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Braids and invariants of 3-manifolds

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Introduction

It is shown that multiplicative invariants of closed 3-manifolds correspond to special trace functionals on the infinite ribbon braid group in the same way as link invariants correspond to special traces on the infinite braid group. Using finite fusion representation rings, which are derived from the K-theory of "nice" braid representations, we obtain invariants of 3-manifolds as a "thermodynamical" limit of cablings of invariants of framed links; the limit can be interpreted as the left regular representation in the representation ring. Applying this to special versions of the Jones polynomial and its generalizations, one obtains a general, simple proof of the existence of invariants of 3-manifolds (predicted by Witten) for loop groups for classical compact Lie groups; they coincide with the invariants constructed using the corresponding quantum groups. In particular, various constants such as the predicted framing anomalies and values for simple 3-manifolds can be checked explicitly. This approach can also be extended to the corresponding projective groups.

As in the work of Reshetikhin and Turaev [RT2] and related work (e.g. [B1], [Li2], [MS], [KM], [TW], [Wa]) invariants of unoriented framed links are constructed, which are invariant under Kirby moves. This yields, by well-known surgery theorems, an invariant of 3-manifolds. The difference between this and their approaches lies in the way this invariant is found:

We construct from a given invariant $\mathscr{L}$ of framed links a sequence of invariants $\mathscr{L}^{\vec{c}}$ via cabling; this means, roughly speaking, if L is a framed link with s components and if $\vec{c} \in \mathbf{N}^s$, the invariant $\mathscr{L}^{\vec{c}}(L)$ is obtained by evaluating $\mathscr{L}$ at the framed link obtained from L by replacing the i-th component of L by c_i parallel components, where "parallel" is determined by the framing (see Section 2). Our observation, slightly oversimplified, now is that under nice conditions, which will be made precise below, suitably normalized cabled invariants will converge to a limiting invariant which is invariant under Kirby moves if the number of cables goes to infinity.

Let us briefly sketch the required conditions. It was observed in [GW] that one can define a ring structure on $K(\rho)$, the K-theory associated to a braid

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representation ρ whose restrictions to finite braid groups are finite dimensional and semisimple. We show that under certain conditions (essentially that the Bratteli inclusion diagram for $\ldots \rho(CB_n) \subset \rho(CB_{n+1}) \ldots$ has period d for some $d \in \mathbf{N}$) this ring has an interesting finite quotient ring $\bar{K}(\rho)$; it was shown in [GW] that for Hecke algebra representations with q a root of unity one obtains the famous so-called Verlinde algebras for Wess-Zumino-Witten models for unitary loop groups. Assume now that ρ is a braid representation derived from the invariant of framed links $\mathscr{L}$ such that $\bar{K}(\rho)$ is finite. Then each of the invariants $\mathscr{L}^{\vec{c}}$ can be decomposed into a linear combination of simpler "irreducible" invariants according to the structure coefficients of the fusion ring $K(\rho)$. It will be shown that by changing the framing anomaly (which is the fixed scalar by which $\mathscr{L}(K)$ changes if the framing of the knot K is increased by 1), if necessary, one can assume that this decompostion is actually well-defined in the finite quotient $\bar{K}(\rho)$, i.e. only finitely many irreducible invariants occur in all these decompositions. As an example, the change of framing anomaly for the Jones polynomial will yield the Kauffman bracket polynomial.

Under additional positivity assumptions (essentially that ρ is unitarizable) and non-singularity conditions (namely that $0 \neq |\mathrm{tr}(\sigma_1)| = \Theta^{-1}$ and that a second constant C depending on $\mathscr{L}$ is well-defined, see the beginning of Section 2.5 for details) we obtain an invariant $\mathscr{L}^{(\infty)}$ of framed 3-manifolds as an average over all possible cablings; more precisely, it is given by the formula

$$\mathscr{L}^{(\infty)}(M(L)) = \lim_{N \to \infty} \frac{1}{N^s} \sum_{\vec{c} \in \{1, 2, \ldots N\}^s} C^s \Theta^{-|\vec{c}|} \mathscr{L}(L^{\vec{c}})$$

provided it does not depend on the choice of orientations in L; here s is the number of components of L, $|\vec{c}|$ is the sum of the coordinates of $\vec{c}$ and $M(L)$ is the 3-manifold obtained from Dehn surgery at L. One obtains from $\mathscr{L}^{(\infty)}$ an invariant $\mathscr{F}$ of 3-manifolds by a now standard renormalization (see [RT2], [Wa], [Li2], [KM]). Although defined by a limit, the invariant $\mathscr{F}$ can be written down explicitly as a linear combination of irreducible invariants of framed links; it can also be expressed in terms of a linear combination of finitely many cablings as it was done in the sl_2 case by Kirby and Melvin. The advantage of the limit definition is that the invariance under Kirby moves follows from a simple Perron-Frobenius argument.

This approach can be used to construct invariants of 3-manifolds for all these specializations of the Jones polynomial and its generalizations, the HOMFLY and the Kauffman polynomial, for which the associated braid representations are unitarizable. We show that these invariants coincide with the ones constructed in [RT2] and [TW], where quantum deformations of the classical Lie algebras were used. It is also possible to obtain the invariants corresponding to projective groups by the same procedure if one requires the number of cables to be divisible by the order of the center of the given compact group, except that we have not checked whether the constant C is well-defined in that case.

This paper is organized as follows: In the first section we establish various correspondences between invariants of framed links and 3-manifolds and special trace functionals on ribbon braid groups. In particular, one obtains from these functionals representations $\rho = \rho_{\mathrm{tr}}$ of ribbon braid groups. In Sect. 2.1 the definitions of the rings $K(\rho)$ and $\bar{K}(\rho)$ are recalled from [GW] and are slightly expanded. Section 2.2 reviews how cabled invariants can be decomposed into "irreducible" invariants according to the structure coefficients of $K(\rho)$ (these results have already more or less appeared in [M2], [Re] and [W2]). In the special case of a framed knot L, it says that the invariant of the c-cabling of L can be written as

$$\mathscr{L}(L^{(c)}) = \sum_{\lambda} a_{\lambda}^{(c)} \mathscr{L}^{\lambda}(L)$$

where λ runs through an index set Λ labelling a basis of $\bar{K}(\rho)$ and $a_{\lambda}^{(c)}$ is the dimension of a simple $\rho(\mathbf{C}B_c)$ module corresponding to λ; it coincides with the coefficient of Φ_λ in the expansion of $\Phi_1^{\otimes f}$, where Φ_1 is the basis element corresponding to the generating representation (or element) in $K(\rho)$. Similar formulas also hold for an s-component link L, where one obtains an irreducible invariant for each assignment of an element Λ_{c_i} to the i-th component of L, $i = 1, 2, \ldots s$, (see Theorem 2.2).

In Sect. 2.3 it is shown that a link invariant can be made rational by choosing a certain framing anomaly if the corresponding braid representation has a periodic Bratteli diagram (see Sect 2.1) and it has a projection p with the contraction property; this means that there exists $d \in \mathbf{N}$ and $p \in \rho(\mathbf{C}B_d)$ such that

$$p\rho(\mathbf{C}B_\infty)p = p\rho(\mathbf{C}B_{d+1,\infty}) \cong \rho(\mathbf{C}B_{d+1,\infty}).$$

In the quantum group picture, p would correspond to a projection onto the trivial representation. Sections 2.4 and 2.5 contain a proof of the invariance of $\mathscr{F}$, defined as above, under the Kirby moves.

In the third section, it is shown that our approach can be applied to unitarizable specializations of the HOMFLY and Kauffman polynomials. Moreover, we show that these invariants are the same as the ones obtained in [RT2] and [TW] for quantizations of classical Lie algebras. This result was also recently obtained by Turaev independently. As an interesting (possibly known) side result, we exhibit the connection between one of the conditions for the universal R-matrix and its behaviour under cabling (Sect. 3.3). We also compute the framing anomaly α_λ for $\mathscr{L}^{\lambda}$ (Proposition 3.2.3) and the constant C, whose absolute value is equal to $\mathscr{F}(S^1 \times S^2)$ (Theorem 3.4). The latter is equal to $1/s_{00}$, the (0, 0)-entry of the so-called S-matrix (see [KP]), as predicted by Witten. We also obtain the spectral decomposition of a full twist in the corresponding representation of the ribbon braid group and a proof of a periodicity phenomenon first observed numerically by Freed and Gompf. In the concluding remarks, we discuss connections between our approach and various other approaches, computability questions and applications.

§1. Topological invariants and functionals on (ribbon) braid groups

For our approach, it will be useful to consider the following simple correspondences between invariants of links (resp. of framed links or of 3-manifolds) and special functionals on the infinite braid groups and ribbon braid groups respectively. The following proposition follows from Jones' discovery of link invariants [J2] and its easy converse [W2], Proposition 3.

Proposition 1.1. *There exists a* 1–1 *correspondence between multiplicative link invariants* $\mathscr{L}$ *(i.e. invariants* $\mathscr{L}$ *such that* $\mathscr{L}(L_1 \cup L_2) = \mathscr{L}(L_1)\mathscr{L}(L_2)$, *where* $L_1 \cup L_2$ *is the disjoint union of the links* L_1 *and* L_2) *such that* $\mathscr{L}$(unknot) $= z \neq 0$ *and local*

Markov traces tr *on the infinite braid group* B_∞, *i.e. functionals* tr: $\mathbf{C}B_\infty \to \mathbf{C}$ *such that*

(M1) $$\mathrm{tr}(\alpha\beta) = \mathrm{tr}(\beta\alpha) \quad \textit{for all}, \ \alpha, \ \beta \in B_\infty$$

(M2) $$\mathrm{tr}(\beta\sigma_n^{\pm 1}) = \mathrm{tr}(\beta)\mathrm{tr}(\sigma_n) \quad \textit{for} \ \beta \in B_n$$

(M3) $$\mathrm{tr}(\alpha\beta) = \mathrm{tr}(\alpha)\mathrm{tr}(\beta) \quad \textit{for disjoint braids} \ \alpha, \ \beta \in B_\infty.$$

Let $(\hat{\beta}, n)$ be the closure of the n-braid β (see [Bi]); then the correspondence is given by the formula

$$\mathrm{tr}(\beta) = \frac{\mathscr{L}(\hat{\beta}, n)}{(\mathscr{L}(\text{unknot}))^n}.$$

Let for $f \in \mathbf{N}$ the element $\sigma_i^{(f)}$ be given by replacing each string of σ_i by f parallel strings (see below)

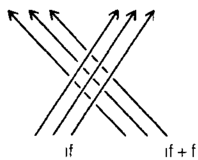


Fig. 1.

Obviously the map $\sigma_i \mapsto \sigma_i^{(f)}$, $i = 1, 2, \ldots, n-1$ extends to an injective homomorphism from B_n into B_{nf}. In order to deal with framed links it will be useful to deal with ribbon braid groups. They can be thought of as braid groups where the strands have been replaced by ribbons which can be twisted. Algebraically, the ribbon braid group RB_n is isomorphic to the semidirect product of $\mathbf{Z}^n$ by B_n, i.e. $RB_n \cong \mathbf{Z}^n \rtimes B_n$, where B_n acts by permutations on $\mathbf{Z}^n$ and $\mathbf{Z}^n$ indicates the number of full twists for each of the n ribbons. The positive full twist of the i-th ribbon will be deonted by τ_i.

It is easy to see that one can obtain any framed link as closure of a ribbon braid. Indeed, using Alexander's theorem, one finds, for a given framed link, a ribbon braid whose closure gives the same link though not necessarily with the same framing; but the framing can be easily adjusted by adding τ_i's (or their inverses) at appropriate places. It is also interesting to observe that full-twists are not necessary for obtaining all the framed links as closures of ribbon braids. Just observe that τ_i is topologically equivalent to the half closure of σ_i (see below).

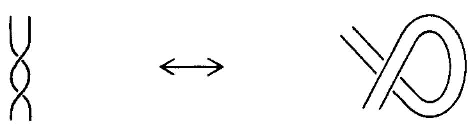


Fig. 2.

It is also easy to formulate the corresponding Markov moves for framed links. First observe that conjugation by a braid (the first Markov move) does not change the framing. The second Markov move for framed links is given by

(RM2) $$(\beta, n) \leftrightarrow (\beta\tau_n^{-1}\sigma_n, n+1) \leftrightarrow (\beta\tau_n\sigma_n^{-1}, n+1).$$

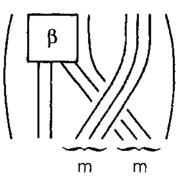
(RM2) is an algebraic rephrasing of what is known to topologists as the C move. Observe that (RM2) does not change the framing of the closure of the corresponding ribbon braids and that it corresponds to the second Markov move in the quotient $B_n \cong RB_n/T_n$. It is an easy corollary of Markov's theorem that 2 ribbon braids give as closure the same framed link if and only if they are equivalent via the moves (RM1) (conjugation by a ribbon braid) and (RM2). One can now formulate similar correspondences as in Proposition 1.1. In the following we shall denote the unknot with 0-framing by 0 – unknot.

Proposition 1.2. (a) *There exists a* 1–1 *correspondence between multiplicative invariants of framed links* $\mathscr{L}$ *such that* $\mathscr{L}(0 - \text{unknot}) \neq 0$ *and local Markov traces* tr *on* RB_∞ *(where* (M2) *is replaced by* $\mathrm{tr}(\beta\sigma_n) = C_0^{-1}\,\mathrm{tr}(\beta\tau_n)$*, for all* $\beta \in RB_n$ *with* $C_0^{-1} = (\mathrm{tr}(\sigma_n)/\mathrm{tr}(\tau_n))$*). It is given by*

$$\mathrm{tr}(\beta) = \frac{\mathscr{L}(\hat{\beta}, n)}{\mathscr{L}(0 - \text{unknot})^n} \qquad (*)$$

where $\mathscr{L}(0 - \text{unknot}) = C_0 = \mathrm{tr}(\tau_i)/\mathrm{tr}(\sigma_i)$.

(b) *There exists a* 1–1 *correspondence between multiplicative invariants* $\mathscr{F}$ *of closed connected oriented* 3*-manifolds with* $\mathscr{F}(S^1 \times S^2) = C_0 \neq 0$ *(multiplicative here means that* $\mathscr{F}(M_1 \# M_2) = \mathscr{F}(M_1)\mathscr{F}(M_2)$*) and local Markov traces on* RB_∞ *satisfying for any* $\beta \in RB_n$ *and any* $m \leq n + 1$

tr 

$= \mathrm{tr}(\beta)C_0^{-m-1} = \mathrm{tr}(\beta)\mathrm{tr}(\sigma_1^{(m)})$

Fig. 3.

and

$$\text{(K2)} \qquad \mathrm{tr}(\sigma_1) = \mathrm{tr}(\sigma_1^{-1}) = 1/C_0^{-2}$$

and such that the corresponding invariant of framed links does not depend on the choice of orientation of the links. The correspondence is given by

$$\mathrm{tr}(\beta) = \frac{\mathscr{F}(M(\beta, n))}{\mathscr{F}(S^1 \times S^2)^n} \qquad (**)$$

where $\mathscr{F}(S^1 \times S^2) = C_0 = \mathrm{tr}(\tau_1)/\mathrm{tr}(\sigma_1)$.

Proof. For (a) this is basically the same proof as the one for Proposition 1.1. Assuming the existence of $\mathscr{L}$ as described above it is easy to see that tr is well-defined independently of the embedding of $\beta \in RB_n \subset RB_{n+1}$. Indeed, if $\beta \in RB_n$ is viewed as an $n + 1$-ribbon braid, one has

$$(\hat{\beta}, n + 1) = (\hat{\beta}, n) \cup (0 - \text{unknot}).$$

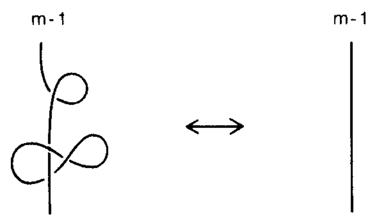
Hence $(*)$ gives the same formula in both cases, using the multiplicativity of $\mathscr{L}$. Observe that $(\hat{\tau}_1, 1)$ and $(\hat{\sigma}_1, 2)$ are both the (1 – unknot). Hence one obtains from $(*)$

$$\operatorname{tr}(\tau_1) = \frac{\mathscr{L}(1-\text{unknot})}{\mathscr{L}(0-\text{unknot})} \quad \text{and} \quad \operatorname{tr}(\sigma_1) = \frac{\mathscr{L}(1-\text{unknot})}{\mathscr{L}(0-\text{unknot})^2}.$$

The expression for $\mathscr{L}(0-\text{unknot})$ follows from these 2 formulas. On the other hand, if we have a local Markov trace tr on the ribbon braid groups, we define $\mathscr{L}$ by solving for the numerator in $(*)$ and replacing $\mathscr{L}(0-\text{unknot})$ by $\operatorname{tr}(\tau_i)/\operatorname{tr}(\sigma_i)$. The invariance of $\mathscr{L}$ under (RM1) and (RM2) follows now immediately from the conditions on tr.

For (b) let $\mathscr{F}$ be a multiplicative invariant of 3-manifolds as in the statement. By the already mentioned surgery theorem, one obtains from this an invariant $\mathscr{L}$ of framed links by $\mathscr{L}(L) = \mathscr{F}(M(L))$, where $M(L)$ is the manifold obtained from surgery at L. Recall the well-known fact that $M(0-\text{unknot}) = S^1 \times S^2$. Hence one obtains from $(**)$ a well-defined trace on RB_∞ as in (a) with $\mathscr{F}(S^1 \times S^2) = \operatorname{tr}(\tau_1)/\operatorname{tr}(\sigma_1)$. By Kirby calculus, in the version of Fenn and Rourke (further simplified by Turaev), $\mathscr{L}$ gives the same value for framed links which are the same except in a small area where they look as sketched below (where $m-1$ indicates that we have $m-1$ parallel ribbons and all the ribbons are supposed to be parallel to the plane of the paper (this is sometimes referred to as "blackboard framing"). It is easy to check, using $(**)$, that the first picture implies the first equality in (K1) and the second picture implies (K2) (where we use the identity $\mathscr{F}(S^3) = 1$; see the corollary below). Also observe that by our first picture $M(\widehat{\sigma_1^{(m)}}, 2m) = M(O^{m-1})$, where the symbol O^{m-1} denotes $m-1$ unlinked unknots, each with 0-framing. The equality $\operatorname{tr}(\sigma_1^{(m)}) = C_0^{-m-1}$ follows now directly from the definition of tr.

To prove the other direction, it suffices to show that any application of Kirby-Fenn-Rourke moves corresponds to replacing the closure of a ribbon braid β by the closure of the braid on the right hand side in Fig. 3. This can be accomplished by conjugating β by an appropriate ribbon braid (which does not change the closure) such that the relevant ribbons are at the end.



or

Fig. 4.

Corollary. *Let* $\mathscr{F}$ *and* tr *be as in Proposition 1.2b. Then*
(a) $\mathscr{F}(S^3) = 1$
(b) $\mathscr{F}(S^1 \times S^2) = 1/\mathrm{tr}(\tau_1)$.

Proof. Observe that $M \# S^3$ is homeomorphic to M for any 3-manifold M. Hence, by multiplicativity of $\mathscr{F}$, one has $\mathscr{F}(S^3) = 1$. It follows from this and the fact that S^3 is obtained by surgery along $(\hat{\tau}_1, 1) = (\hat{\sigma}_1, 2)$ that $\mathrm{tr}(\tau_1) = 1/\mathscr{F}(S^1 \times S^2)$, which shows (b).

Remarks. 1. Conditions (K1) and (K2) are algebraic reformulations of the Kirby moves using the versions of Fenn and Rourke.

2. A very common way of assigning a framing number to a link projection is to assume paper (or blackboard) framing; determine whether a crossing in the link diagram is positive or negative according to the right hand rule (e.g. the crossings in Fig. 1 are all positive; the crossing in σ_1^{-1} would be negative (see also e.g. [Ka])). The framing number of a component of a link is then given by the number of positive selfintersections – number of negative selfintersections (i.e. we only count intersections if both pieces of the string belong to the same component). Also observe that if a knot K is the closure of a braid $\beta = \prod \sigma_{i_j}^{n_j}$, the framing number of K is equal to the exponent sum $e(\beta) = \sum_j n_j$.

3. There is an easy way of producing invariants of framed links from link invariants via some rescaling; the opposite procedure was used by Kauffman to derive the Jones polynomial from his bracket polynomial and also for the derivation of the Kauffman polynomial (see [Ka]). We shall need this procedure later, so we describe it here:

Let $\mathscr{L}$ be a link invariant with Markov trace tr and let $\alpha \in \mathbf{C}$, $\alpha \neq 0$. Then we obtain an invariant $\mathscr{L}_\alpha$ of framed links by $\mathscr{L}_\alpha(\hat{\beta}, n) = \alpha^{e(\beta)} \mathscr{L}(\hat{\beta}, n)$. It is easy to see that whenever one changes the framing of a component of a link by ± 1, its $\mathscr{L}_\alpha$-value changes by $\alpha^{\pm 1}$ (just observe that the Markov move (M2) changes the framing number by ± 1).

On the other hand, we call an invariant $\mathscr{L}$ of framed links an invariant with framing anomaly α if, whenever one changes the framing of one of its component by ± 1, then its $\mathscr{L}$-value changes by $\alpha^{\pm 1}$. We leave it to the reader to check that $\mathscr{L}_{\alpha^{-1}}$ is a link invariant. As $\mathscr{L}$ is essentially already a link invariant, we shall also call $\mathscr{L}$ a *link invariant with framing anomaly* α, although strictly speaking it is not an invariant of links. The following statements can also be checked easily (see also Lemma 2.1.1):

If ρ is the representation of RB_∞ associated to the link invariant $\mathscr{L}$ with framing anomaly α, we have $\rho(\tau_1) = \alpha 1$ and $\alpha = \mathrm{tr}(\sigma_1)\mathscr{L}(0 - \text{unknot})$.

§2. Invariants of 3-manifolds via cabling

2.1. Bratteli diagrams and K-theory for braid representations. We first need some notation for the description of increasing sequences of finite dimensional complex semisimple algebras. Let M_n be the ring of all complex $n \times n$ matrices. Observe that if A and B are semisimple complex algebras, we can write them as $A = \bigoplus A_\lambda$ and $B = \bigoplus B_\mu$ with $A_\lambda \cong M_{a_\lambda}$ and $B_\mu \cong M_{b_\mu}$ for appropriate natural numbers a_λ and b_μ and λ and μ running through index sets labelling the simple components of A and B. The vector $\vec{a}$ whose components are the numbers a_λ is called the *dimension* *vector* of A. The rank of an idempotent $e \in M_n$ is equal to $\mathrm{Tr}(e)$, where Tr is the usual trace on M_n, i.e. the sum of the diagonal elements. If e is an idempotent in the algebra A as described above, we define its rank $K(e)$ (more precisely, it should be $K_0(p)$) to be equal to the vector

$$K(e) = (\mathrm{Tr}_\lambda(e))_\lambda$$

where Tr_λ is the functional which is the usual trace on $A_\lambda \cong M_{a_\lambda}$ and 0 on the other simple components. Observe that if $p_\lambda \in A_\lambda$ is a minimal idempotent, we obtain

$$K(p_\lambda) = e_\lambda = (0, \ldots, 0, 1, 0, \ldots, 0),$$

where the 1 is in the slot labelled by λ. Hence the $\mathbf{Z}$-linear span of $\{K(e), e$ an idempotent in $A\}$ is equal to $\mathbf{Z}^s$, where s is the number of simple components of A and its $\mathbf{N}$-linear span is equal to $\mathbf{N}^s$. These 2 sets (i.e. $\mathbf{Z}^s$ and $\mathbf{N}^s$), denoted by $K(A)$ resp. $K_+(A)$ are called the K-theory of A. Let p be an idempotent in A. Then the set $pAp = \{pap, a \in A\}$ is a subalgebra of A. We leave it to the reader to check (e.g. by writing p in diagonal form) that

$$\overrightarrow{\dim}\, pAp = K(p)$$

where $\overrightarrow{\dim}\, pAp$ is the dimension vector of pAp. If A is a subalgebra of B, any simple B_μ module V_μ is an A module. Let $g_{\mu\lambda}$ be the number of simple A_λ modules in its decomposition into simple A modules. The matrix $G = (g_{\mu\lambda})$ is called the inclusion matrix for $A \subset B$. It is also easy to check that the column $(g_{\mu\lambda})_\mu$ of the inclusion matrix is equal to the dimension vector of $p_\lambda B p_\lambda$, where p_λ is a minimal idempotent in A_λ. Hence it follows from the previous formula that the columns of G are given by

$$(g_{\mu\lambda})_\mu = K(p_\lambda)$$

where p_λ is a minimal idempotent in A_λ. The inclusion of A in B is conveniently described by an inclusion diagram. This is a graph whose vertices are arranged in 2 lines. In one line, the vertices are in 1–1 correspondence with the simple direct summands A_λ of A, in the other one with the summands B_μ of B. Then a vertex corresponding to A_λ is joined with a vertex corresponding to B_μ by $g_{\mu\lambda}$ edges. If A and B have the same identity, there is an easy way of computing the dimension b_μ of a simple B_μ module. We just add up all the dimensions of simple A_λ modules for those λ's which are joined to μ by edges (with multiplicities), or in matrix notation.

$$\vec{b} = G\vec{a}.$$

Examples for inclusion diagrams will be given in Sect. 3. It was observed in [GW] that for sufficiently nice braid representations one can define a multiplication on their K-theory whose structure coefficients are closely related to the entries of inclusion matrices. More precisely, let ρ be a *locally finite* braid representation; this means that $\rho(\mathbf{C}RB_n)$ is finite dimensional and semisimple for all $n \in \mathbf{N}$, i.e. one has

$$\rho(\mathbf{C}RB_n) \cong \bigoplus_{\lambda \in \Lambda_n} M_{a_\lambda}$$

where Λ_n is a finite index set and $a_\lambda \in \mathbf{N}$. The Bratteli diagram of ρ is the sequence of inclusion diagrams $\rho(\mathbf{C}RB_0) \subset \rho(\mathbf{C}RB_1) \subset \rho(\mathbf{C}RB_2) \subset \ldots$, with the vertices standing for the simple components of $\rho(\mathbf{C}RB_n)$ being in the $n+1$-st line. In this context we define the K-theory $K(\rho)$ of our representation as

$$K(\rho) = \bigoplus_{n \geq 0} K(\rho(\mathbf{C}RB_n))$$

where $K(\rho(\mathbf{C}B_0))$ is defined to be equal to $\mathbf{Z}$. To define a multiplication on $K(\rho)$ we use the fact that the algebraic tensor product $\mathbf{C}RB_n \otimes \mathbf{C}RB_m$ can be embedded into $\mathbf{C}RB_{n+m}$ by juxtaposition of the braids (see picture below or [RT1], [FY] or [JS])

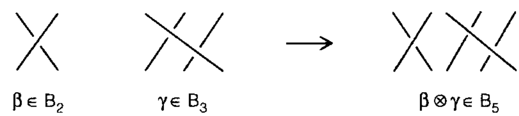


Fig. 5.

This extends to natural embeddings of $\mathbf{C}RB_f^{\otimes n}$ into $\mathbf{C}RB_{nf}$ and to embeddings of finitely many finite ribbon braid groups into $\mathbf{C}RB_\infty$. We define now a multiplication, respecting the natural gradation, by

$$e_\lambda \times e_\mu \in K(\rho(\mathbf{C}RB_n)) \times K(\rho(\mathbf{C}RB_m)) \mapsto K(p_\lambda \otimes p_\mu) \in K(\rho(\mathbf{C}B_{n+m}))$$

where p_λ and p_μ are minimal idempotents in $\rho(\mathbf{C}RB_n)_\lambda$ and $\rho(\mathbf{C}RB_m)_\mu$, i.e. elements in the inverse image of e_λ and e_μ. The structure coefficients $c_{\lambda\mu}^\nu$ of this multiplication are defined by

$$e_\lambda \times e_\mu = \sum_\nu c_{\lambda\mu}^\nu e_\nu$$

where $\nu \in \Lambda_{n+m}$. We have the following

Proposition 2.1.1. (a) *The multiplication above is* $\mathbf{Z}$ *linear and makes* $K(\rho)$ *into an abelian, associative ring.*

(b) *If* 1_r *is the identity in* $\rho(\mathbf{C}RB_r)$, *the multiplication by* $K(1_r)$ *is given by the infinite matrix* G^r, *where we take the natural basis* (e_λ) *of* $K(\rho)$ *as described above, and where* G *is given by*

$$G = \begin{pmatrix} 0 & & & & \\ G_0 & 0 & & & \\ & G_1 & 0 & & \\ & & G_2 & 0 & \\ & & & & \ddots \end{pmatrix}$$

with G_n *being the inclusion for* $\rho(\mathbf{C}RB_n) \subset \rho(\mathbf{C}RB_{n+1})$.

Proof. The first statement has been shown in [GW]. For the second statement, obeserve that for any $p_\lambda \in \rho(\mathbf{C}RB_n)$ the element $p_\lambda \otimes 1_r$ is just the usual embedding of p_λ in $\rho(\mathbf{C}RB_{n+r})$. So for $r = 1$, we obtain

$$K(p_\lambda \otimes 1_1) = (g_{\lambda\mu})_{\mu \in \Lambda_{n+1}} \in K(\rho(\mathbf{C}RB_{n+1})).$$

This shows the claim.

An example for such a ring would be the ring of symmetric functions in d variables, i.e. the representation ring of the unitary group $U(d)$. It can be obtained from the representation ρ_V of the symmetric group S_∞, defined by permuting the factors in $\otimes^\infty$ End V with dim $V = d$. The representation ring of $SU(d)$ appears as quotient of $K(\rho_V)$, defined by the ideal generated by $(1 - K(p_{[1^d]}))$, where $p_{[1^d]}$ is the central idempotent belonging to the Young diagram consisting of 1 column with d boxes. The corresponding representation on the tensor space is of course the determinant representation (see also Sect. 3 for more details).

By definition, the ring $K(\rho)$ is infinite dimensional. In special cases, however, it will be possible to reduce these rings to finite dimensional rings by factoring over an ideal similarly to what was just sketched above. We say that an idempotent $p \in \rho(\mathbf{C}B_d)$ has the *contraction property* if

$$p\rho(\mathbf{C}B_n)p \cong p\rho(\mathbf{C}B_{d+1,n}) \cong \rho(\mathbf{C}B_{d+1,n}) \quad \text{for all } n > d$$

where $B_{d+1,n}$ is the subgroup of B_n generated by $\sigma_{d+1}, \sigma_{d+2} \dots \sigma_{n-1}$.

Lemma 2.1.2. *The following statements are equivalent:*

(a) *There exists $p \in \rho(\mathbf{C}B_d)$ with the contraction property,*

(b) *There exists an injective map j: $\Lambda_n \to \Lambda_{n+d}$ for all $n \in \mathbf{N}_0$ which preserves the structure coefficients of the multiplication, i.e. we have*

$$c_{j(\lambda)\mu}^{j(\nu)} = c_{\lambda j(\mu)}^{j(\nu)} = c_{\lambda\mu}^{\nu} \quad \textit{for all } \lambda \in \Lambda_n,\ \mu \in \Lambda_m \textit{ and } \nu \in \Lambda_{n+m}$$

(c) *There exists an injective map j: $\Lambda_n \to \Lambda_{n+d}$ for all $n \in \mathbf{N}_0$ which preserves the entries of the inclusion matrices i.e. we have*

$$g_{j(\mu)j(\lambda)} = g_{\mu\lambda} \quad \textit{for all } \lambda \in \Lambda_n,\ \mu \in \Lambda_{n+1} \textit{ and } n \in \mathbf{N}_0.$$

Proof. Assume (a). The central idempotents of $p\rho(\mathbf{C}B_{n+d})p$ are given by pz_μ with $\mu \in \Lambda_{n+d}$. Hence the isomorphism between $p\rho(\mathbf{C}B_{n+d})p$ and $\rho(\mathbf{C}B_n)$ defines an injective map j: $\Lambda_n \to \Lambda_{n+d}$. The second statement of (b) follows from the associativity of our multiplication. Indeed, let p_λ and p_μ be minimal idempotents in $\rho(\mathbf{C}B_n)$ and $\rho(\mathbf{C}B_m)$. Then also $p \otimes p_\lambda$ is a minimal idempotent by the contraction property, i.e. $e_0 \times e_\lambda = e_{j(\lambda)}$. But then

$$e_0 \times (e_\lambda \times e_\mu) = e_0 \times \sum_\nu c_{\lambda\mu}^{\nu} e_\nu = \sum_\nu c_{\lambda\mu}^{\nu} e_{j(\nu)}$$

while

$$(e_0 \times e_\lambda) \times e_\mu = e_{j(\lambda)} \times e_\mu = \sum_{\tilde{\nu}} c_{j(\lambda)\mu}^{\tilde{\nu}} e_{\tilde{\nu}}.$$

Setting the last two equations equal yields the claim. (c) follows from (b) immediately, using the fact that $g_{\lambda\mu} = c_{1_1\mu}^{\lambda}$. To prove the last implication, let p be in the inverse image of $j(1) \in K(\rho(\mathbf{C}B_d))$. Then it follows from our assumption that the inclusion matrix of $p\rho(\mathbf{C}B_n)p \subset p\rho(\mathbf{C}B_{n+1})p$ coincides with the inclusion matrix for $\rho(\mathbf{C}B_{n-d}) \subset \rho(\mathbf{C}B_{n+1-d})$. Statement (a) can now be shown easily by induction on n.

The braid representation ρ is said to be *periodic* with period d if there exists $n_0 \in \mathbf{N}$ such that the map j, as defined in the previous Lemma, is surjective for all $n > n_0$. We shall see below that one always obtains a finite dimensional ring from a periodic braid representation by factoring over the ideal generated by $(1 - K(p))$.

It has been already shown in [GW] that one obtains fusion rings of Wess-Zumino-Witten models this way (see also Sect. 3).

To formulate our result, we define the j-orbit of an element $\lambda \in \Lambda_n$ to be the set

$$O_j(\lambda) = \{j^n(\lambda),\ n \in \mathbf{Z} \text{ such that } j^n(\lambda) \text{ is well-defined}\}.$$

Proposition 2.1.3. (a) *If ρ admits a projection p with contraction property, then the quotient $K(\rho)_p = K(\rho)/(1 - K(p))$ has a basis labelled by the j-orbits.*

(b) *If p also makes ρ periodic with period d, then the ring $K(\rho)_p$ is finite dimensional (over $\mathbf{Z}$). One can choose a basis labelled by the elements of $\Lambda_n \cup \Lambda_{n+1} \cup \ldots \cup \Lambda_{n+d-1}$ where $n > n_0$. In this quotient ring, the infinite matrix G of the last lemma will be replaced by the finite dimensional matrix, denoted by the same symbol,*

$$G = \begin{pmatrix} & & & & G_{n+d-1} \\ G_n & & & & \\ & G_{n+1} & & & \\ & & \cdots & & \\ & & & G_{n+d-2} & \end{pmatrix}$$

where we assume $n > n_0$ and n is divisible by d.

(c) *Assume ρ is periodic with period d_1 and with period d_2 where the periodicity is given by the idempotents p_1 and p_2 respectively. Then the quotient over the ideal generated by $(1 - K(p_1))$ and by $(1 - K(p_2))$ has a basis as in* (b) *where* $d = \text{g.c.d}(d_1, d_2)$.

Proof. Let $K(p) = e_0$. Then it follows from $e_0 \times e_\lambda = e_{j(\lambda)}$ and induction that $e_\lambda \equiv e_\mu \bmod (1 - K(p))$ if $\mu \in O_j(\lambda)$. On the other hand, the multiplication does induce any additional relations between these equivalence classes by the last lemma, hence the quotient ring has a basis as in the statement. (b) is essentially just a special case of this. For (c), one applies (a) first to the quotient ring with respect to p_1 and then, in this quotient ring, one applies (a) with respect to $\bar{K}(p_2)$.

2.2. Decomposition of cabled invariants. The rings constructed in the previous section will now be used to express invariants of cablings of framed links in terms of so-called "irreducible" invariants. Let $\mathscr{L}$ be an invariant of framed links and let L be a framed link with s components. Moreover, let $\vec{c} = (c_1, c_2, \ldots, c_s) \in \mathbf{N}^s$. Then it is easy to show that one obtains an invariant $\mathscr{L}^{\vec{c}}$ of framed links by

$$\mathscr{L}^{\vec{c}}(L) = \mathscr{L}(L^{\vec{c}})$$

where $L^{\vec{c}}$ is the framed link obtained by cutting the ribbon of the i-th component of L into c_i parallel ribbons. $\vec{c}$ will also be referred to as a *cabling vector*. Similarly, if β is a (ribbon) braid such that its closure L is an s-component link, one defines $\beta^{\vec{c}}$ to be the braid obtained by replacing a string belonging to the i-th component of L by c_i parallel strings. This definition ensures that $L^{\vec{c}}$ is the closure of $\beta^{\vec{c}}$. This can also be expressed in terms of the cycles of a braid. Recall that any permutation of S_n can be written as a product of disjoint cycles. Similarly, we define the cycles of a braid by its image under the epimorphism $B_n \to S_n$ (e.g. $\sigma_1 \in B_3$ has the cycles $(1,2)$ and (3)). The permutation obtained from $\gamma \in B_n$ is denoted by $\bar{\gamma}$. It is easy to see that strings of a braid are in the same cycle if and only if they are in the same component of the link obtained by the closure of a braid. In the following we shall always assume that the cycles are of the form $(1, 2, \ldots, m_1)$, $(m_1 + 1, m_1 + 2, \ldots, m_1 + m_2) \ldots$, i.e. they can be characterized by a cycle vector $\vec{m} = (m_1, m_2, \ldots, m_s)$. Also observe that the braids which leave invariant the cycle structure $\vec{m}$ form a subgroup $RB_{\vec{m}} \subset RB_n$ of colored ribbon braids, where $n = \sum m_i$.

Recall the definition of $\beta \otimes \gamma$ in Sect. 2.1. This extends to natural embeddings of $\mathbf{C}RB_c^{\otimes n}$ into $\mathbf{C}RB_{nc}$ and to embeddings of finitely many finite ribbon braid groups into $\mathbf{C}RB_\infty$. In particular we define for $\beta \in RB_f$ the elements $\beta^{\otimes s} \in RB_{sf}$ by

$$\beta^{\otimes s} = \beta \otimes \beta \otimes \ldots \otimes \beta \quad (s \text{ times}).$$

Observe the formal similarity to the coproduct for group algebras. Similarly as in that case, one has an interesting action of the symmetric group on such formal tensor powers.

Similarly as the c-th tensor power of a representation of a group can be decomposed into irreducibles, one can also decompose $\mathscr{L}^{\vec{c}}$ into a linear combination of simpler invariants. This will be made precise below where we extend the discussion in [W2] to colored links . Similar results have also been independently obtained, by somewhat different methods, by Reshetikhin [Re] and J. Murakami [M2]. We shall first need the following simple

Observation. Assume that $\gamma \in RB_n$ has cycle structure $\vec{m} \in \mathbf{N}^s$. Fix a cabling vector $\vec{c} \in \mathbf{N}^s$ and choose for the j-th string of γ an element $\beta_j \in RB_{m_i}$, where i is the number of the cycle in which j lies. Then

$$(\beta_1 \otimes \beta_2 \otimes \ldots \otimes \beta_n)\gamma^{\vec{c}} = \gamma^{\vec{c}}(\beta_{\bar{\gamma}(1)} \otimes \beta_{\bar{\gamma}} \otimes \ldots \otimes \beta_{\bar{\gamma}(s)}).$$

In particular, elements of the form $\vec{\beta}^{\otimes \vec{c}}$ commute with $\gamma^{\vec{c}}$.

This can be seen easily at the following example with $\gamma = \sigma_1^{-1}\sigma_2^{-1}$ and $c = 3$.

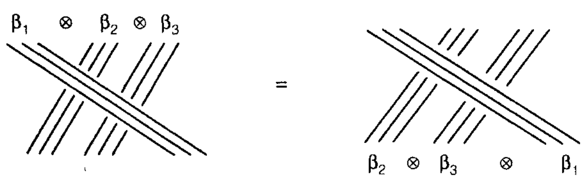


Fig. 6.

Let tr and ρ be as above. Observe that by the multiplicativity property (M3) of tr

$$\rho_{tr}(\mathbf{C}B_n \otimes \mathbf{C}B_m) \cong \rho_{tr}(\mathbf{C}B_n) \otimes \rho_{tr}(\mathbf{C}B_m)$$

so all the relations above also go through in the representation. We can now prove the following theorem (see also [M1], [Re] and [W2]).

Theorem 2.2. *Let notations be as just introduced and let, moreover, $\vec{\lambda}$ be an s-tuple of elements $\lambda_{(i)} \in \Lambda_{c_i}$ (where Λ_m was a labelling set for the simple components of $\rho(\mathbf{C}B_m)$).*

Then there exists an invariant $\mathscr{L}^{\vec{\lambda}}$ of colored framed links with s components such that one has for any such link L

$$\mathscr{L}^{\vec{c}}(L) = \sum_{\vec{\lambda}} a_{\vec{\lambda}}^{\vec{c}}\, \mathscr{L}^{\vec{\lambda}}(L),$$

where the summation goes over all possible s-tuples $\vec{\lambda}$ and $a_{\vec{\lambda}}^{\vec{c}} = \prod_{i=1}^{s} a_{\lambda_i}^{(c_i)}$.

Proof. Let L be a link with s components. If $\gamma \in RB_n$ is a ribbon braid such that $L = (\hat{\gamma}, n)$, then the corresponding permutation $\bar{\gamma}$ under the natural quotient map $RB_n \mapsto S_n$ has exactly s cycles. After conjugating by an appropriate braid, if necessary, one can assume that γ has cycle structure $\vec{m}$ as described above. Let $p_i \in \rho(\mathbf{C}RB_{c_i})$ be a minimal idempotent for $i = 1, 2, \ldots, s$. Then one defines

$$\mathscr{L}^{\vec{p}}(L) = \operatorname{tr}(\gamma^{\vec{c}} \vec{p}^{\otimes \vec{m}})\, \mathscr{L}(0 - \text{unknot})^{\langle \vec{c}, \vec{m} \rangle} \qquad (*)$$

where $\langle \vec{c}, \vec{m} \rangle$ is the number of strings of $\gamma^{\vec{c}}$.

It is easy to show that this defines an invariant of framed links (see [M] or [W2, Proposition 4]). It only depends on the equivalence class of p_i (i.e. if one replaces p_i by another minimal idempotent in the same simple component of $\rho(\mathbf{C}RB_f)$ one obtains the same invariant). Hence one can replace $\vec{p}$ by $\vec{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_s)$ where λ_i labels the simple component to which the idempotent corresponding to the i-th cycle belongs.

Take now any partition of unity $(p_t^{(i)})_t$ of minimal idempotents of $\rho(\mathbf{C}RB_{c_i})$ (i.e. $\sum_t p_t = 1$ and if $t \neq \tilde{t}$, then $p_t p_{\tilde{t}} = 0$). Choose n not necessarily distinct idempotents p_j, $j = 1, 2, \ldots, n$ such that $p_j \in \rho(\mathbf{C}B_{c_i})$ if j is in the i-th cycle of $\gamma \in B_n$. Then it follows, with notations as above,

$$\begin{aligned}\operatorname{tr}((p_1 \otimes p_2 \otimes \ldots \otimes p_n)^2 \gamma^{\vec{c}}) &= \operatorname{tr}((p_1 \otimes p_2 \otimes \ldots \otimes p_n)\gamma^{\vec{c}} \\ &\qquad \cdot (p_{\bar{\gamma}(1)} \otimes p_{\bar{\gamma}(2)} \otimes \ldots \otimes p_{\bar{\gamma}(n)})) \\ &= \operatorname{tr}((p_{\bar{\gamma}(1)} p_1 \otimes p_{\bar{\gamma}(2)} p_2 \otimes \ldots \otimes p_{\bar{\gamma}(n)} p_n)\gamma^{\vec{c}}).\end{aligned}$$

It follows from this easily that the expression above is equal to 0 unless $p_i = p_j$ whenever i and j belong to the same cycle of $\bar{\gamma}$, i.e. one only needs to consider summands of the form $\vec{p}^{\otimes \vec{m}}$, where $\vec{p} = (p_{t_1}, \ldots, p_{t_s})$ with $p_{t_i} \in (p_t)_t$ for $i = 1, 2, \ldots, s$. Hence one obtains by a simple counting argument

$$\begin{aligned}\operatorname{tr}(\gamma^{\vec{c}}) &= \operatorname{tr}\left(\left(\sum_t p_t\right) \otimes \left(\sum_r p_r\right) \otimes \ldots \left(\sum_u p_u\right) \gamma^{\vec{c}}\right) \\ &= \sum_{\vec{\lambda}} a_{\vec{\lambda}}^{(f)} \operatorname{tr}^{\vec{\lambda}}(\gamma) \operatorname{tr}(\vec{p}_{\vec{\lambda}}^{\otimes \vec{m}})\end{aligned}$$

where $\vec{p}_{\lambda} = (p_{t_1}, \ldots, p_{t_s})$ with $p_{t_i} \in \rho(\mathbf{C}RB_f)_{\lambda_i}$. The claim follows now after multiplying both sides of this equation by $\mathscr{L}(0 - \text{unknot})^{\langle \vec{c}, \vec{m} \rangle}$, using $(*)$.

2.3. Rationality and framing anomaly. In analogy of rational conformal field theories we have the following

Definition. (a) We call a Markov trace tr and the corresponding braid representation ρ_{tr} *rational* with periodicity d if there exists an idempotent $p \in \rho(\mathbf{C}B_d)$ with contraction property such that $\bar{K}(\rho) = K(\rho)/(1 - K(\rho))$ has finite $\mathbf{Z}$ rank.

(b) We call an invariant of framed links *rational* if there exists a *finite* set Λ and if there exists for each $s \in \mathbf{N}$ a collection of invariants $\mathscr{L}^{\vec{\lambda}}$ of framed links with s components, labelled by the s-tuples $\vec{\lambda} \in \Lambda^s$ such that $\mathscr{L}^{(\vec{c})}$ is a linear combination of the $\mathscr{L}^{\vec{\lambda}}$'s for all $\vec{c} \in \mathbf{N}^s$.

We shall show in Proposition 2.3(b) that if the braid representation of an invariant $\mathscr{L}$ of framed links is rational, then $\mathscr{L}$ can be made rational by renormalizing its framing anomaly (see end of Sect. 1). For that recall that if ρ is a representation of RB_∞, and if $\alpha \in \mathbf{C}$, $\alpha \neq 0$, then also the map ρ_α: $\sigma_i \to \alpha\rho(\sigma_i)$, $\tau_i \to \alpha\rho(\tau_i)$ defines a representation of RB_∞. Taking the original Markov trace on this representation, one obtains the invariant $\mathscr{L}_\alpha$.

In order to compare the c-cabling with the $(c+d)$-cabling, one needs the following braids $\gamma_{n,m}$ $(n, m \in \mathbf{N})$

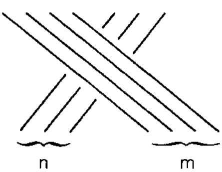


Fig. 7.

Proposition 2.3. *Let ρ be a unitary braid representation with the contraction property for some fixed $d \in \mathbf{N}$ coming from a link invariant $\mathscr{L}$. Then*

(a) *ρ can be suitably scaled by a scalar α such that*

$$p\rho_\alpha(\gamma_{1,d}) = p\rho_\alpha(\gamma_{d,1}^{-1}) \quad and \quad p \otimes p\, \rho_\alpha(\sigma^{(d)}) = \pm\, p \otimes p.$$

(b) *If ρ is rational with periodicity d, then there exists α with $|\alpha| = 1$ such that $\mathscr{L}_\alpha$ is rational; in particular, $\mathscr{L}_\alpha^p(K) = 1$ for all knots K.*

Proof. Observe that the contraction property implies that p is a minimal idempotent in $\rho(\mathbf{C}B_d)$ and $\rho(\mathbf{C}B_{d+1})$. Hence one can find a rescaling of the braid representation such that the first equality in (a) holds. Observe that p is a projection and $\rho(\gamma_{1,d})$ is a unitary. By the uniqueness of the adjoint one has $p\rho(\gamma_{1,d}) = p\rho(\gamma_{d,1})^{-1}$, or in pictures

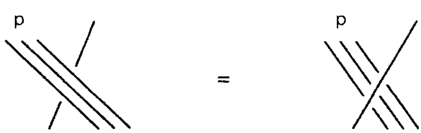


Fig. 8.

This means, whenever we have p together with d strings, we can move it through any other string. Applying this consecutively to $p \otimes p\, \rho(\sigma^{(d)})$, one shows easily that this is equal to $p \otimes p\rho(\sigma^{(d)})^{-1}$. From this follows the second equality in (a).

To show (b), let α be as in (a). We can assume that we have a $+$ sign in the just mentioned equality: this can either be achieved by multiplying the σ_i's by -1 for d odd or by replacing d by $2d$ and p by $p \otimes p$. We shall now show that the identification of an idempotent p_λ with $p \otimes p_\lambda$ in Sect. 2.1 can be carried over to reduced link invariants, as defined in Sect. 2.2. Observe that if we apply p to d cables in $L^{\hat{c}}$, one can separate p-times the closure of these d cables from the rest of the link (by, e.g. replacing any crossing of these d cables with any other string not belonging to these cables by an overcrossing if necessary). For an example, see Fig. 9 (with $d = 3$).

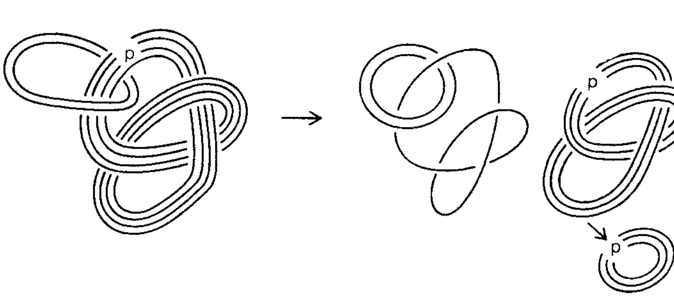


Fig. 9.

Again by using Fig. 8, we can transform the resulting unlinked d cable knot into the unknot. More generally, one shows this way that for any knot K we have $\mathscr{L}_\alpha^p(K) = \mathscr{L}_\alpha^p(0 - \text{unknot}) = 1$, where the last equality is shown in the corollary of Lemma 2.4.2. Using this and the multiplicativity of $\mathscr{L}_\alpha$, one sees that applying p to d cables of $L^{\hat{c}}$ and evaluating it by $\mathscr{L}_\alpha$ gives the same value as if we evaluate it for $L^{\hat{c}}$ with those d cables removed.

In particular, we obtain for a knot K that $\mathscr{L}_\alpha^{p \otimes p_\lambda}(K) = \mathscr{L}_\alpha^{p_\lambda}(K)$ for any minimal idempotent $p_\lambda \in \rho(\mathbf{C}RB_c)$. Hence, if ρ is periodic, the irreducible invariants occurring in a $(d+c)$-cabling of a knot coincide with the ones occurring in the (c)-cabling by Lemma 2.1.2 for c sufficiently large; so all irreducibles already occur in the $c, c+1, \ldots, c+d-1$ cablings of that knot. The periodicity of $\mathscr{L}_\alpha$ in general follows from this argument applied to each component of a link separately.

It is actually quite easy for the Kauffman polynomial to describe explicitly idempotents which satisfy the conditions under (a). The reader who is not familiar with the skein relations of this polynomial and its algebraic description should first check Sect. 3.1 for details. In this particular case, one extends the braid group to a braid monoid by also allowing tangles e_i as shown on the left hand side of Fig. 10 (of unoriented strings). Here, this extension just serves for simplifying computations via pictures; the e_i's can also be written as a linear combination of braids. The contracting idempotent can now be expressed by the tangle $p = (1/x)e_1$, with x the parameter as in [W3, (3.1)]. It is easy to see from the pictures that we have indeed $(p \otimes p)g_1^{(2)} = p \otimes p$. We leave it to the reader to check the transformations of Fig. 9 (with $d = 2$) with p as given below.

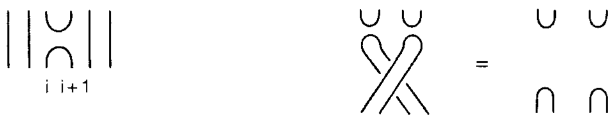


Fig. 10.

2.4. Properties of rational invariants. In the following we assume that $\mathscr{L}$ is a link invariant with framing anomaly α (see end of Sect. 1) such that $\mathscr{L}(0-\text{unknot}) = \Theta$, the Perron Frobenius value of G and that the braid representation corresponding to $\mathscr{L}$ is periodic and unitarizable. To construct Markov traces which are invariant under $K1$ we need some more notations. Recall first that there is up to scalar multiples only one trace of M_n (the algebra of all $n \times n$ matrices), given by the sum of the diagonal entries of a matrix. Hence any trace tr on $B = \oplus B_\mu$ is completely determined by its weight vector $\vec{t} = (t_\mu)$, where $t_\mu = \operatorname{tr}(p_\mu)$ and p_μ is a minimal idempotent of B_μ. A trace tr on B is called *nondegenerate* if its annihilator ideal is equal to 0 or, equivalently, if for any nonzero $b \in B$ there is a $b' \in B$ such that $\operatorname{tr}(bb') \neq 0$. In the semisimple case it is easy to check that tr is nondegenerate if and only if $t_\mu \neq 0$ for each μ. Let $A \subset B$ be algebras with a nondegenerate trace tr such that also its restriction to A is nondegenerate and has the weight vector $\vec{s}$. Then (see e.g. [J1, Eq. (3.2.4)])

$$G^{\mathrm{T}}\vec{t} = \vec{s} \tag{2.4.1}$$

where G^{T} is the transposed of the inclusion matrix G for $A \subset B$. In this set-up one defines the *conditional expectation* ε_A: $B \to A$ to be the orthogonal projection from B onto A with respect to the bilinear form $\langle b_1, b_2 \rangle = \operatorname{tr}(b_1 b_2)$, i.e. $\varepsilon_A(b)$ is the unique element in A for which

$$\operatorname{tr}(a\varepsilon_A(b)) = \operatorname{tr}(ab) \quad \text{for all } a \in A. \tag{2.4.2}$$

In this context we define $\varepsilon_{n,m}$ to be the conditional expectation onto $\rho(\mathbf{C}RB_{n,m})$, the subalgebra of $\rho(\mathbf{C}RB_\infty)$ generated by the image of $\sigma_n, \sigma_{n+1}, \ldots, \sigma_{m-1}$ and τ_n with the simplification $\varepsilon_m = \varepsilon_{1,m}$. The following easy lemma provides simple exercises to get used to working with conditional expectations.

Lemma 2.4.1. *Let* tr *be a Markov trace on* B_∞ *and let* ρ *be the corresponding representation. Then*

(a) $\varepsilon_n(\rho(\sigma_n)) = \operatorname{tr}(\sigma_n)1$,

(b) $\varepsilon_c(\rho(\sigma_1^{(c)})) = \operatorname{tr}(\sigma_1)^c \rho(\Delta_c^2)$, *where* Δ_2^c *is the full-twist of c strings (see e.g.* [Bi] *or Fig.* 2 *(with* $c = 2$*)),*

(c) *Let* tr *be a trace on* RB_∞ *satisfying the Kirby condition* (K1) *of Lemma* 1.1.2. *Then* $\varepsilon_{c-1}(\rho(\sigma_1^{(c)})) = \operatorname{tr}(\sigma_1^{(c)})1$.

(d) *For any inclusion of algebras* $A \subset B \subset C$ *such that* ε_B *is well-defined one has* $\varepsilon_B(A' \cap C) = A' \cap B$

Proof. By definition of Markov trace one has

$$\operatorname{tr}(\beta\sigma_n) = \operatorname{tr}(\beta)\operatorname{tr}(\sigma_n) = \operatorname{tr}(\beta(\operatorname{tr}(\sigma_n))1) \quad \text{for all } \beta \in B_n$$

which shows (a). Claim (c) goes similarly. For claim (b), just observe that for any $\beta \in B_{nc}$ the closures $(\widehat{\beta\sigma_n^{(c)}}, (n+1)c)$ and $(\widehat{\beta\tau_n^{(c)}}, nc)$ are isotopic (see Fig. 2). Hence their traces have to coincide up to a renormalization given by Lemma 1.1. Finally if

$a \in A$ and $c \in A' \cap C$, one has $\mathrm{tr}(cab) = \mathrm{tr}(acb)$ for all $b \in B$. Hence $\varepsilon_B(ca) = \varepsilon_B(ac)$ by (2.4.2). As $a \in B$, one shows easily that $\varepsilon_B(ca) = \varepsilon_B(c)a$ and $\varepsilon_B(ac) = a\varepsilon_B(c)$, which proves (d).

Corollary. *Assume that $\mathscr{L}$ is a link invariant with framing anomaly α and let* tr *be its Markov trace. Then the assignment $\rho^{(c)}$: $\sigma_i \mapsto \sigma_i^{(c)}$ and $\tau_1 \mapsto \alpha^c \Delta_c^2$ extends to an algebra homomorphism from $\mathbf{C}RB_\infty$ into $\mathbf{C}B_\infty$ such that $\mathrm{tr} \circ \rho^{(c)}$ is the Markov trace of $\mathscr{L}^{(c)}$ (where $\mathscr{L}^{(c)}$ is obtained from $\mathscr{L}$ by assigning c cables to each component of a given link).*

Proof. It is easy to see that $\rho^{(c)}$ defines a representation of RB_∞. Obviously $\mathscr{L}^{(c)}(0 - \text{unknot}) = \Theta^c$. If $\beta \in RB_n$ is a product of the σ_i's, it follows from the definitions of $\mathscr{L}^{(c)}$ and $\rho^{(c)}$ that $\mathscr{L}^{(c)} = \Theta^{nc}\mathrm{tr}(\rho^{(c)}(\beta))$. By Proposition 1.2, it remains to check that $\Theta^c \mathrm{tr}(\rho^{(c)}(\beta\sigma_n)) = \mathrm{tr}(\rho^{(c)}(\beta\tau_n))$. This follows from Lemma 2.4.1(b) (with the braids shifted by $(n-1)c$ strings to the left).

In the following we assume that ρ has periodicity d. Then G, as defined in Proposition 2.1.3 is *primitive*; this means that there exists an n such that G^{nd} is a direct sum of d block matrices all of which only have positive entries (see [Ga] for more details) and G permutes these blocks transitively. By the well-known Perron–Frobenius theorem G has an up to scalar multiples unique eigenvector $\vec{v}$ with only positive entries. It can be obtained as $\lim_{n\to\infty}(\Theta^{-1}G)^n\vec{a}$ for any nonzero vector $\vec{a}$ with nonegative coefficients, where Θ is the largest eigenvalue of G. Moreover,

$$\vec{v} = \sum \vec{v}_i \tag{2.4.1}$$

where $\vec{v}_i$ is the Perron–Frobenius vector of the i-th block of G^d and $G\vec{v}_i = \Theta\vec{v}_{i+1}$. In the following we also assume that G^{T} *commutes with* G; so, in particular, G^{T} has the same PerronFrobenius vector as G and we have $\|G\| = \|G^{\mathrm{T}}\| = \Theta$, where $\|\ \|$ is the operator norm with respect to the l_2 norm of vectors. We list several well-known consequences of Perron–Frobenius theory in the following lemma.

Lemma 2.4.2. *With the just introduced notations, we have*

(a) *There exists exactly one positive trace functional* tr *on the algebra $\rho(\mathbf{C}RB_\infty)$ with* $\mathrm{tr}(1) = 1$.

(b) *Let p be as in Sect. 2.3. Then* $\mathrm{tr}(p) = 1/\Theta^d$.

(c) *Let $\lambda \in \Lambda_n$. Define $v_\lambda = \mathrm{tr}(p_\lambda)\Theta^n$, where p_λ is a minimal idempotent in $\rho(\mathbf{C}B_n)_\lambda$. Then $\vec{v} = (v_\lambda)$ is the Perron–Frobenius eigenvector of G and $\langle \vec{a}^{(n)}, \Theta^{-n}\vec{v}_n\rangle = 1$ for all $n > n_0$, where $\vec{v}_n = \vec{v}_i$ if $i \equiv n \bmod d$ and $\vec{v}_i$ is as in* Eq. (2.4.3).

(d) $\|\vec{v}_i\|^2 = \|\vec{v}\|^2/d$ *for all* $i = 1, 2, \dots, d$

(e) $\lim_{n\to\infty}\Theta^{-nd-i}\vec{a}^{(nd+i)} = \vec{v}_i/\|\vec{v}_i\|^2$, *where* $\|\vec{v}_i\|^2 = \langle \vec{v}_i, \vec{v}_i\rangle$.

(f) *Let $\vec{e}_\lambda$ be the vector with* 1 *at the coordinate labelled by λ and* 0 *otherwise. Then*

$$\lim_{n\to\infty} \Theta^{-nd}G^{nd}\vec{e}_\lambda = \frac{dv_\lambda}{\|\vec{v}\|^2}\vec{v}_i$$

where $i \equiv |\lambda| \bmod d$.

Proof. Statement (a) is a well-known result in the study of AF-algebras. Let $\vec{w}_n$ be the weight vector of tr for $\rho(\mathbf{C}B_n)$. By positivity of tr all entries of $\vec{w}_m$ are positive for all $m \in \mathbf{N}$ and $\vec{w}_n = (G^T)^{m-n}\vec{w}_m$ if $m - n$ is divisible by d by (2.4.1). So for $m \to \infty$, we see that $\vec{w}_n$ has to be a multiple of $\vec{v}_n$ (as in Eq. (2.4.3)) by the Perron–Frobenius theorem. The multiple is uniquely determined by $\mathrm{tr}(1) = 1$ (see also statement (c)).

If p has the contraction property and p_λ is a minimal idempotent in $\rho(\mathbf{C}B_n)_\lambda$, then $p \otimes p_\lambda$ is a minimal idempotent in $\rho(\mathbf{C}B_{n+d})$. As $\vec{w}_{n+d}$ is a multiple of $\vec{v}_n$ by the last paragraph, we have $\vec{w}_n = \Theta^d \vec{w}_{n+d}$. Hence

$$w_{n+d,\lambda} = \mathrm{tr}(p \otimes p_\lambda) = \mathrm{tr}(p)\mathrm{tr}(p_\lambda) = \mathrm{tr}(p) w_{n,\lambda}$$

which shows statement (b). Statement (c) is an immediate consequence of $\mathrm{tr}(1) = 1$ and of the fact that $\vec{w}_n = \Theta^{-n}\,\vec{v}_n$ for all $n \in \mathbf{N}$. Statement (c) follows easily from $G^j \vec{v}_i = \Theta^j \vec{v}_{i+j}$ and the fact that G^T commutes with G and statement (e) is a straightforward consequence of the Perron–Frobenius theorem (see e.g. [W1, after (1.10)]).

The limit in (f) obviously has to be a multiple of $\vec{v}_i$ by the Perron–Frobenius theorem. On the other hand, $\vec{v}_i$ is also the Perron–Frobenius vector for $(G^T)^d$. Hence this multiple is given by

$$\langle \lim \Theta^{-nd} G^{nd} \vec{e}_\lambda, \vec{v}_i / \|\vec{v}_i\|^2 \rangle = \langle \vec{e}_\lambda, \vec{v}_i / \|\vec{v}_i\|^2 \rangle = v_\lambda / \|\vec{v}_i\|^2 = d v_\lambda / \|\vec{v}\|^2.$$

Corollary. *Let $\vec{v}$ be as in Lemma 2.4.2* (c) *and let α_λ be the scalar by which $\rho^{(c)}(\tau_i)$ acts in $\rho(\mathbf{C}B_c)_\lambda$ (see the corollary of Lemma* 2.4.1). *If p has contraction property, $\mathscr{L}^p(0 - \text{unknot}) = 1$. In general, if $\lambda \in \Lambda_c$, we have*

$$\mathscr{L}^\lambda(0 - \text{unknot}) = v_\lambda \quad \textit{and} \quad \mathscr{L}^\lambda(1 - \text{unknot}) = \alpha_\lambda v_\lambda.$$

In particular, α_λ is the framing anomaly of $\mathscr{L}^\lambda$.

Proof. By definition of $\mathscr{L}^\lambda$ we have $\mathscr{L}^\lambda(0 - \text{unknot}) = \Theta^c \mathrm{tr}(p_\lambda 1) = v_\lambda$ (by Lemma 2.4.2(c)). The first claim follows from Lemma 2.4.2(b). As Δ_c^2 is in the center of B_c, $\rho^{(c)}(\tau_1)$ acts as a scalar α_λ in $\rho(\mathbf{C}B_c)_\lambda$ (see Corollary of Lemma 2.4.1) and

$$\mathscr{L}^\lambda(1 - \text{unknot}) = \Theta^c \mathrm{tr}(p_\lambda \rho^{(c)}(\tau_1))) = \Theta^c \mathrm{tr}(\alpha_\lambda p_\lambda) = \alpha_\lambda v_\lambda.$$

In the following the symbol $\lim_{\vec{c}_f \to \infty \equiv \vec{c}}$ means that we take the limit with respect to a sequence of vectors $\vec{c}_f$ such that all their coordinates go to ∞ and such that $\vec{c}_f \equiv \vec{c} \bmod d$ for a given vector $\vec{c} \in \mathbf{N}^s$. As a first application of the Perron–Frobenius theory we obtain

Proposition 2.4.3. *Let $\mathscr{L}$ be a periodic invariant with framing anomaly α and with period d. Then one has for any link L with s components that $\lim_{\vec{c}_f \to \infty \equiv \vec{c}} \Theta^{-|\vec{c}_f|} \mathscr{L}(L^{\vec{c}_f})$ exists. Moreover, we also have for the average of all cablings, defined by*

$$av \lim_{\vec{c} \to \infty} \Theta^{-|\vec{c}|} \mathscr{L}(L^{\vec{c}}) = \frac{1}{d} \lim_{N \to \infty} \frac{1}{N^s} \sum_{\vec{c} \in \{1,2,\ldots N\}^s} \Theta^{-|\vec{c}|} \mathscr{L}^{\vec{c}}(L)$$

$$= \sum_{\vec{\lambda} \in \Lambda^s} \frac{v_{\vec{\lambda}}}{\|\vec{v}\|^{2s}} \mathscr{L}^{\vec{\lambda}}(L).$$

In particular, one can define the quantity C, provided the following limit is not equal to 0, *by*

$$C^{-1} = av \lim_{c \to \infty} \Theta^{-c} \mathscr{L}^{(c)}(\hat{\sigma}_1, 2) = av - \lim_{c \to \infty} \Theta^{c} \mathrm{tr}(\sigma_1^{(c)}) = \sum_\lambda \frac{\alpha_\lambda v_\lambda^2}{\|v\|^2},$$

where α_λ is the scalar by which $\rho^{(c)}(\tau_1)$ acts in the irreducible representation of B_c labelled by λ.

Proof. The existence statement follows from Lemma 2.4.2 (e) and Theorem 2.2. For the second statement, one groups together the summands belonging to the $\tilde{c}$s in the same coset of $\mathbf{N}^s$ mod d. The limit of the averages of these partial sums does exist by the first statement. The explicit expression of the limit follows from Lemma 2.4.2(f) and the first statement of this Proposition.

The formula for C^{-1} follows from the just proven formula and the corollary of Lemma 2.4.2. (recall that $(\hat{\sigma}_1, 2) = (\hat{\tau}_1, 1)$).

To prove invariance under Kirby moves we will also need the following well-known consequence of the Perron–Frobenius theorem (where (a) and (b) hold for any C^* algebra coming from a periodic Bratteli diagram).

Lemma 2.4.4. *Let ρ be a unitary braid representation with period d. Let $z_\lambda^{(c)}$ be the central idempotent of the simple component of $\rho(\mathbf{C}B_c)$ which is labelled by λ. Moreover, let (c_f) be an increasing sequence of integers such that $c_f \equiv c \bmod d$ for all $f \in \mathbf{N}$, where c is a fixed integer and let (n_f) be a sequence such that $0 < n_f < c_f$ and both $n_f \to \infty$ and $c_f - n_f \to \infty$ for $f \to \infty$.*

(a) $\lim_{f\to\infty} \mathrm{tr}(z_\lambda^{(c_f)}) = v_\lambda^2/\|\vec{v}_c\|^2$

(b) $\lim_{f\to\infty} \varepsilon_{n_f}(z_\lambda^{(c_f)}) = v_\lambda^2/\|\vec{v}_c\|^2$, *where convergence can be assumed to be uniformly (i.e. with respect to the operator norm on a Hilbert space).*

(c) *Assume* $\lim_{f\to\infty} \Theta^{c_f}\mathrm{tr}(\sigma_1^{(c_f)}) = a$ *exists. Then* $\lim_{f\to\infty} \Theta^{c_f}\varepsilon_{n_f}(\sigma_1^{(c_f)}) = a1$, *with convergence in operator norm.*

Proof. Let notations be as above and let $\lambda, \mu \in \Lambda$. Moreover, let $z_\lambda^{(c_f)}$ and $z_\mu^{(n_f)}$ be the corresponding minimal central idempotents in $\rho(\mathbf{C}RB_{c_f})$ and $\rho(\mathbf{C}RB_{n_f})$ respectively. Observe that

$$\lim_{f\to\infty} \mathrm{tr}(z_\lambda^{(c_f)}) = \lim_{f\to\infty} \tilde{a}_\lambda^{(c_f)}\Theta^{-c_f}v_\lambda = \frac{v_\lambda^2}{\|\vec{v}_c\|^2}. \qquad (*)$$

By Sect. 2.1, the multiplicity of $(z_\mu^{(n_f)})$ in $(z_\lambda^{(c_f)})$ is equal to $\langle G^{c_f-n_f}\hat{e}_\mu, \hat{e}_\lambda\rangle$ and the dimension of $z_\mu^{(n_f)}$ is equal to $a_\mu^{(n_f)}$. Let us assume for the moment also that all the n_f's are congruent to a fixed n mod d. Then

$$\lim_{f\to\infty} \mathrm{tr}(z_\lambda^{(c_f)} z_\mu^{(n_f)}) = \lim_{f\to\infty} \langle G^{c_f-n_f}\hat{e}_\mu, \hat{e}_\lambda\rangle\, a_\mu^{(n_f)}\Theta^{-c_f}v_\lambda = \frac{v_\lambda v_\mu}{\|\vec{v}_n\|^2}\frac{v_\mu}{\|\vec{v}_c\|^2}v_\lambda \qquad (**)$$

where the last equality follows from Lemma 2.4.2 (e) and (f). As $\varepsilon_{n_f}(z_\lambda^{(c_f)})$ is in the center of $\rho(\mathbf{C}RB_{n_f})$, it follows from the definition of the conditional expectation that

$$\varepsilon_{n_f}(z_\lambda^{(c_f)}) = \sum_\mu \frac{\mathrm{tr}(z_\lambda^{(c_f)} z_\mu^{(n_f)})}{\mathrm{tr}\,(z_\mu^{(n_f)})} z_\mu^{(n_f)}.$$

Hence it follows from $(*)$ (also applied to $z_\mu^{(n_f)}$) and $(**)$ that

$$\lim_{f\to\infty} \varepsilon_{n_f}(z_\lambda^{(c_f)}) = \frac{v_\lambda^2}{\|\vec{v}_c\|^2}1.$$

Moreover, one sees from the spectral decomposition of $\varepsilon_{n_f}(z_\lambda^{(c_f)})$ that the convergence is uniformly (i.e. with respect to the operator norm topology). As the limit does not depend on the choice of the coset n of d, the same statement can be shown for arbitrary n_f's by splitting it up into appropriate subsequences which are in the same coset mod d. As $\{z_\lambda^{(c_f)}, \lambda \in \Lambda_{c_f}\}$ spans the center of $\rho(\mathbf{C}RB_{c_f})$ linearly, it follows from (b) that

$$\lim_{f \to \infty} \varepsilon_{n_f}(b_f) = \lim_{f \to \infty} \operatorname{tr}(b_f).$$

for any bounded sequence (b_f), with b_f in the center of $\rho(\mathbf{C}RB_{c_f})$ such that $\operatorname{tr}(b_f)$ converges. By Proposition 2.4.3 $(\Theta^f \operatorname{tr}(\sigma_1^{(f)}))_f$ converges, hence so does

$$(\operatorname{tr}(\Theta^f \varepsilon_f(\sigma_1^{(f)}))_f = (\operatorname{tr}(\alpha^f \Delta_f^2))_f$$

where $\alpha = \operatorname{tr}(\sigma_1)\Theta$. This shows that $\lim_{f\to\infty} \varepsilon_{n_f}(\sigma_1^{(c_f)}) = a$.

We shall need the statements of the last lemma also when the central idempotents are shifted i.e. we also want that $\varepsilon_{s_f+n_f}(1_{s_f} \otimes z_\mu^{(c_f)})$ converges to a scalar. For this it would suffice to prove that the conditional expectation is compatible with the shift. This is easy to check directly for the examples of the HOMFLY and Kauffman polynomials. For the sake of completeness (or generality) we also include the following simple general proof, which uses the theory of von Neumann algebras. As usual, we denote, for any algebra A of bounded operators on a Hilbert space, its commutant by A'. If A is a * algebra, then its double commutant A'' coincides with its weak closure by von Neumann's bicommutant theorem.

Proposition 2.4.5. *Let* tr *be a trace on* B_∞ *such that the corresponding representation* ρ *is periodic and unitarizable. Then* $\rho(\mathbf{C}B_\infty)''$ *is a* II_1 factor such that $\rho(\mathbf{C}B_{m+1,\infty})' \cap \rho(\mathbf{C}B_\infty)'' = \rho(\mathbf{C}B_m)$.

Proof. It is a well-known consequence of the Perron–Frobenius theorem that $\rho(\mathbf{C}B_\infty)$ allows exactly one normalized trace tr, hence its weak closure has to be a factor. It follows from the braid relations that $\rho(\mathbf{C}B_m) \subset \rho(\mathbf{C}B_{m+1,\infty})' \cap \rho(\mathbf{C}B_\infty)''$. On the other hand, choose $f \in \mathbf{N}$ such that $\rho(\mathbf{C}B_{fd})$ is in the periodic part of the Bratteli diagram of ρ. Now observe that if $\tilde{p} \in \rho(\mathbf{C}B_d)$ has the contraction property, then so has $p = \tilde{p} \otimes \ldots \otimes \tilde{p}$ (f times) with d replaced by fd. Hence

$$p\rho(\mathbf{C}B_{m+fd})p \cong \rho(\mathbf{C}B_m).$$

Observe that $\rho(\mathbf{C}B_{fd})$ is mapped onto $\rho(\mathbf{C}B_{m+1,f+md})$ by conjugation by Δ_{m+fd}, the so-called halftwist. Hence there exists $p \in \rho(\mathbf{C}B_{m+1,m+fd})$ such that

$$\dim p\rho\,(\mathbf{C}B_{m+fd})p \leqq \dim \rho(\mathbf{C}B_m),$$

hence, by [W1, Theorem 1.6] one also has that

$$\dim \rho(\mathbf{C}B_{m+1,\infty})' \cap \rho(\mathbf{C}B_\infty)'' \leqq \dim \rho(\mathbf{C}B_m).$$

This shows the claim.

Corollary. *Assume notations and hypotheses of Propositions 2.4.3 and 2.4.5 Then*

$$\lim_{f\to\infty} \Theta^{c_f} \varepsilon_{r_f+n_f}(1_{r_f} \otimes \sigma_1^{(c_f)}) = a1,$$

with convergence in trace norm.

Proof. Let $A = \rho(B_{-\infty,r_f})$, $B = \rho(B_{-\infty,r_f+n_f})$ and let $C = \rho(B_{-\infty,r_f+c_f})$, where $B_{-\infty,n}$ is the group generated by $\sigma_{n-1}, \sigma_{n-2} \ldots \sigma_1, \sigma_0, \sigma_{-1}, \ldots$. It follows from von Neumann algebra theory that one can also define the conditional expectation in this setting as before. Hence, by Lemma 2.4.1(d) one has $\varepsilon_{r_f+n_f}(1_{r_f} \otimes \sigma_1^{(c_f)}) = 1_{r_f} \otimes \varepsilon_{n_f}(\sigma_1^{(c_f)})$. The claim follows now from Proposition 2.4.4.

2.5. Invariants of 3-manifolds. A braid representation ρ corresponding to an invariant of framed links $\mathscr{L}$ is called *positive – rational* if

(a) ρ is unitary,

(b) There exists $d \in \mathbf{N}$ and an idempotent $p \in \rho(\mathbf{C}B_d)$ with contraction property such that $\bar{K}(\rho) = K(p)/(1 - K(p))$ has finite $\mathbf{Z}$-rank,

(c) The Perron–Frobenius eigenvalue of the matrix G (as in Proposition 2.1.3) is equal to $\Theta = |\mathrm{tr}(\sigma_1)|^{-1}$,

(d) $\mathscr{L}$ is *nonsingular*, i.e. $av \lim_{f \to \infty} \Theta^c \mathrm{tr}(\sigma_1^{(c)}) = C^{-1} \neq 0$ (for an explicit expression of C^{-1} see Proposition 2.4.3).

We call $\mathscr{L}$ *positive – rational* if ρ is positive-rational and if $\mathscr{L}$ is rational. It has been shown in Sect. 2.3 that $\mathscr{L}$ can always be made rational by adjusting the framing anomaly if ρ is rational.

Theorem 2.5.1. *Let $\mathscr{L}$ be a positive rational nonsingular invariant of framed links with periodicity d and such that $\mathscr{L}$(0 – unknot) = Θ. Then there exists an invariant $\mathscr{L}^{(\infty)}$ of framed links which is invariant under the Kirby move* (K1). *It is defined for a framed link L with s components by the formula (see also Proposition 2.4.3 for notation)*

$$\mathscr{L}^{(\infty)}(L) = av \lim_{\vec{c} \to \infty} C^s \Theta^{-|\vec{c}|} \mathscr{L}(L^{\vec{c}}) = \sum_{\vec{\lambda} \in \Lambda^s} C^s \frac{v_{\vec{\lambda}}}{\|\vec{v}\|^{2s}} \mathscr{L}^{\vec{\lambda}}(L)$$

where the summation goes over all possible s-tuples $\vec{\lambda}$ with $v_{\vec{\lambda}} = \prod_{i=1}^{s} v_{\lambda_i}$ and where $\vec{v}$ is the Perron–Frobenius vector as in Lemma 2.4.2 (c).

Proof. It follows from the Perron–Frobenius theorem and the discussion above that the limit for $\mathscr{L}^{(\infty)}$ exists and it is easy to see that it defines a multiplicative invariant of framed links. Observe that the c-cabling of the 0-unlink produces c 0-unlinks. As $\mathscr{L}$(0-unlink) = Θ and as $\mathscr{L}$ is multiplicative, we have

$$\mathscr{L}^{(\infty)}(0 - \text{unlink}) = C.$$

We prove now that $\mathscr{L}^{(\infty)}$ is invariant under the first Kirby–Fenn–Rourke move, as described in Sect. 1. If the closure of β has s components, then the closure of $\beta\sigma^{(m)}$ (a braid as in Proposition 1.3) has $s + 1$ components. So their corresponding cabling vectors will be described by $\vec{c}_f$ and $(\vec{c}_f, c_f)$ correspondingly. Moreover, let t_f be the number of cables of $\beta^{\vec{c}_f}$ and let r_f be the number of cables of $(\beta\sigma^{(m)})^{(\vec{c}_f, c_f)}$. Let n' and n be the number of strings of $\beta\sigma^{(m)}$ resp. β. Using notation of Proposition 2.4.2, one has

$$\mathscr{L}^{(\infty)}(\widehat{\beta\sigma^{(m)}}, n') = av \lim_{(\vec{c}_f, c_f) \to \infty} C^{s+1} \Theta^{r_f - |\vec{c}_f| - c_f} \mathrm{tr}((\beta\sigma_i^{(m)})^{(\vec{c}_f, c_f)}).$$

By definition of the conditional expectation and of t_f we have $\mathrm{tr}((\beta\sigma^{(m)})^{(\vec{c}_f, c_f)}) = \mathrm{tr}(\beta^{\vec{c}_f} \varepsilon_{t_f}((\sigma^{(m)})^{(\vec{c}_f, c_f)})$. It follows from the limit formula of Lemma 2.4.4 resp. the corollary of Proposition 2.4.5 that the expression above is equal to

$$\left(av \lim_{\vec{c}_f \to \infty} C^s \Theta^{t_f - |\vec{c}_f|} \mathrm{tr}(\beta^{\vec{c}_f})\right) \times \left(av \lim_{c_f \to \infty} C \Theta^{r_f - t_f - c_f} \mathrm{tr}(\sigma^{(r_f - t_f - c_f)})\right).$$

Observe that the second factor becomes 1 by definition of C, while the first is equal to $\mathscr{L}^{(\infty)}(\hat{\beta}, n)$. This shows that $\mathscr{L}^{(\infty)}$ is invariant under the first Kirby move. The second expression for the limit is a consequence of the Perron–Frobenius theorem.

The same proof also works if one takes for any divisor d' of d the limit only over cablings where the number of cables is divisible by d'.

Corollary. *Let $\mathscr{L}$ be as in the theorem, not necessarily nonsingular and let $d' \in \mathbb{N}$ be such that $d'|d$. Assume that*

$$av \lim_{c\to\infty} \Theta^{d'c} \operatorname{tr}(\sigma_1^{(d'c)}) \neq 0$$

and denote the inverse of this limit by C. Then one obtains an invariant of framed links, invariant under (K1), *by*

$$\mathscr{L}^{(\infty)}(L) = av \lim_{\vec{c}_f\to\infty} C^s \Theta^{-|d'\vec{c}_f|} \mathscr{L}(L^{d'\vec{c}_f}).$$

It can be shown, using the last proposition and the discussion in the first chapter that $\mathscr{L}^{(\infty)}$ defines an invariant of framed 3-manifolds. From this one can obtain an invariant of 3-manifolds by an easy renormalization procedure in the following way (see e.g. [RT2], [Wa] and [Li2]):

Let D be the linking matrix for the link L (where the linking number of a component with itself is the framing number). Let $n_\pm$ (resp. n_0) be the numbers (with multiplicities) of positive/negative eigenvalues (resp. the multiplicity of the eigenvalue 0) of D and let $\sigma(D) = n_+ - n_-$ be the signature of D. Moreover, let $\kappa = C/|C|$.

Theorem 2.5.2. *Let $M(L)$ be the 3-manifold obtained from surgery at the framed link L and let $\mathscr{L}^{(\infty)}$ be an invariant of framed links as obtained in the previous Theorem or its Corollary. If $\mathscr{L}^{(\infty)}(L)$ is independent of the orientation of L for any framed link L then there exists a well-defined invariant of 3-manifolds $\mathscr{F}$ given by*

$$\mathscr{F}(M(L)) = \kappa^{-(2n_- + n_0)} \mathscr{L}^{(\infty)}(L) = \kappa^{\sigma(D)} \sum_{\vec{\lambda}} \frac{|C|^s}{\|\vec{v}\|^s} \frac{v_{\vec{\lambda}}}{\|\vec{v}\|^s} \mathscr{L}^{\vec{\lambda}}(L).$$

Proof. Observe that $\sigma(D)$ does not change under the first Kirby move. Hence $\mathscr{F}$ is invariant under this move. For the second one, observe that the linking matrix for $(\widehat{\sigma_1^{\pm 1}, 2})$ is equal to ± 1. Hence it is enough to check that

$$\mathscr{L}^{(\infty)}(\widehat{\sigma_1^{-1}, 2}) = \kappa^2.$$

As ρ is unitary and as tr is positive one has $\operatorname{tr}(\sigma_1^{-1}) = \overline{\operatorname{tr}(\sigma_1)}$. Hence

$$\lim_{f\to\infty} C\Theta^f \operatorname{tr}(\sigma_1^{-1})^{(f)} = C\bar{C}^{-1} = k^2.$$

Hence $\mathscr{F}$ is also invariant under $(K2)$. The expression of $\mathscr{F}$ in terms of irreducible link invariants follows from the one in Theorem 2.5.1 and from the fact that $s = n_+ + n_0 + n_-$.

Remarks 1. To show invariance of orientations for $\mathscr{L}^{(\infty)}$ in general, one would have to extend the ribbon braid group to a more general object, the tangle algebra (see e.g. [RT1, 2]). More on this in the remarks at the end of Sect. 3.3 and the concluding remarks. For the Kauffman polynomial and the Kauffman bracket, this question will not be relevant as already $\mathscr{L}$ is independent of orientations.

2. In all our examples we have $|C| = \|\tilde{v}\|$ (see Sect. 3.4).

3. We have not checked whether the framing chosen here coincides with the canonical framing as in [At]. More about this can be found in [Wa].

§3. Examples

3.1. Braid representations corresponding to the HOMFLY and the Kauffman polynomial. The Homfly polynomial $\mathscr{H}$, which depends on 2 parameters r and s is defined by the following skein relations:

(H1)
$$\mathscr{H}_{\bigcirc} = \frac{r - r^{-1}}{s - s^{-1}}$$

(H2)
$$r\mathscr{H}_{\times} - r^{-1}\mathscr{H}_{\times} = (s - s^{-1})\mathscr{H}_{)(}.$$

Here the second line relates the invariants of any links which are the same everywhere except in a small square where they look like the indicated pictures. As can easily be seen, this is the case for the closures of braids in the formula below, where $\beta \in B_n$ and $i < n$.

$$r\mathscr{H}(\beta\hat{\sigma}_i, n) - r^{-1}\mathscr{H}(\beta\hat{\sigma}_i^{-1}, n) = (s - s^{-1})\mathscr{H}(\hat{\beta}, n).$$

Hence, by definition of tr, one has

$$\mathrm{tr}(\beta(r\sigma_i - r^{-1}\sigma_i^{-1} - (s - s^{-1})1)) = 0 \quad \text{for all } \beta \in B_\infty.$$

It follows now from the definition of ρ_{tr} that $\tilde{g}_i = r\rho_{\mathrm{tr}}(\sigma_i)$ satisfies the relation

$$\tilde{g}_i - \tilde{g}_i^{-1} = s - s^{-1}.$$

On the other hand, observe that the Iwahori–Hecke algebra $H_n(q)$ of type A is given by generators $1, g_1, g_2, \ldots, g_{n-1}$, which satisfy the braid relations (B1) and (B2) (with σ_i replaced by g_i) and the additional relation

(H)
$$g_i^2 = (q - 1)g_i + q, \quad i = 1, 2, \ldots n - 1.$$

It is easy to check that the elements $sr\rho_{\mathrm{tr}}(\sigma_i)$ satisfy the relations of the Hecke algebra $H_n(q)$ (with $q = s^2$), hence $\rho_{\mathrm{tr}}(CB_n)$ is a quotient of $H_n(q)$. It is well-known that $H_n(q)$ is semisimple except if q is an l-th root of unity, $l = 2, 3, \ldots, n$ and that it is isomorphic to the group algebra of the symmetric group in this case (see e.g. [W1]).

Recall that a Young diagram $\lambda = [\lambda_1, \lambda_2, \ldots \lambda_k]$ is given by an array of boxes such that λ_i boxes are in the i-th row and $\lambda_i \geq \lambda_{i+1}, i = 1, 2, \ldots, k - 1$. Moreover, a standard tableau t of shape λ is a filling of the diagram λ with the numbers $1, 2, \ldots, n$ such that the numbers increase if one goes to the right in a row or one goes downwards in a column. Then the Bratteli diagram for the Hecke algebras with q not a root of unity is given by Young's lattice. This is the graph whose vertices on the n-th line are labelled by Young diagrams with n boxes. Such a vertex is connected with one on the n + 1st line if the corresponding Young diagram is contained in the other one. The crucial theorem for our purposes is the following (where k has nothing to do with the level of a representation of a loop group (which will turn out to be $l-k$ in this context))

Theorem 3.1.1 (see [W-1]). *The braid representation coming from the HOMFLY polynomial is unitarizable if $q = e^{\pm 2\pi i/l}$ and $r = q^k$. The Bratteli diagram corresponding to this representation is the subgraph of Young's lattice obtained by only allowing diagrams with k rows at the most and such that the first and the k-th row differ by $l-k$ at the most. This Bratteli diagram has periodicity k.*

A major role in the proof of this theorem is played by certain representations of Hecke algebras which can be considered as q-analogous of Young's orthogonal representations of the symmetric groups (see [H], [W1] or [W4]). We shall need them for the special diagram $\lambda = [2, 1^{n-2}]$ in order to compute the correct framing anomaly (see Lemma 2.3.1). This representation is equivalent, up to some renormalization to the reduced Burau representation. We label a basis for the representation space by $t_1, t_2, \ldots, t_{n-1}$, where t_i is the unique standard tableau of shape $[2, 1^{n-2}]$ with the number $i+1$ in the second column (which only contains one box). Then g_i (with $i > 1$) acts on this vector space via multiplication by -1 on all basis vectors t_j with $j \neq i-1, i$ and on the subspace spanned by t_{i-1} and t_i via the matrix

$$\frac{1}{1-q^i}\begin{pmatrix} -(1-q) & \sqrt{(1-q^{i+1})(q-q^i)} \\ \sqrt{(1-q^{i+1})(q-q^i)} & q^i(1-q) \end{pmatrix}.$$

In particular, g_1 and g_2 act on the subspace spanned by t_1 and t_2 via the matrices

$$\begin{pmatrix} q & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad \frac{1}{1+q}\begin{pmatrix} -1 & \sqrt{q+q^2+q^3} \\ \sqrt{q+q^2+q^3} & q^2 \end{pmatrix}.$$

A simple computation shows that $g_2 g_1^2 g_2$ is given by the diagonal matrix diag $(q, q^3, 1, 1, \ldots)$. By definition of our representation, all other g_i's with $i > 2$ multiply by -1 the tableau t_1. B_{n-1} acts irreducibly on the subspace spanned by the tableaux $t_1, t_2, \ldots, t_{n-2}$ (this representation is equivalent to the one labelled by the diagram $[2, 1^{n-3}]$). Hence, as $\gamma_{1,n-1}\gamma_{n-1,1}$ commutes with B_{n-1}, we obtain

$$\pi_\lambda(\gamma_{1,n-1}\gamma_{n-1,1}) = \operatorname{diag}(q, q, \ldots q, x).$$

To determine x, just observe that the determinant of $\pi_\lambda(\gamma_{1,n-1}\gamma_{n-1,1})$ has to be equal to q^{2n-2}. Hence $x = q^n$. We have almost shown

Lemma 3.1.2. *The HOMFLY polynomial can be made rational with periodicity $d = k$ for $s^2 = q$ an l-th root of unity and $r = s^k$. Its rational version is obtained via rescaling by the framing anomaly $q^{-(k+1)/2k}$.*

Proof. It follows directly from the Bratteli diagram that the idempotent corresponding to $[1^k]$ will also be a minimal idempotent in $\pi_{tr}(\mathbf{C}B_{k+1})$ for the HOMFLY polynomial at the given parameters, i.e. it satisfies the contraction property. Moreover, it projects onto the last basis vector in the representation labelled by $[2, 1^{k-1}]$, as defined above. It follows now immediately from the computation before and Lemma 2.3.1 that the representation

$$\sigma_i \mapsto q^{-(k+1)/2k} g_i$$

induces a rational link invariant.

To show that the periodicity is indeed equal to k one has to rule out the possibility of a minus sign in the second equation of Lemma 2.3.1 (a). Observe that the representation above depends continuously on $q^{1/2k}$. Hence the sign is determined as soon as one knows it for one value of q. For $q = 1$, this scalar coincides with the action of $\sigma_1^{(k)}$ on the vector

$$(v_1 \wedge v_2 \wedge \ldots \wedge v_k) \otimes (v_1 \wedge v_2 \wedge \ldots \wedge v_k)$$

where $\{v_i\}$ is a basis for V and where the action is given by the symmetric group (i.e. by permuting the vectors). It is easy to see that here $\sigma_1^{(k)}$ acts as the identity. □

Analogous statements also hold for the second important 2-variable link invariant, the Kauffman polynomial. It is defined by a renormalization (i.e. by multiplication by a power of r) of an invariant $\mathcal{K}$ of regular isotopy. By this one means an invariant of (unoriented) link diagrams (i.e. projections of links into a plane without triple points, where at each crossing point it is indicated which string goes over the other one). Two such link projections are considered to be equivalent with respect to regular isotopy if one of them can be obtained from the other one only by moves within the plane (i.e. without the first Reidemeister move; see [Ka] or [BW] for details). We will only deal with this invariant of regular isotopy which will be relevant for the construction of invariants of 3-manifolds. The invariant $\mathcal{K}$, which depends on 2 parameters r and s, is defined inductively by the following relations (which are a slight renormalization of Kauffman's Dubrovnik version)

(Ka1), $$\mathcal{K}_{\bigcirc} = x$$

where

$$x = \frac{r - r^{-1}}{s - s^{-1}} + 1$$

(Ka2) $$\mathcal{K}_{|} = r^{-1}\mathcal{K}_{\text{(positive kink)}} = r\mathcal{K}_{\text{(negative kink)}}$$

(Ka3) $$\mathcal{K}_{\times_+} - \mathcal{K}_{\times_-} = (s - s^{-1})(\mathcal{K}_{)(} - \mathcal{K}_{\asymp})$$

Here, the last 2 lines relate the invariants of link diagrams which are identical everywhere except in a small square where they look like the pictures indicated. It follows from Kauffman's work that (Ka1)–(Ka3) determine a well-defined invariant of link diagrams. The corresponding link invariant is obtained by multiplying this invariant by r taken to the power of the number of positive crossings minus the negative crossings (the first crossing in (Ka3) would be positive, and the second one negative).

In this case, the Hecke algebra H_n is replaced by a q-deformation $D_n(r, s)$ of Brauer's centralizer algebra. It is given by generators $g_1, g_2, \ldots, g_{n-1}$ and the following relations: The similarity between the crossings on the left hand side of (K3) and the standard generators of the braid group suggests the definition of elements $e_i \in D_n(r, s)$ by

(D) $$e_i = 1 - \frac{1}{s - s^{-1}}(g_i - g_i^{-1}).$$

Then it follows from [BW] and [W3, § 3] (see also [M])

Theorem 3.1.3. *The braid representation induced by the Markov trace of $\mathcal{K}$ factors through the algebra $D_n(r, s)$ given by generators $g_1, g_2, \ldots, g_{f-1}$ satisfying the braid relations and*
(R1) $e_i g_i = r^{-1} e_i$,
(R2) $e_i g_{i-1}^{\pm 1} e_i = r^{\pm 1} e_i$.
The structure of D_n remains unchanged under the following changes of parameters:

$$(r, s) \leftrightarrow (r, -s^{-1}) \leftrightarrow (-r, -s) \leftrightarrow (r^{-1}, s^{-1}).$$

The main theorems in [W3] state the following

Theorem 3.1.4. (a) *For generic values of r and s (i.e. if s is not a root of unity and if $r \neq s^n$ for any $n \in \mathbf{Z}$) one has the isomorphism*

$$D_n(r, s) \cong I_n(r, s) \oplus H_n(s^2)$$

where I_n is the ideal generated by e_1.

(b) *Its simple components are labelled by the Young diagrams with n (for the Hecke algebra part), $n-2$, $n-4, \ldots, 1$ resp. 0 boxes. In its Bratteli diagram, 2 Young diagrams are connected if they differ by exactly one box.*

(c) *The Bratteli diagram of $\rho(\mathbf{C}B_\infty)$ becomes periodic if and only if s is a root of unity and $r = s^m$ for some $m \in \mathbf{Z}$ with $m \neq -1$. In all these cases one has periodicity 2 or 1.*

The following table describes all unitary representations of B_∞ which factor over the infinite Hecke algebra (first line) or over D_∞. The second column lists the quantum groups whose R-matrices (for the fundamental representation) provide representations of these algebras, where in the BC case it can occur in the $U_q so_{2m+1}$ series as well as in the $U_q sp_{2m'}$ series for suitable choices of m and m' (for more details see Sect.3.3 and, e.g. [Re], [T1] or [TW]). In the column below P_l^+ we list the diagrams occurring in the Bratteli diagrams of the corresponding braid representations (i.e. in the BCD cases, we have the same rule as in the generic case, as outlined above, except we are now only allowed to use the diagrams listed under P_l^+). Observe that although we listed infinitely many diagrams in the $U_q sl_m$ case, the cabling and reduction process only produces finitely many irreducible colors, as shown in Lemma 3.1.2. Also observe that the matrices G, as defined in Sect. 2.1 are all selfadjoint except for $\mathcal{H}(s^m, s)$ with $m \geq 3$; indeed, in the BCD cases, G is just the incidence matrix of the part of Young's lattice given by P_l^+, while the sl_2 matrices coincide with the sp_2 matrices. This reflects the fact that the corresponding representations of Lie algebras are self-conjugate. The transpose of G in the $U_q sl_m$-case with $m \geq 3$ corresponds to the dual representation of the fundamental representation of $U_q sl_m$. Finally, we also mention that in all these cases $|\mathrm{tr}(\sigma_1)| = \Theta^{-1}$, which was shown for our cases in [W1] and [W3, Sect. 3]. We can subsume our findings in the following

Theorem 3.1.5. *The braid representations ρ associated to the link invariants in the table below satisfy conditions* (a), (b) *and* (c) *of positive rationality, as defined at the beginning of Sect. 2.5. Moreover, the matrix G as defined in Sect. 2.1 commutes with its transposed and satisfies the conditions for the Perron–Frobenius theorem.*

3.2 Computation of the framing anomalies. In this subsection, we compute the framing anomaly α_λ according to Sect. 2.3. Here we already use the correspondence

Table 3.1

Polynomial	$U_q g$	s	P_+^l	Θ
$\mathscr{H}(s^m, s)$	$U_q sl_m$	$e^{\pm \pi i/l}$	$\{\lambda, \lambda_1 - \lambda_m \leqq l - m$ $\lambda'_1 \leqq m\}$	$\dfrac{\sin m\pi/l}{\sin \pi/l}$
$\mathscr{K}(s^{m-1}, s)$	$U_q so_m$	$e^{\pm \pi i/l}$	$\{\lambda, \lambda_1 + \lambda_2 \leqq l - m + 2$ $\lambda'_1 + \lambda'_2 \leqq m\}$ $\cup\{[l - m, + 2, 1^{m-2}]\}$	$\dfrac{\sin(m-1)\pi/l}{\sin \pi/l} + 1$
$\mathscr{K}(s^{-2m-1}, s)$	$U_q sp_{2m}$	$e^{\pm \pi i/2l}$	$\{\lambda, \lambda'_1 \leqq l - m - 1$ $\lambda_1 \leqq m\}$	$\dfrac{\sin(2m-1)\pi/2l}{\sin \pi/2l} - 1$
$\mathscr{K}(s^{-2m}, s)$	BC-case	$-e^{\pm 2\pi i/(2l+1)}$	$\{\lambda, \lambda_1 + \lambda_2 \leqq 2m + 1$ $\lambda'_1 \leqq l - m\}$	$\dfrac{\sin 4m\pi/(2l+1)}{\sin 2\pi/(2l+1)} + 1$

between special values of the HOMFLY resp. Kauffman polynomials and classical Lie algebras which will be described in more detail in the next subsection. In accordance to notation in our previous work, k here denotes the number of rows of a Young diagram; it should not be confused with the level of a representation of an affine Lie algebra.

Lemma 3.2.1. *Let $\tilde{\alpha}_\lambda$ be the scalar by which the full twist acts in the irreducible Hecke algebra representation labelled by λ. Then*

$$\tilde{\alpha}_\lambda = q^{n(n-1) - \sum_{i<j}(\lambda_i + 1)\lambda_j}\,.$$

Proof. We need a character formula for a simple reflection s_i in the symmetric group, due to Frobenius (see e.g. [M, I.7 ex.7]). It states

$$\frac{\chi_\lambda(s_i)}{d_\lambda} = \frac{\sum_i (a_i + 1/2)^2 - (b_i + 1/2)^2}{n(n-1)}$$

where the summation goes over the diagonal elements of λ and a_i resp. b_i are the lengths indicated below.

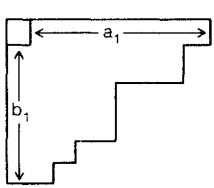


Fig. 11.

Let us assume for the moment that λ has k rows and the k-th row has at least k boxes. Then $a_i = \lambda_i - i$ and $b_i = k - i$. Plugging this into the formula above, one obtains for the numerator

$$\sum_{i=1}^{k} (\lambda_i - i + 1/2)^2 - (k - i + 1/2)^2 = \sum \lambda_i^2 - (2i + 1)\lambda_i.$$

Let r_λ (resp. l_λ) be the rank of the spectral projection for -1 (resp. for q). Using $n = \sum \lambda_i$ and $r_\lambda = (d_\lambda - \chi_\lambda(s_i))/2$, we get

$$\frac{2n(n-1)r_\lambda}{d_\lambda} = 2\sum_{i<j} \lambda_i\lambda_j + 2\sum_i (i-1)\lambda_i.$$

Now observe that $q^{n(n-1)l_\lambda}$ is the determinant of the image $\pi_\lambda(\Delta_n^2)$ of the full twist. Hence it has to act by a d_λ-th root of that number on that irreducible representation. Using $l_\lambda = d_\lambda - r_\lambda$, we see that this scalar has to be the number given in the statement up to multiplication by a d_λ-th root of unity. But for $q = 1$ the full twist becomes the identity, hence the scalar has to be the given number by continuity (see also the Lemma below for a similar argument or [J2]). For diagrams of other shapes, the claim will be shown in the next proposition.

Proposition 3.2.2. *Let $l_i = \lambda_i + k - i$. The framing anomalies in type A_{k-1} are given by*

$$\alpha_\lambda = \zeta^{|\lambda|^2} \exp_q\left(\left[\frac{1}{2k}\sum_{i<j}(l_i - l_j)^2\right] - k(k^2-1)/24\right)$$
$$= \zeta^{|\lambda|^2} \exp_q((\lambda + 2\rho, \lambda)/2)$$

where $\exp_q n = q^n$ and ζ is a $2k$-th (k-th) root of unity for k even (odd); *in the second formula, we take the usual scalar product on the weight space of A_{k-1} with ρ being half the sum of the positive roots and λ being identified with the image of the vector $(\lambda_1, \ldots, \lambda_k)$ in the plane orthogonal to* $(1, 1, \ldots, 1)$.

Proof. Observe that with respect to the Markov trace for $r = q^k$ we have

$$\mathrm{tr}(g_i) = \frac{q^k(1-q)}{1-q^k} = q^{(k+1)/2}\frac{q^{1/2} - q^{-1/2}}{q^{k/2} - q^{-k/2}}$$

hence

$$q^{-(k+1)/2k}\,\mathrm{tr}(g_i) = \frac{q^{(k^2-1)/2k}(q^{1/2} - q^{-1/2})}{q^{k/2} - q^{-k/2}}.$$

Recall that α_λ is the scalar by which $\Theta^m \varepsilon_m(\sigma_1^{(m)}) = \Theta^m \mathrm{tr}(\sigma_1)^m \Delta_m^2$ acts in the irreducible representation of B_m labelled by λ. Hence its exponent (with respect to basis q) is equal to (up to multiplication by ζ^{n^2})

$$n(k+1)/2 - n^2(k+1)/2k + n(n-1) - \sum_{i<j} (\lambda_i + 1)\lambda_j.$$

Again using $n = \sum \lambda_i$, one shows by a direct computation that the quantity above is equal to

$$\frac{1}{2k}\sum_{i<j} (\lambda_i - \lambda_j)^2 + k(\lambda_i - \lambda_j). \qquad (*)$$

It can now be checked by elementary computations that this expression coincides with the first expression for α_λ in the statement. To obtain the second one, define for any vector $\vec{v} \in \mathbf{C}^n$ the vector $av(\vec{v}) = ((1/k) \sum v_i)(1, 1, \ldots, 1)$. Then it is well-known and easy to check that

$$\| \vec{v} - av(\vec{v}) \|^2 = \frac{1}{k} \sum_{i<j} (v_i - v_j)^2.$$

The weight corresponding to $\lambda + \rho$, where $\rho = (-(k-1)/2, -(k-3)/2, \ldots (k-1)/2)$, is obtained by projecting the vector (l_i) into the plane orthogonal to $(1, 1, \ldots, 1)$, i.e. it is equal to $(l_i) - av(l_i))$. Hence, by the previous formulas, the quantity in $(*)$ is equal to

$$(\lambda, \lambda)/2 + (\rho, \lambda) = (\lambda + 2\rho, \lambda)/2.$$

Finally, let us prove the previous lemma also for diagrams which do not have at least k boxes in the k-th row. First observe that we also know α_λ for such diagrams, as it only depends on the differences between the various rows. From this one obtains $\tilde{\alpha}_\lambda$ by just reversing the computation above. Nowhere in this computation was used the special form of the shape. Hence the exponent of $\tilde{\alpha}_\lambda$ has to be $n(n-1) - \sum (\lambda_i + 1)\lambda_j$ in general.

It is now easy to compute α_λ for the Kauffman polynomial. By 2-periodicity (see Theorem 3.1.4), it is enough to compute the scalar by which the full twist acts in $D_{n,\lambda} \cong H_{n,\lambda}(s^2)$, where $n = |\lambda|$. Observe that the q above is the square of the q in the parametrization of [W3]. Hence the eigenvalues of the g_i's in [W3] are $q^{1/2}$ and $-q^{-1/2}$ and the full-twist acts in $D_{n,\lambda}$ by $q^{-n(n-1)/2}\tilde{\alpha}_\lambda$, i.e. the q-exponent of this scalar is

$$n(n-1)/2 - \sum (\lambda_i + 1)\lambda_j = \frac{1}{2}\left(\sum_i \lambda_i^2 + (1 - 2i)\lambda_i \right),$$

where again we used $n = \sum \lambda_i$. Moreover, one has $\mathrm{tr}(g_i) = r|\mathrm{tr}(g_i)|$ in the unitary case. Recall that $r = q^{(m-1)/2}$ corresponds to quantizations of $O(m)$ for $m > 0$ and $Sp(|m|)$ for $m < -1$, m even. For this special case of r, the exponent of α_λ is equal to

$$\frac{1}{2}\left(\sum_i \lambda_i^2 + (1 - 2i + (n-1))\lambda_i \right) = \frac{1}{2}\left(\sum_i \lambda_i(\lambda_i + n - 2i) \right). \quad \square \qquad (3.2.2)$$

Proposition 3.2.3. *Assume the correspondences of specializations of the HOMFLY and Kauffman polynomials with classical Lie algebras as in Sect. 3.1. Then*

$$\alpha_\lambda = \exp_q((\lambda + 2\rho, \rho)/2)$$

where ρ is half the sum of the positive roots of $\mathbf{g}$ *and $(,)$ is the usual invariant bilinear form on the weight space of* $\mathbf{g}$ *(see e.g.* [Hu], *or for explicit formulas see the proof below).*

Proof. The claim has already been shown in the previous proposition for type A. The other cases follow essentially directly from formula (3.2.2); we include here some details about the correspondence between diagrams and weights for the reader's convenience. For the orthogonal groups $O(m)$ where $m = 2p + 1$ resp $m = 2p$ it suffices to show the claim for diagrams λ with p rows at the most (this will be explained in Sect. 3.3 before Proposition 3.3.3). In this case, the diagram λ can be identified with the weight vector $(\lambda_i)_{1\leq i\leq p}$, which will also be denoted by λ. ρ is now given by the vector $(m/2-i)_{1\leq i\leq p}$ (see [Hu]), hence

$$(\lambda+2\rho,\lambda)=\sum_i \lambda_i(\lambda_i+m-2i)$$

which, together with (3.2.2), shows the claim. For $Sp(2p)$ one has $\rho=(p+1-i)_i$ and the weight corresponding to λ is given by the vector $(\lambda_i')_i$, where λ_i' is the number of boxes in the i-th column of λ. Now observe that the exponent sum of $\tilde{\alpha}_\lambda$ is also given by the expression $\sum_{i<j}(\lambda_i'+1)\lambda_j'$. One obtains from this, by essentially the same computations as before formula (3.2.2)

$$\alpha_\lambda=\exp_q\left(\sum_i l_i^2/2-p(p+1)(2p+1)/12\right)=\exp_q((\lambda+2\rho,\lambda)/2).$$

3.3. Comparison with quantum group approach. Here we shall show that our invariants $\mathscr{L}^\lambda$ coincide with invariants obtained using the theory of quantum groups for q not a root of unity (see e.g. [T1], [Re], [W2, 3]). This will be useful in explaining the sets P_+^l in terms of Weyl alcoves and for showing that our invariants of 3-manifolds coincide with the ones constructed in [RT2] and [TW].

We shall need some notations about quasitriangular Hopf algebras and quantum groups (see e.g. [Dr], [Ji], [KaR], [Lu], [TW]). For any vector space V we denote by P_V the "flip operator" $V\otimes V\to V\otimes V$ transforming $v_1\otimes v_2$ into $v_2\otimes v_1$.

A Hopf algebra A with unity is called *quasitriangular* if there exists an invertible $R\in A\otimes A$ (referred to as *universal R-matrix*) satisfying

(Q1) $$\bar{\Delta}(a)=R\Delta(a)R^{-1}\quad \text{for all } a\in A,$$

where $\bar{\Delta}=P_A\circ\Delta$ is the opposite comultiplication and

(Q2a) $$(\Delta\otimes id)R=R_{13}R_{23}$$

(Q2b) $$(id\otimes\Delta)R=R_{13}R_{12}.$$

Here Δ is the comultiplication $A\mapsto A\otimes A$ and R_{ij} is the embedding of R into the i-th and j-th factor of $A\otimes A\otimes A$, i.e.

$$R_{12}=R\otimes 1,\quad R_{23}=1\otimes R\quad\text{and}\quad R_{13}=(1\otimes P_A)(R_{12}).$$

It is easy to show that R satisfies the quantum Yang–Baxter equation (QYBE):

$$R_{12}R_{13}R_{23}=R_{23}R_{13}R_{12}. \tag{3.3.1}$$

Let $\check{R}$ be the operator acting on $A\otimes A$ via left multiplication by $\check{R}=P_AR$. Then it is easy to show that (with $\check{R}_{ij}$ embedded in $\mathrm{End}(A\otimes A\otimes A)$ as above) the *QYBE* is equivalent to

$$\check{R}_{12}\check{R}_{23}\check{R}_{12}=\check{R}_{23}\check{R}_{12}\check{R}_{23}. \tag{3.3.2}$$

Let V, W be A-modules. Then we denote by $R^{V,W}$ resp. $\check{R}^{V,W}$ the linear operators in $\mathrm{End}(V\otimes W)$ which are given by the action of R on $V\otimes W$ resp. the element $P^{V,W}R^{V,W}$, where $P^{V,W}$ maps $v\otimes w\in V\otimes W$ to $w\otimes v\in W\otimes V$. If $V=W$, we just write R^V, $\check{R}^V$ etc. The elements R_{ij}^V resp $\check{R}_{ij}^V$ are defined as linear operators on $V^{\otimes n}$

accordingly. As an immediate consequence of the *QYBE*, Eq. 3.2.1, one obtains for each A-module V a representation of the braid group B_n in End $V^{\otimes n}$ via the map

$$\rho V: \sigma_i \mapsto R^V_{i,i+1} \in \text{End } V^{\otimes n}. \tag{3.3.3}$$

More generally, we can define a representation $\rho(W)$ of colored ribbon braids with n ribbons on $W = \bigotimes_{i=1}^n V_i$, where the V_i's are A-modules, in the same fashion: the crossing of 2 ribbons labelled by V_i and V_j is described by the operator $(\check{R}^{V_i,V_j})^{\pm 1}$ with the sign depending on the crossing (see [RT1, 2] for more details); if $V_i = V$ for all $1 \leq i \leqslant n$, we get the representation π_V as defined in (3.3.3).

To define link invariants, we need a so-called q-trace. In the following we assume that A is a q-deformation of a classical Lie algebra, usually referred to as "quantum group" (see [Dr], [Ji] or [Lu]). Let W be an A-module and let $q^\rho \in A$ be the element in A which acts on a weight vector w_μ belonging to the weight μ by $q^\delta w_\mu = q^{(\rho,\mu)} w_\mu$; here δ is half the sum of all positive roots of the Lie algebra and $(\,,)$ is the usual inner product on the weight lattice of the Lie algebra (see also Sect. 3.2 or [Hu]). Then we define the functional $\mathrm{Tr}_{(W)}$ on $\mathrm{End}_A W$, where $\mathrm{End}_A W$ is the set of elements in End W which commute with the action of A, by

$$\mathrm{Tr}_{(W)}(b) = \mathrm{Tr}(q^\delta b) \qquad b \in \mathrm{End}_A W,$$

where Tr is the usual trace on End W. In [RT1, 2], invariants of colored links were constructed in the following way (adapted to our context):

Let L be a (framed) link with s components and let γ be a (ribbon) braid such that its closure is equal to L. Choosing a suitable conjugate, if necessary, we can assume that γ has cycle structure $\vec{m}$, as explained before Observation 1 in Sect. 2.2. We choose as "color" of the i-th component of L an A-module $V_{\lambda(i)}$; this means we represent γ on the vector space $W = \bigoplus_i V_{\lambda(i)}^{\otimes m_i}$ by a linear operator $\rho_{\vec{V}}(\gamma) = \rho_{(W)}$ as explained above, where $\vec{V}$ stands for the choice of colors $(V_{\lambda(i)})$. The invariant $\mathscr{L}^{\vec{V}}$ is defined by taking the q-trace of this operator, i.e.

$$\mathscr{L}^{\vec{V}}(L) = \mathrm{Tr}_W(\rho_{\vec{V}}(\gamma)). \tag{3.3.4}$$

In the rest of this subsection we shall relate these invariants to the ones constructed in Sect. 2.2. The main observation is that relations ($Q2a$, b) enable us to construct R-matrices for tensor powers of a module V from the one for V by cabling.

Lemma 3.3.1. *With notations as above we have* $\check{R}^{V^{\otimes n}, V^{\otimes m}} = \rho_V(\gamma_{n,m})$, *with* $\gamma_{n,m}$ *as in Fig. 7, and*

$$R^{V^{\otimes n}, V^{\otimes m}} = (R^V_{1,n+m} R^V_{2,n+m} \dots R^V_{n,n+m})(R^V_{1,n+m-1} \dots R^V_{n,n+m-1}) \\ \dots (R^V_{1,n+1} \dots R^V_{n,n+1}).$$

Proof. It is easy to show by induction on n, using (Q2a, b), that

$$(\Delta^{(n)} \otimes id)(R) = R_{1,n+1} R_{2,n+1} \dots R_{n,n+1}$$

and

$$(id \otimes \Delta^{(n)})(R) = R_{1,n+1} R_{1,n} \dots R_{1,2}.$$

Let ρ be the representation of A on V. Then the second claim follows from these identities, from $(\Delta^{(n)} \otimes \Delta^{(m)}) = (\Delta^{(n)} \otimes id^{\otimes m})\,(id \otimes \Delta^{(m)})$ and from

$$R^{V^{\otimes n}, V^{\otimes m}} = (\rho^{\otimes n} \otimes \rho^{\otimes m})(\Delta^{(n)} \otimes \Delta^{(m)})\,(R).$$

It is not hard to derive the first claim from this by induction on n and m. Observe that $P^V_{i,i+1}R^V_{r,s}P^V_{i,i+1} = R^V_{s_i(r),s_i(s)}$, where s_i is the permutation $(i, i+1)$. Moreover, $P^{V^{\otimes n},V^{\otimes m}}$ is the permutation corresponding to the braid $\gamma_{n,m}$ of Fig. 7. So then we get

$$P^{V^{\otimes n},V^{\otimes 1}}R^{V^{\otimes n},V^{\otimes 1}} = (P^V_{1,2}P^V_{2,3}\dots P^V_{n,n+1})(R^V_{1,n+1}\dots R^V_{n-2,n+1}(P^V_{n,n+1}P^V_{n,n+1})R^V_{n,n+1})$$
$$= (P^V_{1,2}\dots P^V_{n-1,n})(R^V_{1,n}R^V_{2,n}\dots R^V_{n-1,n})\check{R}^V_{n,n+1} = \rho_V(\gamma_{n,1})$$

where the last equality followed from the induction assumption for $n-1$. We leave it to the reader to work out the induction for m.

Corollary. *Let V, V_λ, V_μ be A-modules such that $V_\lambda \cong p_\lambda V^{\otimes n}$ and $V_\mu \cong p_\mu V^{\otimes m}$ for some A-invariant idempotent $p_\lambda \in$ End $V^{\otimes n}$ resp $p_\mu \in$ End $V^{\otimes m}$. Then one can identify $V_\lambda \otimes V_\mu$ with $(p_\lambda \otimes p_\mu)V^{\otimes(n+m)}$ and R^{V_λ,V_μ} resp $\check{R}^{V_\lambda,V_\mu}$ with $R^{V^{\otimes n},V^{\otimes m}}(p_\lambda \otimes p_\mu)$ resp with $\check{R}^{V^{\otimes n},V^{\otimes m}}(p_\lambda \otimes p_\mu)$.*

The corollary above allows us to derive the operators R^{V_λ,V_μ} and $\check{R}^{V_\lambda,V_\mu}$ from R^V for any choice of submodules V_λ, V_μ of any tensor powers of V. This is analogous to our construction of new link invariants from a given one via the cabling procedure, as described in Sect. 2.2.. Let V be the fundamental module of a classical Lie algebra and let $\mathscr{L}^V$ be the invariant of framed links, where one chooses the color V for all its components.

Lemma 3.3.2. *$\mathscr{L}^V$ coincides with the specializations of the HOMFLY polynomial as given in Table 3.1, with framing anomaly as in Lemma 3.1.2 for Lie type A and with specializations of Kauffman's invariant of regular isotopy (as indicated in Table 3.1) for Lie types B, C and D.*

Proof. It is well-known that after correcting by the framing anomaly $\mathscr{L}^V$ is a specialization of the HOMFLY or the Kauffman polynomial, depending on the Lie algebra, as given in Table 3.1 (see e.g. [T1], [Re], [W2, 3]). It only remains to check the correct framing anomaly. Observe that if V_0 is the trivial 1-dimensional A module, we have $V_0 \cong V_0 \otimes V_0 \cong V_0^{\otimes 3}$. It follows easily from this and Lemma 3.3.1 that $\check{R}$ has to act as the identity on $V_0 \otimes V_0$. It is well-known that V_0 occurs as a submodule in $V^{\otimes k}$, if V is the fundamental module of sl_k. Lemma 3.3.1 tells us that $\check{R}^{V^{\otimes k}}$ is obtained from R^V via cabling. This determines the framing anomaly, as it was done in Lemma 3.1.2 (the fact that this is only unique up to the choice of a $2k$-th root of q is reflected in the formula for R, where one also obtains $2k$ in the denominator of an exponent of q; this comes from the fact that $(\,,)$ has values in $\mathbf{Z}/k$ (see [KiR] or [TW, Sects. 3.4 and 3.5])).

For Lie types B, C and D, this can be done in the same way or one checks the claim by using Jimbo's explicit R-matrices for these cases (this was done, e.g. in [W3, Lemma 5.1] for type B).

We can now easily show that the invariants $\mathscr{L}^{\hat{V}}$ coincide with the invariants $\mathscr{L}^{\lambda}$, constructed in Sect. 2.2 up to some minor modifications. To do so let us first describe the connection between the set of elements in $P^\infty_\mp$ (Table 3.1) and the set P of irreducible modules occurring in tensor powers of the fundamental representation of a classical Lie group (or resp. its Lie algebra) (see [Wy] or also e.g. [TW, Sect. 5]). Here we use the fact that for q not a root of unity, the representation theory of quantized $U_q\mathbf{g}$ is 'the same' as the one of the classical Lie algebra; this means we have a 1–1 correspondence between the irreducible modules of $\mathbf{g}$ and the ones of $U_q\mathfrak{g}$ which induces an isomorphism between their respective representation rings. Hence it is enough to do this for the classical Lie algebras (or, respectively, the classical compact Lie groups $SU(m)$, $Sp(2m)$ and $SO(m)$). The correspondence comes from associating to a minimal idempotent $p_\lambda \in \pi(\mathbf{C}B_n)_\lambda$ the A module $\pi_V(p_\lambda)V^{\otimes n}$.

(a) For Lie type A_{m-1} (corresponding to $SU(m)$), the set P_+^∞ consists of all diagrams with m rows at the most. Here a minimal idempotent p_λ in $\pi(\mathbf{C}B_n)_\lambda$ corresponds to the irreducible highest weight module $V_{\tilde{\lambda}}$, where $\tilde{\lambda} = [\lambda_1 - \lambda_m, \lambda_2 - \lambda_m, \ldots, \lambda_{m-1} - \lambda_m, 0]$.

(b) For Lie type C_m we have a 1–1 correspondence between the elements of P_+^∞ (the set of diagrams with m columns at most) and the elements $V_{\lambda'}$ in P, where λ' is the diagram obtained from λ by interchanging rows with columns (see also proof of Proposition 3.2.3).

(c) In case of the orthogonal groups we also have a 1–1 correspondence between the elements $\lambda \in P_+^\infty$ and the irreducible $O(m)$ modules V_λ occurring in tensor powers of the fundamental representation of $O(m)$. It will be more convenient, however, to consider the corresponding $SO(m)$ module. Here one has the following restriction rules: If $m = 2p + 1$ is odd and if $\lambda \in P_+^\infty$ such that $\lambda_1' \leqq p$ then we associate to λ the module V_λ labelled by the same diagram; if $\lambda_1' > p$, we associate to λ the module $V_{\tilde{\lambda}}$, where $\tilde{\lambda}$ coincides with λ except in the first column, where it only has $m - \lambda_1'$ boxes (so the irredicuble $SO\,(m)$-modules appearing in tensor powers of V are labelled by the diagrams with p rows). Finally, if $m = 2p$ is even, we use the same rules except if $\lambda_1' = p$; in this case, the $O(m)$ module $\rho_V(p_\lambda)V^{\otimes n}$ (with p_λ a minimal idempotent in $\pi_V(\mathbf{C}B_n)$) decomposes into the direct sum $V_{\lambda^{(1)}} \oplus \tilde{V}_{\lambda^{(2)}}$ of 2 irreducible $SO(m)$ modules.

Proposition 3.3.3. *Using the identifications between the elements of P_+^∞ and the irredicuble A modules as described above, we have $\mathscr{L}^{\vec{\lambda}} = \mathscr{L}^{\vec{V}}$.*

Proof. We have already shown in Lemma 3.3.2 that $\mathscr{L}^V$ coincides with $\mathscr{L}$, where $\mathscr{L}$ is the invariant of framed links obtained from the framed version of the invariants listed in the first column of Table 3.1 and V is the fundamental module of the (quantum) Lie algebra in the second column. The claim is now basically just a consequence of the definitions. Let $\theta = \mathscr{L}(0 - \text{unknot})$. By Theorem 2.2 we have for $\vec{\lambda} = (\lambda(i))_{i=1}^s$ an invariant $\mathscr{L}^{\vec{\lambda}}$ of colored s-component links defined by

$$\mathscr{L}^{\vec{\lambda}}(\hat{\gamma}, n) = \theta^{\langle \vec{c}, \vec{m} \rangle}\operatorname{tr}(\gamma^{\vec{c}}\vec{p}^{\otimes \vec{m}})$$

where p_i is a minimal idempotent in $\rho(\mathbf{C}B_{c_i})_{\lambda(i)}$ and $\vec{p}^{\otimes \vec{m}} = \otimes p_i^{\otimes m_i}$ and where we assume for γ a cycle structure as in Sect. 2.2. Let $V_i = \rho(p_i)V^{\otimes c_i}$. Then

$$W = \bigotimes_{i=1}^{s} V_i^{\otimes m_i} \cong \rho_V(\vec{p}^{\otimes \vec{m}})V^{\otimes \langle \vec{c}, \vec{m} \rangle}. \qquad (*)$$

By Lemma 3.3.1 and its corollary, the operators R^{V_i, V_j} are obtained by the cabling and reduction procedure as used in Sect. 2.2. Hence, using the identification $(*)$ of W with a submodule of $V^{\otimes \langle \vec{c}, \vec{m} \rangle}$, we obtain $\rho_{\vec{V}}(\gamma) = \rho_V(\gamma^{\vec{c}}\vec{p}^{\otimes \vec{m}})$. Now observe that $\operatorname{Tr}_{(V^{\otimes n})} = \theta^n \operatorname{tr}_{|B_n}$, by Lemma 3.3.2, (3.3.4) and Proposition 1.2. Combining these last 2 observations, we obtain $\operatorname{Tr}_{(W)}(\gamma) = \theta^{\langle \vec{c}, \vec{m} \rangle}\operatorname{tr}(\gamma^{\vec{c}}\vec{p}^{\otimes m})$, which implies the claim.

Corollary. If q is a primitive l-th root of unity, Proposition 3.3.3 still holds if all the components of $\vec{\lambda}$ are in P_+^l. The corresponding set $P(l)$ of irreducible modules consists of all highest weight modules V_λ such that $(\lambda, \theta) \leqq l - \check{h}$, where $\check{h}$ is the dual Coxeter number of the Lie algebra $\mathbf{g}$ (see [Kc] or the beginning of Sect. 3.4), θ is its highest root and $(,)$ is as in Sect. 3.2.

Proof. The first statement was shown in [TW] (where the nontrivial part is to show the existence of the modules V_λ) and in greater generality and with additional results by Andersen [An]. The second statement is just a restatement of the inequality involving l in the 4-th column of Table 3.1 (see e.g. [Hu] or [V]).

Remarks 1. We defined the invariants of colored framed links only by using the operators R^{V_λ, V_μ}. This is sufficient for the definition and for showing that they are identical to the ones obtained from our cabling process. A more general framework has been used in [RT1, 2], with parts of ribbons as described in [RT2, Fig. 3 and

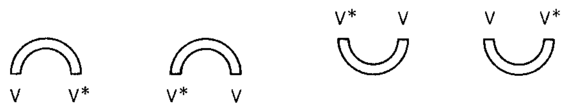


Fig. 12]. The additional ribbons, notably the ones in the second line of [RT, Fig. 3] can be interpreted as contraction operations $V \otimes V^* \to \mathbf{1}$ resp. $V^* \otimes V \to \mathbf{1}$ and as inclusion operations $\mathbf{1} \to V^* \otimes V$ resp. $\mathbf{1} \to V \otimes V^*$, where $\mathbf{1}$ denotes the module corresponding to the trivial representation of A. In this context the trace appears as a contraction operation, i.e. we can define for $b \in \operatorname{End} V$ the trace $\operatorname{Tr}_{(V)}$ as concatenation of the maps

$$\mathbf{1} \to V \otimes V^* \to V \otimes V^* \to V^* \otimes V \to \mathbf{1}.$$

For more on this see [RT2], [JS], [KaR], [T3].

2. The more general framework in [RT2] is needed there to show the change of orientation of a component of a link does not change the value of $\mathscr{L}^{\check{V}}$ provided one replaces the color $V_{\lambda(i)}$ for that component by $V^*_{\lambda(i)}$; this in turn shows that the resulting invariant of 3-manifolds is indeed independent of the choice of orientations of the components of the link.

3. The additional tangles in the figure above can be understood in our context using projections with the contraction property. Indeed, let p_λ and p_{λ^*} be idempotents corresponding to the modules V_λ and $V_\lambda^* \cong V_{\lambda^*}$, as explained before Proposition 3.3.3. Then $p_\lambda \otimes p_{\lambda^*}$ contains a unique subprojection p with contraction property (see Sect. 2.1). Using results of this section, the morphisms in Remark 1 can be translated into our setting by $\alpha \in \mathbf{C} \mapsto \alpha p$ and by projecting down by p (where $\mathbf{C}$ is identified with $\mathbf{C}p$ and where one needs to keep track of the orientations).

3.4. Examples of invariants. The computation of C follows now easily from important representations of $SL_2(\mathbf{Z})$ on characters of affine Kac–Moody algebras, found by Kac and Peterson [KP] (see also [KW] and [K, §13.8]); more recently, these representations (at least on the projective level) were also found in the context of link diagrams and quantum groups (see e.g. [MS] and [TW, Sect. 3.5]). Recall that

$SL_2(\mathbf{Z})$, the group of all 2×2 matrices with integer entries and determinant equal to 1, is generated by the matrices

$$T=\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix} \quad\text{and}\quad S=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}.$$

It is easy to check that

$$STS^{-1}=T^{-1}S^{-1}T^{-1}. \tag{3.4.1}$$

Let now for a classical Lie algebra $\mathbf{g}$ the vector space V^l be given by an orthonormal basis $\{e_\lambda, \lambda\in P(l)\}$, where $P(l)$ is as in the corollary of Proposition 3.3.3. Moreover, let $\check{h}$ be the dual Coxeter number of the corresponding untwisted affine Lie algebra. So we have $\check{h}=m$ for $\mathbf{g}=sl_m$, $\check{h}=m-2$ for $\mathbf{g}=so_m$ and $\check{h}=m/2+1$ for $\mathbf{g}=sp_m$. In our notations, the representations are defined as follows (see [Kc, 12.8.10–12 and Theorem 13.8]):

$$T=\operatorname{diag}(\alpha_\lambda e^{-c(l)2\pi i/l})$$

where $c(l)=(l-\check{h})\dim\mathbf{g}/24$. The entries of the so-called S-matrix (i.e. the image of S under this representation) are given by

$$s_{\lambda\mu}=\sqrt{\check{h}/l}^{\,p}\,\frac{\sum_{w\in W}\varepsilon(w)e^{-(\lambda+\delta,w(\mu+\delta))2\pi i/l}}{\sum_{w\in W}\varepsilon(w)e^{-(\delta,w(\delta))2\pi i/\check{h}}} \tag{3.4.2}$$

where W is the Weyl group of the Lie algebra, $\varepsilon(w)$ is the sign of w, p the rank of the root lattice of $\mathbf{g}$ and δ is half the sum of all positive roots of $\mathbf{g}$. This representation has the following properties:

(a) S is a unitary matrix (we denote the image of S also by S following usual conventions) such that $S=S^{\mathrm{T}}$ and $S^2=\delta_{\lambda,\lambda^*}$ where δ is (only in this sentence!) the Kronecker delta and λ^* labels the module dual to V_λ.

(b) The first row vector (corresponding to the trivial representation) is equal to the vector $\vec{d}/\|\vec{d}\|$, where $\vec{d}=(\dim_q V_\lambda)_\lambda=(\chi^\lambda(q^\delta))_\lambda$, with χ^λ the classical character of V_λ and $q=e^{\pm 2\pi i/l}$. Hence it follows from this and (3.4.1), with $q=e^{2\pi i/l}$ that

$$\sum_{\lambda\in P(l)}\frac{\alpha_\lambda d_\lambda^2}{\|\vec{d}\|^2}=q^{c(l)}(STS^{-1})_{00}=q^{c(l)}(T^{-1}S^{-1}T^{-1})_{00}=q^{3c(l)}s_{00}, \tag{3.4.3}$$

where one observes that the first row and the first column of both S and S^{-1} are given by the vector $\vec{d}/\|\vec{d}\|$ and that $t_{00}=q^{-c(l)}$.

Theorem 3.4. *The invariants $\mathscr{L}$ constructed from unitarizable specializations of the HOMFLY and Kauffman polynomials satisfy the nonsingularity condition* (d) *of Sect. 2.5. In particular, after renormalization by the constant $\kappa=q^{-3c(l)}$, one obtains an invariant $\mathscr{F}$ of* 3*-manifolds. It satisfies*

$$\mathscr{F}(M(L))=q^{-3\sigma(D)c(l)}\sum_{\vec{\lambda}\in(P(l))^s}\frac{d_{\vec{\lambda}}}{\|\vec{d}\|^s}\mathscr{L}^{\vec{\lambda}}(L)$$

where $\sigma(D)$ is the signature of the linking matrix, as defined before Theorem 2.5.2, s is the number of components of L, where $P(l)$ is as in the Corollary of Proposition 3.3.3 and where $d_{\vec{\lambda}}=\prod_{i=1}^s d_{\lambda(i)}$. In particular, one has

$$\mathscr{F}(S^1\times S^2)=\frac{1}{s_{0,0}}=\sqrt{l/\check{h}}^{\,p}\prod_{\gamma\in\Delta_+}\frac{\sin((\delta,\gamma)\pi/\check{h})}{\sin((\delta,\gamma)\pi/l)}. \tag{3.4.4}$$

Proof. We have already checked in Theorem 3.1.4 that if $\mathscr{L}$ is one of the link invariants listed in Table 3.1, it satisfies properties (a), (b) and (c) of a positive rational invariant, as listed at the beginning of Sect. 2.5. To check property (d), first observe that s_{00} can be written as in (3.4.4) (which follows from (3.4.2) by the Weyl denominator identity); from this it is easy to see that $s_{00} \neq 0$. For Lie types A and C, $P(l)$ and P^l_+ coincide (see discussion before Proposition 3.3.3) and its corollary); hence C^{-1} is equal to the expressions in (3.4.3). In the odd dimensional orthogonal case, P^l_+ is just twice $P(l)$, with $d_\lambda = d_{\tilde\lambda}$ and with $\alpha_\lambda = \alpha_{\tilde\lambda}$ by Proposition 3.3.3; this shows again that C^{-1} coincides with (3.4.3). We leave it to the reader to check the $2p$ dimensional case, using $d_\lambda = d_{\lambda^{(1)}} + d_{\lambda^{(2)}}$ if λ has p rows. In all these cases, s_{00} is real, hence $\kappa = q^{-3c(\lambda)}$.

It follows from [W1, Theorem 3.6] and [W3, Theorem 6.4] that the vector $\vec{V}$ in Section 2 is equal to $\vec{d}$ in this setting. To get the formula for $\mathscr{F}$ (Theorem 2.5.2), it suffices to show that $\|\vec{d}\| = |C|$, by remark (b) before (3.4.3). An elementary proof for that can be found in the thesis of Erlijman [E], where it was obtained in order to compute the indices of certain subfactors. Using the S-matrix, one just needs to observe, by Weyl's character formula, that $d_\lambda = s_{0\lambda}/s_{00}$; hence

$$\|\vec{d}\|^2 = \sum (s_{0\lambda}/s_{00})^2 = 1/s_{00}^2$$

using remark (a) before (3.4.3).

By Theorems 2.5.1 and 2.5.2, $\mathscr{F}$ is an invariant of 3-manifolds provided that $\mathscr{L}^{(\infty)}(L)$ does not depend on the choice of orientations of L for any link L. In the case of the Kauffman polynomial and the Kauffman bracket polynomial (the sl_2 case) we started with an invariant $\mathscr{L}$ of unoriented links, hence so is the limit $\mathscr{L}^{(\infty)}$. To prove the same fact for Lie type A_m , $m \geq 2$, observe that $d_\lambda = d_{\lambda^*}$ for all $\lambda \in P^l_+$. Moreover, $\mathscr{L}^\lambda(L)$ does not change if one changes the orientation of the i-th component of L provided one replaces its color $\lambda(i)$ by $\lambda(i)^*$; this follows for $\mathscr{L}^V(L)$ from results in [RT2, Sect. 5] and [TW] and for $\mathscr{L}^\lambda(L)$ from this and Proposition 3.3.3. The last 2 observations together with (3.4.4) imply that $\mathscr{F}(L)$ does not depend on the choice of orientations of L. This finishes the proof of the Theorem. Comparison of (3.4.4) with [RT2] and [TW, 1.5 and Theorem 5.4], using Proposition 3.3.3 also yields

Corollary 1. *The invariants constructed here coincide with the ones constructed in* [RT2] *and* [TW].

Corollary 2. *Let* $\rho^{(\infty)}$ *be the representation of* RB_∞ *coming from* $\mathscr{L}^{(\infty)}$.

(a) $\rho^{(\infty)}(\tau_1)$ *has the spectral decomposition* $\rho^{(\infty)}(\tau_1) = \sum_\lambda \alpha_\lambda z_\lambda$, *where the* z_λ*'s are mutually orthogonal idempotents with* $\mathrm{tr}(z_\lambda) = s_{0\lambda}^2$.

(b) *A change of framing by m in the j-th component in a framed link* $\hat\beta$ *corresponds to multiplication by* $\rho^{(\infty)}(\tau_i)^m$, *where i is the number of a ribbon which is in the j-th component of* $\hat\beta$.

(c) $\mathscr{L}^{(\infty)}(m - \text{unknot}) = (\sum_\lambda \alpha_\lambda^m v_\lambda^2)/\bar{C}$ *where* $m -$ unknot *is the unknot with framing m.*

(d) $\mathscr{L}^{(\infty)}$ *has the same values for 3-manifolds obtained from identical links, whose framings differ only by multiples of 2kl, where k is the index of the root lattice in the weight lattice of* $\mathfrak{g}$.

Proof. The first statement follows from the definition of $\mathscr{L}^{(\infty)}$. Statement (b) just is a consequence of the definition of $\rho^{(\infty)}$. For (c) observe that $\|\vec{d}\| = |C|$ by the proof of the last theorem; hence $\bar{C} = \sum \alpha_\lambda d_\lambda^2$ from which one can easily deduce the claim.

Finally $\alpha_\lambda^{2kl} = 1$ for all $\lambda \in P_+^l$ by the formula in Proposition 3.2.3; this together with (a) shows (d).

3.5. The example sl_2 in detail. The image of the braid representation coming from the Jones polynomial has been completely determined by Vaughan Jones already before the discovery of his invariant. If q is not a root of unity, its Bratteli diagram is essentially one half of Pascal's triangle (see e.g. [J1]). So it does not satisfy the periodicity assumption of Theorem 2.5.1. However, if $q = s^2$ is a primitive l-th root of unity, it does become periodic with periodicity 2. The rule is that one does not add any more vertices on the right hand side of the Bratteli diagram as soon as the number of vertices in 2 consecutive lines is equal to $l - 1$. We give below the diagrams for q a primitive 5-th and 6-th root of unity.

Fig. 12.

The matrix G, as defined in Sect. 2.1, can be written in the following form (with respect to the basis $[0], [1], \ldots, [l-1]$)

$$\begin{pmatrix} 0 & 1 & & & & \\ 1 & 0 & 1 & & & \\ & \cdots & \cdots & \cdots & & \\ & & \cdots & \cdots & \cdots & \\ & & & 1 & 0 & 1 \\ & & & & 1 & 0 \end{pmatrix}.$$

To get the block structure as in Sect. 2.1, one would have to write the diagrams with an even number of boxes first and then the ones with an odd number of boxes (or vice versa). The braid representations are unitary if and only if $q = e^{\pm 2\pi i/l}$. In this case, the Perron–Frobenius vector is equal to

$$(\sin(j\pi/l))_j .$$

Using this and Lemma 3.2.1 and Proposition 3.2.2, (with $j = l_1$), one has

$$C^{-1} = \frac{\sum_{j=1}^{l-1} q^{-(j^2-1)/4} \sin^2(j\pi/l)}{\sum_{j=1}^{l-1} \sin^2(j\pi/l)} .$$

The computation of C^{-1} follows now easily from the computations in [RT2, Sect. 8.3] (see also [Ko]); it can also be easily done using the Poisson summation formula following Dirichlets method of computing Gauss sums (see [D, p. 13 ff]). We leave it to the reader to check (or look up the publications just quoted) that

$$C^{-1} = \frac{q^{3/4} e^{-3\pi i/4} \sin(\pi/l)}{\sqrt{l/2}}.$$

§4. Connections to other approaches and further remarks

In the following remarks we discuss connections of our approach to some other approaches; at the current level of activity in this field it would be impossible to discuss all of them. In particular, Witten's paper itself and several other important papers which are closer to his approach are not mentioned here.

1. (a) Our method here was inspired by the approach of Reshetikhin and Turaev [RT] and its generalization in [TW]. There, a system of linear equations was derived from the Kirby moves from which the authors obtained a linear combination of invariants which coincides with (3.4.4) for our examples.

(b) It seems to be impossible (or at least very hard) to prove independence of orientations for the limiting invariant $\mathscr{L}^{(\infty)}$ for Lie type A_m, $m \geqq 2$ by just using ribbon braids for the construction of framed links. This was the reason for Reshetikhin and Turaev to introduce the more general tangle algebra (see [RT1, 2]). It seems to be possible to carry over our approach to cablings of tangle algebras; this is discussed in the remarks at the end of Sect. 3.3. Further approaches in this direction can be found in category theoretic formulations (see [FY], [JS], [KaR] and [T3]).

(c) There are several papers on the surgery approach in the sl_2 case, among them [Li2], [MS2], [KM] and [B1]. Moreover, after hearing a talk on the contents of this paper, a simple proof of our limit formula for the special case $\mathbf{g} = sl_2$ was found by Blanchet et al. in [B2].

(d) Connections between the surgery approach (and also the one discussed in remark 3) and the notion of a topological quantum field theory, as axiomatized by Atiyah and Segal, are discussed in [B1] for the sl_2 case, in [Wa] and in the recent preprint [T3], among others.

2. It is also possible to formulate our algebraic approach in the language of braid type factor extensions of hyperfinite II_1 factors (see [W5]). Here tensoring by the same representation would correspond to a new extension by the shift $\sigma_i \mapsto \sigma_{i+1}$, while tensoring with the dual representation would be implemented by Jones' extension. Although our presentation here uses von Neumann algebras only occasionally (where one could also find different proofs, at least for our examples) its most natural framework would seem to be in this category of braid type factor extensions.

3. A different approach to invariants of 3-manifolds can be found in the paper [TV] by Turaev and Viro. Here surgery and representations of braid groups are replaced by triangulations and assignments of numbers to the simplices of the triangulations which satisfy certain conditions. They then show that the quantum $6j$ symbols of $U_q sl_2$ satisfy these conditions. It has been shown by Walker and by Turaev that the Turaev–Viro invariant gives the square of the absolute value of the invariant of Reshetikhin and Turaev in [RT]. One can find a generalization of this approach, using the $6j$ symbols for representation rings $K(\rho)$ as studied here (i.e. the Verlinde algebras associated to WZW models of loops over classical compact

Lie groups) in work by Duurhus, Jakobsen and Nest [DJN]). Work in that direction has also been done by several other researchers, among them Karowski, Müller and Schrader (e.g. [KS]).

In another interesting development, Ocneanu has announced that his analogue of 6*j*-symbols (called 'cells' in his terminology) by which he characterizes finite depth subfactors also satisfy the conditions in [TV] hence give rise to an invariant of 3-manifold. It would be interesting to understand the connection between his and our approach.

4. A third approach uses the presentation of 3-manifolds via Heegard decomposition and representations of mapping class groups of orientable surfaces which again depend on a Lie algebra and a level. This approach was studied in the paper by Kohno [Ko] for the Lie algebra sl_2.

5. Our invariants appear to have various symmetry properties corresponding to symmetries of the various Weyl alcoves, which generalize the Kirby–Melvin symmetry principle. It should be possible to prove this essentially only by using properties of the fusions rings. Using the structure coefficients of these rings, it is also possible to compute the invariants of 3-manifolds in terms of cablings. This has been done by Kirby and Melvin for the sl_2 case. To generalize this, one only needs to write down the formula of Theorem 2.2 for sufficiently many different cablings so that one has enough linearly independent equations to solve for the irreducible invariants. The resulting formulas would of course be simplified by the generalized symmetry principle. Very recently (December 1992), the author received a preprint [KT] where some or all of this program was carried out for sl_n.

6. Examples of manifolds have been produced by Kania–Bartoszynska and by Lickorish which can not be distinguished by the $U_q sl_2$ invariants. The idea was to find pairs of links for which the Kauffman bracket polynomial coincides also for some of their cablings (see [KB] and [Li3]). It would be interesting to see whether they can be distinguished by the invariants in this paper.

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