both resonant and non resonant cases. The use of complex canonical coordinates in $\mathbf{C}^{2n}$ requires the generating function to have a set of reality conditions to insure that the transformation of variables is real.

In section 4 we present the recurrent equations which have to be implemented in a computer code to calculate the normal form and the associated canonical transformation.

1-Normal form for symplectic maps

Let $\mathbf{x}$ be an element of $\mathbf{R}^{2n}$:

$$\mathbf{x} = \sum_{i=1}^{2n} x_i \mathbf{e}_i = \sum_{j=1}^{n} p_j \mathbf{e}_{2j-1} + q_j \mathbf{e}_{2j}$$

where $\mathbf{e}_i$ is the standard basis. We shall say that the map:

$$\mathbf{x}' = \mathbf{F}(\mathbf{x}) \tag{1.1}$$

is symplectic if the Jacobian matrix $\mathbf{F}_*(\mathbf{x})$ has the following property:

$$\widetilde{\mathbf{F}_*}(\mathbf{x})\mathbf{J}\mathbf{F}_*(\mathbf{x}) = \mathbf{J} \tag{1.2}$$

where $\mathbf{J}$ is the $2n \times 2n$ matrix:

$$\mathbf{J} = \prod_{k=1}^{n} \otimes \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and the tilde denotes the transposed matrix.

Equation (1.2) is equivalent to usual conditions for the symplecticity:

$$\left\{p'_j, p'_k\right\} = \left\{q'_j, q'_k\right\} = 0$$

$$\left\{p'_j, q'_k\right\} = \delta_{jk} \tag{1.3}$$

$\forall j, k = 1..n$ where we use the usual Poisson Bracket operator:

$$\left\{a_j, b_k\right\} = \sum_{l=1}^{n} \frac{\partial a_j}{\partial p_l}\frac{\partial b_k}{\partial q_l} - \frac{\partial b_k}{\partial p_l}\frac{\partial a_j}{\partial q_l}$$

We shall suppose from now on that the symplectic map(1.1) is an analytic mapping with a fixed point at the origin.

We characterize the normal forms for the maps by means of the property of symmetry with respect to some particular groups. Let $\mathbf{L}$ be a subspace of $\mathbf{Z}^n$, we define the rotational group $G_{\mathbf{L}}$, whose elements are the direct product of n rotations in the plane $< p_j, q_j >$ with angle α_j such that the n-nuple α satisfies the resonant condition with respect to $\mathbf{L}$:

$$\mathbf{k} \cdot \alpha = 0 \mod 2\pi \quad \forall \quad \mathbf{k} \in \mathbf{L} \tag{1.4}$$

In particular when $\mathbf{L} = \{0\}$ and the condition (1.4) is satisfied for any α, we shall speak of the group G of rotations and of the non resonant case.

We say that map (1.1) is a normal form of elliptic type with respect to $\mathbf{L}$ if:

$$\mathbf{R}(\alpha)\circ\mathbf{F}\circ\mathbf{R}(-\alpha)=\mathbf{F}\quad\forall\quad\mathbf{R}(\alpha)\in G_{\mathbf{L}} \tag{1.5}$$

In the non resonant case the definition (1.5) provides n independent groups of symmetry to map $\mathbf{F}$, one for each parameter α_j and Noether's theorem (see appendix 1) allows us to find n independent first integrals of motion for map $\mathbf{F}$:

$$\rho_j=\frac{p_j^2+q_j^2}{2}\quad\text{with}\quad j=1..n \tag{1.6}$$

Moreover these integrals are in involution because the groups of symmetry commute each other.

Generally with $\mathbf{L}\subseteq\mathbf{Z}^n$ and $\dim\mathbf{L}=r$, we can always construct a matrix $\mathbf{M}$ with integer entries and $\det\mathbf{M}=1$ such that:

$$\mathbf{M}(\mathbf{L})\subseteq\operatorname{span}\{\mathbf{e}_1....\mathbf{e}_r\} \tag{1.7}$$

and the linear space $\mathbf{M}(\mathbf{L})$ is isomorphic to $\mathbf{Z}^r$.

Now if $\alpha\in\mathbf{R}^n$ is resonant with respect to $\mathbf{L}$ then it follows that the n-nuple β:

$$\beta=\widetilde{\mathbf{M}}^{-1}(\alpha)$$

is resonant with respect to $\mathbf{M}(\mathbf{L})$:

$$\mathbf{h}\cdot\beta=0\mod 2\pi\quad\forall\quad\mathbf{h}\in\mathbf{M}(\mathbf{L}) \tag{1.8}$$

and vice versa (we show an example with $n=2$ and $r=1$ in Appendix 2). As one can see from the property (1.7), the parameters $\beta_{r+1}..\beta_n$ in (1.8) are completely free and Noether's theorem allows one to construct $n-r$ first integrals in involution for the normal form $\mathbf{F}$. Such integrals read:

$$I_l=\sum_{j=1}^{n}\widetilde{\mathbf{M}}_{lj}\left(\frac{p_j^2+q_j^2}{2}\right)=\sum_{j=1}^{n}\widetilde{\mathbf{M}}_{lj}\rho_j\quad l=r+1....n$$

The normal form mappings have a simple expression in a set of complex variables which reduce $\mathbf{R}(\alpha)$ to diagonal form.

Let us enlarge our real space to the complex space $\mathbf{C}^{2n}$ defining the variables (π,χ) such that:

$$\begin{cases}\mathbf{p}=Re\pi\\ \mathbf{q}=Re\chi\end{cases} \tag{1.9}$$

The extension of a real map in $\mathbf{C}^{2n}$ will be characterized by the invariance of the subspace:

$$\begin{cases}Im\pi=0\\ Im\chi=0\end{cases} \tag{1.10}$$

We then perform the following change of variables in $\mathbf{C}^{2n}$ which preserves the symplecticity:

$$\begin{cases} \mathbf{z} = \pi + i\chi \\ \mathbf{w} = \pi - i\chi \end{cases} \tag{1.11}$$

so that a generic $\mathbf{R}(\alpha)$ reads:

$$\mathbf{R}(\alpha) = \prod_{j=1}^{n} \otimes \begin{pmatrix} e^{\alpha_j} & 0 \\ 0 & e^{-\alpha_j} \end{pmatrix} \tag{1.12}$$

and the equation of the invariant subspace (1.10) becomes:

$$\mathbf{w} = \mathbf{z}^* \tag{1.13}$$

In the non resonant case definition (1.5) and the invariance property of subspace (1.13) reduce map $\mathbf{F}$ to a completely integrable form:

$$\begin{cases} \mathbf{z}' = \mathbf{z}e^{i\Omega(\mathbf{zw})} \\ \mathbf{w}' = \mathbf{w}e^{-i\Omega(\mathbf{zw})} \end{cases} \tag{1.14}$$

where $\Omega(\mathbf{zz}^*)$ is real and:

$$\Omega_j(\rho) = \frac{\partial S}{\partial \rho_j}(\rho) \qquad \text{with} \quad j = 1...n$$

In the resonant case a normal form with respect to $\mathbf{L}$ reads:

$$\begin{cases} \mathbf{z}' = e^{i\omega}\mathbf{z} + \sum_{\mathbf{k}\geq 2} \mathbf{F}_{\mathbf{k}}(\mathbf{z},\mathbf{w}) \\ \mathbf{w}' = e^{-i\omega}\mathbf{w} + \sum_{\mathbf{k}\geq 2} \mathbf{G}_{\mathbf{k}}(\mathbf{z},\mathbf{w}) \end{cases} \tag{1.15}$$

where the homogeneous polynomials $\mathbf{F}_k$ and $\mathbf{G}_k$ satisfy the following equations:

$$\begin{aligned} \mathbf{G}_k^{(\mathbf{z})}(e^{i\alpha}\mathbf{z}, e^{-i\alpha}\mathbf{w}) &= e^{i\alpha}\mathbf{G}_k^{(\mathbf{z})}(\mathbf{z},\mathbf{w}) \\ \mathbf{G}_k^{(\mathbf{w})}(e^{i\alpha}\mathbf{z}, e^{-i\alpha}\mathbf{w}) &= e^{-i\alpha}\mathbf{G}_k^{(\mathbf{w})}(\mathbf{z},\mathbf{w}) \end{aligned} \tag{1.16}$$

$\forall\, \alpha \in \mathbf{R}^n$ resonant with respect to $\mathbf{L}$ and have precise constraints due to the conditions for symplecticity. Indeed we can represent map (1.15) by means of a complex generating function :

$$G(\mathbf{z},\mathbf{w}') = e^{i\omega}\mathbf{z}\mathbf{w}' + \sum_{k\geq 1} G_k(\mathbf{z},\mathbf{w}') \tag{1.17}$$

where the homogeneous polynomials $G_k(\mathbf{z},\mathbf{w})$ are invariant under the action of group $\mathbf{G_L}$:

$$G_k(e^{i\alpha}\mathbf{z}, e^{-i\alpha}\mathbf{w}) = G_k(\mathbf{z},\mathbf{w}) \tag{1.18}$$

It must be pointed out that the invariance property (1.18) of the generating function (1.17) implies the commutative property (1.5) for the corresponding transformation because we are using the complex coordinates (1.11). In the original variables $(\mathbf{p}, \mathbf{q})$ this is no longer true as may easily be checked. The reason is that the generating function G is not a geometrical object: i.e. G depends on the coordinates we are using.

We recall that $\mathbf{z} = \mathbf{w}^*$ is an invariant subspace for map (1.15)

$$\left[\mathbf{G}_k^{(\mathbf{z})}(\mathbf{z}, \mathbf{z}^*)\right]^* = \mathbf{G}_k^{(\mathbf{w})}(\mathbf{z}, \mathbf{z}^*)$$

and consequently the generating function $G(\mathbf{z}, \mathbf{w}')$ has some special properties which we discuss in appendix 4.

2-Conjugation theory

Let us consider two symplectic maps $\mathbf{F}$ and $\mathbf{G}$:

$$\mathbf{x}' = \mathbf{F}_1\mathbf{x} + \mathbf{F}_2(\mathbf{x}) + \mathbf{F}_3(\mathbf{x}) + \ldots.. \tag{2.1}$$

$$\mathbf{X}' = \mathbf{G}_1\mathbf{X} + \mathbf{G}_2(\mathbf{X}) + \mathbf{G}_3(\mathbf{X}) + \ldots.. \tag{2.2}$$

where $\mathbf{F}_k(\mathbf{x})$ represents a $2n$-vector whose components are homogeneous polynomials in $\mathbf{x}$ and $\mathbf{F}_1 = \mathbf{F}_*(0)$ and the same for $\mathbf{G}$.

We say that maps (2.1) and (2.2) are conjugated to each other up to terms of order N if there exists a canonical change of variables $\mathbf{T} : \mathbf{x} \longmapsto \mathbf{X}$ defined in a neighbourhood of the origin such that:

$$\mathbf{G} = \mathbf{T} \circ \mathbf{F} \circ \mathbf{T}^{-1} + o(N) \tag{2.3}$$

Establishing the conditions for the existence of transformation $\mathbf{T}$ is a necessary step in order to construct a normal form theory for the maps.

The main problem is to exhibit explicitly the symplectic character of the maps. We overcome this difficulty using the representation of a symplectic map by means of a generating funtion: a real function $S(\mathbf{x})$ in $\mathbf{R}^{2n}$ such that:

$$S(\mathbf{0}) = 0 \quad \text{and} \quad \frac{\partial S}{\partial \mathbf{x}}(\mathbf{0}) = 0$$

defines a symplectic map:

$$\Phi : (\mathbf{p}, \mathbf{q}) \longmapsto (\mathbf{p}', \mathbf{q}')$$

by means of the equations:

$$\begin{cases} \mathbf{p} = \mathbf{p}' + \dfrac{\partial S}{\partial \mathbf{q}}(\mathbf{p}', \mathbf{q}) \\ \mathbf{q}' = \mathbf{q} + \dfrac{\partial S}{\partial \mathbf{p}'}(\mathbf{p}', \mathbf{q}) \end{cases} \tag{2.4}$$

wherever the implicit function theorem is satisfied. We shall use the following notation:

$$\mathbf{x}' = \Phi(\mathbf{x}) = \left(\mathbf{1} + \mathbf{M}_S\right)(\mathbf{x}) \tag{2.5}$$

Let us introduce the symbol $\underset{k}{=}$ defining:

$$A(\mathbf{x}) \underset{k}{=} B(\mathbf{x}) \Longrightarrow A(\mathbf{x}) = B(\mathbf{x}) + o(k+1)$$

We recall the properties of symplectic maps close to the identity (for a proof see appendix 3):

1) Given a symplectic map in $\mathbf{R}^{2n}$:

$$\mathbf{x}' = \mathbf{x} + \mathbf{F}_k(\mathbf{x}) + \mathbf{F}_{k+1}(\mathbf{x}) + \ldots\ldots \tag{2.6}$$

it is possible to find a homogeneous polynomial $S_{k+1}(\mathbf{x})$ such that:

$$\mathbf{x}' \underset{k}{=} \left(\mathbf{1} + \mathbf{M}_{S_{k+1}}\right)(\mathbf{x})$$

differs from (2.6) by terms of degree larger than k.

2) if $\mathbf{A}$ is a symplectic map in $\mathbf{R}^{2n}$:

$$\mathbf{A}(\mathbf{x}) = \mathbf{A}_1\mathbf{x} + \mathbf{A}_2(\mathbf{x}) + \ldots.$$

then:

$$\mathbf{A} \circ \left(\mathbf{1} + \mathbf{M}_{S_k}\right) \circ \mathbf{A}^{-1} \underset{k-1}{=} \left(\mathbf{1} + \mathbf{M}_{S_k \circ \mathbf{A}_1^{-1}}\right)$$

Moreover one can easily check that the following equations hold $\forall\ k$:

$$\left(\mathbf{1} + \mathbf{M}_{T_k}\right) \circ \left(\mathbf{1} + \mathbf{M}_{S_k}\right) \underset{k-1}{=} \left(\mathbf{1} + \mathbf{M}_{T_k + S_k}\right)$$

$$\left(\mathbf{1} + \mathbf{M}_{S_k}\right)^{-1} \underset{k-1}{=} \left(\mathbf{1} + \mathbf{M}_{-S_k}\right)$$

$$\left(\mathbf{1} + \mathbf{M}_{\sum_{j=3}^{k+1} T_j}\right) \underset{k}{=} \left(\mathbf{1} + \mathbf{M}_{T_{k+1}}\right) \circ \left(\mathbf{1} + \mathbf{M}_{\sum_{j=3}^{k} T_j}\right) \tag{2.7}$$

We now proceed computing order by order the right hand side, $\mathbf{T} \circ \mathbf{F} \circ \mathbf{T}^{-1}$, of equation (2.3). The linear part implies that $\mathbf{F}_1$ and $\mathbf{G}_1$ are similar matrices and we can assume $\mathbf{G}_1 = \mathbf{F}_1$ without loss of generality.

Then transformation $\mathbf{T}$ can be written in the form:

$$\mathbf{T} = \left(\mathbf{1} + \mathbf{M}_{\sum_{k \geq 3} T_k}\right) \tag{2.8}$$

In order to find an explicit expression for the second order terms we write maps $\mathbf{F}(\mathbf{x})$ and $\mathbf{G}(\mathbf{X})$ in the following form:

$$\mathbf{F}(\mathbf{x}) = \mathbf{R}_F^{(2)} \circ \mathbf{F}_1\mathbf{x} = \mathbf{R}_F^{(3)} \circ \left(\mathbf{1} + \mathbf{M}_{F_3}\right) \circ \mathbf{F}_1\mathbf{x} \underset{2}{=} \left(\mathbf{1} + \mathbf{M}_{F_3}\right) \circ \mathbf{F}_1\mathbf{x}$$

$$\mathbf{G}(\mathbf{X}) = \mathbf{R}_G^{(2)} \circ \mathbf{F}_1\mathbf{X} = \mathbf{R}_G^{(3)} \circ \left(\mathbf{1} + \mathbf{M}_{G_3}\right) \circ \mathbf{F}_1\mathbf{X} \underset{2}{=} \left(\mathbf{1} + \mathbf{M}_{G_3}\right) \circ \mathbf{F}_1\mathbf{X} \tag{2.9}$$

where the homogeneous polynomials F_3 and G_3 are calculated using property **1**:

$$\mathbf{R}_F^{(2)}(\mathbf{x}) = \mathbf{F}(\mathbf{F}_1^{-1}\mathbf{x}) \underset{2}{=} \left(\mathbf{1} + \mathbf{M}_{F_3}\right)(\mathbf{x})$$

$$\mathbf{R}_G^{(2)}(\mathbf{X}) = \mathbf{G}(\mathbf{G}_1^{-1} \underset{2}{=} \left(\mathbf{1} + \mathbf{M}_{G_3}\right)\mathbf{X} \tag{2.10}$$

and as a consequence:

$$\mathbf{R}_F^{(3)}(\mathbf{x}) = \underset{2}{=} \mathbf{x}$$

$$\mathbf{R}_G^{(3)}(\mathbf{X}) = \underset{2}{=} \mathbf{X} \tag{2.11}$$

Now we substitute eq. (2.9) and (2.11) in the conjugation equation (2.3) obtaining:

$$\mathbf{R}_G^{(3)} \circ \left(\mathbf{1} + \mathbf{M}_{G_3}\right) \circ \mathbf{F}_1\mathbf{X} = \left(\mathbf{1} + \mathbf{M}_{\sum_{k\geq 3} T_k}\right) \circ \mathbf{R}_F^{(3)} \circ \left(\mathbf{1} + \mathbf{M}_{F_3}\right) \circ \mathbf{F}_1 \circ \left(\mathbf{1} + \mathbf{M}_{\sum_{k\geq 3} T_k}\right)^{-1}(\mathbf{X})$$

Therefore retaining only the second order terms we have:

$$\left(\mathbf{1} + \mathbf{M}_{G_3}\right) \circ \mathbf{F}_1\mathbf{X} \underset{2}{=} \left(\mathbf{1} + \mathbf{M}_{T_3}\right) \circ \left(\mathbf{1} + \mathbf{M}_{F_3}\right)\mathbf{F}_1 \circ \left(\mathbf{1} + \mathbf{M}_{T_3}\right)^{-1}(\mathbf{X}) \tag{2.12}$$

or equivalently:

$$\left(\mathbf{1} + \mathbf{M}_{G_3}\right)\mathbf{X} \underset{2}{=} \left(\mathbf{1} + \mathbf{M}_{T_3}\right) \circ \left(\mathbf{1} + \mathbf{M}_{F_3}\right)\mathbf{F}_1 \circ \left(\mathbf{1} + \mathbf{M}_{T_3}\right)^{-1} \circ \mathbf{F}_1^{-1}\mathbf{X} \tag{2.13}$$

Then thanks to property **2**) and equalities (2.7), equation (2.13) takes the form:

$$\left(\mathbf{1} + \mathbf{M}_{G_3}\right)(\mathbf{X}) \underset{2}{=} \left(\mathbf{1} + \mathbf{M}_{T_3 + F_3 - T_3(\mathbf{F}_1^{-1})}\right)(\mathbf{X}) \tag{2.14}$$

Thus maps (2.1) and (2.2) are conjugated to each other up to order 2 if a homogeneous polynomial T_3 exists such that:

$$T_3(\mathbf{x}) - T_3(\mathbf{F}_*^{-1}(0)\mathbf{x}) + F_3(\mathbf{x}) = G_3(\mathbf{x}) \tag{2.15}$$

where F_3 and G_3 are defined in (2.9).

Using an inductive argument we shall show the explicit form of the generic order N of eq. (2.3). We assume by hypothesis that we have just calculated the polynomials $T_3(\mathbf{x})...T_N(\mathbf{x})$ such that:

$$\mathbf{G}(\mathbf{X}) \underset{N-1}{=} \left(\mathbf{1}+\mathbf{M}_{\sum_{k=3}^{N} T_k}\right) \circ \mathbf{F} \circ \left(\mathbf{1}+\mathbf{M}_{\sum_{k=3}^{N} T_k}\right)^{-1} \tag{2.16}$$

and that we also know the homogeneous polynomials $G_3(\mathbf{x})...G_N(\mathbf{x})$ which verify the following equality:

$$\mathbf{G}(\mathbf{X}) \underset{N-1}{=} \left(\mathbf{1}+\mathbf{M}_{G_N}\right)....\left(\mathbf{1}+\mathbf{M}_{G_3}\right) \circ \mathbf{F}_1\mathbf{X} = \mathbf{G}^{(N-1)}(\mathbf{X}) \tag{2.17}$$

Therefore according to eq. (2.16) and (2.17) we can write:

$$\left(\mathbf{1}+\mathbf{M}_{\sum_{k=3}^{N} T_k}\right) \circ \mathbf{F} \circ \left(\mathbf{1}+\mathbf{M}_{\sum_{k=3}^{N} T_k}\right)^{-1} = \mathbf{R}_F^{(N)} \circ \mathbf{G}^{(N)} \tag{2.18}$$

$$\mathbf{G}(\mathbf{X}) = \mathbf{R}_G^{(N)}\mathbf{G}^{(N-1)}(\mathbf{X}) \tag{2.19}$$

where:

$$\mathbf{R}_F^{(N)}(\mathbf{x}) \underset{N-1}{=} \mathbf{x}$$

$$\mathbf{R}_G^{(N)}(\mathbf{X}) \underset{N-1}{=} \mathbf{X}$$

We observe that the remainder $\mathbf{R}_F^{(N)}$ depends not only on $\mathbf{F}$ but also on the polynomials T_k with $k = 3...N$.

Then we can calculate two homogeneous polynomials F_{N+1} and G_{N+1} with the following property:

$$\begin{cases} \mathbf{R}_F^{(N)} \underset{N}{=} \left(\mathbf{1}+\mathbf{M}_{F_{N+1}}\right) \\ \mathbf{R}_G^{(N)} \underset{N}{=} \left(\mathbf{1}+\mathbf{M}_{G_{N+1}}\right) \end{cases} \tag{2.20}$$

so that equation (2.3) reads:

$$\begin{aligned} \left(\mathbf{1}+\mathbf{M}_{G_{N+1}}\right) \circ \mathbf{G}^{(N-1)}(\mathbf{X}) &\underset{N}{=} \mathbf{G}(\mathbf{X}) = \left(\mathbf{1}+\mathbf{M}_{\sum_{k\geq 3} T_k}\right) \circ \mathbf{F} \circ \left(\mathbf{1}+\mathbf{M}_{\sum_{k\geq 3} T_k}\right)^{-1}(\mathbf{X}) \underset{N}{=} \\ &\underset{N}{=} \left(\mathbf{1}+\mathbf{M}_{T_{N+1}}\right)\left(\mathbf{1}+\mathbf{M}_{\sum_{k=3}^{N} T_k}\right) \circ \mathbf{F} \circ \left(\mathbf{1}+\mathbf{M}_{\sum_{k=3}^{N} T_k}\right)^{-1} \circ \left(\mathbf{1}+\mathbf{M}_{T_{N+1}}\right)^{-1}(\mathbf{X}) \underset{N}{=} \\ &\underset{N}{=} \left(\mathbf{1}+\mathbf{M}_{T_{N+1}}\right) \circ \left(\mathbf{1}+\mathbf{M}_{F_{N+1}}\right) \circ \mathbf{G}^{(N-1)}\left(\mathbf{1}+\mathbf{M}_{T_{N+1}}\right)^{-1}(\mathbf{X}) \end{aligned} \tag{2.21}$$

Applying map $\left(\mathbf{G}^{(N-1)}\right)^{-1}$ to both sides of eq. (2.21), we obtain:

$$\left(\mathbf{1}+\mathbf{M}_{G_{N+1}}\right) \underset{N}{=} \left(\mathbf{1}+\mathbf{M}_{T_{N+1}}\right)\circ\left(\mathbf{1}+\mathbf{M}_{F_{N+1}}\right)\circ\mathbf{G}^{(N-1)}\circ\left(\mathbf{1}+\mathbf{M}_{T_{N+1}}\right)^{-1}\circ(\mathbf{G}^{(N-1)})^{-1}$$

Finally we recover the same form of equation (2.14):

$$\left(\mathbf{1}+\mathbf{M}_{G_{N+1}}\right) \underset{N}{=} \left(\mathbf{1}+\mathbf{M}_{T_{N+1}+F_{N+1}-T_{N+1}\circ\mathbf{F}_*^{-1}(0)}\right)$$

i.e. :

$$T_{N+1}(\mathbf{x})-T_{N+1}(\mathbf{F}_*^{-1}(0)\mathbf{x})+F_{N+1}(\mathbf{x})=G_{N+1}(\mathbf{x}) \tag{2.22}$$

Therefore we can formulate the **Theorem**:

Two analytic maps **F** and **G** with a fixed point at the origin are conjugated to each other up to terms of order N:

$$\mathbf{G} \underset{N}{=} \mathbf{T}\circ\mathbf{F}\circ\mathbf{T}^{-1}$$

where:

$$\mathbf{T}=\left(\mathbf{1}+\mathbf{M}_{\sum_{k\geq 3}T_k}\right)$$

if the homogeneous polynomials T_k satisfy the recurrent equations:

$$T_k(\mathbf{x})-T_k(\mathbf{F}_1^{-1}\mathbf{x})+F_k(\mathbf{x})=G_k(\mathbf{x})$$

with $k=3...N+1$

where the polynomials G_k give a factorization of map **G** according to:

$$\mathbf{G} \underset{N}{=} \left(\mathbf{1}+\mathbf{M}_{G_{N+1}}\right)...\left(\mathbf{1}+\mathbf{M}_{G_3}\right)\circ\mathbf{F}_1$$

while the polynomials F_k depend on map **F** and on the polynomials T_j with $j=3...k-1$. The complexity of the computation of G_k and F_k are irrelevant because other algorithms will be proposed when the existence of the solution is garanteed.

3-Reduction to normal form of a symplectic map

Let ξ be an element of $\mathbf{C}^{2n}$:

$$\xi=\sum_{i=1}^{2n}\xi_i\mathbf{e}_i=\sum_{j=1}^{n}z_j\mathbf{e}_{2j-1}+w_j\mathbf{e}_{2j}$$

with $\mathbf{z}$ and $\mathbf{w}$ defined by (1.11). We consider a map $\mathbf{F}$:

$$\xi' = \mathbf{F}(\xi) = \mathbf{R}(\omega)\xi + \mathbf{F}_2(\xi) + \mathbf{F}_3(\xi) + ... \tag{3.1}$$

which is the image of a real map: i.e. $\mathbf{z}^* = \mathbf{w}$ is an invariant subspace for $\mathbf{F}$.

We say that the n-nuple ω is non resonant with respect to $\mathbf{M}$ if:

$$\mathbf{k} \cdot \omega = 2\pi n \quad \text{with} \quad n \quad \text{integer} \quad \Longrightarrow \quad \mathbf{k} \in \mathbf{M} \tag{3.2}$$

If (3.2) holds with $\mathbf{M} = \{\mathbf{0}\}$ in (3.2) we simply say that ω is non resonant.

We shall construct a canonical trasformation $\mathbf{T}$ on $\mathbf{C}^{2n}$:

$$\mathbf{T} = \left(\mathbf{1} + \mathbf{M}_{\sum_{k \geq 3} T_k}\right)$$

such that if ω is non resonant with respect to $\mathbf{M}$ then map $\mathbf{G}$:

$$\mathbf{G} = \mathbf{T} \circ \mathbf{F} \circ \mathbf{T}^{-1} \tag{3.3}$$

is the image of a real normal form of elliptic type with respect to $\mathbf{M}$.

Then we must not only solve the conjugation equation (3.3) but also verify that transformation $\mathbf{T}$ is the image of a real transformation of variables; this can be achieved by requiring the reality conditions for T_k (see appendix 4):

$$T_k(\mathbf{z}, \mathbf{w}) + T_k^*(\mathbf{w}, \mathbf{z}) = R_k(\mathbf{z}, \mathbf{w}) \tag{3.4}$$

These equations are not trivial and in effect play the role of symplectic conditions for transformation $\mathbf{T}$.

Let us consider order $k-1$ of equation (3.3) (see eq.2.22):

$$T_k(\xi) - T_k(\mathbf{R}(-\omega)\xi) + F_k(\xi) = G_k(\xi) \tag{3.5}$$

It is useful to introduce the $\Pi_{\mathbf{M}}$ projector in the space of complex polynomials $P(\xi)$ which are invariant under the action of group $\mathbf{G_M}$:

$$\Pi_{\mathbf{M}} P(\mathbf{R}(\alpha)\xi) = \Pi_{\mathbf{M}} P(\xi) \quad \forall \quad \mathbf{R}(\alpha) \in \mathbf{G_M}$$

We recall that the condition:

$$\Pi_{\mathbf{M}} G_k = G_k \quad \forall \quad k \tag{3.6}$$

garantees that map $\mathbf{G}$ (see eq. (2.17)) is a normal form with respect to $\mathbf{M}$.

Therefore we split equation (3.5) into the system:

$$\begin{cases} (\mathbf{1} - \Pi_{\mathbf{M}}) \left[T_k(\xi) - T_k(\mathbf{R}(-\omega)\xi) + F_k(\xi)\right] = 0 \\ \Pi_{\mathbf{M}} \left[T_k(\xi) - T_k(\mathbf{R}(-\omega)\xi)\right] + \Pi_{\mathbf{M}}(F_k(\xi)) = G_k(\xi) \end{cases} \tag{3.7}$$

Now we substitute T_k ,F_k and G_k with their explicit expressions:

$$T_k(\xi) = \sum_{\mathbf{l},\mathbf{m}\in\mathbf{Z}^n,|\mathbf{l}|+|\mathbf{m}|=k} t_{\mathbf{l},\mathbf{m}}\xi^{[\mathbf{l},\mathbf{m}]}$$

with the convention:

$$\xi^{[\mathbf{l},\mathbf{m}]} = \prod_{j=1}^{n} z_j^{l_j} w_j^{m_j}$$

obtaining the following equations:

$$\begin{cases} (1 - e^{i\omega\cdot(\mathbf{m}-\mathbf{l})})t_{\mathbf{l},\mathbf{m}} + f_{\mathbf{l},\mathbf{m}} = 0 & \text{if} \quad \mathbf{l}-\mathbf{m} \notin \mathbf{M} \\ (1 - e^{i\omega\cdot(\mathbf{m}-\mathbf{l})})t_{\mathbf{l},\mathbf{m}} + f_{\mathbf{l},\mathbf{m}} = g_{\mathbf{l},\mathbf{m}} & \text{if} \quad \mathbf{l}-\mathbf{m} \in \mathbf{M} \end{cases} \tag{3.8}$$

If ω is non resonant with respect to $\mathbf{M}$, we know that:

$$\mathbf{k} \notin \mathbf{M} \Longrightarrow (1 - e^{i\omega\cdot\mathbf{k}}) \neq 0$$

and consequently we can always solve the first equation of system (3.8) letting:

$$t_{\mathbf{l},\mathbf{m}} = \frac{-f_{\mathbf{l},\mathbf{m}}}{(1 - e^{i\omega\cdot(\mathbf{m}-\mathbf{l})})} \tag{3.9}$$

Conversely the second equation of system (3.8) determines the coefficients $g_{\mathbf{l},\mathbf{m}}$ while the remaining coefficients $t_{\mathbf{l},\mathbf{m}}$ are free.

The proof that we can always find a T_k solution for equations (3.7) consistently with reality conditions (3.4) makes use of an inductive argument.

Let us consider the case $k = 3$. Then eq.(3.4) has the simple form:

$$T_3(\mathbf{z},\mathbf{w}) + T_3^*(\mathbf{w},\mathbf{z}) = 0 \tag{3.10}$$

or more explicitly:

$$t_{\mathbf{l},\mathbf{m}} + t^*_{\mathbf{m},\mathbf{l}} = 0 \quad \text{with} \quad |\mathbf{l}| + |\mathbf{m}| = 3 \tag{3.11}$$

Replacing $t_{\mathbf{l},\mathbf{m}}$ with the right hand side of (3.9), eq. (3.11) reads:

$$t_{\mathbf{l},\mathbf{m}} + t^*_{\mathbf{m},\mathbf{l}} = \frac{1}{(1 - e^{i\omega\cdot(\mathbf{m}-\mathbf{l})})}\left(f_{\mathbf{l},\mathbf{m}} + f^*_{\mathbf{m},\mathbf{l}}\right) = 0 \quad \text{with} \quad (\mathbf{l}-\mathbf{m}) \notin \mathbf{M}$$

so that if F_3 satisfies the reality conditions, the same holds for $(\mathbf{1} - \Pi_{\mathbf{M}})T_3$. Recalling that the initial map $\mathbf{F}$ is the image of a real map and that F_3 is defined by:

$$\mathbf{F} \underset{2}{=} \left(\mathbf{1} + \mathbf{M}_{F_3}\right) \circ \mathbf{R}(\omega)$$

one can easily check that F_3 has the required property.

This shows the consistency of $(\mathbf{1}-\Pi_{\mathbf{M}})T_3$ with the reality conditions; thus if we choose the remaining coefficients of $\Pi_{\mathbf{M}}T_3$ according to:

$$t_{\mathbf{l},\mathbf{m}} + t^*_{\mathbf{m},\mathbf{l}} = 0 \quad \text{with} \quad (\mathbf{l}-\mathbf{m}) \in \mathbf{M}$$

we complete the proof in the case $k=3$.

Let us consider the generic order k, assuming by inductive hypothesis that we have just solved our problem at every order $< k$. We choose a homogeneous polynomial T'_k that satisfies equation (3.4) and we calculate the polynomial G'_k:

$$G'_k(\xi) = T'_k(\xi) - T'_k(\mathbf{R}(-\omega)\xi) + F_k(\xi) \tag{3.12}$$

In this way we conjugate the initial map $\mathbf{F}$ with a map $\mathbf{G}'$ of the form:

$$\mathbf{G}' \underset{k-1}{=} \left(\mathbf{1}+\mathbf{M}_{G'_k}\right) \circ \left(\mathbf{1}+\mathbf{M}_{G_{k-1}}\right) \dots \left(\mathbf{1}+\mathbf{M}_{G_3}\right) \circ \mathbf{R}(\omega)$$

which is the image of a real map up to the terms of order $k-1$. Moreover by our inductive hypothesis the map:

$$\mathbf{G}^{(k-2)} = \left(\mathbf{1}+\mathbf{M}_{G_{k-1}}\right) \dots \left(\mathbf{1}+\mathbf{M}_{G_3}\right) \circ \mathbf{R}(\omega)$$

is a normal form with respect to $\mathbf{M}$.

It is easy to see that at this perturbative order map $\mathbf{G}'$ can be read in the form:

$$\mathbf{G}' \underset{k-1}{=} \left(\mathbf{1}+\mathbf{M}_{\left(\mathbf{1}-\Pi_{\mathbf{M}}\right)G'_k}\right) \circ \left(\mathbf{1}+\mathbf{M}_{\Pi_{\mathbf{M}}G'_k}\right) \circ \mathbf{G}^{(k-2)} \tag{3.13}$$

From the group's property of the normal forms with respect to $\mathbf{M}$, we see that all the terms not in normal form in the right hand side of (3.13) come from

$$\left(\mathbf{1}+\mathbf{M}_{\left(\mathbf{1}-\Pi_{\mathbf{M}}\right)G'_k}\right)$$

By our choice, map $\mathbf{G}'$ is the image of a real map up to terms of order $k-1$ then $(\mathbf{1}-\Pi_{\mathbf{M}})G'_k$ satisfies the reality conditions which read:

$$(\mathbf{1}-\Pi_{\mathbf{M}})G'_k(\mathbf{z},\mathbf{w}) + (\mathbf{1}-\Pi_{\mathbf{M}})G'^*_k(\mathbf{w},\mathbf{z}) = 0 \tag{3.14}$$

We replace in (3.14) G'_k with the right hand side of (3.12) obtaining:

$$\begin{aligned}&(\mathbf{1}-\Pi_{\mathbf{M}})\left[(T'_k - T'_k \circ \mathbf{R}(-\omega))(\mathbf{z},\mathbf{w}) + (T'^*_k - T'^*_k \circ \mathbf{R}(\omega))(\mathbf{w},\mathbf{z})\right] + \\ &+ (\mathbf{1}-\Pi_{\mathbf{M}})\left[F_k(\mathbf{z},\mathbf{w}) + F^*(\mathbf{w},\mathbf{z})\right] = 0\end{aligned} \tag{3.15}$$

Now, our initial hypothesis was that T'_k satisfies reality conditions (3.4); consequently equation (3.15) reduces to:

$$(\mathbf{1}-\Pi_{\mathbf{M}})\left[F_k(\mathbf{z},\mathbf{w}) + F^*_k(\mathbf{w},\mathbf{z})\right] + (\mathbf{1}-\Pi_{\mathbf{M}})\left[(R_k - R_k \circ \mathbf{R}(-\omega))(\mathbf{z},\mathbf{w})\right] = 0 \tag{3.16}$$

Replacing F_k and R_k with their explicit expressions we obtain the system

$$f_{\mathbf{l},\mathbf{m}} + f^*_{\mathbf{m},\mathbf{l}} + \left(1 - e^{i\omega\cdot(\mathbf{m}-\mathbf{l})}\right) r_{\mathbf{l},\mathbf{m}} = 0 \quad \text{with} \quad \mathbf{l} - \mathbf{m} \notin \mathbf{M} \tag{3.17}$$

So if we consider solution (3.9) for $\left(\mathbf{1} - \Pi_{\mathbf{M}}\right)T_k$, from eq. (3.17) it follows easily that

$$t_{\mathbf{l},\mathbf{m}} + t^*_{\mathbf{m},\mathbf{l}} = \frac{-(f_{\mathbf{l},\mathbf{m}} + f^*_{\mathbf{m},\mathbf{l}})}{\left(1 - e^{i\omega\cdot(\mathbf{m}-\mathbf{l})}\right)} = r_{\mathbf{l},\mathbf{m}} \quad \text{with} \quad \mathbf{l} - \mathbf{m} \notin \mathbf{M} \tag{3.18}$$

which proves the reality conditions for $\left(\mathbf{1} - \Pi_{\mathbf{M}}\right)T_k$.

We then choose the remaining part, $\Pi_{\mathbf{M}}T_k$, consistently with reality conditions (3.4) (this is always possible as one can check) and we calculate G_k from the equation:

$$G_k(\xi) = \Pi_{\mathbf{M}}F_k(\xi) + \Pi_{\mathbf{M}}\left(T_k(\xi) - T_k(\mathbf{R}(-\omega)\xi)\right) \tag{3.19}$$

This completes the reduction into normal form of map $\mathbf{F}$ up to terms of order $k-1$.

Freedom in the choice of $\Pi_{\mathbf{M}}T_k$ means that the normal form $\mathbf{G}$ is defined except for a similarity relation:

$$\mathbf{G}' = \mathbf{U} \circ \mathbf{G} \circ \mathbf{U}^{-1} \tag{3.20}$$

where $\mathbf{U}$ is a canonical transformation in normal form with respect to $\mathbf{M}$; i.e. the initial map $\mathbf{F}$ has in general a whole family of normal forms with respect to a subspace $\mathbf{M}$. The only exception is the non resonant case, when the set of the normal forms is an abelian group so that equation (3.20) reduces to:

$$\mathbf{G}' = \mathbf{G}$$

We summarize the previous results in the following **Theorem**:

Given a symplectic map $\mathbf{F}$ in $\mathbf{R}^{2n}$

$$\xi' = \mathbf{R}(\omega)\xi + \mathbf{F}_2(\xi) + \mathbf{F}_3(\xi) + \ldots.$$

if ω is non resonant modulo $\mathbf{M}$,subspace of $\mathbf{Z}^{2n}$, it is possible to find a canonical trasformation $\mathbf{T}$ in a neighbourhood of the origin such that the map:

$$\mathbf{G} = \mathbf{T} \circ \mathbf{F} \circ \mathbf{T}^{-1}$$

is in normal form with respect to $\mathbf{M}$ up to terms of order N. Transformation $\mathbf{T}$ is determined except for a further change of variables in normal form with respect to $\mathbf{M}$. Moreover in the non resonant case,the part in normal form of map $\mathbf{G}$ depends only on the initial map $\mathbf{F}$.

4-Algorithms for the calculation of normal forms

We shall present the recurrent equation it is necessary to implement in a computer code, calculating the normal forms for a map.

The analytic expansions of map $\mathbf{F}$ and the normal form $\mathbf{G}$ read:

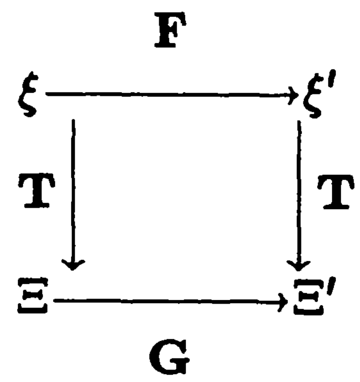
$$\begin{cases} \mathbf{z}' = e^{i\omega}\mathbf{z} + \sum\limits_{k\geq 2} \mathbf{f}_{(k)}(\mathbf{z},\mathbf{w}) \\ \mathbf{w}' = e^{-i\omega}\mathbf{w} + \sum\limits_{k\geq 2} \mathbf{g}_{(k)}(\mathbf{z},\mathbf{w}) \end{cases} \tag{4.1}$$

$$\begin{cases} \mathbf{Z}' = e^{i\omega}\mathbf{Z} + \sum\limits_{k\geq 2} \mathbf{u}_{(k)}(\mathbf{Z},\mathbf{W}) \\ \mathbf{W}' = e^{-i\omega}\mathbf{W} + \sum\limits_{k\geq 2} \mathbf{v}_{(k)}(\mathbf{Z},\mathbf{W}) \end{cases} \tag{4.2}$$

while the canonical transformation $\mathbf{T} : \xi \longmapsto \Xi$ is given in implicit form by the equations:

$$\begin{cases} \mathbf{z} = \mathbf{Z} + \sum\limits_{k\geq 3} \frac{\partial T_k}{\partial \mathbf{w}}(\mathbf{Z},\mathbf{w}) \\ \mathbf{W} = \mathbf{w} + \sum\limits_{k\geq 3} \frac{\partial T_k}{\partial \mathbf{Z}}(\mathbf{Z},\mathbf{w}) \end{cases} \tag{4.3}$$

Using the commutativity of diagram



we can construct a set of recurrent equations in order to calculate the expansions (4.2) and (4.3).

We start from the right hand side of the diagram, represented by the equations

$$\begin{cases} \mathbf{z}' = \mathbf{Z}' + \sum\limits_{k\geq 3} \frac{\partial T_k}{\partial \mathbf{w}'}(\mathbf{Z}',\mathbf{w}') \\ \mathbf{W}' = \mathbf{w}' + \sum\limits_{k\geq 3} \frac{\partial T_k}{\partial \mathbf{Z}'}(\mathbf{Z}',\mathbf{w}') \end{cases} \tag{4.4}$$

and we substitute in the order: **z**' and **w**' from equations (4.1), **W**' and **Z**' from equations (4.2) and finally **w** and **Z** from equations (4.3).

In this way we obtain a set of polynomial equations in $(\mathbf{Z},\mathbf{w})$ which must be identically satisfied. The generic perturbative order k reads:

$$\begin{cases} e^{-i\omega}\frac{\partial}{\partial \mathbf{z}}\Big(T_{k+1}(\mathbf{z},\mathbf{w}) - T_{k+1}(e^{i\omega}\mathbf{z}, e^{-i\omega}\mathbf{w})\Big) - \mathbf{u}_k(\mathbf{z},\mathbf{w}) = \mathbf{C}_k^{(\mathbf{z})}(\mathbf{z},\mathbf{w}) \\ e^{i\omega}\frac{\partial}{\partial \mathbf{w}}\Big(T_{k+1}(\mathbf{z},\mathbf{w}) - T_{k+1}(e^{i\omega}\mathbf{z}, e^{-i\omega}\mathbf{w})\Big) + \mathbf{v}_k(\mathbf{z},\mathbf{w}) = \mathbf{C}_k^{(\mathbf{w})}(\mathbf{z},\mathbf{w}) \end{cases} \tag{4.5}$$

Moreover the polynomial T_{k+1} must verify the reality conditions

$$\begin{cases} \dfrac{\partial}{\partial \mathbf{z}}\Big(T_{k+1}(\mathbf{z},\mathbf{w}) + T^*_{k+1}(\mathbf{w},\mathbf{z})\Big) = R^{(\mathbf{z})}_{k+1}(\mathbf{z},\mathbf{w}) \\ \dfrac{\partial}{\partial \mathbf{w}}\Big(T_{k+1}(\mathbf{z},\mathbf{w}) + T^*_{k+1}(\mathbf{w},\mathbf{z})\Big) = R^{(\mathbf{w})}_{k+1}(\mathbf{z},\mathbf{w}) \end{cases} \tag{4.6}$$

Clearly equations (4.5) and (4.6) form an overconstrained system but we know a priori of the existence of a solution for T_{k+1} , $\mathbf{u}_k$ and $\mathbf{v}_k$. Consequently it is sufficient to retain only a part of system (4.5) and (4.6) in order to avoid useless repetition of calculations.

The remainders $\mathbf{C}_k$ and R_{k+1} are constructed by using only the composition of polynomials. Thus the main limitations for a computer code based on eq. (4.5) and (4.6) derive from the quantity of storage we need when working with polynomials in more than two variables.

APPENDIX 1

We shall present a proof of Noether's theorem for symplectic maps.

Let g^s with $s \in \mathbf{R}$ be a monoparametric group of symplectic maps in $\mathbf{R}^{2n}$ with $g^0 = \mathbf{E}_{2n}$, we say that g^s is a group of symmetries for a mapping $\mathbf{F}$:

$$\begin{cases} \mathbf{p}' = \mathbf{f}(\mathbf{p},\mathbf{q}) \\ \mathbf{q}' = \mathbf{g}(\mathbf{p},\mathbf{q}) \end{cases}$$

if:

$$g^s \circ \mathbf{F} - \mathbf{F} \circ g^s = 0 \quad \forall s \in \mathbf{R} \tag{1}$$

Deriving both sides of equation (1) for s, we obtain:

$$\frac{\partial}{\partial s}\Big(g^s \circ \mathbf{F} - \mathbf{F} \circ g^s\Big)\Big|_0 = 0 \tag{2}$$

The symplectic nature of g^s implies the following expansion in s:

$$g^s(\mathbf{p},\mathbf{q}) = \begin{cases} \mathbf{p} - s\dfrac{\partial G}{\partial \mathbf{q}}(\mathbf{p},\mathbf{q}) + o(s^2) \\ \mathbf{q} + s\dfrac{\partial G}{\partial \mathbf{p}}(\mathbf{p},\mathbf{q}) + o(s^2) \end{cases} \tag{3}$$

Then equation (2) takes the form:

$$\begin{cases} \dfrac{\partial \mathbf{g}}{\partial q_j}\dfrac{\partial G}{\partial p_j} - \dfrac{\partial \mathbf{g}}{\partial p_j}\dfrac{\partial G}{\partial q_j} = -\Big\{\mathbf{g}, G\Big\} = \dfrac{\partial G}{\partial \mathbf{p}} \circ \mathbf{F} \\ -\dfrac{\partial \mathbf{f}}{\partial q_j}\dfrac{\partial G}{\partial p_j} + \dfrac{\partial \mathbf{f}}{\partial p_j}\dfrac{\partial G}{\partial q_j} = \Big\{\mathbf{f}, G\Big\} = \dfrac{\partial G}{\partial \mathbf{q}} \circ \mathbf{F} \end{cases} \tag{4}$$

On the left hand side of (4) we recognize the matrix:

$$-\mathbf{J}\mathbf{F}_*\mathbf{J} = \begin{pmatrix} \frac{\partial \mathbf{g}}{\partial \mathbf{q}} & -\frac{\partial \mathbf{g}}{\partial \mathbf{p}} \\ -\frac{\partial \mathbf{f}}{\partial \mathbf{q}} & \frac{\partial \mathbf{f}}{\partial \mathbf{p}} \end{pmatrix}$$

Now we recall that the Jacobian matrix $\mathbf{F}_*$ has the property

$$-(\mathbf{J}\mathbf{F}_*\mathbf{J})^{-1} = \widetilde{\mathbf{F}}_*$$

so that we can explicitly invert equations (4) obtaining:

$$\begin{cases} \frac{\partial f^j}{\partial \mathbf{p}} \frac{\partial G}{\partial p_j} \circ \mathbf{F} + \frac{\partial g_j}{\partial \mathbf{p}} \frac{\partial G}{\partial q_j} \circ \mathbf{F} = \frac{\partial G}{\partial \mathbf{p}} \\ \frac{\partial f_j}{\partial \mathbf{q}} \frac{\partial G}{\partial p_j} \circ \mathbf{F} + \frac{\partial g_j}{\partial \mathbf{q}} \frac{\partial G}{\partial q_j} \circ \mathbf{F} = \frac{\partial G}{\partial \mathbf{q}} \end{cases}$$

Now it is easy to recognize that the previous equations can be written in the form:

$$\begin{cases} \frac{\partial}{\partial \mathbf{p}} (G \circ \mathbf{F} - G) = 0 \\ \frac{\partial}{\partial \mathbf{q}} (G \circ \mathbf{F} - G) = 0 \end{cases} \tag{5}$$

or equivalently:

$$G(\mathbf{F}(\mathbf{x})) - G(\mathbf{x}) = const. \quad \forall \quad \mathbf{x} \in \mathbf{R}^{2n}$$

Recalling that by our hypothesis map $\mathbf{F}$ has a fixed point in the origin, the previous equation reduces to:

$$G(\mathbf{F}(\mathbf{x})) - G(\mathbf{x}) = 0 \quad \forall \quad \mathbf{x} \in \mathbf{R}^{2n} \tag{6}$$

This means that the real funtion G defined in (3) is a first integral of motion for map $\mathbf{F}$ and we refer to it as the integral associated to the group of symmetries g^s.

APPENDIX 2

Let $\mathbf{L}$ be a subgroup of $\mathbf{Z}^2$ with $\dim \mathbf{L} = 1$; then two integers p and q having no common divisors, exist such that;

$$\mathbf{L} = \left\{ \mathbf{k} \in \mathbf{Z}^2 / \mathbf{k} = l(rp, rq) \right\} \tag{1}$$

where r is a fixed integer while $l \in \mathbf{Z}$.

We shall construct the 2×2 matrix $\mathbf{M}$ with integer entries:

$$\mathbf{M} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$$

such that $\mathbf{M}(\mathbf{L}) \subseteq \mathbf{Z}$ and $\det \mathbf{M} = 1$. We shall determine the entries of $\mathbf{M}$ by means of the equations

$$\begin{cases} m_{11}p + m_{12}q = 1 \\ m_{21}p + m_{22}q = 0 \\ m_{11}m_{22} - m_{12}m_{21} = 1 \end{cases} \tag{2}$$

If we choose $m_{21} = -q$ and $m_{22} = p$ the second equation will be satisfied. Consequently the remaining equations are reduced to the single equation

$$m_{11}p + m_{12}q = 1 \tag{3}$$

Now it is well known that if p and q have no common divisor, then it is always possible to find two integers such that equation (3) holds. Using definition (1),we see that by our choice:

$$\mathbf{M}(\mathbf{L}) = lr\mathbf{M}\begin{pmatrix} p \\ q \end{pmatrix} = lr\begin{pmatrix} 1 \\ 0 \end{pmatrix} = l\begin{pmatrix} r \\ 0 \end{pmatrix} \quad \forall \quad l \in \mathbf{Z}$$

i.e. $\mathbf{M}(\mathbf{L})$ is the subgroup of $\mathbf{Z}$ which contains all the multiples of r.

APPENDIX 3

We shall show some properties of the symplectic maps close to the identity. Let us consider a map $\mathbf{F}$ in $\mathbf{R}^{2n}$ whose analytical expansion reads:

$$\mathbf{x}' = \mathbf{x} + \mathbf{F}_k(\mathbf{x}) + \mathbf{F}_{k+1}(\mathbf{x}) + ... \tag{1}$$

Using the symplectic conditions (1.2) on the Jacobian matrix, we can write:

$$\left(1 + \frac{\partial \widetilde{\mathbf{F}_k}}{\partial \mathbf{x}}\right)\mathbf{J}\left(1 + \frac{\partial \mathbf{F}_k}{\partial \mathbf{x}}\right) \underset{k-1}{=} \mathbf{J} \tag{2}$$

so that the following equation holds:

$$\frac{\partial \widetilde{\mathbf{F}_k}}{\partial \mathbf{x}}\mathbf{J} + \mathbf{J}\frac{\partial \mathbf{F}_k}{\partial \mathbf{x}} = 0 \tag{3}$$

Recalling that $\widetilde{\mathbf{J}} = -\mathbf{J}$ we deduce from equation (3) that:

$$\mathbf{J}\frac{\partial \mathbf{F}_k}{\partial \mathbf{x}}$$

is a symmetric matrix. Thus we can always find a homogeneous polynomial $S_{k+1}(\mathbf{x})$ such that

$$-\mathbf{J}\frac{\partial \mathbf{F}_k}{\partial \mathbf{x}} = \frac{\partial^2 S_{k+1}}{\partial \mathbf{x}\partial \mathbf{x}} \tag{4}$$

If we consider the map:

$$\left(\mathbf{1} + \mathbf{M}_{S_{k+1}}\right) \tag{5}$$

by definition we have:

$$\left(\mathbf{1} + \mathbf{M}_{S_{k+1}}\right)(\mathbf{x}) \underset{k}{=} \mathbf{x} + \mathbf{J}\frac{\partial S_{k+1}}{\partial \mathbf{x}}(\mathbf{x}) = \mathbf{x} + \mathbf{F}_k(\mathbf{x}) \underset{k}{=} \mathbf{F}(\mathbf{x})$$

This proves point **1**).

In order to prove the second point we write map (5) in the form:

$$\mathbf{x}' \underset{k}{=} \mathbf{x} + \mathbf{J}\frac{\partial S_{k+1}}{\partial \mathbf{x}}(\mathbf{x}) \tag{6}$$

Then if we perform the change of variables:

$$\mathbf{X} = \mathbf{A}(\mathbf{x})$$

on eq. (6), we obtain the equation:

$$\mathbf{X}' \underset{k}{=} \mathbf{A}\left(\mathbf{A}^{-1}(\mathbf{X}) + \mathbf{J}\frac{\partial S_{k+1}}{\partial \mathbf{x}}(\mathbf{A}^{-1}(\mathbf{X}))\right) \tag{7}$$

Now it is easy to check that

$$\frac{\partial S_{k+1}}{\partial \mathbf{x}}(\mathbf{A}^{-1}(\mathbf{X})) = \widetilde{\mathbf{A}_*}(\mathbf{x})\frac{\partial}{\partial \mathbf{X}}\left(S_{k+1}(\mathbf{A}^{-1}(\mathbf{X}))\right)$$

Thus we can write eq. (7) in the form:

$$\begin{aligned}
\mathbf{X}' &\underset{k}{=} \mathbf{X} + \mathbf{A}_*(\mathbf{x})\mathbf{J}\widetilde{\mathbf{A}_*}(\mathbf{x})\frac{\partial}{\partial \mathbf{X}}\left(S_{k+1}(\mathbf{A}^{-1}(\mathbf{X}))\right) \\
&\underset{k}{=} \mathbf{X} + \mathbf{A}_*(\mathbf{x})\mathbf{J}\widetilde{\mathbf{A}_*}(\mathbf{x})\frac{\partial}{\partial \mathbf{X}}\left(S_{k+1}(\mathbf{A}_1^{-1}\mathbf{X})\right) \\
&\underset{k}{=} \mathbf{X} + \mathbf{J}\frac{\partial}{\partial \mathbf{X}}\left(S_{k+1}(\mathbf{A}_1^{-1}\mathbf{X})\right)
\end{aligned} \tag{8}$$

Then the thesis follows from equation (8) recalling the definition of map (5).

APPENDIX 4

We shall characterize the complex generating functions which represent a symplectic map in $\mathbf{C}^{2n}$ with an invariant subspace of equation $\mathbf{w} = \mathbf{z}^*$.

Let us consider the complex generating function

$$S(\mathbf{z},\mathbf{w}) = \mathbf{z}\cdot\mathbf{w} + \sum_{k\geq 3} S_k(\mathbf{z},\mathbf{w}) \tag{1}$$

which represents a map in $\mathbf{C}^{2n}$:

$$\mathbf{M} : (\mathbf{z},\mathbf{w}) \longrightarrow (\mathbf{z}',\mathbf{w}')$$

by means of the equations:

$$\begin{cases} \mathbf{z} = \mathbf{z}' + \sum\limits_{k\geq 3} \dfrac{\partial S_k}{\partial \mathbf{w}}(\mathbf{z}',\mathbf{w}) \\ \mathbf{w}' = \mathbf{w} + \sum\limits_{k\geq 3} \dfrac{\partial S_k}{\partial \mathbf{z}}(\mathbf{z}',\mathbf{w}) \end{cases} \tag{2}$$

We require that:

$$\mathbf{w} = \mathbf{z}^* \Longleftrightarrow \mathbf{w}' = \mathbf{z}'^* \tag{3}$$

Let us take the complex conjugates of equations (2)

$$\begin{cases} \mathbf{z}^* = \mathbf{z}'^* + \sum\limits_{k\geq 3} \dfrac{\partial S_k^*}{\partial \mathbf{w}^*}(\mathbf{z}'^*,\mathbf{w}^*) \\ \mathbf{w}'^* = \mathbf{w}^* + \sum\limits_{k\geq 3} \dfrac{\partial S_k^*}{\partial \mathbf{z}^*}(\mathbf{z}'^*,\mathbf{w}^*) \end{cases} \tag{4}$$

then request (3) implies that the following equations must identically hold:

$$\begin{cases} \mathbf{w} \equiv \left.\left(\mathbf{z}'^* + \sum\limits_{k\geq 3} \dfrac{\partial S_k^*}{\partial \mathbf{w}^*}(\mathbf{z}'^*,\mathbf{z})\right)\right|_{\mathbf{z}'^*=\mathbf{w}'} \\ \mathbf{z}' \equiv \left.\left(\mathbf{w}^* + \sum\limits_{k\geq 3} \dfrac{\partial S_k^*}{\partial \mathbf{z}^*}(\mathbf{w}',\mathbf{w}^*)\right)\right|_{\mathbf{w}^*=\mathbf{z}} \end{cases} \tag{5}$$

Replacing $\mathbf{z}$ and $\mathbf{w}'$ with the right hand side of equation (2), we obtain the identities in $\mathbf{z}$ and $\mathbf{w}$

$$\begin{cases} \sum\limits_{k\geq 3} \dfrac{\partial S_k}{\partial \mathbf{z}} + \dfrac{\partial S_k^*}{\partial \mathbf{w}^*}\left(\mathbf{w} + \sum\limits_{k\geq 3} \dfrac{\partial S_k}{\partial \mathbf{z}}, \mathbf{z} + \sum\limits_{k\geq 3} \dfrac{\partial S_k}{\partial \mathbf{w}}\right) \equiv 0 \\ \sum\limits_{k\geq 3} \dfrac{\partial S_k}{\partial \mathbf{w}} + \dfrac{\partial S_k^*}{\partial \mathbf{z}^*}\left(\mathbf{w} + \sum\limits_{k\geq 3} \dfrac{\partial S_k}{\partial \mathbf{z}}, \mathbf{z} + \sum\limits_{k\geq 3} \dfrac{\partial S_k}{\partial \mathbf{w}}\right) \equiv 0 \end{cases} \tag{6}$$

We shall refer to equations (6) as to the reality conditions for $S_k(\mathbf{z},\mathbf{w})$.

At the generic order $k-1$ the reality conditions read:

$$\begin{cases} \frac{\partial S_k}{\partial \mathbf{z}}(\mathbf{z},\mathbf{w}) + \frac{\partial S_k^*}{\partial \mathbf{w}^*}(\mathbf{w},\mathbf{z}) = R_k^{(\mathbf{z})}(\mathbf{z},\mathbf{w}) \\ \frac{\partial S_k}{\partial \mathbf{w}}(\mathbf{z},\mathbf{w}) + \frac{\partial S_k^*}{\partial \mathbf{z}^*}(\mathbf{w},\mathbf{z}) = R_k^{(\mathbf{w})}(\mathbf{z},\mathbf{w}) \end{cases} \tag{7}$$

where the remainders R_k depend on S_j with $j < k$. Now we replace the left hand side of eq. (7) with the following equalities:

$$\frac{\partial S_k}{\partial \mathbf{z}}(\mathbf{z},\mathbf{w}) + \frac{\partial S_k^*}{\partial \mathbf{w}^*}(\mathbf{w},\mathbf{z}) = \frac{\partial}{\partial \mathbf{z}}\Big(S_k(\mathbf{z},\mathbf{w}) + S_k^*(\mathbf{w},\mathbf{z})\Big)$$

$$\frac{\partial S_k}{\partial \mathbf{w}}(\mathbf{z},\mathbf{w}) + \frac{\partial S_k^*}{\partial \mathbf{z}^*}(\mathbf{w},\mathbf{z}) = \frac{\partial}{\partial \mathbf{w}}\Big(S_k(\mathbf{z},\mathbf{w}) + S_k^*(\mathbf{w},\mathbf{z})\Big) \tag{8}$$

and system (8) takes the form:

$$\begin{cases} \frac{\partial}{\partial \mathbf{z}}\Big(S_k(\mathbf{z},\mathbf{w}) + S_k^*(\mathbf{w},\mathbf{z})\Big) = R_k^{(\mathbf{z})}(\mathbf{z},\mathbf{w}) \\ \frac{\partial}{\partial \mathbf{w}}\Big(S_k(\mathbf{z},\mathbf{w}) + S_k^*(\mathbf{w},\mathbf{z})\Big) = R_k^{(\mathbf{w})}(\mathbf{z},\mathbf{w}) \end{cases} \tag{9}$$

This is clearly an overconstrained system, but we know a priori that this system is integrable so that the reality conditions for S_k reduce to:

$$S_k(\mathbf{z},\mathbf{w}) + S_k^*(\mathbf{w},\mathbf{z}) = R_k(\mathbf{z},\mathbf{w}) \tag{10}$$

where as usual remainders R_k depend on S_j with $j < k$. When $k = 3$ we have simply:

$$S_3(\mathbf{z},\mathbf{w}) + S_3^*(\mathbf{w},\mathbf{z}) = 0$$

i.e. the real part of $S_3(\mathbf{z},\mathbf{z}^*)$ is zero.

Aknowledgements

I wish to thank Professor G.Turchetti for his constant support and useful remarks and Professor M.Pusterla for his encouragement and stimulation,which made this work possible. Discussions with Dr. A.Pisent were most helpful.

REFERENCES

[1] G.D. Birkhoff: *"Dynamical system "* – Am. Math. Soc. (1927)
[2] F. Gustavson: "" – Astronomical Journal **71** 670-686 (1964)
[3] K. Meyer: – Celestial Mechanics **9** 517-522 (1974)
[4] G.D. Birkhoff: "" – Acta Math. **43** 1 (1920)
[5] G.Servizi,G.Turchetti: – Computer Physic Comunication **32** 201 (1984)
[6] A.J. Dragt : *"Lectures in non linear orbit dynamics"* – A.I.P. conference proceedings **87** (1982)
[7] J.C.Herrera : *"Nonlinear dynamics and the beam-beam interaction"* – American Institute of Physics 232 (1979)
[8] C.l.Siegel : *"On the integrals of canonical systems"* – Ann. Math. **42** 806-822 (1941)
[9] A. Bazzani: *"Metodi perturbativi per mappe hamiltoniane"* – Tesi della Università di Padova (1987)
[10] A. Giorgilli,L. Galgani: – Celestial Mechanics **37** 95-112 (1985)
[11] G. Servizi,G. Turchetti: – Nuovo Cimento **95B** 121 (1986)
[12] M.Pusterla,G.Servizi,G.Turchetti: "" – IEE Trans. Nucl. Sc. **30n.4** 2642 (1983)
[13] V.I. Arnold : *"Chapitres supplementaires a la theories des equations differentielles"* – ed. MIR (1977)
[14] C.L. Siegel, J. Moser : *"Lectures in celestial mechanics"* – Springer Verlag (1971)
[15] G. Turchetti: *"Perturbative methods for hamiltonian maps"* – Proceedings of the Summer School of Medellin 1986 Ed by W.A. Saenz, World Scientific, to appear (1987)
[16] J. Moser: *"Lectures in hamiltonian mechanics"* – Am. Math. Soc. (1972)
[17] D.R.Douglas: *"Lie algebraic methods for particle accelerator"* – thesis of the University of Maryland (1982)