# STATIONARY SOLUTIONS AND THEIR CHARACTERISTIC EXPONENTS IN THE RESTRICTED THREE-BODY PROBLEM WHEN THE MORE MASSIVE PRIMARY IS AN OBLATE SPHEROID

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**Abstract.** This paper deals with the numerical investigations of the locations of the five equilibrium points by taking into consideration the effect of oblateness of the more massive primary for some systems of astronomical interest. This note is further concerned with the periodic solutions of the linearized equations of motion around the five equilibrium points. Interesting differences in the trends of the angular frequencies of these motions have been noticed.

## 1. Introduction

The restricted three-body problem possesses five stationary solutions called Lagrangian points, three of which, called collinear equlibria, lie on the line joining the primaries, and the other two, called equilateral equilibria, make equilateral triangles with the primaries. In general the collinear equilibria are unstable while the equilateral points are stable in the Lyapunov sense only in a certain region for the mass parameter. However under certain initial conditions periodic solutions to the first variational equations that represent the infinitesimal (linearized) periodic orbits around the collinear equilibria can be established.

In the present note the authors propose to study the above problem treating the more massive primary as an oblate spheroid with its equatorial plane coincident with the plane of motion. Locations of the equilibrium points as well as the characteristic exponents and eccentricities of the infinitesimal periodic orbits around them have been presented for some systems of astronomical interest. We shall be taking into consideration only the secular effects (McCuskey, 1963) of the oblateness on the motion of the primaries and shall adopt the notation and terminology of Szebehely (1967). Accordingly the distance between the primaries experiences no variation and is taken equal to unity; the sum of masses of the primaries is unity. The unit of time is chosen so as to make the unperturbed mean motion $n_0$ and the gravitational constant unity. The perturbed mean motion $n$ however is constant and is greater than $n_0$.

In an earlier report the authors (1975) studied the problem for the case of collinear equilibria when both the primaries are oblate spheroids, neglecting the effect of oblateness on the motion of the primaries themselves. It might be pointed out that

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recently Vidyakin (1974) too studied the problem in a general way considering also the higher order terms in the potential function. Results arrived at herein concerning stability are in agreement with those of Vidyakin.

## 2. Equations of Motion

The equations of motion in the dimensionless synodic coordinate system $(x, y)$ are

$$\begin{aligned} \ddot{x} - 2n\dot{y} &= \partial\Omega/\partial x, \\ \ddot{y} + 2n\dot{x} &= \partial\Omega/\partial y, \end{aligned} \tag{1}$$

where

$$\Omega = (n^2/2)[(1-\mu)r_1^2 + \mu r_2^2] + (1-\mu)/r_1 + \mu/r_2 + (1-\mu)A_1/2r_1^3, \tag{2}$$

$$r_1^2 = (x-\mu)^2 + y^2, \qquad r_2^2 = (x+1-\mu)^2 + y^2, \tag{3}$$

$n$, the perturbed mean motion of the primaries is given by

$$n^2 = 1 + 3A_1/2, \tag{4}$$

$$A_1 = (AE^2 - AP^2)/5R^2, \tag{5}$$

$AE$, $AP$ being the dimensional equatorial and polar radii of the larger primary and $R$ is the distance between the primaries. The presence of $n$ in (1) and (2) is due to the modification of the force between the primaries.

The Jacobi integral, as usual, takes the form

$$\dot{x}^2 + \dot{y}^2 = 2\Omega - C.$$

The libration points will now be determined from $\partial\Omega/\partial x = 0 = \partial\Omega/\partial y$, i.e.

$$\begin{aligned} &(1 + 3A_1/2)x - (x-\mu)(1-\mu)/r_1^3 - \mu(x+1-\mu)/r_2^3 - \\ &\quad - 3A_1(x-\mu)(1-\mu)/2r_1^5 = 0, \end{aligned} \tag{6}$$

$$y[(1 + 3A_1/2) - (1-\mu)/r_1^3 - \mu/r_2^3 - 3A_1/2r_1^5] = 0. \tag{7}$$

## 3. Equilibrium Points Location

When $y=0$, Equation (6) determines the location of the collinear points $L_1(x_1, 0)$, $L_2(x_2, 0)$ and $L_3(x_3, 0)$, where

$$\begin{aligned} x_1 &= \mu - 1 - \xi_1, \\ x_2 &= \mu - 1 + \xi_2, \\ x_3 &= \mu + \xi_3, \end{aligned} \tag{8}$$

$\xi_1, \xi_2, \xi_3$ satisfying

$$(2 + 3A_1)\xi_1^7 + (2 + 3A_1)(5 - \mu)\xi_1^6 + (2 + 3A_1)(10 - 4\mu)\xi_1^5 + \\ + [(2 + 3A_1)(10 - 6\mu) - 2]\xi_1^4 + [(2 + 3A_1)(5 - 4\mu) - \\ - 4(1 + \mu)]\xi_1^3 - 12\mu\xi_1^2 - 8\mu\xi_1 - 2\mu = 0, \tag{9}$$

$$(2 + 3A_1)\xi_2^7 - (2 + 3A_1)(5 - \mu)\xi_2^6 + (2 + 3A_1)(10 - 4\mu)\xi_2^5 - \\ - [(2 + 3A_1)(10 - 6\mu) - (2 - 4\mu)]\xi_2^4 + [(2 + 3A_1)(5 - 4\mu) - \\ - 4 + 12\mu]\xi_2^3 - 12\mu\xi_2^2 + 8\mu\xi_2 - 2\mu = 0, \tag{10}$$

$$(2 + 3A_1)\xi_3^7 + (2 + 3A_1)(2 + \mu)\xi_3^6 + (2 + 3A_1)(1 + 2\mu)\xi_3^5 + \\ + [(2 + 3A_1)\mu - 2]\xi_3^4 - 4(1 - \mu)\xi_3^3 - (2 + 3A_1)(1 - \mu)\xi_3^2 - \\ - 6(1 - \mu)A_1\xi_3 - 3(1 - \mu)A_1 = 0. \tag{11}$$

For $y \neq 0$, Equations (6) and (7) readily give

$$r_1 = 1, \qquad r_2^3 = 2/(2 + 3A_1) < 1. \tag{12}$$

Equations (12) locate the other two points $L_4$ and $L_5$. As they form only near equilateral triangles with the primaries, we call them triangular points.

Equations (8) to (11) are solved for some systems of astronomical interest viz., Earth–Moon, Jupiter its moon, Saturn its moon systems, and are presented in Table II. The necessary data have been borrowed from Duncombe *et al.* (1973) and Morrison and Gruikshank (1974). However, the mass parameter $\mu$ and the oblateness coefficient $A_1$ of the systems under study are presented in Table I. As can be seen from Table II, points $L_2$ and $L_3$ get shifted away from the more massive primary

TABLE I

Systems and their parameters

| S. No. | System | $\mu$ | $A_1$ |
|---|---|---|---|
| 1 | Earth–Moon | 0.012 150 299 4 | 0.000 000 368 6 |
| 2 | Jupiter–Io | 0.000 041 528 3 | 0.000 670 142 1 |
| 3 | Jupiter–Europa | 0.000 025 079 4 | 0.000 264 638 2 |
| 4 | Jupiter–Ganymede | 0.000 080 783 5 | 0.000 104 040 1 |
| 5 | Jupiter–Callisto | 0.000 047 967 7 | 0.000 033 701 7 |
| 6 | Saturn–Mimas | 0.000 000 065 9 | 0.004 234 999 6 |
| 7 | Saturn–Enceladus | 0.000 000 148 0 | 0.002 586 576 7 |
| 8 | Saturn–Tethys | 0.000 001 095 0 | 0.001 683 585 7 |
| 9 | Saturn–Dione | 0.000 002 039 0 | 0.001 030 852 6 |
| 10 | Saturn–Rhea | 0.000 003 200 0 | 0.000 527 543 2 |
| 11 | Saturn–Titan | 0.000 246 129 4 | 0.000 098 115 3 |
| 12 | Saturn–Hyperion | 0.000 000 200 0 | 0.000 066 798 9 |
| 13 | Saturn–Iapetus | 0.000 003 940 0 | 0.000 011 560 6 |
| 14 | Saturn–Phoebe | 0.000 000 052 0 | 0.000 000 876 4 |

TABLE II

Locations of collinear equilibria

| S. No. of the system | With oblateness | | | Without oblateness | | |
|---|---|---|---|---|---|---|
| | $L_1$ | $L_2$ | $L_3$ | $L_1$ | $L_2$ | $L_3$ |
| 1 | −1.155 681 017 9 | −0.836 916 586 1 | 1.005 062 528 6 | −1.155 681 064 7 | −0.836 916 534 0 | 1.005 062 526 6 |
| 2 | −1.024 146 874 7 | −0.976 154 598 5 | 1.000 017 816 2 | −1.024 160 164 5 | −0.976 141 095 2 | 1.000 017 303 4 |
| 3 | −1.020 402 394 0 | −0.979 822 094 9 | 1.000 010 452 8 | −1.020 406 838 3 | −0.979 817 590 2 | 1.000 010 449 7 |
| 4 | −1.030 186 863 3 | −0.970 250 477 6 | 1.000 033 663 6 | −1.030 189 436 4 | −0.970 247 852 6 | 1.000 033 659 8 |
| 5 | −1.025 354 130 3 | −0.974 973 025 7 | 1.000 019 987 3 | −1.025 354 832 0 | −0.974 972 312 1 | 1.000 019 986 5 |
| 6 | −1.002 793 363 8 | −0.997 211 751 2 | 1.000 000 027 6 | −1.002 803 169 6 | −0.997 201 927 6 | 1.000 000 027 5 |
| 7 | −1.003 664 052 8 | −0.996 344 637 5 | 1.000 000 061 8 | −1.003 671 914 9 | −0.996 336 756 5 | 1.000 000 061 7 |
| 8 | −1.007 152 482 3 | −0.992 879 423 5 | 1.000 000 457 1 | −1.007 162 457 4 | −0.992 869 401 3 | 1.000 000 456 2 |
| 9 | −1.008 808 354 9 | −0.991 239 145 8 | 1.000 000 850 5 | −1.008 815 873 1 | −0.991 231 583 7 | 1.000 000 849 6 |
| 10 | −1.010 244 462 4 | −0.989 818 764 5 | 1.000 001 334 1 | −1.010 248 935 2 | −0.989 814 261 3 | 1.000 001 333 3 |
| 11 | −1.043 825 279 6 | −0.956 940 989 4 | 1.000 102 565 0 | −1.043 828 782 7 | −0.956 937 383 4 | 1.000 102 553 9 |
| 12 | −1.004 059 848 9 | −0.995 950 712 6 | 1.000 000 083 3 | −1.004 060 074 3 | −0.995 950 486 6 | 1.000 000 083 3 |
| 13 | −1.010 986 893 6 | −0.989 085 176 9 | 1.000 001 641 7 | −1.010 986 998 7 | −0.989 085 071 0 | 1.000 001 641 7 |
| 14 | −1.002 590 155 4 | −0.997 414 205 7 | 1.000 000 021 7 | −1.002 590 157 3 | −0.997 414 203 8 | 1.000 000 021 7 |

while $L_1$ moves towards it with the inclusion of the oblateness effect. It is noted that the shifts in $L_1$ and $L_2$ due to oblateness are of the same order while in the case of $L_3$ they are negligible. It is further noted that the shifts get reduced with $\mu$ and $A_1$. Though the numerical details have not been mentioned herein, it is observed that at $L_{4,5}$ shifts are comparatively more than those at collinear points.

## 4. Variational Equations and Characteristic Exponents

For studying the motion near any of the equilibrium points $L(a, b)$, the introduction of $\xi = x - a$, $\eta = y - b$ brings the equations of motion in linear analysis, to the form

$$\begin{aligned} \ddot{\xi} - 2n\dot{\eta} &= \Omega_{xx}(a, b)\xi + \Omega_{xy}(a, b)\eta, \\ \ddot{\eta} + 2n\dot{\xi} &= \Omega_{xy}(a, b)\xi + \Omega_{yy}(a, b)\eta, \end{aligned} \tag{13}$$

the characteristic equation of which is

$$\begin{aligned} &\lambda^4 + [4n^2 - \Omega_{xx}(a, b) - \Omega_{yy}(a, b)]\lambda^2 + \\ &\quad + [\Omega_{xx}(a, b)\Omega_{yy}(a, b) - (\Omega_{xy}(a, b))^2] = 0. \end{aligned} \tag{14}$$

At collinear points, we have

$$\begin{aligned} \Omega_{xx} &= n^2 + 2(1 - \mu)/r_1^3 + 2\mu/r_2^3 + 6(1 - \mu)A_1/r_1^5 > 0, \\ \Omega_{xy} &= 0, \\ \Omega_{yy} &= n^2 - (1 - \mu)/r_1^3 - \mu/r_2^3 - 3(1 - \mu)A_1/2r_1^5 < 0. \end{aligned} \tag{15}$$

Consequently

$$\Omega_{xx}\Omega_{yy} - (\Omega_{xy})^2 < 0.$$

It is easy to note that the roots $\lambda_i$ $(i = 1, 2, 3, 4)$ of (14) are

$$\begin{aligned} \lambda_{1,2} &= \pm[-\beta_1 + (\beta_1^2 + \beta_2^2)^{1/2}]^{1/2} = \pm\lambda, \\ \lambda_{3,4} &= \pm[-\beta_1 - (\beta_1^2 + \beta_2^2)^{1/2}]^{1/2} = \pm is, \end{aligned} \tag{16}$$

where

$$\begin{aligned} \beta_1 &= 2n^2 - (\Omega_{xx} + \Omega_{yy})/2, \\ \beta_2^2 &= -\Omega_{xx}\Omega_{yy} > 0. \end{aligned}$$

The general solution of Equations (13) takes the form

$$\xi = \sum_{i=1}^{4} \alpha_i e^{\lambda_i t}, \qquad \eta = \sum_{i=1}^{4} \gamma_i e^{\lambda_i t},$$

and (17)

$$(\lambda_i^2 - \Omega_{xx})\alpha_i = (2n\lambda_i + \Omega_{xy})\gamma_i.$$

TABLE III

Root '$\lambda$' of the characteristic equation

| S. No. of the system | With oblateness | | | Without oblateness | | |
|---|---|---|---|---|---|---|
| | $L_1$ | $L_2$ | $L_3$ | $L_1$ | $L_2$ | $L_3$ |
| 1 | 2.158 677 953 9 | 2.941 682 487 5 | 0.177 873 463 9 | 2.158 676 926 2 | 2.932 052 389 9 | 0.177 873 279 9 |
| 2 | 2.453 936 614 4 | 2.569 748 720 9 | 0.010 460 714 8 | 2.451 616 407 2 | 2.567 197 976 5 | 0.010 440 714 4 |
| 3 | 2.461 170 217 3 | 2.558 944 565 0 | 0.008 119 839 1 | 2.460 247 941 2 | 2.557 937 638 5 | 0.008 113 703 2 |
| 4 | 2.438 218 908 4 | 2.582 616 796 6 | 0.014 566 075 9 | 2.437 862 167 1 | 2.582 164 715 8 | 0.014 561 747 5 |
| 5 | 2.448 997 008 4 | 2.570 314 816 2 | 0.011 222 084 5 | 2.448 880 509 1 | 2.570 154 507 6 | 0.011 221 004 1 |
| 6 | 2.516 730 777 4 | 2.530 356 554 6 | 0.000 420 967 8 | 2.501 563 599 0 | 2.515 041 666 5 | 0.000 415 917 6 |
| 7 | 2.508 748 904 1 | 2.526 517 882 9 | 0.000 627 913 7 | 2.499 488 725 3 | 2.517 139 109 2 | 0.000 623 297 6 |
| 8 | 2.497 187 363 4 | 2.531 732 366 7 | 0.001 703 563 2 | 2.491 192 300 4 | 2.525 585 997 7 | 0.001 695 396 6 |
| 9 | 2.490 946 221 7 | 2.533 375 250 0 | 0.002 320 335 8 | 2.487 284 219 9 | 2.529 598 297 0 | 0.002 313 516 0 |
| 10 | 2.485 778 034 1 | 2.535 021 664 8 | 0.002 902 637 8 | 2.483 907 935 1 | 2.533 081 784 8 | 0.002 898 267 4 |
| 11 | 2.407 512 350 4 | 2.617 021 392 4 | 0.025 423 204 5 | 2.407 183 407 3 | 2.616 464 849 2 | 0.025 416 081 8 |
| 12 | 2.498 802 373 9 | 2.518 319 812 6 | 0.000 724 707 0 | 2.498 562 972 7 | 2.518 076 876 7 | 0.000 724 568 7 |
| 13 | 2.482 213 897 3 | 2.534 923 244 6 | 0.003 216 071 1 | 2.482 172 953 3 | 2.534 878 038 5 | 0.003 215 964 9 |
| 14 | 2.502 076 120 9 | 2.514 530 887 3 | 0.000 369 460 0 | 2.502 072 971 9 | 2.514 527 675 0 | 0.000 369 459 0 |

TABLE IV

Root '$s$' of the characteristic equation

| S. No. of the system | With oblateness | | | Without oblateness | | |
|---|---|---|---|---|---|---|
| | $L_1$ | $L_2$ | $L_3$ | $L_1$ | $L_2$ | $L_3$ |
| 1 | 1.862 648 075 6 | 2.334 384 626 0 | 1.010 419 397 2 | 1.862 647 387 2 | 2.334 383 652 1 | 1.010 419 654 7 |
| 2 | 2.038 614 523 2 | 2.109 081 630 6 | 0.999 533 728 2 | 2.037 179 517 6 | 2.107 569 605 0 | 1.000 036 334 6 |
| 3 | 2.042 977 767 9 | 2.102 496 213 9 | 0.999 823 475 3 | 2.042 408 737 7 | 2.101 901 463 0 | 1.000 021 943 5 |
| 4 | 2.029 077 744 4 | 2.116 977 096 7 | 0.999 992 680 7 | 2.028 856 299 9 | 2.116 740 709 8 | 1.000 070 675 7 |
| 5 | 2.035 595 088 3 | 2.109 456 442 0 | 1.000 016 699 0 | 2.035 522 990 1 | 2.109 380 275 9 | 1.000 041 968 3 |
| 6 | 2.076 762 458 4 | 2.085 026 639 1 | 0.996 818 748 9 | 2.067 501 341 5 | 2.075 709 065 8 | 1.000 000 057 7 |
| 7 | 2.071 894 314 3 | 2.082 688 120 2 | 0.998 058 313 4 | 2.066 238 757 9 | 2.076 987 300 3 | 1.000 000 129 5 |
| 8 | 2.064 861 797 4 | 2.085 864 082 4 | 0.998 737 479 1 | 2.061 192 840 3 | 2.082 137 640 3 | 1.000 000 958 1 |
| 9 | 2.061 060 560 0 | 2.086 872 073 4 | 0.999 228 355 1 | 2.058 817 349 2 | 2.084 585 517 9 | 1.000 001 784 1 |
| 10 | 2.057 912 329 4 | 2.087 883 811 7 | 0.999 607 071 9 | 2.056 765 839 7 | 2.086 711 525 0 | 1.000 002 800 0 |
| 11 | 2.010 539 883 1 | 2.138 031 549 8 | 1.000 141 791 8 | 2.010 333 974 7 | 2.137 805 198 3 | 1.000 215 271 7 |
| 12 | 2.065 821 682 7 | 2.077 706 353 1 | 0.999 950 074 6 | 2.065 675 509 2 | 2.077 558 882 5 | 1.000 000 175 0 |
| 13 | 2.055 737 000 3 | 2.087 833 786 5 | 0.999 994 777 2 | 2.055 711 889 7 | 2.087 808 069 6 | 1.000 003 447 5 |
| 14 | 2.067 813 260 3 | 2.075 397 797 2 | 0.999 999 388 2 | 2.067 811 339 5 | 2.075 395 865 6 | 1.000 000 045 5 |

TABLE V

Eccentricity of the orbits around collinear points

| S. No. of the system | With oblateness | | | Without oblateness | | |
|---|---|---|---|---|---|---|
| | $L_1$ | $L_2$ | $L_3$ | $L_1$ | $L_2$ | $L_3$ |
| 1 | 0.939 212 986 2 | 0.960 342 338 3 | 0.866 072 067 6 | 0.939 212 968 1 | 0.960 342 313 3 | 0.866 071 910 8 |
| 2 | 0.948 610 444 1 | 0.951 814 565 5 | 0.866 315 225 3 | 0.948 572 597 7 | 0.951 774 838 9 | 0.866 025 404 4 |
| 3 | 0.948 835 777 2 | 0.951 543 342 9 | 0.866 139 938 1 | 0.948 820 759 2 | 0.951 527 695 1 | 0.866 025 404 0 |
| 4 | 0.948 179 910 5 | 0.952 177 122 7 | 0.866 070 442 6 | 0.948 174 062 7 | 0.952 170 908 8 | 0.866 025 405 9 |
| 5 | 0.948 495 531 4 | 0.951 855 411 3 | 0.866 039 995 9 | 0.948 493 627 4 | 0.951 853 407 5 | 0.866 025 404 5 |
| 6 | 0.950 230 312 0 | 0.950 605 266 2 | 0.867 845 723 4 | 0.949 988 157 0 | 0.950 361 767 4 | 0.866 025 403 8 |
| 7 | 0.950 078 732 7 | 0.950 569 087 0 | 0.867 140 377 6 | 0.949 930 329 0 | 0.950 419 592 8 | 0.866 025 403 8 |
| 8 | 0.949 794 746 4 | 0.950 749 490 1 | 0.866 752 275 1 | 0.949 698 263 8 | 0.950 651 613 5 | 0.866 025 403 8 |
| 9 | 0.949 647 558 6 | 0.950 821 478 2 | 0.866 470 970 8 | 0.949 588 481 9 | 0.950 761 345 4 | 0.866 025 403 8 |
| 10 | 0.949 523 625 5 | 0.950 887 224 6 | 0.866 253 624 8 | 0.949 493 397 5 | 0.950 856 365 7 | 0.866 025 403 8 |
| 11 | 0.947 276 727 0 | 0.953 069 051 7 | 0.866 067 885 2 | 0.947 271 288 9 | 0.953 063 113 4 | 0.866 025 423 8 |
| 12 | 0.949 908 356 9 | 0.950 449 307 2 | 0.866 054 325 2 | 0.949 904 500 6 | 0.950 445 419 1 | 0.866 025 403 8 |
| 13 | 0.949 445 111 9 | 0.950 905 950 2 | 0.866 030 409 5 | 0.949 444 449 1 | 0.950 905 272 5 | 0.866 025 403 8 |
| 14 | 0.950 002 391 5 | 0.950 347 634 9 | 0.866 025 783 3 | 0.950 002 340 8 | 0.950 347 584 0 | 0.866 025 403 8 |

It may be noted that $\lambda_{1,2}$ are real while $\lambda_{3,4}$ are pure imaginary. Hence the collinear equilibria are unstable in general. However, it is possible to choose the initial conditions $(\xi_0, \eta_0)$ such that $\alpha_{1,2} = 0$ and then Equations (17) represent an ellipse whose eccentricity and semimajor axis are $\sqrt{1-\beta_3^{-2}}$ and $\sqrt{(\beta_3^2\xi_0^2+\eta_0^2)}$ respectively, where

$$\beta_3 = (s^2 + \Omega_{xx})/2ns.$$

The characteristic exponents $\lambda$ and $s$ at $L_i$ $(i=1, 2, 3)$ appearing in (16) for all the 14 systems reported in Table I have been presented in Tables III and IV. It may be noted that $\lambda$ and $s$ increase at all the points $L_i$, except that at $L_3$ $s$ decreases with the inclusion of oblateness effect. The eccentricities of the resulting conditional infinitesimal (linearized) periodic orbits at $L_i$ are presented in Table V. It is noted that the oblateness effect increases the eccentricity at all the points $L_i$ $(i=1, 2, 3)$.

At triangular points $L_4$ and $L_5$, we have

$$\begin{aligned} \Omega_{xx} &= (x - \mu)^2 f + (x + 1 - \mu)^2 g > 0, \\ \Omega_{xy} &= y[(x - \mu) f + (x + 1 - \mu) g], \\ \Omega_{yy} &= y^2(f + g) > 0, \end{aligned} \tag{18}$$

where

$$\begin{aligned} f &= (1 - \mu)(3 + 15A_1/2) > 0, \\ g &= 3\mu/r_2^5 > 0. \end{aligned} \tag{19}$$

The characteristic Equation (14) now takes the form

$$\Lambda^2 + (4n^2 - f - g r_2^2)\Lambda + y^2 f g = 0, \tag{20}$$

where we have replaced $\lambda^2$ by $\Lambda$.

When the discriminant $D$ of the Equation (20) is positive it can be noted that the roots $\Lambda_{1,2}$ satisfy

$$-\tfrac{1}{2} \leqslant \Lambda_1 \leqslant 0, \qquad -\tfrac{1}{2} > \Lambda_2 \geqslant -1,$$

i.e.

$$\begin{aligned} \lambda_{1,2} &= \pm i(-\Lambda_1)^{1/2} = \pm i s_4, \\ \lambda_{3,4} &= \pm i(-\Lambda_2)^{1/2} = \pm i s_5, \end{aligned} \tag{21}$$

showing that the triangular points are linearly stable.

However this belongs to the category of special cases of Lyapunov in which stability problem cannot be solved by considering the first approximation alone as it depends upon the terms of higher powers appearing in the expansions on the right side of (13).

The solution of Equations (13) in this case can be easily seen as consisting of short- and long-period terms with angular frequencies $s_5$ and $s_4$ respectively. With proper selection of initial conditions the long- or short-period terms can be eliminated from the solution, as in Szebehely (1967). In both the cases the motion is along a retrograde

TABLE VI

Characteristic exponents at $L_{4,5}$

| S. No. of the system | With oblateness $s_4$ | With oblateness $s_5$ | Without oblateness $s_4$ | Without oblateness $s_5$ | Angular frequency in the $z$-direction (with oblateness) $s_z$ |
|---|---|---|---|---|---|
| 1 | 0.298 204 690 8 | 0.954 501 662 1 | 0.298 204 329 0 | 0.954 502 057 7 | 1.000 000 822 5 |
| 2 | 0.016 777 394 0 | 0.999 356 487 6 | 0.016 744 635 9 | 0.999 859 798 8 | 1.001 506 643 0 |
| 3 | 0.013 021 972 0 | 0.999 716 705 3 | 0.013 011 924 3 | 0.999 915 341 3 | 1.000 595 248 8 |
| 4 | 0.023 363 933 6 | 0.999 648 984 3 | 0.023 356 840 8 | 0.999 727 191 8 | 1.000 234 050 1 |
| 5 | 0.017 998 195 2 | 0.999 812 741 1 | 0.017 996 425 5 | 0.999 838 051 2 | 1.000 075 823 6 |
| 6 | 0.000 675 219 9 | 0.996 818 461 6 | 0.000 666 952 1 | 0.999 999 777 6 | 1.009 483 777 8 |
| 7 | 0.001 007 057 1 | 0.998 057 674 7 | 0.000 999 500 2 | 0.999 999 500 5 | 1.005 802 959 7 |
| 8 | 0.002 732 061 5 | 0.998 732 778 5 | 0.002 718 692 4 | 0.999 996 304 3 | 1.003 780 917 5 |
| 9 | 0.003 721 067 7 | 0.999 219 636 1 | 0.003 709 903 4 | 0.999 993 118 3 | 1.002 316 731 6 |
| 10 | 0.004 654 769 8 | 0.999 593 429 0 | 0.004 647 615 3 | 0.999 989 199 8 | 1.001 186 266 1 |
| 11 | 0.040 800 574 7 | 0.999 093 695 6 | 0.040 788 878 1 | 0.999 167 787 5 | 1.000 220 698 9 |
| 12 | 0.001 162 121 9 | 0.999 949 224 3 | 0.001 161 895 6 | 0.999 999 325 0 | 1.000 150 286 2 |
| 13 | 0.005 157 256 2 | 0.999 978 030 8 | 0.005 157 082 3 | 0.999 986 702 2 | 1.000 026 010 9 |
| 14 | 0.000 592 454 1 | 0.999 999 167 2 | 0.000 592 452 6 | 0.999 999 824 5 | 1.000 001 971 8 |

TABLE VII

Eccentricities of long- and short-period orbits around $L_{4,5}$

| S. No. of the system | With oblateness Long-period | With oblateness Short-period | Without oblateness Long-period | Without oblateness Short-period |
|---|---|---|---|---|
| 1 | 0.980 843 294 3 | 0.870 862 307 0 | 0.980 843 314 7 | 0.870 862 127 5 |
| 2 | 0.999 937 600 9 | 0.866 328 834 3 | 0.999 937 698 5 | 0.866 038 899 1 |
| 3 | 0.999 962 354 4 | 0.866 148 113 0 | 0.999 962 377 6 | 0.866 033 551 7 |
| 4 | 0.999 878 761 0 | 0.866 096 742 6 | 0.999 878 790 5 | 0.866 051 671 5 |
| 5 | 0.999 928 030 7 | 0.866 055 591 1 | 0.999 928 036 4 | 0.866 040 993 2 |
| 6 | 0.999 999 900 2 | 0.867 845 746 0 | 0.999 999 901 1 | 0.866 025 425 2 |
| 7 | 0.999 999 776 7 | 0.867 140 427 2 | 0.999 999 778 0 | 0.866 025 451 8 |
| 8 | 0.999 998 351 0 | 0.866 752 638 3 | 0.999 998 357 5 | 0.866 025 759 4 |
| 9 | 0.999 996 934 1 | 0.866 471 641 7 | 0.999 996 941 5 | 0.866 026 066 0 |
| 10 | 0.999 995 194 0 | 0.866 254 671 0 | 0.999 995 200 0 | 0.866 026 443 1 |
| 11 | 0.999 630 401 9 | 0.866 148 199 6 | 0.999 630 486 8 | 0.866 105 638 9 |
| 12 | 0.999 999 700 0 | 0.866 054 390 2 | 0.999 999 700 0 | 0.866 025 468 7 |
| 13 | 0.999 994 089 8 | 0.866 031 689 3 | 0.999 994 089 9 | 0.866 026 683 4 |
| 14 | 0.999 999 922 0 | 0.866 025 800 1 | 0.999 999 922 0 | 0.866 025 420 7 |

ellipse whose eccentricity and the orientation of the major axis are independent of the initial conditions. Angular frequencies $s_4$ and $s_5$ with and without oblateness effect for the fourteen systems are provided in Table VI. As can be seen $s_5$ decreases while $s_4$ increases with oblateness effect. Eccentricities of the long- and short-period orbits are presented in Table VII. It is observed that the eccentricity of the short-period orbits increases while that of the long-period orbits decreases with the inclusion of oblateness effect.

However, when $D<0$ the roots of (10) are given by

$$\lambda_{1,2} = \pm[f + gr_2^2 - 4n^2 + i\sigma]^{1/2}/\sqrt{2} = \pm(a_1 + ib_1),$$
$$\lambda_{3,4} = \pm[f + gr_2^2 - 4n^2 - i\sigma]^{1/2}/\sqrt{2} = \pm(a_3 + ib_3),$$

where

$$\sigma = [4y^2fg - (4 - f - gr_2^2)^2]^{1/2},$$

we easily see that $a_1 = a_3 > 0$ and $b_1 = -b_3 > 0$ and it follows that the real parts of two of the four roots are positive and equal and hence the triangular equilibria, in this case, are unstable. However by suitable selection of the initial conditions infinitesimal (linearized) periodic motion can be achieved which approaches the equilibrium point asymptotically.

When $D=0$ two roots of (20) will be equal to the other two which give rise to secular terms in the solution of the variational Equations (13). The equilibria in this case are unstable.

Thus we note that the stability characteristics (linear) of the equilibria are not changed with the inclusion of the oblateness considerations.

## 5. Three-Dimensional Case

Equations of motion in this case are

$$\begin{aligned} \ddot{x} - 2n\dot{y} &= \partial\Omega/\partial x, \\ \ddot{y} + 2n\dot{x} &= \partial\Omega/\partial y, \\ \ddot{z} &= \partial\Omega/\partial z, \end{aligned} \tag{22}$$

where

$$\begin{aligned} \Omega = {} & (n^2/2)(x^2 + y^2) + (1 - \mu)/r_1 + \mu/r_2 + (1 - \mu)A_1/2r_1^3 \\ & - 3(1 - \mu)A_1 z^2/2r_1^5, \end{aligned} \tag{23}$$

with

$$\begin{aligned} r_1^2 &= (x - \mu)^2 + y^2 + z^2, \\ r_2^2 &= (x + 1 - \mu)^2 + y^2 + z^2. \end{aligned} \tag{24}$$

Jacobi's integral is

$$\dot{x}^2 + \dot{y}^2 + \dot{z}^2 = 2\Omega - C. \tag{25}$$

TABLE VIII

Angular frequency $s_z$ at collinear equilibria

| S. No. of the system | With oblateness | | | Without oblateness | | |
|---|---|---|---|---|---|---|
| | at $L_1$ | at $L_2$ | at $L_3$ | at $L_1$ | at $L_2$ | at $L_3$ |
| 1 | 1.786 178 535 5 | 2.275 032 718 1 | 1.005 332 142 4 | 1.786 177 701 9 | 2.268 828 812 8 | 1.005 331 301 2 |
| 2 | 1.966 689 136 4 | 2.038 950 635 2 | 1.001 524 926 8 | 1.964 770 373 5 | 2.036 824 933 8 | 1.000 018 168 8 |
| 3 | 1.970 886 753 0 | 2.031 860 377 9 | 1.000 606 248 6 | 1.970 123 468 4 | 2.031 023 191 0 | 1.000 010 972 3 |
| 4 | 1.956 544 676 9 | 2.046 578 572 4 | 1.000 269 428 4 | 1.956 249 999 4 | 2.046 211 960 5 | 1.000 035 343 5 |
| 5 | 1.963 170 927 2 | 2.038 805 346 4 | 1.000 096 816 4 | 1.963 074 605 0 | 2.038 678 259 3 | 1.000 020 986 1 |
| 6 | 2.008 406 456 9 | 2.016 946 478 0 | 1.009 483 807 7 | 1.995 810 272 2 | 2.004 212 179 0 | 1.000 000 028 8 |
| 7 | 2.002 207 467 8 | 2.013 321 207 7 | 1.005 803 026 0 | 1.994 517 807 3 | 2.005 520 642 9 | 1.000 000 064 7 |
| 8 | 1.994 327 265 7 | 2.015 908 899 2 | 1.003 781 404 1 | 1.989 352 445 5 | 2.010 792 748 9 | 1.000 000 479 1 |
| 9 | 1.989 958 543 6 | 2.016 443 443 1 | 1.002 317 632 3 | 1.986 920 711 3 | 2.013 298 478 3 | 1.000 000 892 1 |
| 10 | 1.986 371 543 5 | 2.017 090 265 4 | 1.001 187 673 1 | 1.984 820 624 3 | 2.015 474 718 2 | 1.000 001 400 0 |
| 11 | 1.937 560 177 7 | 2.068 201 862 1 | 1.000 328 486 9 | 1.937 289 154 0 | 2.067 771 080 4 | 1.000 107 688 0 |
| 12 | 1.994 140 034 1 | 2.006 307 814 2 | 1.000 150 373 7 | 1.993 941 227 6 | 2.006 105 741 6 | 1.000 000 087 5 |
| 13 | 1.983 775 667 6 | 2.016 634 276 4 | 1.000 027 734 8 | 1.983 741 716 2 | 2.016 597 167 1 | 1.000 001 723 7 |
| 14 | 1.996 130 222 2 | 2.003 894 235 5 | 1.000 001 994 6 | 1.996 127 606 4 | 2.003 891 571 3 | 1.000 000 022 7 |

The singularities of the manifold of the state of motion are obtained from the equations

$$\dot{x} = 0 = \dot{y} = \dot{z}, \qquad \Omega_x = 0 = \Omega_y = \Omega_z .$$

It can be proved with some effort that here too the five points of libration are in the $xy$-plane located at exactly the same places as in the planar case. As such at the equilibria $\Omega_{xz}=0=\Omega_{yz}$ and the variational equations in the linear analysis become

$$\begin{aligned} \ddot{\xi} - 2n\dot{\eta} &= \Omega^0_{xx}\xi + \Omega^0_{xy}\eta , \\ \ddot{\eta} + 2n\dot{\xi} &= \Omega^0_{xy}\xi + \Omega^0_{yy}\eta , \\ \ddot{\zeta} &= \Omega^0_{zz}\zeta , \end{aligned} \tag{26}$$

where $x=a+\xi$, $y=b+\eta$, $z=c+\zeta$ and the superscript '0' indicates that the second derivatives are to be evaluated at the points $L_i(a, b, c)$. The motion, therefore, in the $xy$-plane does not influence the motion in the $z$-direction at the five libration points. At the collinear points, since we have

$$\Omega_{xx} + \Omega_{yy} + \Omega_{zz} = 2n^2,$$

the mean motion in the $z$-direction is

$$s_z = (-\Omega_{zz})^{1/2} = (\Omega_{xx} + \Omega_{yy} - 2n^2)^{1/2}.$$

However, at the triangular points

$$s_z = [n^2 + 3A_1(1 - \mu)]^{1/2} > n,$$

the mean motion of the primaries.

The angular velocities $s_z$ at the three collinear points are listed for the 14 systems in Table VIII while those at the triangular points are presented in the last column of Table VI. At $L_1$ and $L_2$, $s_z$ is less than the corresponding value of $s$ even after the inclusion of oblateness effect and in fact become much closer than in the unperturbed case. $s_z$ increases at all the points $L_i$ $(i=1,\ldots, 5)$ after the inclusion of oblateness effect.

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