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l-adic representations associated to modular forms over imaginary quadratic fields

I. Lifting to $GSp_4(\mathbb{Q})$

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Introduction

It is now well known that if f is a holomorphic elliptic modular newform then one can associate to f a compatible system of λ-adic representations. These occur naturally in the l-adic cohomology of certain sheaves on modular curves. In the case of weight 2, trivial character and q-expansion with rational coefficients there is an elliptic curve $A/\mathbb{Q}$ such that the l-adic representations associated to f are the dual of the Tate modules of A. For the most part these results have been generalised to the case of holomorphic Hilbert modular newforms (except that one does not know how to construct the elliptic curve A in all cases where it should exist).

It has been suggested for some time that similar results hold for certain modular forms over an imaginary quadratic field K. One way to think of these is as classes in the first homology of certain compactifications of $\Gamma\backslash\mathscr{X}$, where $\mathscr{X}$ is hyperbolic three space and Γ is a congruence subgroup of $\mathrm{SL}_2(\mathcal{O}_K)$. These can be effectively computed and there is a lot of numerical evidence for such conjectures (see for instance [EGM] and [Cr]). The first difficulty in attacking this problem is that there is no obvious link to algebraic geometry and hence to arithmetic. The locally symmetric spaces are three manifolds and hence not varieties.

It is with this difficulty that this paper is concerned. We shall think of modular forms over K as cuspidal automorphic representations of $\mathrm{GL}_2(\mathbb{A}_K)$. If the central character of π factors through the norm map $\mathbb{A}_K^\times \to \mathbb{A}^\times$ then Langlands' philosophy implies that there should be two near equivalence classes of automorphic representations of $GSp_4(\mathbb{A})$ derived from π. On the Galois side one induces to $\mathbb{Q}$ and notes that this four dimensional representation can be polarised in two distinct ways. The two classes correspond to the two grossen-characters of $\mathbb{A}^\times/\mathbb{Q}^\times$ through which the central character of π factors. A closer analysis shows that one of the near equivalence classes should contain holomorphic elements, i.e. elements that contribute to H^0 of certain automorphic coherent sheaves on Siegel threefolds or equivalently to classical holomorphic Siegel modular forms.

The main aim of this paper is to construct such holomorphic automorphic representations of $GSp_4(\mathbb{A})$. Combining these results with an analysis of the l-adic cohomology of Siegel threefolds (see [Ta2]), certain congruence arguments using pseudo-representations (see [Ta1] and [Ta2]) can be used to associate λ-adic representations to many modular forms over K (see [Ta3]). Precise statements are given in Sect. 5 of this paper. In specific cases where a suitable elliptic curve A/K can be found by some means one can use our results and the Faltings-Serre method to prove that A corresponds to a given modular form over K (see Sect. 5 for an example). As an example of the sort of results so obtained we give the following rather concrete special case.

Theorem A *Assume that K has class number 1. Let $\mathfrak{n}$ be an ideal of $\mathcal{O}_K$, let $\Gamma_0(\mathfrak{n})$ denote those elements of $\mathrm{GL}_2(\mathcal{O}_K)$ which reduce to an upper triangular matrix modulo $\mathfrak{n}$ and let $X_\mathfrak{n}^* = \Gamma_0(\mathfrak{n})\backslash(\mathscr{Z} \cup \mathbb{P}^1(K))$. Also let θ be a system of eigenvalues of the Hecke operators $T_v(v \nmid \mathfrak{n})$ on $H_1(X_\mathfrak{n}^*, \mathbb{C})$ (see [Cr] for the definition). Then the field $F_\theta = \mathbb{Q}(\theta(T_v)|(v \nmid \mathfrak{n}))$ is a number field. Let $c(\theta)$ be the largest ideal of $\mathcal{O}_K$ for which the same system θ of eigenvalues occurs on $H_1(X_{c(\theta)}^*, \mathbb{C})$ and suppose moreover that for some prime v of K we have $v(c(\theta)) = 1$. Then there is an extension E/F_θ of degree at most four and for every prime λ of E there is a continuous irreducible representation*

$$\rho\colon \mathrm{Gal}(\bar{K}/K) \to \mathrm{GL}_2(E_\lambda)$$

such that if v is an unramified prime of K which does not divide $\mathfrak{n}l$ (where l is the residue characteristic of λ) then ρ is unramified at v and either $\rho(\mathrm{Frob}_v)$ has characteristic polynomial

$$X^2 - \theta(T_v)X + Nv$$

or $\theta(T_{c_v}) = 0$ and $\rho(\mathrm{Frob}_v)$ has characteristic polynomial $X^2 + Nv$. The first possibility occurs outside a set of Dirichlet density zero. This completely determines ρ.

The main method of this paper is the theory of theta series. GL_2/K is closely related to a four variable orthogonal similitude group $GO/\mathbb{Q}$. There is a theta lifting from cuspidal automorphic representations of $GO(\mathbb{A})$ to automorphic representations of $GSp_4(\mathbb{A})$. However it was not clear that this lifting could produce holomorphic automorphic representations of $GSp_4(\mathbb{A})$, because such representations are not generic. The results of [KRS] convinced us that it must be possible to get non-generic theta lifts and our main discovery is that this is the case. It is closely related to the disconnectedness of GO. In fact our results are very suggestive that the ways of extending a cuspidal automorphic representation from GO° to GO exactly reflect the structure of certain L-packets on GSp_4.

In the first section we explain the relation between GL_2/K and $GO/\mathbb{Q}$ and the structure of cuspidal automorphic representations of $GO(\mathbb{A})$. In the second we discuss some generalities on similitude theta liftings. The literature on these seems somewhat incomplete as most authors work with symmetry groups, however the same methods usually work. In the third section we calculate explicitly the local unramified theta liftings from GO to GSp_4. It results from this that the unramified lifting is unique in the similitude case. Is this a general phenomenon? In this section we also make the crucial calculation of the local theta lift at infinity.

In the fourth section we consider the global theta lift from GO to GSp_4. In particular we obtain a non-vanishing condition by evaluating a Fourier coefficient. We also discuss when this non-vanishing condition can be fulfilled. In section five we discuss the case of interest for attaching λ-adic representations to regular algebraic π (whose central character factors through the norm) and state the main theorems from [Ta 3] for which this paper is one of the crucial ingredients. In section six we discuss the relationship of our results to Langlands' philosophy.

1 Lifting from $\mathrm{GL}_2(K)$ to $GO(3, 1)$

Before starting on the subject of this section it is convenient to fix once and for all some additive characters. For a rational prime p let ψ_p: $\mathbb{Q}_p/\mathbb{Z}_p \hookrightarrow \mathbb{C}^\times$ be the standard character. Also let ψ_∞: $\mathbb{R}/\mathbb{Z} \hookrightarrow \mathbb{C}^\times$ be the standard continuous character ($x \mapsto e^{2\pi i x}$). If F is a number field and v a place of F above a rational place w let ψ_v: $F \to \mathbb{C}^\times$ denote $\psi_w \circ \mathrm{tr}_{F_w/\mathbb{Q}_v}$. Let ψ_F: $\mathbb{A}_F/F \to \mathbb{C}^\times$ denote $\prod_v \psi_v$.

We write ψ for $\psi_{\mathbb{Q}}$. The choice of ψ probably makes no real difference, but it is important that $\psi_F = \psi \circ \mathrm{tr}$.

Fix an imaginary quadratic field K. Let c denote the non-trivial element of $\mathrm{Gal}(K/\mathbb{Q})$ and let ε_K denote its non-trivial character. Let G denote the restriction of scalars of GL_2 from K to $\mathbb{Q}$. Let W_1 denote the space of hermitian ($x = {}^{ct}x$) matrices in $M_2(K)$. Then $-\det$: $W_1 \to \mathbb{Q}$ is a quadratic form. Let GO denote the group of orthogonal similitudes of W_1 and let ν: $GO \to \mathbb{G}_m$ denote the multiplier character. Also let det: $GO \to \mathbb{G}_m$ denote the determinant of the action on W_1 and set $sg = \det \nu^{-2}$: $GO \to \{\pm 1\}$. Then sg is surjective and its kernel is the identity component of GO, which we shall denote $GO°$. We shall also let t denote the element of GO with action $x \mapsto {}^t x$. Then $GO = GO° \rtimes \{1, t\}$.

We have a commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & G & \xrightarrow{\sigma} & GO° & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & A & \longrightarrow & C & \xrightarrow{N} & \mathbb{G}_m & \longrightarrow & 0 \end{array}$$

where C is the restriction of scalars from K to $\mathbb{Q}$ of $\mathbb{G}_m$, N is the norm map and A its kernel. Moreover the vertical maps are the natural inclusions of the centres and the map σ arises from the action of G on W_1 via $g\colon x \mapsto g x {}^{ct}g$. If F is a field of characteristic zero then $H^1(\mathrm{Gal}(\bar{F}/F), \mathrm{GL}_2(\bar{F} \otimes_{\mathbb{Q}} K))$ and $H^1(\mathrm{Gal}(\bar{F}/F), (\bar{F} \otimes_{\mathbb{Q}} K)^\times)$ vanish so we have a commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A(F) & \longrightarrow & G(F) & \xrightarrow{\sigma} & GO^\circ(F) & \longrightarrow & H^1(\mathrm{Gal}(\bar{F}/F), A(F)) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow{\scriptstyle =} & & \\
0 & \longrightarrow & A(F) & \longrightarrow & (F \otimes_{\mathbb{Q}} K)^\times & \xrightarrow{N} & F^\times & \longrightarrow & H^1(\mathrm{Gal}(\bar{F}/F), A(F)) & \longrightarrow & 0.
\end{array}$$

We see that $GO^\circ(F) = F^\times(\sigma G(F))$.

For any place v of $\mathbb{Q}$, we have $H^1(\mathrm{Gal}(\overline{\mathbb{Q}_v}/\mathbb{Q}_v), A(\mathbb{Q}_v)) = (0)$ or C_2 depending on whether v splits in K or not. We also have an exact sequence:

$$0 \to H^1(\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}), A(\mathbb{Q}_v)) \to \bigoplus_v H^1(\mathrm{Gal}(\overline{\mathbb{Q}_v}/\mathbb{Q}_v), A(\mathbb{Q}_v)) \to C_2 \to 0$$

where the last map is the product map. All the groups considered come with structures over $\mathbb{Z}$ defined by taking the lattices $\mathcal{O}_K^2$ in K^2 and $M_2(\mathcal{O}_K) \cap W_1$ in W_1. If p does not ramify in K then $H^1(\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p), A(\mathbb{Z}_p)) = (0)$ and so $G(\mathbb{Z}_p) \twoheadrightarrow GO^\circ(\mathbb{Z}_p)$. Thus we see that:

- $GO^\circ(\mathbb{A}) = \mathbb{A}^\times(\sigma G(\mathbb{A}))$,
-

$$\begin{array}{ccccccccccc}
 & & & & & & & & 0 & & \\
 & & & & & & & & \downarrow & & \\
0 & \longrightarrow & A(\mathbb{Q}) & \longrightarrow & G(\mathbb{Q}) & \xrightarrow{\sigma} & GO^\circ(\mathbb{Q}) & \longrightarrow & \mathbb{Q}^\times/NK^\times & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A(\mathbb{A}) & \longrightarrow & G(\mathbb{A}) & \xrightarrow{\sigma} & GO^\circ(\mathbb{A}) & \longrightarrow & \mathbb{A}^\times/N\mathbb{A}^\times & \longrightarrow & 0 \\
 & & & & & & & & \downarrow & & \\
 & & & & & & & & \mathrm{Gal}(K/\mathbb{Q}) & & \\
 & & & & & & & & \downarrow & & \\
 & & & & & & & & 0 & &
\end{array}$$

- and $0 \to GO^\circ(\mathbb{Q})(\sigma G(\mathbb{A})) \to GO^\circ(\mathbb{A}) \to \mathrm{Gal}(K/\mathbb{Q}) \to 0$.

Proposition 1 *There is a bijection between cuspidal automorphic representations $\tilde{\pi}$ of $GO^\circ(\mathbb{A})$ and pairs $(\pi, \tilde{\chi})$ of a cuspidal automorphic representation π of $G(\mathbb{A})$ and a grossencharacter $\tilde{\chi}\colon \mathbb{Q}^\times \backslash \mathbb{A}^\times \to \mathbb{C}^\times$ such that $\tilde{\chi} \circ N$ is the central character of π.*

Proof. This follows easily from the above remarks. The bijection sends $\tilde{\pi}$ to $(\{f \circ \sigma \mid f \in \tilde{\pi}\}, \chi_{\tilde{\pi}})$, where $\chi_{\tilde{\pi}}$ denotes the central character of $\tilde{\pi}$. In the other direction it sends the pair $(\pi, \tilde{\chi})$ to the set of functions from $GO^\circ(\mathbb{Q})\backslash GO^\circ(\mathbb{A})$ to $\mathbb{C}$ such that:

- $f \circ \sigma \in \pi$,
- $f(zg) = \tilde{\chi}(z) f(g)$ for all $z \in \mathbb{A}^\times$ and all $g \in GO^\circ(\mathbb{A})$.

Note that the second set in the proposition maps 2-1 to the set of cuspidal automorphic representations of $G(\mathbb{A})$ whose central character factors through the norm map. Also note that the proposition implies that $GO^\circ(\mathbb{A})$ satisfies strong multiplicity one:

Corollary 1 *If $\tilde{\pi}_1$ and $\tilde{\pi}_2$ are two cuspidal automorphic representations of $GO^\circ(\mathbb{A})$ and if $\tilde{\pi}_{1v} \cong \tilde{\pi}_{2v}$ for all but finitely many places v then $\tilde{\pi}_1 = \tilde{\pi}_2$.*

The same considerations apply locally because $GO^\circ(\mathbb{Q}_v) = \mathbb{Q}_v^\times (\sigma G(\mathbb{Q}_v))$. We see that:

Lemma 1 *There is a bijection between irreducible admissible representations $\tilde{\pi}_v$ of $GO^\circ(\mathbb{Q}_v)$ and pairs $(\pi_v, \tilde{\chi}_v)$ of an irreducible admissible representation π_v of $G(\mathbb{Q}_v)$ and a character $\tilde{\chi}_v$: $\mathbb{Q}_v^\times \to \mathbb{C}^\times$ such that $\tilde{\chi}_v \circ N$ is the central character of π_v. This correspondence is compatible with the global correspondence.*

We now compare GO° and GO. Let c denote the automorphism of GO° induced by conjugation by t. It extends the action of $c \in \mathrm{Gal}(K/\mathbb{Q})$ on σG, and so no confusion should arise. Let $\tilde{\pi}_v$ be an irreducible admissible representation of $GO^\circ(\mathbb{Q}_v)$. Then either:

1. $\tilde{\pi}_v^c \not\cong \tilde{\pi}_v$: in this case $\mathrm{Ind}_{GO^\circ(\mathbb{Q}_v)}^{GO(\mathbb{Q}_v)}(\tilde{\pi}_v)$ is irreducible and we shall denote it $\tilde{\pi}_v^+$.
2. $\tilde{\pi}_v^c \cong \tilde{\pi}_v$: in this case $\mathrm{Ind}_{GO^\circ(\mathbb{Q}_v)}^{GO(\mathbb{Q}_v)}(\tilde{\pi}_v) = \hat{\pi}_v^+ \oplus \hat{\pi}_v^-$ is the sum of two irreducible representations. $\hat{\pi}_v^\pm$ may be realised as $\tilde{\pi}_v$ with the action extended to $GO(\mathbb{Q}_v)$ by letting t act by $\theta^\pm$, where $\theta^\pm$ are the two linear maps from $\tilde{\pi}_v$ to itself satisfying $(\theta^\pm)^2 = \mathrm{Id}$ and $\theta \circ g = {}^c g \circ \theta$ for all $g \in GO^\circ(\mathbb{Q}_v)$. $\tilde{\pi}_v$ corresponds to some pair $(\pi_v, \tilde{\chi}_v)$ and π_v has a unique Whittaker model $\mathscr{W}_{\pi_v}$, i.e. a unique realisation in the space of functions f: $G(\mathbb{Q}_v) \to \mathbb{C}$ such that:

$$f\left(\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} g\right) = \psi_v(\mathrm{tr}\, a) f(g)$$

for all $a \in K_v$ and all $g \in G(\mathbb{Q}_v)$. Then the action of $\theta^\pm$ on $\mathscr{W}_{\pi_v}$ is by $f \mapsto \pm f \circ c$. Let θ^+ be chosen so as to correspond to $f \mapsto f \circ c$.

Note that if $\tilde{\pi}_v$ is unramified then so is $\hat{\pi}_v^+$ but not $\hat{\pi}_v^-$. Any irreducible admissible representation of $GO(\mathbb{Q}_v)$ arises in this way.

We now consider the global problem. If $\hat{\pi}$ is a cuspidal automorphic representation of $GO(\mathbb{A})$ we shall denote by $\hat{\pi}^\circ$ the space of functions $f|_{GO^\circ(\mathbb{A})}$ with $f \in \pi_h$. It is contained in the space of cuspidal automorphic forms on $GO^\circ(\mathbb{A})$ (of the same central character) and so is a direct sum of cuspidal automorphic representations of $GO^\circ(\mathbb{A})$. We have:

Lemma 2 *If $\hat{\pi}$ is a cuspidal automorphic representation of $GO(\mathbb{A})$ then either $\hat{\pi}^\circ = \tilde{\pi}$ with $\tilde{\pi} = \tilde{\pi}^c$ irreducible or $\hat{\pi}^\circ = \tilde{\pi} \oplus \tilde{\pi}^c$ with $\tilde{\pi}$ irreducible. Thus we obtain a map from cuspidal automorphic representations of $GO(\mathbb{A})$ to cuspidal automorphic representations of $GO^\circ(\mathbb{A})$ modulo the action of $\{1, c\}$.*

Proof. First note that $\hat{\pi}^\circ = \hat{\pi}^{\circ c}$ because:

$$f({}^c g) = f(t\,g\,t) = f(g\,t) = (f\,|\,t)(g)$$

for $f \in \hat{\pi}$ and $g \in GO^\circ(\mathbb{A})$. Also all constituents of $\hat{\pi}^\circ$ have the same central character. So what we must show is that if $\tilde{\pi}_1$ and $\tilde{\pi}_2$ are two irreducible constituents of $\hat{\pi}^\circ$ and if $\tilde{\pi}_1$ and $\tilde{\pi}_2$ give rise to representations π_1 and π_2 of $G(\mathbb{A})$ then either $\pi_1 = \pi_2$ or $\pi_1 = \pi_2^c$. For any place v $\tilde{\pi}_{1v}$ and $\tilde{\pi}_{2v}$ are irreducible $GO^\circ(\mathbb{Q}_v)$ submodules of $\hat{\pi}_v$, and so $\tilde{\pi}_{1v} = \tilde{\pi}_{2v}$ or $\tilde{\pi}_{1v} = \tilde{\pi}^c_{2v}$. Thus $(\pi_1 \oplus \pi_1^c)_v = (\pi_2 \oplus \pi_2^c)_v$ and so by [JS, II, Theorem 4.2] we see that $\pi_1 = \pi_2$ or $\pi_1 = \pi_2^c$ as desired.

We now describe the fibres of the above map. Let $\tilde{\pi}$ be a cuspidal automorphic representation of $GO^\circ(\mathbb{A})$. Let T denote the set of places for which $\tilde{\pi}_v \cong \tilde{\pi}_v^c$. Let $\hat{\pi}$ now denote the sum of the cuspidal automorphic representations of $GO(\mathbb{A})$ lying above $\tilde{\pi}$. Fix a finite set S of places containing at least ∞, the primes which ramify in K and the primes for which $\tilde{\pi}_v$ is ramified. Set:

- $U^S = \prod_{v \notin S} GO^\circ(\mathbb{Z}_v)$,
- $G_S = \prod_{v \in S} GO^\circ(\mathbb{Q}_v)$,
- $\hat{U}^S = \prod_{v \notin S} GO(\mathbb{Z}_v)$,
- $\hat{G}_S = \prod_{v \in S} GO(\mathbb{Q}_v)$.

We shall describe $\hat{\pi}^{\hat{U}^S}$. Note that $\mathrm{Ind}_{G_S}^{\hat{G}_S} \tilde{\pi}^{U^S} \cong \bigoplus_{\delta: S \cap T \to \{\pm 1\}} \prod_{v \in S} \hat{\pi}_v^{\delta(v)}$ where $\delta(v) = +$ if $v \in S - T$. Let $(\mathrm{Ind}_{G_S}^{\hat{G}_S} \tilde{\pi}^{U^S})^+$ denote $\bigoplus_{\delta: S \cap T \to \{\pm 1\}} \prod_{v \in S} \hat{\pi}_v^{\delta(v)}$ where the sum is restricted to those δ such that $\prod_{v \in S} \delta(v) = +$. We shall prove:

Lemma 3 1. *If* $\tilde{\pi} \neq \tilde{\pi}^c$ *then* $\hat{\pi}^{\hat{U}^S} \cong \mathrm{Ind}_{G_S}^{\hat{G}_S} \tilde{\pi}^{U^S}$,
2. *if* $\tilde{\pi} = \tilde{\pi}^c$ *then* $\hat{\pi}^{\hat{U}^S} \cong (\mathrm{Ind}_{G_S}^{\hat{G}_S} \tilde{\pi}^{U^S})^+$.

Proof. For any set R of places let $t_R \in GO(\mathbb{A})$ be defined by $(t_R)_v = 1$ if $v \notin R$ and $= t$ if $v \in R$. If ρ is a representation of G_S we shall identify $\mathrm{Ind}_{G_S}^{\hat{G}_S} \rho$ with $\rho^{\wp(S)}$ by:

- $(f_R)_{R \subset S} | g = (f_R | t_R\, g\, t_R)_{R \subset S}$ if $g \in G_S$,
- $(f_R)_{R \subset S} | t_v = (f_{R \Delta \{v\}})_{R \subset S}$.

In the case that $\tilde{\pi} = \tilde{\pi}^c$ we check that $(\mathrm{Ind}_{G_S}^{\hat{G}_S} \tilde{\pi}^{U^S})^+$ corresponds to those elements satisfying $f_{S-R} = f_R \circ c$. We introduce $\hat{\pi}^\circ$ as above, so that $\hat{\pi}^\circ = \tilde{\pi}$ if $\tilde{\pi} \cong \tilde{\pi}^c$ and $\hat{\pi}^\circ = \tilde{\pi} \oplus \tilde{\pi}^c$ otherwise. We define $(\mathrm{Ind}_{G_S}^{\hat{G}_S} \hat{\pi}^{\circ U^S})^+$ to be those $(f_R)_{R \subset S}$ which satisfy $f_{S-R} = f_R \circ c$. This coincides with our previous definition if $\tilde{\pi} = \tilde{\pi}^c$. In the case $\tilde{\pi} \neq \tilde{\pi}^c$ we have $\mathrm{Ind}_{G_S}^{\hat{G}_S} \tilde{\pi}^{U^S} \cong (\mathrm{Ind}_{G_S}^{\hat{G}_S} \hat{\pi}^{\circ U^S})^+$ via the map $(f_R)_{R \subset S} \mapsto (f_R + f_{S-R} \circ c)_{R \subset S}$. Thus in either case what we must prove is:

$$\hat{\pi}^{\hat{U}^S} \cong (\mathrm{Ind}_{G_S}^{\hat{G}_S} \hat{\pi}^{\circ U^S})^+ .$$

Now we have maps between $\hat{\pi}^{\hat{U}^S}$ and $(\mathrm{Ind}_{G_S}^{\hat{G}_S} \hat{\pi}^{\circ U^S})^+$ given by:

- $f \mapsto (g \mapsto f(g\, t_R))_{R \subset S}$,
- $(f_R)_{R \subset S} \mapsto (f : g\, t_R \mapsto f_{R \cap S}(g))$ for $g \in GO^\circ(\mathbb{A})$.

These maps are mutually inverse because elements of $\hat{\pi}^{U^S}$ are right invariant by t_R for $R \cap S = \emptyset$. The second is well defined because:

$$f(t\,g\,t_R) = f({}^c g\,t_{R^c}) = f_{S \cap R^c}({}^c g) = f_{S-R}({}^c g) = f_{S \cap R}(g) = f(g\,t_R)$$

for $g \in GO^\circ(\mathbb{A})$. These maps are easily checked to be $\hat{G}_S$ equivariant.

On taking a direct limit over S we deduce:

Proposition 2 *There is a bijection between cuspidal automorphic representations $\hat{\pi}$ of $GO(\mathbb{A})$ and triples $(\pi, \tilde{\chi}, \delta)$ modulo the action of $\{1, c\}$. Here:*

- *π is a cuspidal automorphic representation of $G(\mathbb{A})$,*
- *$\tilde{\chi}$ is a grossencharacter of $\mathbb{Q}^\times \backslash \mathbb{A}^\times$ such that $\tilde{\chi} \circ N$ is the central character of π,*
- *δ is a map from the places of $\mathbb{Q}$ to $\{\pm 1\}$ which is 1 at all but finitely many places, 1 at v if $\pi_v \not\cong \pi_v^c$ and in the case $\pi \cong \pi^c$ satisfies $\prod_v \delta(v) = 1$.*

Also c maps $(\pi, \tilde{\chi}, \delta)$ to $(\pi^c, \tilde{\chi}, \delta)$.

We remark that all lifts of $(\pi, \tilde{\chi})$ are twists of each other by characters $\varepsilon \circ s g$ where $\varepsilon\colon C_2 \backslash \bigoplus_v C_2 \to \{\pm 1\}$. If $\hat{\pi}$ is one such lift so is $\hat{\pi} \otimes (\varepsilon \circ s g)$ for any ε of the above form. We also remark that $GO(\mathbb{A})$ has the weak multiplicity one property:

Corollary 2 *If $\hat{\pi}_1$ and $\hat{\pi}_2$ are two cuspidal automorphic representations of $GO(\mathbb{A})$ with $\hat{\pi}_1 \cong \hat{\pi}_2$ then $\hat{\pi}_1 = \hat{\pi}_2$.*

However the strong multiplicity one theorem will fail for $GO(\mathbb{A})$.

We now look at the local behaviour. For the rest of this section induction will mean unitary induction. Let B_G denote the Borel subgroup of upper triangular matrices in G. Two characters χ_1, χ_2 of $(K \otimes \mathbb{Q}_v)^\times$ give rise to a character (χ_1, χ_2) of $B_G(\mathbb{Q}_v)$ by:

$$(\chi_1, \chi_2)\begin{pmatrix} d_1 & * \\ 0 & d_2 \end{pmatrix} = \chi_1(d_1)\,\chi_2(d_2).$$

We let T_G denote the torus of diagonal matrices. Consider the flag:

$$W_1 \supset W_1' = \left\{\begin{pmatrix} * & * \\ * & 0 \end{pmatrix}\right\} \supset W_1'' = \left\{\begin{pmatrix} * & 0 \\ 0 & 0 \end{pmatrix}\right\} \supset (0).$$

Let B_{GO} denote the Borel subgroup of GO consisting of elements preserving this flag. Let T_{GO} denote the Levi component consisting of elements preserving the decomposition:

$$W_1 = \begin{pmatrix} * & 0 \\ 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 \\ 0 & * \end{pmatrix}.$$

We identify T_{GO}° with $C \times \mathbb{G}_m$ by letting (x, r) act by $\operatorname{diag}(r, x, r^{-1} N x)$. Then $\sigma B_G \subset B_{GO}^\circ$, $\sigma T_G \subset T_{GO}^\circ$ and $\sigma \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix} = (d_1\,{}^c d_2, N d_1)$. Note that $T_{GO} = T_{GO}^\circ \rtimes \{1, t\}$ where $t^2 = 1$ and $t(x, r)\,t = ({}^c x, r)$.

Let $(\pi, \tilde{\chi}, \delta)$ be a triple as above corresponding to $\hat{\pi}$. Suppose that π_v is principal series corresponding to a character (χ_1, χ_2) of $B_G(\mathbb{Q}_v)$. Then $\chi_1\chi_2 = \tilde{\chi}_v \circ N$. If $\pi_v \not\cong \pi_v^c$ (i.e. $\{\chi_1, \chi_2\} \neq \{\chi_1^c, \chi_2^c\}$) then $\hat{\pi}_v$ is the representation induced from the character of B_{GO}° which is trivial on the unipotent radical and which sends $(x, r) \mapsto (\chi_1/\tilde{\chi}_v)(r)\,\chi_2({}^c x)$. Suppose on the other hand that $\pi_v \cong \pi_v^c$. Consider first the case that $v \neq \infty$ and $\chi_i = \chi_i^c$ for $i=1,2$. Then using the isomorphism $\mathrm{Ind}_{B_G(\mathbb{Q}_v)}^{G(\mathbb{Q}_v)}(\chi_1, \chi_2) \cong \mathscr{W}_{\pi_v}$ described in Proposition 3.2 of Chap. 1 of [JL] we see that t acts on $\mathrm{Ind}_{B_G(\mathbb{Q}_v)}^{G(\mathbb{Q}_v)}(\chi_1, \chi_2)$ by $f \mapsto \delta(v) f \circ c$. Thus $\hat{\pi}_v$ is induced from the character of B_{GO} which is trivial on the unipotent radical and which sends $(x, r) \mapsto (\chi_1/\tilde{\chi}_v)(r)\,\chi_2({}^c x)$ and $t \mapsto \delta(v)$. If v is split or if v is inert and (χ_1, χ_2) is unramified this will be the case.

Suppose now that $v=\infty$ and that $\chi_1 = \chi_2^c$. To describe $\hat{\pi}_\infty$ we shall describe the K-types occurring in it. Let $W_1(\mathbb{R}) = W_1^{(3)} \oplus W_1^{(1)}$ where $W_1^{(3)}$ consists of matrices of the form $\begin{pmatrix} x & z \\ \bar{z} & -x \end{pmatrix}$ and $W_1^{(1)}$ consists of matrices of the form $\begin{pmatrix} y & 0 \\ 0 & y \end{pmatrix}$. Using this decomposition we identify the maximal compact subgroup of $GO(\mathbb{R})$ with $O_3 \times O_1$. Let s_3 (resp. s_1) denote $-1 \in O_3$ (resp. $\in O_1$). Then $O_3 \times O_1 \cong SO_3 \times \langle s_3 \rangle \times \langle s_1 \rangle$. We shall denote the irreducible representations of $O_3 \times O_1$ by triples $(n, \pm, \pm)$, where $(n, \varepsilon_3, \varepsilon_1)$ is the irreducible $2n+1$ dimensional representation, σ_n, of SO_3 on which s_3 acts by ε_3 and s_1 acts by ε_1. Suppose that π_∞ has Langlands parameter:

$$z \mapsto \begin{pmatrix} |z|^s z^N & 0 \\ 0 & |z|^{sc} z^N \end{pmatrix}$$

with $N \in \mathbb{Z}_{>0}$. Then $\pi_\infty|_{SO_3} \cong \sum_{n=N}^{\infty} \sigma_n$, and so $\hat{\pi}_\infty|_{O_3 \times O_1} \cong \sum_{n=N}^{\infty} (n, \varepsilon_3^{(n)}, \varepsilon_1^{(n)})$ and we have $\varepsilon_3^{(n)} \varepsilon_1^{(n)} = \tilde{\chi}_\infty(-1)$ for all n. We also have that $t = \sigma(w) s_3$ where $w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. The map from $\mathscr{W}_{\pi_\infty}$ to itself given by $W \mapsto \sigma(w)(W \circ c)$ is SO_3 equivariant and so acts as $r_n = \pm 1$ on $\sigma_n \subset \mathscr{W}_{\pi_\infty}$. With this notation the $O_3 \times O_1$-types occurring in $\hat{\pi}_\infty$ are $(n, \delta(\infty) r_n, \tilde{\chi}_\infty(-1)\,\delta(\infty) r_n)$ for $n \geq N$.

It remains to calculate r_n. We use the notation of paragraphs 1 and 6 of Chap. 1 of [JL]. $\mathscr{W}_{\pi_\infty}$ is spanned by functions W_Φ where

$$W_\Phi(g) = \mu_1(g)\,|\det g| \int_{\mathbb{C}^\times} (r(g)\Phi)(t, t^{-1})(\mu_1/\mu_2)(t)\, d^\times t,$$

Φ runs over certain $SU(2)$ finite functions in $\mathscr{S}(\mathbb{C}^2)$, r is the representation of $\mathrm{GL}_2(\mathbb{C})$ on $\mathscr{S}(\mathbb{C}^2)$ described in Propositions 1.3 and 1.6 of [JL], and $\mu_1(t) = |t|^s t^N$ and $\mu_2(t) = |t|^{sc} t^N$. For such a Φ set

$$\tilde{\Phi}(a, b) = \int_{\mathbb{C}} \Phi(a, y)\, \psi_\infty(b y)\, dy\, d\bar{y}$$

so by Proposition 1.6 of [JL] we have $(r(g)\Phi)(x) = \tilde{\Phi}(x g)$. Using this we see that $r({}^c g)\Phi = (r(g)\Phi \circ c) \circ c$. Also if $i(x, y) = (y, x)$ we see from Proposition 1.3

of [JL] that for $g \in SL_2(\mathbb{C})$ we have $r(g)(\Phi \circ i) = (r(g)\Phi) \circ i$. Thus letting $g' = \begin{pmatrix} \det g^{-1} & 0 \\ 0 & 1 \end{pmatrix} g$, we have

$$\begin{aligned} W_\Phi({}^c g) &= \mu_1(\det {}^c g)\,|\det g| \int_{\mathbb{C}^\times} (r(g)\Phi \circ c)({}^c t, {}^c t^{-1})(\mu_1/\mu_2)(t)\, d^\times t \\ &= \mu_2(\det g)\,|\det g| \int_{\mathbb{C}^\times} (r(g')\Phi \circ c)((\det g)^c t, {}^c t^{-1})(\mu_1/\mu_2)(t)\, d^\times t \\ &= \mu_2(\det g)\,|\det g| \int_{\mathbb{C}^\times} (r(g')\Phi \circ c)(s^{-1}, s \det g)(\mu_1/\mu_2)(s \det g)\, d^\times s \\ &= \mu_1(\det g)\,|\det g| \int_{\mathbb{C}^\times} (r(g')\Phi \circ c \circ i)(s \det g, s^{-1})(\mu_1/\mu_2)(s)\, d^\times s \\ &= W_{\Phi \circ c \circ i}(g). \end{aligned}$$

We deduce that $W_\Phi({}^c g\, w) = W_{\Phi' \circ i \circ c \circ i}(g) = W_{\Phi' \circ c}(g)$. (Here $'$ is as on p. 3 of [JL], i.e. $\Phi'(a, b) = \int_{\mathbb{C} \oplus \mathbb{C}} \Phi(x, y)\, \psi_\infty(ax + by)\, dx\, d\bar{x}\, dy\, d\bar{y}$.) We see easily that $\Phi' \circ c = (\tilde{\Phi} \circ c)'$. Thus if we can find $\tilde{\Phi} \in \mathscr{S}(\mathbb{C}^2)$ with:

- $f_\Phi \neq 0$ (see p. 233 of [JL], this implies that $W_\Phi \in \mathscr{W}_{\pi_\infty} - \{0\}$),
- $\tilde{\Phi}$ transforms under $\rho SU(2)$ by σ_n, and
- $(\tilde{\Phi} \circ c)' = \pm \tilde{\Phi}$,

then we see that $r_n = \pm$ respectively. For instance take:

$$\tilde{\Phi}(x, y) = e^{-2\pi(|x|^2 + |y|^2)c}\, y^{2N}.$$

This satisfies the first two conditions for $n = N$ (pp. 233, 234 of [JL]) and $(\tilde{\Phi} \circ c)' = (-1)^N \tilde{\Phi}$. Thus $\hat{\pi}_\infty$ has lowest $O_3 \times O_1$-type:

$$(N, (-1)^N \delta(\infty), (-1)^N \delta(\infty)\, \tilde{\chi}_\infty(-1)).$$

2 Generalities on similitude theta liftings

In this section we shall describe some generalities about theta liftings from general orthogonal groups to general symplectic groups. The analogous results for orthogonal and symplectic groups are well known. In some cases where the proofs are identical but the similitude case does not seem to be implied by the symmetry case, we refer the reader to the symmetry case rather than repeating arguments that need a lot of notation. To treat similitude groups we have followed the approach of Harris and Kudla [HK] rather than that of introducing an extra variable.

Let F be a field. Let $V_n = X_n \oplus X_n^*$ be $2n$-dimensional symplectic space, i.e. $\langle (x, x^*), (y, y^*) \rangle = y^*(x) - x^*(y)$. Let J_n: $V_n \to V_n^*$ denote the corresponding skew-symmetric linear map $J_n(x, x^*) = (x^*, -x)$. We denote by $\varepsilon_1, \ldots, \varepsilon_n$ the standard basis of X_n. Let W be an m-dimensional orthogonal space over F. We shall assume throughout that m is even. Let S_W: $W \to W^*$ be the corresponding symmetric linear map and let W_r denote the $m + 2r$ dimensional orthogonal space in the same Witt class as W. Then $W_r = Y_r \oplus W \oplus Y_r^*$ with Y_r an r dimensional space over F. Let $f_1, \ldots, f_r$ denote the standard basis of Y_r. We shall let GSp_{2n} (resp. GO_{W_r}) denote the group of similitudes of V_n (resp. W_r). Let μ (resp. ν) denote their multiplier characters and let Sp_{2n} (resp. O_{W_r}) denote their kernels. Let R denote the subgroup of $GSp_{2n} \times GO_{W_r}$ which is the kernel of $\mu\nu$. We have an isomorphism $V_n \otimes W_r \cong V_{n(m+2r)}$ given explicitly by:

$$V_n \otimes W_r \cong (X_n \otimes W_r) \oplus (X_n^* \otimes W_r) \xrightarrow{\mathrm{Id} \oplus (\mathrm{Id} \otimes S_{W_r})} (X_n \otimes W_r) \oplus (X_n \otimes W_r)^* \cong V_{n(m+2r)}.$$

Thus we have a map $i: R \to Sp_{2n(m+2r)}$.

Now suppose that F is the reals or a finite extension of the p-adic numbers. $H^2_{cts}(Sp_{2n}(F), \mathbb{Q}/\mathbb{Z})$ has a unique element of order two. Thus for any even integer N we get a non-trivial central extension of $Sp_{2n}(F)$ by the N^{th} roots of unity. For $N=2$ denote this $\widetilde{Sp_{2n}}(F)$. For computational purposes it will be convenient also to introduce the notation $Mp_{2n}(F)$ for the case $N=8$. Thus $\widetilde{Sp_{2n}}(F) \hookrightarrow Mp_{2n}(F)$ (uniquely as a map over Sp_{2n}) and $Mp_{2n}(F) = (\widetilde{Sp_{2n}}(F) \times \mu_8)/\{\pm 1\}$. We can give an explicit co-cycle giving the multiplication in Mp_{2n}. Choose a maximal isotropic subspace H of V_n, then $c_H(g_1, g_2) = \gamma(q(H, g_1^{-1} H, g_2 H))$ will do. Here q is the Leray invariant and γ is the Weil invariant (see [Pe] or [RR], note that γ depends on ψ_v). We remark that the cohomology class of c_H is independent of H as it should be. Explicitly

$$c_{kH}(g_1, g_2) = c_H(k^{-1} g_1 k, k^{-1} g_2 k) = a(g_1 g_2)\, a(g_1)^{-1}\, a(g_2)^{-1}\, c_H(g_1, g_2)$$

where $a(g) = c_H(k^{-1}, gk)\, c_H(g, k) = c_{kH}(k^{-1}, (kgk^{-1})k)\, c_{kH}(kgk^{-1}, k)$. If F is non-archimedean with residue characteristic greater than two there are unique liftings $Sp_{2n}(\mathcal{O}_F) \hookrightarrow \widetilde{Sp_{2n}}(F)$ and $Mp_{2n}(F)$. In fact if F has residue characteristic greater than 2 and if H is a maximal isotropic subspace conjugate to X_n by an element of $Sp_{2n}(\mathcal{O}_F)$ then c_H is trivial on $Sp_{2n}(\mathcal{O}_F)^2$. Let ω_n denote the Weil representation of Mp_{2n} (again it depends on the choice of ψ_v). We will also write $\omega_{n,v}$, where v is the prime of F. We consider ω_n as an admissible representation, so in the case $F = \mathbb{R}$ it is really a $(\mathfrak{sp}_{2n}, K)$ module, where K is a maximal compact subgroup of $Mp_{2n}(\mathbb{R})$. ω_n consists of the K finite smooth vectors in a unitary representation ω_n^H and we will let ω_n^S denote the representation on the smooth vectors of ω_n^H. If F is non-archimedean, unramified and of residue characteristic greater than two, then ω_n has a unique line invariant by $Sp_{2n}(\mathcal{O}_F)$. If H is a maximal isotropic subspace of V_n and $V_n = H \oplus H^*$ is a decomposition of V_n, then ω_n^S can be realised on the space of Schwartz functions on H^*. Let P_H denote the parabolic subgroup stabilising H, let $U_H \cong \mathrm{Hom}^+(H^*, H)$ denote its unipotent radical (the $+$ indicates symmetric homomorphisms) and let $L \cong GL_H$ denote the Levi component preserving the decomposition $V_n = H \oplus H^*$. If we write Mp_{2n} as pairs in $Sp_{2n} \times \mu_8$ with respect to c_H then we have the following explicit formulae

- if $u \in U_H(F)$ then $((u, 1)\phi)(x) = \psi_v(\langle ux, x\rangle/2)\, \phi(x)$;
- if $g \in L_H(F)$ then $((g, 1)\phi)(x) = |\det g|^{1/2}\, \phi(g^* x)$;
- if $e \in \mu_8$ then $((1, e)\phi)(x) = e\, \phi(x)$.

There are two liftings $\tilde{i}: R(F) \to Mp_{n(m+2r)}$ of i and they differ by the quadratic character of GO_{W_r}. We fix one which is given with respect to $c_{X_n \otimes W_r}$ by:

$$(g, h) \mapsto (i(g, h), d_W(g)),$$

where d_W is given as follows. Let $e_1, \ldots, e_n$ be a basis of X_n; let $e_1^*, \ldots, e_n^*$ be the dual basis of X_n^*; for $S \subset \{1, \ldots, n\}$ let w_S denote the element of Sp_n given by $w_S e_i = e_i$ if $i \notin S$ and $= e_i^*$ if $i \in S$ and $w_S e_i^* = e_i^*$ if $i \notin S$ and $= -e_i$ if $i \in S$; let $(\, , \,)$ denote the norm-residue for F; and let D_W denote the discriminant of W; then

$$d_W\left(\begin{pmatrix} a_1 & b_1 \\ 0 & a_1^{*-1}\end{pmatrix} w_S \begin{pmatrix} a_2 & b_2 \\ 0 & \mu a_2^{*-1}\end{pmatrix}\right) = \gamma(W)^{\# S}(\det a_1 a_2, (-1)^{\dim W/2} D_W).$$

We shall say that admissible representations Π and π of $GSp_n(F)$ and $GO_{W_r}(F)$ respectively are associated if there is a non-trivial map of $R(F)$-modules $\omega_{n(m+2r)} \to \Pi \otimes \check{\pi}$. Note that if π and Π are associated then their central characters are related by $\chi_\Pi = \chi_\pi^{-1}$.

Assume now that F is a finite extension of $\mathbb{Q}_p$ for some p. Let Q denote the parabolic subgroup of GO_{W_r} preserving the flag $\langle f_1 \rangle \subset \langle f_1, f_2 \rangle \subset \ldots \subset \langle f_1, \ldots, f_r \rangle$ and let $R_Q = R \cap (GSp_{2n} \times Q)$. Let N_Q denote the unipotent radical of Q (and R_Q). Let N_Q' denote the subgroup $\mathrm{Hom}^-(Y_r^*, Y_r)$ of N_Q, where $-$ denotes the antisymmetric homomorphisms. Let P_i be the parabolic subgroup of GSp_{2n} which is the stabiliser of the flag $\langle e_1 \rangle \subset \langle e_1, e_2 \rangle \subset \ldots \subset \langle e_1, \ldots, e_i \rangle$. Let $R_{P_i,Q} = R \cap (P_i \times Q)$ and let $N_{P_i,Q}$ denote its unipotent radical. Then $R_{P_i,Q}/N_{P_i,Q} \cong \mathbb{G}_m^i \times R_i \times \mathbb{G}_m^r$, where R_i is defined in the same way as R, but with reference to $GSp_{2(n-i)}$ and GO_W. We denote a typical element of this quotient $(\alpha_1, \ldots, \alpha_i, (g, h), \beta_1, \ldots, \beta_r)$. The following lemma follows from the method of [K].

Lemma 4 *The Jacquet module $\omega_{n(m+2r), N_Q'}$ has a filtration with steps $\omega_{n(m+2r),i}$ for $i = 0, \ldots, r$. Let $\chi = \chi_1 \ldots \chi_r$ be a continuous character of $(F^\times)^r$ and let $\omega_{n(m+2r),i,\chi}$ denote the maximal quotient of the Jacquet module $\omega_{n(m+2r),i,N_Q}$ on which $(F^\times)^r \subset Q(F)$ acts by χ. ($(F^\times)^r \hookrightarrow Q(F)$ so that the j^{th} factor acts on f_j by multiplication and acts trivially on the other basis elements.) Then $\omega_{n(m+2r),i,\chi} = (0)$ unless $\chi_j = | \ |^n$ for $j = 1, \ldots, r-i$. If these conditions are met then $\omega_{n(m+2r),i,\chi}$ is the unnormalised induction from $R_{P_i,Q}$ of the representation of $R_{P_i,Q}/N_{P_i,Q}$ given by*

$$(\alpha_1, \ldots, \alpha_i, (g, h), \beta_1, \ldots, \beta_r) \mapsto |\mu(g)|^{nr/2 - ni - mi/4} \, |\alpha|^{n+m/2} (\alpha, (-1)^{\dim W/2} D_W) \, |\beta|^n$$
$$\cdot \prod_{j=1}^{i} \chi_{r-i+j}(\alpha_i^{-1} \beta_{r-i+j} \, \mu(g)) \, \omega_{(n-i)m} \circ \tilde{i}(g, h),$$

where $\alpha = \alpha_1 \ldots \alpha_i$, $\beta = \beta_1 \ldots \beta_{r-i}$, D_W is the discriminant of W and $(\, , \,)$ is the Hilbert symbol.

Now suppose that $F = \mathbb{R}$. Let $J \in Sp_{2n}(\mathbb{R})$ satisfy $J^2 = -1_{2n}$ and $\langle Jv, v \rangle > 0$ for all non-zero $v \in V_n$. For instance take $J = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}$. Then V_n becomes an n dimensional complex vector space isomorphic to $H \otimes \mathbb{C}$ $(h_1 + \sqrt{-1}\, h_2 \mapsto h_1 + J h_2)$. Let $\mathscr{F}$ denote the space of polynomial functions on the complex vector space $H^* \otimes \mathbb{C}$. Write $\mathscr{F} = \bigoplus_{d=0}^{\infty} \mathscr{F}_d$ where $\mathscr{F}_d$ denotes those of degree d. Let U denote the centraliser of J, it is a maximal compact subgroup of $Sp_{2n}(\mathbb{R})$ isomorphic to $U(n)$. We can also think of $U \subset GL_{H\otimes\mathbb{C}}$. Let $\tilde{U}$ denote the preimage of U in $\widetilde{Sp_{2n}}$. Then $\tilde{U} \cong \{(U, \lambda) \in U \times \mathbb{C}^\times \mid \det u = \lambda^2\}$. ω_n can be realised on $\mathscr{F}$ so that $\tilde{U}$ acts by $((u, \lambda) f)(z) = \lambda f(u^* z)$. If we choose a basis $h_1, \ldots, h_n$ of H which is orthonormal with respect to the form $\langle J, \rangle$ then we can associate $f \in \mathscr{F}$ with a polynomial in n variables $z_1, \ldots, z_n$. We can also consider elements of the Schwartz space $\mathscr{S}(H^*)$ as functions of n real variables $x_1, \ldots, x_n$. There is a map $\mathscr{F} \to \mathscr{S}(H^*)$ given by

$$\prod_{i=1}^{n} z_i^{m_i} \mapsto \left(\prod_i (2\pi)^{-m_i/2} \left(-\frac{\partial}{\partial x_i} + \pi x_i\right)^{m_i}\right) e^{-(\pi/2)\sum_i x_i^2}.$$

(See for instance [P1, 1.4.20]. The difference in powers of two from this reference is due to the difference between $\psi_\infty(x)$ and $e^{4\pi i x}$. We use exactly the same action on $\mathscr{F}$ as in [P1].) This identifies the admissible $(\mathfrak{sp}_{2n}, (\tilde{U} \times \mu_8)/\{\pm 1\})$ module $\mathscr{F}$ with the $(\tilde{U} \times \mu_8)/\{\pm 1\}$ finite vectors of $\mathscr{S}(H^*)$ in the smooth representation described above. The space $\mathscr{F}$ is called the Fock model for ω_n.

Let $O_W = O(p, q)$ (so $p+q$ is even). Then R has maximal compact subgroup $U(n) \times O(p) \times O(q)$. Let $f_1, \ldots, f_p, g_1, \ldots, g_q$ be an orthogonal basis of W for which each f_i has length 1 and each g_i has length -1. Then the following vectors form a standard symplectic basis of $V_n \otimes W$: $e_i \otimes f_j$, $e_i \otimes g_j$, $e_i^* \otimes f_j$ and $-e_i^* \otimes g_j$. Thus we get

$$i\left(\begin{pmatrix} a & b \\ -b & a \end{pmatrix}, (\alpha, \beta)\right) = \begin{pmatrix} a \otimes (\alpha \oplus \beta) & b \otimes (\alpha \oplus -\beta) \\ b \otimes (-\alpha \oplus \beta) & a \otimes (\alpha \oplus \beta) \end{pmatrix}.$$

Thus as a map $U(n) \times O(p) \times O(q) \to U(n(p+q))$ we see that $i(u, \alpha, \beta) = (u \otimes \alpha) \oplus ({}^t u^{-1} \otimes \beta)$. Now consider $\tilde{i}$: $U(n) \times O(p) \times O(q) \to \widetilde{(U(n(p+q)) \times \mu_8)}/\{\pm 1\}$. We see that there is only one possibility for $\tilde{i}$: $U(n) \to \widetilde{(U(n(p+q)) \times \mu_8)}/\{\pm 1\}$, namely $\tilde{i}(u, 1_p, 1_q) = ((u^{\oplus p} \oplus ({}^t u^{-1})^{\oplus q}, (\det u)^{(p-q)/2}), 1)$. Take H to be the space spanned by the $e_i \otimes f_j$ and $e_i \otimes g_j$. Then we see that $\tilde{i}(1, \alpha, \beta)$ acts on the functions $2 x_k e^{-\pi \sum_i x_i^2}$ by $((\det \alpha \det \beta)^n, (-1)^{(p-q)/2})(\alpha \oplus \beta)^{\oplus n}$. Thus $\tilde{i}(1, \alpha, \beta) = (((\alpha \oplus \beta)^{\oplus n}, 1), 1)$ if $(\det \alpha \det \beta)^n = 1$ and $= (((\alpha \oplus \beta)^{\oplus n}, -\sqrt{-1}), (-1, (-1)^{(p-q)/2})\sqrt{-1})$ if $(\det \alpha \det \beta)^n = -1$. Thus $U(n) \times O(p) \times O(q)$ acts on $\mathscr{F}_d$ via

$$(u, \alpha, \beta) \mapsto (((\det \alpha \det \beta)^n, (-1)) \det u)^{(p-q)/2}) \operatorname{Symm}^d(u \otimes \alpha \oplus {}^t u^{-1} \otimes \beta).$$

Now let F be a number field. There are unique central extensions $\widetilde{Sp_{2n}}(\mathbb{A}_F)$ and $Mp_{2n}(F)$ of $Sp_{2n}(\mathbb{A}_F)$ of degrees two and eight respectively which are surjective images of the restricted products over all places v of $\widetilde{Sp_{2n}}(F_v)$ and $Mp_{2n}(F_v)$. The restriction is with respect to the subgroups $Sp_{2n}(\mathcal{O}_{F_v})$. It is a theorem of Weil that there is a unique homomorphism $Sp_{2n}(F) \hookrightarrow \widetilde{Sp_{2n}}(\mathbb{A}_F)$ lifting the diagonal embedding into $Sp_{2n}(\mathbb{A}_F)$. We let ω_n also denote the Weil representation of $Mp_{2n}(\mathbb{A}_F)$, i.e. the restricted tensor product of the $\omega_{n,v}$. Again we have a corresponding unitary representation ω_n^H and smooth representation ω_n^S. We will give a description of ω_n^S below. Moreover there is a continuous linear form

θ on ω_n^S such that for $g \in Sp_{2n}(F)$ we have $\theta \circ \omega_n^S(g) = \theta(g)$. If $\phi \in \omega_n$ define θ_ϕ to be the function on $Mp_{2n}(\mathbb{A}_F)$ which sends g to $\theta(\omega_n^S(g)\phi)$. This gives an automorphic form on $Mp_{2n}(\mathbb{A}_F)$.

If A is a ring let $GSp_{2n}^+(A)$ denote the elements in $GSp_{2n}(A)$ such that $\mu(g) = \nu(h)$ for some $h \in GO_{W_r}(A)$. If f is a cuspidal automorphic form on $GO_{W_r}(\mathbb{A}_F)$ we define a function $\theta_\phi(f)$ on $GSp_{2n}^+(F)\backslash GSp_{2n}^+(\mathbb{A}_F)$ by

$$\theta_\phi(f)(g) = \int_{O_{W_r}(F)\backslash O_{W_r}(\mathbb{A}_F)} \theta_\phi(r(g, h h_0)) f(h h_0)\, dh,$$

where h_0 is chosen with $\nu(h_0) = \mu(g)$ and the measure dh is as described after formula 5.1.11 in [HK]. The definition is easily checked to be independent of h_0 and to be left $GSp_{2n}^+(F)$ invariant. Extend $\theta_\phi(f)$ to a function on $GSp_{2n}(F)\backslash GSp_{2n}(\mathbb{A}_F)$ by insisting that it is left $GSp_{2n}(F)$ invariant and zero outside $GSp_{2n}(F)\, GSp_{2n}^+(\mathbb{A}_F)$. It is an automorphic form on $GSp_{2n}(\mathbb{A}_F)$. Now let π be a cuspidal automorphic representation of $GO_{W_r}(\mathbb{A}_F)$. Define $\Theta_n(\pi)$ to be the admissible $GSp_{2n}(\mathbb{A}_F)$ module within the space of automorphic forms generated by the $\theta_\phi(f)$ for $\phi \in \omega_{n(m+2r)}$ and $f \in \pi$. Suppose that Π is an irreducible quotient of $\Theta_n(\pi)$. Then we get a non-trivial intertwining operator of $R(\mathbb{A}_F)$ modules $\omega_n \to \Pi \otimes \check{\pi}$. In particular for all places v of F we have that π_v and Π_v are associated.

We describe ω_n^S. Let H be a maximal isotropic subspace of V_n and let $V_n = H \oplus H^*$ be a decomposition of V_n. Write elements of $Mp_{2n}(\mathbb{A}_F)$ as pairs $(g, e) \in Sp_{2n}(\mathbb{A}_F) \times \mu_8$ with multiplication defined by the cocycle c_H (for all but finitely many v, $c_{H(F_v)}$ vanishes on $Sp_{2n}(\mathcal{O}_{F_v})^2$ and so $c_H = \prod_v c_{H(F_v)}$ makes sense).

The representation space of ω_n^S is the space of Schwartz functions on $H^*(\mathbb{A}_F)$. Let P_H denote the parabolic subgroup of Sp_{2n} of elements which stabilise H, let U_H denote its unipotent radical (it is isomorphic to the space of symmetric homomorphisms $H^* \to H$) and let L_H denote the Levi component consisting of elements which preserve a decomposition $V_n = H \oplus H^*$. Then the action of the inverse image of P_H in Mp_{2n} is given by

- if $u \in U_H(\mathbb{A}_F)$ then $((u, 1)\phi)(x) = \psi(\langle u x, x\rangle/2)\, \phi(x)$;
- if $g \in L_H(\mathbb{A}_F)$ then $((g, 1)\phi)(x) = \|\det g\|^{1/2}\, \phi(g^* x)$;
- if $e \in \mu_8$ then $((1, e)\phi)(x) = e\,\phi(x)$.

The linear form θ takes ϕ to $\sum_{x \in H^*(F)} \phi(x)$.

The following lemma is proved in exactly the same way as Theorem I.1.1 in [Ra].

Lemma 5 *Either $\Theta_n(\pi)$ is contained in the space of cusp forms or $\Theta_{n-1}(\pi) \neq (0)$.*

If T is a symmetric $n \times n$ matrix over F and if f is an automorphic form on $GSp_{2n}(\mathbb{A}_F)$ we define the T^{th} Fourier coefficient f_T of f by

$$f_T(g) = \int_{U(F)\backslash U(\mathbb{A}_F)} f\left(\begin{pmatrix} 1_n & u \\ 0 & 1_n \end{pmatrix} g\right) \psi^{-1}((\operatorname{tr} Tu)/2)\, du,$$

where U denotes the unipotent radical of the Siegel parabolic. Note that $g(f)_T = g(f_T)$. Now let π be a cuspidal automorphic representation of GO_{W_r}, let $f \in \pi$

and $\phi \in \omega_{n(m+2r)}$. We calculate $\theta_\phi(f)_T$. We shall assume that T is non-degenerate. Let $g \in GSp_{2n}^+(\mathbb{A}_F)$ and let $h_0 \in GO_{W_r}$ with $\nu(h_0)=\mu(g)$. Let $\phi' = r(g, h_0)\phi$. Then

$$\begin{aligned}\theta_\phi(f)_T(g) &= \int_{U(F)\backslash U(\mathbb{A}_F)} \int_{O_{W_r}(F)\backslash O_{W_r}(\mathbb{A}_F)} \theta(r(ug, hh_0)\phi) f(hh_0)\psi^{-1}((\operatorname{tr} Tu)/2)\, dh\, du \\ &= \int_{O_{W_r}(F)\backslash O_{W_r}(\mathbb{A}_F)} f(hh_0)\Big(\sum_{x \in (X_n^\# \otimes W_r^\#)(F)} (r(1,h)\phi')(x) \\ &\qquad \int_{U(F)\backslash U(\mathbb{A}_F)} \psi(\langle (u\otimes 1)x, x\rangle/2)\psi^{-1}((\operatorname{tr} Tu)/2)\, du\Big)\, dh \\ &= \int_{O_{W_r}(F)\backslash O_{W_r}(\mathbb{A}_F)} f(hh_0)\Big(\sum_{x \in \operatorname{Hom}(X_n, W)(F)} (r(1,h)\phi')(x) \\ &\qquad \int_{U(F)\backslash U(\mathbb{A}_F)} \psi(\operatorname{tr}({}^t x Q x - T)u/2)\, du\Big)\, dh \\ &= \int_{O_{W_r}(F)\backslash O_{W_r}(\mathbb{A}_F)} f(hh_0) \sum_{x \in \operatorname{Hom}(X_n, W)(F)_T} \phi'(h\circ x)\, dh.\end{aligned}$$

Here Q denotes a matrix representing the quadratic form on W_r and $\operatorname{Hom}(X_n, W)(F)_T$ denotes those homomorphisms x such that $\langle S_{W_r} x(e_i), x(e_j)\rangle = T_{ij}$. By Witt's theorem this set is either empty, in which case $\theta_\phi(f)_T = 0$, or it forms a single orbit under $O_{W_r}(F)$. In the latter case $W_r \cong W_T \oplus W_T'$ for some quadratic space W_T' (we have written W_T for the n dimensional quadratic space corresponding to T). The stabiliser of W_T is isomorphic to $O_{W_T'}$. Let x_0: $X_2 \xrightarrow{\sim} W_T$ such that $\langle S_{W_r} x_0(e_i), x_0(e_j)\rangle = T_{ij}$. In this case we get

$$\theta_\phi(f)_T(g) = \int_{O_{W_T'}(\mathbb{A}_F)\backslash O_{W_r}(\mathbb{A}_F)} \phi'(h\circ x_0) \int_{O_{W_T'}(F)\backslash O_{W_T'}(\mathbb{A}_F)} f(h' h h_0)\, dh'\, dh.$$

In particular $\theta_\phi(f)_T \equiv 0$ for all $\phi \in \omega_{n(m+2r)}$ and $f \in \pi$ if and only if for all $f \in \pi$

$$\int_{O_{W_T'}(F)\backslash O_{W_T'}(\mathbb{A}_F)} f(h)\, dh = 0.$$

We also deduce the following lemma.

Lemma 6 *Suppose that $W_r = A \oplus B$ for non-degenerate quadratic spaces A and B with $\dim A = n$. Let π be a cuspidal automorphic representation of $GO_{W_r}(\mathbb{A}_F)$. If for some $f \in \pi$ we have*

$$\int_{O_B(F)\backslash O_B(\mathbb{A}_F)} f(h)\, dh \neq 0,$$

then $\Theta_n(\pi) \neq (0)$.

3 The local theta lift

In this section we specialise the discussion of the last section to the case of the orthogonal similitude group GO of the first section and GSp_4.

First we consider the non-archimedean case. Before considering the liftings of interest to us we record two lemmas that will be helpful. The first is a standard calculation. The second follows from Rodier's classification [Ro].

Lemma 7 *The L-group of GSp_4 is $GSp_4(\mathbb{C})$. If Π is the unramified sub-quotient of the representation of $GSp_4(\mathbb{Q}_v)$ unitarily induced from the character of $P_2(\mathbb{Q}_v)$ which is trivial on the unipotent radical and sends:*

$$\mathrm{diag}(a, b, \mu a^{-1}, \mu b^{-1}) \mapsto \chi_1(a)\,\chi_2(b)\,\chi_3(\mu),$$

then Π has Langlands parameter $(\chi_3(v), \chi_3\chi_1(v), \chi_3\chi_2(v), \chi_3\chi_1\chi_2(v)) \in GSp_4(\mathbb{C})$.

We remark that we use $\chi(v)$ as an abreviation for the value of χ at a uniformiser of $\mathbb{Q}_v$, when χ is unramified.

Lemma 8 *Suppose Π is an irreducible pre-unitary representation of $GSp_4(\mathbb{Q}_v)$ which is a subquotient of an unramified principal series representation with Langlands parameter* $\mathrm{diag}(\alpha, \beta, \gamma, \delta) \in GSp_4(\mathbb{C})$, *then either Π is the full induced representation or the absolute values of α, β, γ, δ are, up to the action of the Weyl group, Nv to the power $(-1/2, -r, r, 1/2)$ with $0 \leqq r \leqq 1/4$, or $(-1/2, -1/2, 1/2, 1/2)$, or $(-3/2, -1/2, 1/2, 3/2)$.*

Now let v be a prime of $\mathbb{Q}$ which is inert and unramified in K. Let W denote the quadratic space such that $W(\mathbb{Q}_v) = K_v$ with quadratic form equal to minus the norm form. Then $GO = GO_{W_1}$. We start by proving the following result.

Lemma 9 *Let R denote the subgroup of Sp_4 associated to GSp_2 and GO_W. Then the $O_W(\mathbb{Q}_v)$ coinvariants of $\omega_2 \circ \tilde{\imath}$ are the representation of $GSp_2^+(\mathbb{Q}_v)$ induced (via unnormalised induction) from the character*

$$\begin{pmatrix} b & * \\ 0 & ab^{-1} \end{pmatrix} \mapsto \varepsilon_M(b)\,|b|\,|a|^{-1/2}$$

of the Borel of upper triangular matrices.

Proof. $\omega_2 \circ \tilde{\imath}$ can be realised on the Schwartz space $\mathcal{S}(K_v)$, so that $R_{P_1}(\mathbb{Q}_v)$ acts by:

$$\left(\begin{pmatrix} b & n \\ 0 & (b\,\nu(h))^{-1} \end{pmatrix}, h\right)(f)(x) = (b, -D_W)\,|b|\,|\nu(h)|^{1/2}\,\psi(b\,n\,\nu(h)(N\,x)/2)\,f(b\,h^*\,x).$$

As $O_W(\mathbb{Q}_v)$ is compact and acts transitively on the elements of K_v of given norm, the co-invariants can be described as $\mathcal{S}(\mathbb{Q}_v^+, \mathbb{C})$ (where $+$ denotes the elements of even norm) with the action:

$$\left(\begin{pmatrix} b & n \\ 0 & (b\,\nu(h))^{-1} \end{pmatrix}, h\right)(f)(x) = (b, -D_W)\,|b|\,|\nu(h)|^{1/2}\,\psi(b\,n\,\nu(h)(N\,x)/2)\,f(b^2\,\nu(h)\,x).$$

One can see that under the upper triangular matrices the subspace of functions that vanish at 0 is irreducible. Looking at the action of $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ one sees that the whole representation is irreducible (see for instance [B] for the action of this element). Moreover the Jacquet module with respect to the upper triangular unipotent matrices has a one dimensional quotient ($f \mapsto f(0)$) on which elements of the form

$$\begin{pmatrix} b & 0 \\ 0 & (b\,\nu(h))^{-1} \end{pmatrix}$$

acts as $(b, -D_W)\,|b|\,|\nu(h)|^{1/2}$. The result follows.

Lemma 10 *Suppose that v is a finite rational prime inert and unramified in K. Suppose that π is an unramified irreducible pre-unitary principal series representation of $\mathrm{GL}_2(K_v)$ with Langlands parameters (α, β). Suppose that Π is a pre-unitary admissible representation of $GSp_2(\mathbb{Q}_v)$ which is associated to the representation $(\pi, \tilde{\chi}, +)$ of $GO(\mathbb{Q}_v)$. Then Π is an unramified irreducible principal series representation of $GSp_2(\mathbb{Q}_v)$ with Langlands parameter $\mathrm{diag}(\sqrt{\alpha^{-1}}, -\sqrt{\alpha^{-1}}, \sqrt{\beta^{-1}}, -\sqrt{\beta^{-1}}) \in GSp_2(\mathbb{C})$, where $\sqrt{\alpha}\sqrt{\beta} = \tilde{\chi}(v)$. The same remains true if π and Π are pre-unitary only up to a twist.*

Proof. $\tilde{\pi}$ is unitarily induced from two characters of the form $\chi_1 \circ N$, $\chi_2 \circ N$ where each χ_i is a character of $\mathbb{Q}_v^\times$ and $\chi_1 \chi_2 = \tilde{\chi}$. Note that $\chi_1 \varepsilon_K$, $\chi_2 \varepsilon_K$ is also a possible pair. We consider the pair GSp_4 and $GO = GO_{W_1}$. Then $L_Q(\mathbb{Q}_v) = T(\mathbb{Q}_v)$ acts on $(\pi, \tilde{\chi}, +)_Q$ so that t acts trivially and $T^\circ(\mathbb{Q}_v)$ acts by:

$$(r, x) \mapsto \chi_2(Nx)\,|Nx|^{-1/2} (\chi_1/\chi_2)(r)\,|r|$$

or the character obtained by swapping χ_1 and χ_2. Note that $\chi_1/\chi_2 \neq |\ |$ and so there must be a non-trivial intertwining operator to $\Pi \otimes (\pi, \tilde{\chi}, +)_Q$ from the unnormalised induction from $R_{P_1,Q}(\mathbb{Q}_v)$ to R_Q of the representation which is trivial on the unipotent radical and which sends (in the notation of the last section):

$$(a, (g, h), b) \mapsto |\mu(g)|^{-1/2} (\chi_1/\chi_2)(\mu(g))\,|a|^2 (\varepsilon_K \chi_2/\chi_1)(a)\,\omega_2(\tilde{i}(g, h)),$$

or of the representation obtained by swapping χ_1 and χ_2. Let $P_2^+(\mathbb{Q}_v) = P_2(\mathbb{Q}_v) \cap GSp_2^+(\mathbb{Q}_v)$. Using Lemma 9 we see that $(\Pi \otimes (\chi_2^{-1}|\ |^{1/2}) \circ \mu)|_{GSp_4^+(\mathbb{Q}_v)}$ must contain a non-trivial quotient of the unnormalised induction from $B^+(\mathbb{Q}_v)$ of the character which is trivial on the unipotent radical and sends

$$\mathrm{diag}(a, b, \mu^{-1}a, \mu^{-1}b) \mapsto |\mu|^{-1} (\chi_1/\chi_2)(\mu)\,|a|^2 (\varepsilon_K \chi_2/\chi_1)(a)\,|b|\,\varepsilon_K(b),$$

or of the character obtained by swapping χ_1 and χ_2. Thus Π must be a quotient of the unnormalised induction from $B(\mathbb{Q}_v)$ to $GSp_2(\mathbb{Q}_v)$ of one of the characters:

$$\mathrm{diag}(a, b, \mu^{-1}a, \mu^{-1}b) \mapsto |\mu|^{-3/2}\,\chi_1(\mu)\,|a|^2 (\varepsilon_K \chi_2/\chi_1)(a)\,|b|\,\varepsilon_K(b),$$

$$\mathrm{diag}(a, b, \mu^{-1}a, \mu^{-1}b) \mapsto |\mu|^{-3/2} (\varepsilon_K(\mu)\chi_1)(\mu)\,|a|^2 (\varepsilon_K \chi_2/\chi_1)(a)\,|b|\,\varepsilon_K(b),$$

or of the same with χ_1 and χ_2 swapped. By Lemma 7 all these four representations have unramified subquotients with Langlands parameters:

$$\mathrm{diag}(\chi_1(v), \chi_2\,\varepsilon_K(v), \chi_1\,\varepsilon_K(v), \chi_2(v)).$$

The result now follows from the fact that $|\chi_i(v)| < |Nv|^{1/2}$ and Lemma 8.

We remark that as $GSp_4^+(\mathbb{Q}_v)$ is properly contained in $GSp_4(\mathbb{Q})$ it seems not to be obvious that the lift be unique. We wonder in what generality such uniqueness statements hold.

Now suppose that v is a rational prime which splits in K. We fix an identification $K_v \cong \mathbb{Q}_v \oplus \mathbb{Q}_v$. We have $GO(\mathbb{Q}_v) \cong GO_{(0)_2}(\mathbb{Q}_v)$ and we can take

$$f_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad f_2 = \begin{pmatrix} 0 & (1,0) \\ (0,1) & 0 \end{pmatrix}.$$

We shall use the basis f_1, f_2, f_1^*, f_2^* of $(0)_2$.

Lemma 11 *Suppose that v is a finite rational prime which splits $v = w^c w$ in K. Suppose that $\pi = \pi_1 \otimes \pi_2$ is an unramified irreducible pre-unitary principal series representation of $\mathrm{GL}_2(K_v) \cong \mathrm{GL}_2(\mathbb{Q}_v)^2$ with Langlands parameters (α_1, β_1) and (α_2, β_2). Suppose that Π is a pre-unitary admissible representation of $GSp_2(\mathbb{Q}_v)$ which is associated to the representation $(\pi, \tilde{\chi}, +)$. Then Π is an unramified irreducible principal series representation of $GSp_2(\mathbb{Q}_v)$ with Langlands parameter $\mathrm{diag}(\alpha_1, \alpha_2, \beta_1, \beta_2) \in GSp_2(\mathbb{C})$. The same remains true for π and Π pre-unitary only up to a twist.*

Proof. $\tilde{\pi}$ is induced from two pairs of characters (χ_{11}, χ_{21}) and (χ_{12}, χ_{22}) with $\tilde{\chi} = \chi_{11}\chi_{21} = \chi_{12}\chi_{22}$. Then $T^\circ(\mathbb{Q}_v) = L_Q(\mathbb{Q}_v)$ acts on $(\pi, \tilde{\chi}, +)$ by some of the characters

$$\mathrm{diag}(t_1, t_2, t_3 t_1^{-1}, t_3 t_2^{-1}) \mapsto (\chi_{11}/\chi_{12})(t_1)\,|t_2|(\chi_{12}/\chi_{21})(t_2)\,|t_3|^{-1/2}\,\chi_{21}(t_3)$$

or one of its conjugates under the group $\mathscr{W}$ of order eight which is generated by the elements σ_1, which switches χ_{11} and χ_{21}, and τ which switches χ_{j1} and χ_{j2} for $j = 1, 2$. Because π is unitary and irreducible principal series we have that $\chi_{ij} \neq \chi_{i'j'}|\ |$. Thus, for one of the characters $\vec{\chi}$ above, $\Pi \otimes \vec{\chi}$ must be a quotient of the induction from $R_{P_2,Q}(\mathbb{Q}_v)$ to $R_Q(\mathbb{Q}_v)$ of the character which is trivial on the unipotent radical and sends

$$(\mathrm{diag}(a, b, \mu a^{-1}, \mu b^{-1}), \mathrm{diag}(t_1, t_2, (\mu t_1)^{-1}, (\mu t_2)^{-1}))$$

onto:

$$|\mu|^{-2}\,|ab|^2\,\vec{\chi}(a^{-1} t_1 \mu, b^{-1} t_2 \mu, 1).$$

Thus $\Pi \otimes \vec{\chi}(1, 1, \mu^{-1})$ must be a quotient of the un-normalised induction from $P_2(\mathbb{Q}_v)$ to $GSp_2(\mathbb{Q}_v)$ of a character which is trivial on unipotents and sends:

$$\mathrm{diag}(a, b, \mu a^{-1}, \mu b^{-1}) \mapsto |\mu|^{-1/2}\,|b|\,\vec{\chi}(a^{-1}\mu, b^{-1}\mu, 1)$$

for one of the characters $\vec{\chi}$. Thus Π is a quotient of the un-normalised induction from $P_2(\mathbb{Q}_v)$ of the character which sends

$$\mathrm{diag}(a, b, \mu a^{-1}, \mu b^{-1}) \mapsto (\chi_{12}/\chi_{11})(a)(\chi_{21}/\chi_{12})(b)\,\chi_{11}(\mu)$$

or one of its conjugates by $\mathscr{W}$. The un-normalised induction of all these characters have unramified subquotients with Langlands parameters

$$\mathrm{diag}(\chi_{11}(v), \chi_{12}(v), \chi_{21}(v), \chi_{22}(v)) \in GSp_2(\mathbb{C}).$$

The result now follows from Lemma 8 as in the inert case.

Finally in this section we turn to the archimedean case. Consider the representation $\hat{\pi}=(\pi_{N,s}, \varepsilon|\ |^s, \delta)$ of $GO(\mathbb{R})$, where $\varepsilon=1$ or sgn, $\delta=\pm$ and $\pi_{N,s}$ has Langlands parameter:

$$z \mapsto \begin{pmatrix} |z|^s z^N & 0 \\ 0 & |z|^{s\,c} z^N \end{pmatrix}$$

($N\in\mathbb{Z}_{>0}$ and $s\in\mathbb{C}$). We label the irreducible representations of $U(2)$ (respectively $U(1)$) by pairs of integers (x, y) with $x\geqq y$ (respectively by integers x). These are the usual highest weights. We will identify the group of characters on the diagonal maximal torus in GSp_4 with triples $(a, b; c)\in\mathbb{Z}^3$ with $a+b\equiv c \bmod 2$ via

$$\mathrm{diag}(x, y, zx^{-1}, zy^{-1})\mapsto x^a y^b z^{(c-a-b)/2}.$$

Lemma 12 1. *If* $\varepsilon=\mathrm{sgn}^{N+1}$ *or if* $\delta=-1$ *then* $\hat{\pi}$ *is associated to no representation of* $GSp_2(\mathbb{R})$.
2. *If* $\varepsilon=\mathrm{sgn}^N$ *and* $\delta=-1$ *then* $\hat{\pi}$ *is associated to no representation of* $GSp_2(\mathbb{R})$ *or* $GSp_4(\mathbb{R})$.
3. *If* Π *is a representation of* $GSp_4(\mathbb{R})$ *associated to* $\hat{\pi}$ *then*

- Π *contains the* $U(2)$ *type indicated by the following table*

$(\varepsilon(-1), \delta)$	$U(2)$-*type*
$((-1)^N, +)$	$(N+1, 1)$
$((-1)^{N+1}, +)$	$(N+1, 0)$
$((-1)^{N+1}, -)$	$(N+1, 2)$

- $\chi_\Pi=\varepsilon|\ |^{-s}$;
- Π *has infinitesimal character with Harish Chandra parameter* $(N, 0; -(N+s))$.

Proof. We will use Howe's theory of K-types, see [H] for an account of this theory. Consider the Fock model $\mathscr{F}=\bigoplus_0^\infty \mathscr{F}_d$ for the lift from $GO(\mathbb{R})$ to $GSp_2(\mathbb{R})$. Then $U(1)\times O(3)\times O(1)$ acts on $\mathscr{F}_d$ via the representation

$$(1)\otimes \mathrm{Symm}^d((1)\otimes(1, -, +)\oplus(-1)\otimes(0, +, -))$$
$$\cong \bigoplus_{a=0}^{d} \bigoplus_{0\leqq 2b\leqq a} (2a+1-d)\otimes(a-2b, (-)^a, (-)^{d-a}).$$

The Howe minimal K-types occurring are then $(0)\otimes(0, +, -)$ and $(1+d)\otimes(d, (-)^d, +)$ for $d\in\mathbb{Z}_{\geqq 0}$. These have degrees 1 and d respectively. If $(1+d)\otimes(d, (-)^d, +)$ corresponds to $\Pi\otimes\check{\hat{\pi}}$ then the only other $O(3)\times O(1)$-types occurring in $\check{\hat{\pi}}$ can be $(e, (-)^e, +)$ with $e>d$ (and $(0, +, -)$ if $d=0$). Thus we must have $d=N$, $\delta=1$ and $\varepsilon(-1)=(-1)^N$. The first part follows.

Now consider the Fock model $\mathscr{F}=\bigoplus_{0}^{\infty}\mathscr{F}_d$ for the lift from $GO(\mathbb{R})$ to $GSp_4(\mathbb{R})$. Then $U(2)\times O(3)\times O(1)$ acts on $\mathscr{F}_d$ via the representation

$$(1,1)\otimes \mathrm{Symm}^d((1,0)\otimes(1,-,+)\oplus(0,-1)\otimes(0,+,-))$$
$$\cong\bigoplus_{a=0}^{d}(1,1)\otimes\mathrm{Symm}^a((1,0)\otimes(1,-,+))\otimes\mathrm{Symm}^{d-a}((0,-1))\otimes(0,+,(-)^{d-a}).$$

We have

$$\mathrm{Symm}^a((1,0)\otimes(1,-,+))=(a,0)\otimes(a,(-)^a,+)\oplus(a-1,1)\otimes(a-1,(-)^a,+)\oplus\ldots$$

where the omitted terms only involve $(e,(-)^a,+)$ for $e<a-1$. We must omit the last term for $a=0$ or 1. Thus

$$\mathscr{F}_d\cong(d+1,1)\otimes(d,(-)^d,+)\oplus(d,2)\otimes(d-1,(-)^d,+)$$
$$\oplus((d,0)\oplus(d-1,1))\otimes(d-1,(-)^d,-)\oplus\ldots$$

where the omitted terms involve $(e,(-)^a,(-)^{d-a})$ for $e<d-1$. For $d=0$ we just get the first term and for $d=1$ we just get the first and third terms. Thus the Howe minimal $O(3)\times O(1)$ types are as follows.

- $(e,(-)^e,+)$ for $e\geqq 0$ which has degree e and occurs in $\mathscr{F}_e$ as $(e+1,1)\otimes(e,(-)^e,+)$.
- $(e,(-)^e,-)$ for $e\geqq 0$ which has degree $e+1$ and occurs in $\mathscr{F}_{e+1}$ as $(e+1,0)\otimes(e,(-)^e,-)$.
- $(e,(-)^{e+1},+)$ for $e\geqq 1$ which has degree $e+1$ and occurs in $\mathscr{F}_{e+1}$ as $(e+1,2)\otimes(e,(-)^{e+1},+)$.
- $(e,(-)^{e+1},-)$ does not occur.
- $(0,-,+)$ does not occur.

(If $(0,-,+)$ did occur then the trivial representation of $SO(3)$ would occur in $S^a(X_1^2)$ where $a>0$ is an odd integer and X_d denotes the $2d+1$ dimensional representation of $SO(3)$. Thus the trivial representation occurs in $S^b(X_1)\otimes S^c(X_1)$ for some non-negative integers b, c with $b+c>0$ and odd. Thus it also occurs in $X_d\otimes X_e$ for non-negative integers d, e of different parity. This is a contradiction.)

Suppose that $\hat{\pi}$ is associated to some Π. Then the Howe minimal $O(3)\times O(1)$ type of $\hat{\pi}$ must be $(N,(-1)^N\delta,(-1)^N\delta\varepsilon(-1))$ (all other $O(3)\times O(1)$ types occurring in $\hat{\pi}$ have degree $\geqq N+1$). Thus $\delta=-1$, $\varepsilon(-1)=(-1)^N$ does not occur. In the other cases we see that Π must have Howe minimal $U(2)$ type as listed in the table.

The calculation of the infinitesimal character of Π follows from the results of [P2].

Corollary 3 *Keep the notation of the lemma. Assume that $N>1$, that $\varepsilon=\mathrm{sgn}^{N+1}$ and that Π is unitary. If $\delta=1$ then Π is a non-holomorphic limit of discrete series representation. If $\delta=-1$ then Π is a holomorphic limit of discrete series representation.*

Proof. Combine the lemma with the list of unitary representations for $Sp_4(\mathbb{R})$ in [P1].

Presumably this corollary remains true for $N=1$ but we have not checked this.

4 The global theta lift

We now consider the global theta lifting from $GO(\mathbb{A})$ to $GSp_4(\mathbb{A})$. Let $\hat{\pi}=(\pi, \tilde{\chi}, \delta)$ denote a cuspidal automorphic representation of $GO(\mathbb{A})$. Thus π is a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_K)$ with central character χ with $\chi^c=\chi$, $\tilde{\chi}$ is a grossencharacter over $\mathbb{Q}$ such that $\chi=\tilde{\chi}\circ N$ and δ maps the set of rational places to $\{\pm 1\}$ and satisfies the conditions described in section one. Let $\Theta(\hat{\pi})$ denote the theta lift of $\hat{\pi}$ to $GSp_4(\mathbb{A})$. If w is a place of K at which π is unramified let π_v have Langlands parameters $\{\alpha_w, \beta_w\}$. Then we have the following result.

Proposition 3 1. *Let S denote the set of rational primes which do not ramify in K and above which π is unramified and $\delta=+1$. Suppose that Π is an irreducible quotient of $\Theta(\hat{\pi})$ and $v\notin S$. Then Π_v is an unramified irreducible principal series representation with Langlands parameters*

- $\mathrm{diag}(\sqrt{\alpha_v^{-1}}, -\sqrt{\alpha_v^{-1}}, \sqrt{\beta_v^{-1}}, -\sqrt{\beta_v^{-1}})\in GSp_2(\mathbb{C})$ *where* $\sqrt{\alpha_v}\sqrt{\beta_v}=\tilde{\chi}(v)$ *if v is inert in K;*
- $\mathrm{diag}(\alpha_w^{-1}, \alpha_{c_w}^{-1}, \beta_w^{-1}, \beta_{c_w}^{-1})\in GSp_2(\mathbb{C})$ *if v splits as $w^c w$ in K.*

2. *Suppose that π_∞ has Langlands parameter*

$$z\mapsto\begin{pmatrix}|z|^s z^N & 0\\ 0 & |z|^s {}^c z^N\end{pmatrix}$$

with $N\in\mathbb{Z}_{>0}$ and $s\in\mathbb{C}$. Also suppose that $\tilde{\chi}_\infty(-1)=(-1)^{N+1}$. Then $\Theta(\hat{\pi})$ is contained in the space of cusp forms. Moreover if Π is an irreducible constituent of $\Theta(\hat{\pi})$ then Π_∞ has infinitesimal character with Harish Chandra parameter $(N, 0; -(N+s))$ and contains the $U(2)$ type $(N+1, 1-\delta_\infty)$. If $N>1$ and $\delta_\infty=1$ then Π_∞ is a non-holomorphic limit of discrete series representation; if $N>1$ and $\delta_\infty=-1$ then Π_∞ is a holomorphic limit of discrete series representation.
3. *If there is a grossencharacter ϕ of $\mathbb{A}_K^\times/K^\times$ whose restriction to $\mathbb{A}^\times$ is $\tilde{\chi}$ and such that*

- *for all places v of $\mathbb{Q}$ with $\pi_v\cong\pi_v^c$ we have:*

$$0\neq 1+\delta(v)\,\tilde{\chi}_v(-1)\,\varepsilon(\pi_v\otimes\psi_v^{-1}, 1/2),$$

- $L(\pi\otimes\psi^{-1}, 1/2)\neq 0,$

then $\Theta(\hat{\pi})\neq(0)$.

Proof. Parts 1 and 2 follow from the compatibility of the local and global lifts, the results of the last section and Lemma 5.

Let W denote the two dimensional quadratic space associated to minus the norm from K to $\mathbb{Q}$, so that $GO = GO_{W_1}$. $W_1 \cong W \oplus H$, where H denotes the hyperbolic plane. By Lemma 6, $\Theta(\hat{\pi})$ is non-zero if for some $f \in \hat{\pi}$ we have

$$\int_{O_H(\mathbb{Q})\backslash O_H(\mathbb{A})} f(h)\, dh \neq 0.$$

For ϕ a character as in the proposition and for $y \in \mathbb{A}_K^\times$ define

$$C_{f,\phi}(y) = \phi(y)^{-1} \int_{O_H(\mathbb{Q})\backslash O_H(\mathbb{A})} f\left(x\left(\sigma\begin{pmatrix} y & 0 \\ 0 & 1\end{pmatrix}\right)\right) dx.$$

We must show that for some y, ϕ and f, $C_{f,\phi}(y) \neq 0$. We calculate $C_{f,\phi}(y)$. Let S be a finite set of places so that $f \in \hat{\pi}^{\hat{U}^S}$, y is a unit outside S, $\infty \in S$ and primes ramifying in K are in S. Let $w' = \begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}$. For a set R of places define $w'_R \in G(\mathbb{A})$ by $w'_{R_v} = 1_2$ if $v \notin R$ and $= w'$ if $v \in R$. Let $(\tilde{f}_R)_{R \subset S} \in (\mathrm{Ind}_{G_S}^{\hat{G}_S} \hat{\pi}^\circ)^+$ correspond to f as in section one. Then:

$$\begin{aligned}
C_{f,\phi}(y) &= 2^{-|S|-1} \sum_{R\subset S} \int_{O_H^\circ(\mathbb{Q})\backslash O_H^\circ(\mathbb{A})} \phi(y)^{-1} \tilde{f}\left(x\, t_R\, \sigma\left(w'_R \begin{pmatrix} y & 0 \\ 0 & 1\end{pmatrix}\right)\right) dx \\
&= 2^{-|S|-1} \sum_{R\subset S} \int_{O_H^\circ(\mathbb{Q})\backslash O_H^\circ(\mathbb{A})} \phi(y)^{-1} \tilde{f}\left(x\, \sigma\left(\begin{pmatrix} y^R & 0 \\ 0 & {}^c y_R\end{pmatrix} w'_R\right) t_R\right) dx \\
&= 2^{-|S|-1} \sum_{R\subset S} \int_{\mathbb{Q}^\times\backslash\mathbb{A}^\times} \phi(y)^{-1} \tilde{f}\left(x^{-1}\, \sigma\left(\begin{pmatrix} x\, y^R & 0 \\ 0 & {}^c y_R\end{pmatrix} w'_R\right) t_R\right) dx \\
&= 2^{-|S|-1} \sum_{R\subset S} \int_{\mathbb{Q}^\times\backslash\mathbb{A}^\times} \phi(x\,y)^{-1} \tilde{f}_R \circ \sigma\left(\begin{pmatrix} x\, y^R & 0 \\ 0 & {}^c y_R\end{pmatrix} w'_R\right) dx \\
&= 2^{-|S|-1} \sum_{R\subset S} \int_{\mathbb{Q}^\times\backslash\mathbb{A}^\times} \phi(x\,y)^{-1} \tilde{f}_R \circ \sigma\left({}^c y_R \begin{pmatrix} x\, y/N\, y_R & 0 \\ 0 & 1\end{pmatrix} w'_R\right) dx \\
&= 2^{-|S|-1} \sum_{R\subset S} \int_{\mathbb{Q}^\times\backslash\mathbb{A}^\times} \phi(x\,y/N\,y_R)^{-1} \tilde{f}_R \circ \sigma\left(\begin{pmatrix} x\, y/N\, y_R & 0 \\ 0 & 1\end{pmatrix} w'_R\right) dx \\
&= 2^{-|S|-1} \sum_{R\subset S} \int_{\mathbb{Q}^\times\backslash\mathbb{A}^\times} \phi(x\,y)^{-1} \tilde{f}_R \circ \sigma\left(\begin{pmatrix} x\, y & 0 \\ 0 & 1\end{pmatrix} w'_R\right) dx.
\end{aligned}$$

So we see that if $z \in \mathbb{A}^\times K^\times$ then $C_{f,\phi}(z\,y) = C_{f,\phi}(y)$. Thus we have to show that for some character λ on $K^\times \mathbb{A}^\times \backslash \mathbb{A}_K^\times$ we have:

$$0 \neq \int_{K^\times \mathbb{A}^\times\backslash\mathbb{A}_K^\times} C_{f,\phi}(y)\, \lambda(y)\, dy = \int_{K^\times \mathbb{A}^\times\backslash\mathbb{A}_K^\times} C_{f,\phi\lambda^{-1}}(y)(y)\, dy,$$

i.e. for some choice of ϕ:

$$0 \neq \sum_{R\subset S} \int_{K^\times\backslash\mathbb{A}_K^\times} \phi(x)^{-1} \tilde{f}_R \circ \sigma\left(\begin{pmatrix} x & 0 \\ 0 & 1\end{pmatrix} w'_R\right) dx.$$

In the case $\pi = \pi^c$ let $f_R = \tilde{f}_R \circ \sigma \in \pi$. In the case $\pi \neq \pi^c$ we have $\tilde{f}_R \circ \sigma = f_R + f_{S-R} \circ c$ for some $f_R \in \pi$. In the latter case our integral becomes:

$$\sum_{R \subset S} \int_{K^\times \backslash \mathbb{A}_K^\times} \left(\phi(x)^{-1} f_R\left(\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} w'_R \right) + \phi(x)^{-1} f_{S-R}\left(\begin{pmatrix} {}^c x & 0 \\ 0 & 1 \end{pmatrix} w'_R \right) \right) dx$$

$$= \sum_{R \subset S} \int_{K^\times \backslash \mathbb{A}_K^\times} \left(\phi(x)^{-1} f_R\left(\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} w'_R \right) + \phi(x)^{-1} f_R\left(\begin{pmatrix} {}^c x & 0 \\ 0 & 1 \end{pmatrix} w'_{S-R} \right) \right) dx$$

$$= \sum_{R \subset S} \int_{K^\times \backslash \mathbb{A}_K^\times} \left(\phi(x)^{-1} f_R\left(\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} w'_R \right) + \phi(x)^{-1} f_R\left(\begin{pmatrix} 1 & 0 \\ 0 & {}^c x \end{pmatrix} w'_R \right) \right) dx$$

$$= \sum_{R \subset S} \int_{K^\times \backslash \mathbb{A}_K^\times} \left(\phi(x)^{-1} f_R\left(\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} w'_R \right) + \phi(x)^{-1} \phi(N x) f_R\left(\begin{pmatrix} {}^c x^{-1} & 0 \\ 0 & 1 \end{pmatrix} w'_R \right) \right) dx$$

$$= \sum_{R \subset S} 2 \int_{K^\times \backslash \mathbb{A}_K^\times} \phi(x)^{-1} f_R\left(\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} w'_R \right) dx.$$

Thus in either case if we let f_R correspond to a function W_R in the Whittaker model we must show that:

$$\sum_{R \subset S} \int_{\mathbb{A}_K^\times} \phi(x)^{-1} W_R\left(\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} w'_R \right) |x|^{(s-1/2)} dx$$

which makes sense for Re s sufficiently large, is non-zero at $s = 1/2$ (defined by analytic continuation).

Write $S = S_1 \cup S_2$ where for $v \in S_1$ $\pi_v \cong \pi_v^c$ and for $v \in S_2$ $\pi_v \not\cong \pi_v^c$. Recall that the only constraint on the choice of W_R is that $W_{R \Delta \{v\}} = \theta_v^{\delta(v)} W_R$ for $v \in S_1$. Thus we must show that for some W in the Whittaker model of π we have:

$$\sum_{R \subset S_1} \int_{\mathbb{A}_K^\times} \phi(x)^{-1} \prod_{v \in R} \theta_v^{\delta(v)} W\left(\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} w'_R \right) |x|^{(s-1/2)} dx$$

does not vanish at $s = 1/2$. We lose no generality in assuming that $W = \prod_v W_v$. Then for Re s sufficiently large the integral becomes:

$$\prod_{v \notin S_1} \int_{K_v^\times} \phi_v(x)^{-1} W_v \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} |x|_v^{(s-1/2)} dx$$

$$\prod_{v \in S_1} \int_{K_v^\times} \phi_v(x)^{-1} \left(W_v \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} + \delta(v) W_v \left(\begin{pmatrix} {}^c x & 0 \\ 0 & 1 \end{pmatrix} w' \right) \right) |x|_v^{(s-1/2)} dx$$

$$= \prod_{v \notin S_1} \int_{K_v^\times} \phi_v(x)^{-1} W_v \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} |x|_v^{(s-1/2)} dx$$

$$\prod_{v \in S_1} \int_{K_v^\times} \left(\phi_v(x)^{-1} W_v \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} + \delta(v) \phi_v({}^c x)^{-1} W_v \left(\begin{pmatrix} -x & 0 \\ 0 & 1 \end{pmatrix} w \right) \right) |x|_v^{(s-1/2)} dx$$

$$= \prod_{v \notin S_1} \int_{K_v^\times} \phi_v(x)^{-1} W_v \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} |x|_v^{(s-1/2)} dx$$

$$\prod_{v \in S_1} \int_{K_v^\times} \left(\phi_v(x)^{-1} W_v \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} + \delta(v) \phi_v(-1) \phi_v({}^c x)^{-1} W_v \left(\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} w \right) \right) |x|_v^{(s-1/2)} dx,$$

where $w=\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Now let π' denote $\pi \otimes \phi^{-1}$ and let W' denote $W\phi^{-1}$. Let:

$$\zeta(g, \lambda, W_v', s) = \int_{K_v^\times} \lambda(x)\, W_v'\left(\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} g\right) |x|_v^{(s-1/2)}\, dx.$$

We may choose W_v so that $\zeta(1, 1, W_v', s) = L(\pi_v', s)$. Note that π' has central character $\phi^c \phi^{-1}$. Then our integral is:

$$\prod_{v \notin S_1} L(\pi_v', s) \prod_{v \in S_1} (L(\pi_v', s) + \delta(v)\, \phi_v(-1)\, \zeta(w, (\phi^c \phi^{-1})^{-1}, W_v', s))$$

$$= \prod_{v \notin S_1} L(\pi_v', s) \prod_{v \in S_1} (L(\pi_v', s) + \delta(v)\, \phi_v(-1)\, \varepsilon(\pi_v', 1-s)\, L((\pi_v')^c, s))$$

$$= L(\pi', s) \prod_{v \in S_1} (1 + \delta(v)\, \phi_v(-1)\, \varepsilon(\pi_v', 1-s))$$

where we have used the local functional equation and the fact that $L((\pi_v')^c, s) = L(\pi_v', s)$. So what we require is:

$$0 \neq L(\pi', 1/2) \prod_{v \in S_1} (1 + \delta(v)\, \phi_v(-1)\, \varepsilon(\pi_v', 1/2)).$$

Noting that $\phi_v(-1) = \tilde{\chi}_v(-1)$ we see that the final part of the proposition follows.

We will finish this section by making some remarks about the conditions in part three of the lemma. Our guess is that one can remove the condition of the L-function not vanishing, it is probably an artifact of the particular Fourier coefficient we have decided to evaluate. We also guess that the local conditions are exactly the conditions for a local theta lift to exist (and hence are necessary conditions).

Let π be a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_K 7$ with central character χ satisfying $\chi^c = \chi$. Let $\tilde{\chi}$ be a grossencharacter over $\mathbb{Q}$ with $\chi = \tilde{\chi} \circ N$. Then we have the following observations.

Lemma 13 *Suppose that π_∞ has Langlands parameter*

$$\mathbb{C}^\times \to \mathrm{GL}_2(\mathbb{C})$$

$$z \mapsto \begin{pmatrix} |z|^s z^N & 0 \\ 0 & |z|^{tc} z^N \end{pmatrix}$$

with $s, t \in \mathbb{C}$, $N \in \mathbb{Z}_{\geq 0}$. Let ϕ_∞ denote a character of $\mathbb{C}^\times$ with $\phi_\infty|_{\mathbb{R}^\times} = \tilde{\chi}$. Then we have.

1. *If $N=0$ or if $\tilde{\chi}_\infty(-1) = (-1)^N$ then $\varepsilon(\pi_\infty \phi_\infty^{-1}, 1/2) = \tilde{\chi}_\infty(-1)$.*
2. *If $N>0$ and $\tilde{\chi}_\infty(-1) = (-1)^{N+1}$ then the set of such characters ϕ_∞ with $\varepsilon(\pi_\infty \phi_\infty^{-1}, 1/2) = -\tilde{\chi}_\infty(-1)$ is non-empty but finite. In fact it consists of the characters $z \mapsto |z|^{N+s}(z/|z|)^M$ for $|M| \leq N$ and $M \equiv N-1 \bmod 2$.*

Proof. Let $\tilde{\chi}_\infty(-1)=(-1)^\beta$ with $\beta=0$ or 1. Then $\phi_\infty(z)=z^{-M}|z|^{N+M+(s+t)/2}$ for some $M\in\mathbb{Z}$ with $M\equiv\beta \bmod 2$. Then we have

$$\begin{aligned}\varepsilon(\pi_\infty\phi_\infty^{-1},1/2)&=\varepsilon(|z|^{(s-t)/2-N-M}z^{N+M},1/2)\,\varepsilon(|z|^{(t-s)/2+N-M}z^{M-N},1/2)\\&=\sqrt{-1}^{\,|N+M|+|N-M|}\\&=\begin{cases}\tilde{\chi}_\infty(-1) & \text{if } N\leqq|M|\\(-1)^N & \text{if } N\geqq|M|,\end{cases}\end{aligned}$$

and the lemma follows.

Notice that this lemma is consistent with the guesses made above and the calculations on the local lifting at infinity made in section three.

Lemma 14 *Let v be a place such that $\pi_v\cong\pi_v^c$. Let ϕ_v be a character of $K_v^\times$ with $\phi_v|_{\mathbb{Q}_v^\times}=\tilde{\chi}_v$. Then we have.*
1. *For all places v, $\varepsilon(\pi_v\otimes\phi_v^{-1},1/2)$ does not depend on the choice of additive character ψ_v as long as $\psi_v=\psi_v^c$.*
2. $\varepsilon(\pi_v\otimes\phi_v^{-1},1/2)=\pm1.$
3. $\varepsilon(\pi_v\otimes\phi_v^{-1},1/2)=\tilde{\chi}_v(-1)$ *in the following cases.*

- *π_v unramified;*
- *v split;*
- *π_v principal series corresponding to two characters χ_1,χ_2 with $\chi_i^c=\chi_i$;*
- *ϕ_v is sufficiently ramified.*

The first and last assertions do not require that $\pi_v\cong\pi_v^c$.

4. *Suppose that v is not split and that π_v is the base change of a discrete series representation σ of $\mathrm{GL}_2(\mathbb{Q}_v)$ with central character χ_σ (note that this depends only on π_v). Let D_v denote the non-split quaternion algebra with centre $\mathbb{Q}_v$ so that $K_V\hookrightarrow D_v$. Let σ^D denote the representation of $D^\times$ corresponding to σ by the Jacquet-Langlands correspondence. If $\tilde{\chi}_v\neq\chi_\sigma$ and $v\nmid 2$ then $\varepsilon(\pi_v\otimes\phi_v^{-1},1/2)=\tilde{\chi}_v(-1)$. If $\tilde{\chi}_v=\chi_\sigma$ then the set of ϕ_v as above with $\varepsilon(\pi_v\otimes\phi_v^{-1},1/2)=-\tilde{\chi}_v(-1)$ is equal to the set of characters of $K_v^\times$ occurring in $\sigma^D|_{K_v^\times}$. In particular it is non-empty but finite.*

Proof. (1) If $a\in\mathbb{Q}_v^\times$ we have

$$\begin{aligned}\varepsilon(\pi_v\otimes\phi_v^{-1},1/2,\psi_{v}a)&=(\tilde{\chi}_v\circ N)\,\phi_v^{-2}(a)\,\varepsilon(\pi_v\otimes\phi_v^{-1},1/2,\psi_v)\\&=(\phi_v^c/\phi_v)(a)\,\varepsilon(\pi_v\otimes\phi_v^{-1},1/2,\psi_v)\\&=\varepsilon(\pi_v\otimes\phi_v^{-1},1/2,\psi_v).\end{aligned}$$

For the subsequent parts of this lemma we assume that v is finite, the case v infinite following from the last lemma.

(2)

$$\begin{aligned}\varepsilon(\pi_v\otimes\phi_v^{-1},1/2)^2&=\varepsilon(\pi_v\otimes\phi_v^{-1},1/2)\,\varepsilon(\pi_v\otimes\phi_v^{-c},1/2)\\&=\varepsilon(\pi_v\otimes\phi_v^{-1},1/2)\,\varepsilon(\check{\pi}_v\otimes\phi_v,1/2)\\&=((\tilde{\chi}_v\circ N)\,\phi_v^{-2})(-1)\\&=1.\end{aligned}$$

(3) For the first assertion note that as π_v is unramified so is $\tilde{\chi}_v \circ N$ and hence χ_v (as K_v is non-ramified). Thus $\tilde{\chi}(-1)=1$. Now let f denote the conductor of ϕ_v. Then we have

$$\begin{aligned}\varepsilon(\pi_v \otimes \phi_v^{-1}, 1/2) &= \chi_v(f)\, \varepsilon(\phi_v^{-1}, 1/2)^2 \\ &= \chi_v(f)\, \varepsilon(\phi_v^{-1}, 1/2)\, \varepsilon(\chi_v^{-1}\phi_v, 1/2) \\ &= \varepsilon(\phi_v^{-1}, 1/2)\, \varepsilon(\phi_v, 1/2) \\ &= \phi_v(-1) \\ &= \tilde{\chi}_v(-1) \\ &= 1.\end{aligned}$$

For the second assertion let $v = w^c w$. Then

$$\begin{aligned}\varepsilon(\pi_v \otimes \phi_v^{-1}, 1/2) &= \varepsilon(\pi_w \phi_w^{-1}, 1/2)\, \varepsilon(\pi_w \phi_w \tilde{\chi}_v^{-1}, 1/2) \\ &= \varepsilon(\pi_w \phi_w^{-1}, 1/2)\, \varepsilon(\check{\pi}_w \phi_w, 1/2) \\ &= (\tilde{\chi}_v \phi_w^2)(-1) \\ &= \tilde{\chi}(-1).\end{aligned}$$

For the third assertion

$$\begin{aligned}\varepsilon(\pi_v \otimes \phi_v^{-1}, 1/2) &= \varepsilon(\chi_1 \phi_v^{-1}, 1/2)\, \varepsilon(\chi_2 \phi_v^{-1}, 1/2) \\ &= \varepsilon(\chi_1 \phi_v^{-1}, 1/2)\, \varepsilon(\chi_2 \phi_v^{-c}, 1/2) \\ &= \varepsilon(\chi_1 \phi_v^{-1}, 1/2)\, \varepsilon(\chi_2 (\tilde{\chi}_v \circ N)^{-1} \phi_v, 1/2) \\ &= \varepsilon(\chi_1 \phi_v^{-1}, 1/2)\, \varepsilon(\chi_1^{-1} \phi_v, 1/2) \\ &= (\chi_1 \phi_v^{-1})(-1) \\ &= \tilde{\chi}(-1),\end{aligned}$$

because $\chi_1 = \tilde{\chi}_1 \circ N$ and $N(-1)=1$.

For the fourth assertion note that (for ϕ_v sufficiently ramified) there exists $y \in K_v^\times$ such that

$$\begin{aligned}\varepsilon(\pi_v \otimes \phi_v^{-1}, 1/2) &= \chi_v(y)\, \varepsilon(\phi_v^{-1}, 1/2)^2 \\ &= \chi_v(y)\, \varepsilon(\phi_v^{-1}, 1/2)\, \varepsilon(\chi_v^{-1} \phi_v, 1/2) \\ &= \varepsilon(\phi_v^{-1}, 1/2)\, \varepsilon(\phi_v, 1/2) \\ &= \phi_v(-1) \\ &= \tilde{\chi}_v(-1).\end{aligned}$$

(4) In the case $\tilde{\chi}_v = \chi_\sigma$ this follows from the main theorem of [Tu] plus the remarks on p. 1297 of that paper if $v \nmid 2$ and from [Sa] if $v | 2$. Thus we suppose that $\tilde{\chi}_v \neq \chi_\sigma$. We first consider the case σ is special. Then σ is a subquotient of the induction to GL_2 of the character of the Borel of upper triangular matrices defined by a pair $(\lambda|\ |^{-1/2}, \lambda|\ |^{1/2})$ of characters of $\mathbb{Q}_v^\times$. Let $\phi'_v = \phi_v^{-1}(\lambda \circ N)$, so that $\phi'_v|_{\mathbb{Q}_v^\times}$ is the quadratic character corresponding to $K_v/\mathbb{Q}_v$. Then

$$\varepsilon(\pi_v \circ \phi_v^{-1}, 1/2) = \varepsilon(|\ |^{-1/2} \phi'_v, 1/2)\, \varepsilon(|\ |^{1/2} \phi'_v, 1/2)\, \alpha,$$

where $\alpha=1$ if ϕ'_v is ramified and $=-\phi'_v(\varpi_v)^{-1}$ if ϕ'_v is unramified. If ϕ'_v is unramified then $K_v/\mathbb{Q}_v$ is unramified and so ϕ'_v is the unramified quadratic character and in any case $\alpha=1$. Thus

$$\begin{aligned}\varepsilon(\pi_v\circ\phi_v^{-1}, 1/2) &= \varepsilon(|\ |^{-1/2}\phi'_v, 1/2)\,\varepsilon(|\ |^{1/2}(\phi'_v)^c, 1/2)\\ &= \varepsilon(|\ |^{-1/2}\phi'_v, 1/2)\,\varepsilon(|\ |^{1/2}(\phi'_v)^{-1}, 1/2)\\ &= \phi'_v(-1)\\ &= \tilde{\chi}_v(-1).\end{aligned}$$

Next suppose that π_v is principal series corresponding to a pair of characters (λ, λ^c) with $\lambda \neq \lambda^c$. Set $\phi'_v=\lambda\,\phi_v^{-1}$ so that $\phi'_v|_{\mathbb{Q}_v^\times}=1$. Then

$$\begin{aligned}\varepsilon(\pi_v\circ\phi_v^{-1}, 1/2) &= \varepsilon(\phi'_v, 1/2)\,\varepsilon((\lambda^c/\lambda)\,\phi'_v, 1/2)\\ &= k^2((\phi'_v)^2(\lambda^c/\lambda))(\Delta)\\ &= k^2\,\lambda(-1),\end{aligned}$$

where ${}^c\Delta=-\Delta$ and k are independent of ϕ_v (see Theorem 3.2 of [D]). Passing to a very ramified ϕ_v we get the result.

Finally suppose that there is a quadratic extension $L/\mathbb{Q}_v$ different from K_v and a character λ of $L^\times$ such that π_v is the automorphic induction from LK_v of $\lambda\circ N_{LK_v/L}$ and π_v is supercuspidal. Let M denote the third quadratic extension of $\mathbb{Q}_v$ in LK_v. Given ϕ_v set $\mu_{\phi_v}=(\lambda\circ N_{LK_v/L})(\phi_v^{-1}\circ N_{LK_v/K_v})$. Then

$$\mu_{\phi_v}|_{M^\times}=(\theta|_{\mathbb{Q}_v^\times}\,\phi_v^{-1}|_{\mathbb{Q}_v^\times})\circ N_{M/\mathbb{Q}_v}=\delta_M\circ N_{M/\mathbb{Q}_v}=1,$$

where δ_M is the quadratic character corresponding to $M/\mathbb{Q}_v$. Thus we can find k and $\Delta\in LK_v$ which are independent of ϕ_v and with ${}^\tau\Delta=-\Delta$, where τ is the non-trivial automorphism of LK_v fixing M, such that

$$\begin{aligned}\varepsilon(\pi_v\otimes\phi_v^{-1}, 1/2) &= \varepsilon(\mu_{\phi_v}, 1/2)\\ &= k\,\mu_{\phi_v}(\Delta)\\ &= k\,\theta(-\Delta')\,\phi_v^{-1}(\Delta')\\ &= k\,\delta_L(-1)\,\delta_M(\Delta')\,\tilde{\chi}_v(-1)\end{aligned}$$

where $\Delta'=N_{LK_v/K_v}\Delta\in\mathbb{Q}_v^\times$. Thus $\varepsilon(\pi_v\otimes\phi_v^{-1}, 1/2)$ is independent of ϕ_v and so identically 1. This completes the proof of the lemma.

Having analysed the local conditions we turn to the global condition. Despite the fact that we expect no global obstruction to the lifting our results are much less complete. We shall say that a pair $(\pi, \tilde{\chi})$ as above admits δ if for all places v such that $\pi_v\cong\pi_v^c$ there exists a character ϕ_v of $K_v^\times$ such that $\phi_v|_{\mathbb{Q}_v^\times}=\tilde{\chi}_v$ and $\varepsilon(\pi_v\otimes\phi_v^{-1}, 1/2)\,\tilde{\chi}_v(-1)=\delta_v$. If $\pi\cong\pi^c$ we also insist that $\prod_v\delta_v=1$.

Proposition 4 *Let $\tilde{\pi}=(\pi, \tilde{\chi})$ be a cuspidal automorphic representation of $GO^0(\mathbb{A})$. Suppose moreover that*

- *there is a grossencharacter ϕ of $\mathbb{A}^\times/\mathbb{Q}^\times$ with $\phi^2=\tilde{\chi}$;*
- *either $\varepsilon(\pi\otimes\phi^{-1}\circ N)=1$ or π_v is special for some finite place v and $\pi\not\cong\pi^c$.*

If η is a quadratic character of $\mathbb{A}_K^\times/K^\times$ and if v is a rational place with $\pi_v \cong \pi_v^c$ set $\delta_{\eta,v} = \varepsilon(\pi_v \otimes \eta(\phi_v \circ N)^{-1}, 1/2)$ (then $(\pi \otimes \eta, \tilde{\chi})$ admits δ). Then there is a set $\mathscr{H}$ of quadratic characters of $\mathbb{A}_K^\times/K^\times$ with the following properties.

1. *If R is a finite set of finite places of K disjoint from the primes where π, π^c or K are ramified and if for $v \in R$, η_v is an unramified character of $K_v^\times$ with $\eta_v^2 = 1$, then there is a character $\eta \in \mathscr{H}$ which localises to η_v for $v \in R$.*
2. $\Theta(\pi \otimes \eta, \tilde{\chi}, \delta_\eta) \neq (0)$.

Proof. By the previous proposition it is enough to prove the proposition with the second condition replaced by $L(\pi \otimes \eta(\phi \circ N)^{-1}, 1/2) \neq 0$. By Theorem 4 of [W] it is enough to prove it with the third condition replaced by $\varepsilon(\pi \otimes \eta(\phi \circ N)^{-1}, 1/2) = 1$. Now let $\mathscr{H}$ denote the set of quadratic characters η of $\mathbb{A}_K^\times/K^\times$ for which $\varepsilon(\pi \otimes \eta(\phi \circ N)^{-1}, 1/2) = 1$. We show that this set has the first property of the proposition. For v above a prime of S define quadratic characters of $K_v^\times$ as follows. If $\varepsilon(\pi \otimes (\phi \circ N)^{-1}) = 1$ then set $\eta_v = 1$ for all such v. If not choose one such place v_0 such that π_{v_0} is special. Set $\eta_v = 1$ if $v \neq v_0$. $\pi_{v_0} \otimes (\phi_{v_0} \circ N)^{-1}$ is associated to two characters $\lambda |\,|^{1/2}$ and $\lambda |\,|^{-1/2}$ with $\lambda^2 = 1$. We have that $\varepsilon(\pi_{v_0} \otimes (\phi_{v_0} \circ N)^{-1}, 1/2) = \lambda(-1)$ if λ is non-trivial and -1 if λ is trivial. If $\varepsilon(\pi_{v_0} \otimes (\phi_{v_0} \circ N)^{-1}, 1/2) = 1$ take $\eta_{v_0} = \lambda$. If $\varepsilon(\pi_{v_0} \otimes (\phi_{v_0} \circ N)^{-1}, 1/2) = -1$ take η_{v_0} to be the product of λ and the non-trivial unramified quadratic character of $K_{v_0}^\times$. Now choose a quadratic character η of $\mathbb{A}_K^\times/K^\times$ which localises to η_v for all $v \in R$ or above an element of S. (If η_v corresponds to $K_v(\sqrt{x_v})$, choose $x \in K$ sufficiently close to x_v v-adically for all such v and let η correspond to $K(\sqrt{x})$.) Then it is easy to check that η will do.

We can do somewhat better if we assume the following result which has been announced by Bump et al. [BFH], but not written.

Conjecture/Theorem 1 *Let π be a unitary cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_K)$ such that $\varepsilon(\pi, 1/2) = 1$. Let S' denote the set of primes of K which are ramified or for which π_v is ramified. Let R be a disjoint set of K and for $v \in R$ let η_v be an unramified quadratic character of $K_v^\times$. Then there is a quadratic character η of $\mathbb{A}_K^\times/K^\times$ such that*

- *η restricts to η_v on $K_v^\times$;*
- *$\eta_v = 1$ for $v \in S'$;*
- *$L(\pi \otimes \eta, 1/2) \neq 0$.*

Granted this we have the following result.

Proposition 5 *Let $\tilde{\pi} = (\pi, \tilde{\chi})$ be a cuspidal automorphic representation of $GO^\circ(\mathbb{A})$ which admits δ. Suppose moreover that the above conjecture/theorem is true. Then there is a set $\mathscr{H}$ of quadratic characters of $\mathbb{A}_K^\times/K^\times$ with the following properties.*

1. *If $\eta \in \mathscr{H}$ and v is a place with K, π or π^c ramified at v then $\eta_v = 1$.*
2. *If R is a finite set of finite places of K disjoint from the primes where π, π^c or K are ramified and if for $v \in R$, η_v is an unramified quadratic character of $K_v^\times$ with $\eta_v^2 = 1$, then there is a character $\eta \in \mathscr{H}$ which localises to η_v for $v \in R$.*
3. $\Theta(\pi \otimes \eta, \tilde{\chi}, \delta) \neq (0)$.

Proof. Suppose we are given R and η_v for $v \in R$. We will construct η with the desired properties. Note that by enlarging R if necessary we may ensure that $\pi \otimes \eta \not\cong (\pi \otimes \eta)^c$. Also note that it is sufficient to find a quadratic character η_1 and a grossencharacter ϕ such that

- $\eta_{1,v} = 1$ for $v \in R$ or for π, π^c or K ramified at v;

- $\phi|_{\mathbb{A}^\times} = \tilde{\chi}$;
- if $\pi_v \cong \pi_v^c$ then $\delta_v = \tilde{\chi}_v(-1)\,\varepsilon(\pi_v \otimes \phi_v^{-1}, 1/2)$;
- $\varepsilon(\pi \otimes \phi^{-1}, 1/2) = 1$.

Let S denote the set of rational places for which π or K are ramified. For $v \in S$, v not split in K and $\pi_v \cong \pi_v^c$ choose ϕ_v a character of $K_v^\times$ such that $\varepsilon(\pi_v \otimes \phi_v^{-1}, 1/2) = \delta_v\, \tilde{\chi}_v(-1)$ and $\phi_v|_{\mathbb{Q}_v^\times} = \tilde{\chi}$. For the other $v \in S$ choose ϕ_v a character of $K_v^\times$ such that $\phi_v|_{\mathbb{Q}_v^\times} = \tilde{\chi}_v$ and ϕ_v is sufficiently ramified that we must have $\varepsilon(\pi_v \otimes \phi_v^{-1}, 1/2)\,\tilde{\chi}_v(-1) = 1$. If $\pi_\infty \not\cong \pi_\infty^c$ choose ϕ_∞ so that $\varepsilon(\pi_\infty \otimes \phi_\infty^{-1}, 1/2) = 1$. Let $A = \mathbb{A}^\times (\prod_{v \in S \cup R - \{\infty\}} \mathcal{O}_{K,v}^\times)\mathbb{C}^\times \subset \mathbb{A}_K^\times$. Then there is a character ϕ of A which restricts to $\tilde{\chi}$ on $\mathbb{A}^\times$, to ϕ_v on $\mathbb{Q}_v^\times \mathcal{O}_{K,v}$ if $v \in S - \{\infty\}$, to ϕ_∞ on $\mathbb{C}^\times$ and to 1 on $\mathcal{O}_{K,v}^\times$ if $v \in R$. Choose an open subgroup W of $\prod_{v \notin S \cup R} \mathcal{O}_{K,v}^\times$ such that $W \cup \mathcal{O}_K^\times = \{1\}$. Extend ϕ to AW by making it trivial on W. Then extend ϕ to a character on $K^\times AW$ which is trivial on $K^\times$. This character is continuous because AW is open. Finally extend it to a character ϕ on $\mathbb{A}_K^\times / K^\times$. Then $\phi|_{\mathbb{A}^\times} = \tilde{\chi}$; ϕ_v is unramified for $v \in R$; if $v \in S$ and $\pi_v \cong \pi_v^c$ then $\varepsilon(\pi_v \otimes \phi_v^{-1}, 1/2)\,\tilde{\chi}(-1) = \delta_v$; and for all other v, $\varepsilon(\pi_v \otimes \phi_v^{-1}, 1/2)\,\tilde{\chi}(-1) = 1$. Thus we have that $\varepsilon(\pi \otimes \phi^{-1}, 1/2) = \prod_v \delta_v$ (where the product is over those v with $\pi_v \cong \pi_v^c$). If $\prod_v \delta_v = 1$ then we are done.

Thus assume that $\prod_v \delta_v = -1$. Now choose a rational prime w which splits in K and such that $w \notin S$, $w \neq 2$ and ϕ_w is unramified. Choose a quadratic character η_1 with $\eta_{1,v} = 1$ if $v \in S \cup R$, η_1 unramified at one prime, w_1, above w and ramified at the other, w_2. Now choose a grossencharacter λ of $\mathbb{A}_K^\times / K^\times$ such that

- $\lambda|_{\mathbb{A}^\times} = 1$;
- $\lambda_v = 1$ for $v \in S \cup R$;
- if $v \notin S \cup \{w\}$ and $\eta_{1,v}$ is ramified then λ_v is very ramified;
- $\lambda_{1,w}$ is unramified and $\lambda_{1,w_2}(\varpi_{w_2})^2 = -\eta_{1,w_2}(-1)\,\phi_{w_2}^2\,\tilde{\chi}_w^{-1}(\varpi_{w_2})$.

We have the following values for $\varepsilon(\pi_v \otimes \eta_{1,v}(\phi_v \lambda_v)^{-1}, 1/2)\,\tilde{\chi}_v(-1)$.

- δ_v if $v \in S$ and $\pi_v \cong \pi_v^c$.
- 1 for all other $v \in S$ with $\pi_v \not\cong \pi_v^c$.
- 1 for $v \notin S \cup \{w\}$.
- -1 for $v = w$, for in this case

$$\varepsilon(\pi_v \otimes \eta_{1,v}(\phi_v \lambda_v)^{-1}, 1/2)\,\tilde{\chi}_v(-1) = \varepsilon(\eta_{1,w_2}, 1/2)^2 (\tilde{\chi}_w \phi_{w_2}^{-2} \lambda_{w_2}^{-2})(\varpi_{w_2}) = -1.$$

Thus $\varepsilon(\pi \otimes \eta_1 \phi^{-1} \lambda^{-1}, 1/2) = 1$ and again we are done.

5 An arithmetic application

In this section let π be a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_K)$ with central character χ satisfying $\chi^c = \chi$. We shall suppose that $\pi \not\cong \pi^c$. We shall also suppose that π_∞ has Langlands parameter

$$W_{\mathbb{C}} = \mathbb{C}^\times \to \mathrm{GL}_2(\mathbb{C})$$
$$z \mapsto \begin{pmatrix} z^{1-k} & 0 \\ 0 & {}^c z^{1-k} \end{pmatrix},$$

where $k \in \mathbb{Z}_{\geqq 2}$. Let $\tilde{\chi}$ be the grossencharacter of $\mathbb{A}^\times/\mathbb{Q}^\times$ such that $\chi = \tilde{\chi} \circ N$ and $\tilde{\chi}_\infty(-1) = (-1)^k$. Let S denote the set of rational primes which ramify in K or for which π_v is ramified. For w a prime of K above no prime of S, let $\{\alpha_w, \beta_w\}$ denote the Langlands parameters of π_w. Then we have the following result.

Theorem 1 *Keep the above notations and assumptions. Suppose moreover that*

- *k is even;*
- *there is a grossencharacter ϕ of $\mathbb{A}^\times/\mathbb{Q}^\times$ with $\phi^2 = \tilde{\chi}$;*
- *and either $\varepsilon(\pi \otimes \phi^{-1}) = 1$ or π_v is special for some finite place v.*

Then there is a set $\mathscr{H}$ of quadratic characters of $\mathbb{A}_K^\times/K^\times$ with the following properties.

1. *If R is a finite set of finite places of K disjoint from the primes above S and if for $v \in R$, η_v is an unramified quadratic character of $K_v^\times$ with $\eta_v^2 = 1$, then there is a character $\eta \in \mathscr{H}$ which localises to η_v for $v \in R$.*
2. *If $\eta \in \mathscr{H}$ then there is a cuspidal automorphic representation Π_η of $GSp_4(\mathbb{A}_\mathbb{Q})$ with the following properties.*

- *Π_η has central character $\tilde{\chi} \| \ \|^3$.*
- *$\Pi_{\eta,\infty}$ has infinitesimal character $(k-1, 0; 4-k)$ and contains the $U(2)$-type $(k, 2)$.*
- *If $p \notin S$ is a rational prime lying over a split prime v of K with η_v and η_{c_v} unramified then $\Pi_{\eta,v}$ is unramified with Langlands' parameter*

$$\mathrm{diag}(p^{-3/2}\alpha_v\eta(v), p^{-3/2}\alpha_{c_v}\eta(^c v), p^{-3/2}\beta_v\eta(v), p^{-3/2}\beta_{c_v}\eta(^c v)) \in GSp_4(\mathbb{C}).$$

- *If $p \notin S$ is a rational prime lying over an inert prime v of K with η_v and η_{c_v} unramified then $\Pi_{\eta,v}$ is unramified with Langlands' parameter*

$$\mathrm{diag}(\sqrt{p^{-3}\alpha_v\eta(v)}, -\sqrt{p^{-3}\alpha_v\eta(v)}, \sqrt{p^{-3}\beta_v\eta(v)}, -\sqrt{p^{-3}\beta_v\eta(v)}) \in GSp_4(\mathbb{C}),$$

where the square roots are chosen so that $\sqrt{\alpha_v\eta(v)}\sqrt{\beta_v\eta(v)} = \tilde{\chi}(v)p^{-3}$.

Proof. For δ as described in Proposition 4 we see that $\Theta(\pi, \tilde{\chi}, \delta)$ is non-trivial. We can check using the calculation in the proof of Lemma 13 that $\delta_\infty = -1$. Let Π be an irreducible constituent of $\| \ \|^{3/2}\tilde{\chi}\,\Theta(\pi, \tilde{\chi}, \delta)$. Then this result follows from Proposition 3.

We remark that it would follow from the result announced by Bump, Friedberg and Hoffstein (see Conjecture/Theorem 1) that we can remove the three additional assumptions of this theorem (by using Proposition 5).

The crucial point in this theorem is that the lift Π is holomorphic, i.e. it corresponds to a classical holomorphic Siegel modular form. In the case $k=2$ the classical Siegel modular form is scalar valued of weight 2. To achieve this holomorphicity it is essential that $\delta_\infty = -1$, i.e. we are making essential use of the disconnectedness of GO. Because Π is holomorphic it can not have a Whittaker model. Of course π does have a Whittaker model and it is a general principle that theta lifts preserve the property of having a Whittaker model. However again it is the disconnectedness of GO which accounts for this.

Because Π is holomorphic one can start to apply methods of algebraic geometry to it. In fact combining this result with the results of [Ta1] and [Ta2] it is shown in [Ta3] how to attach compatible systems of l-adic representations to π. We have the following result (see [Ta3]).

Theorem 2 *Let K be an imaginary quadratic field, let c denote its non-trivial automorphism and let π be a cuspidal automorphic representation of* $\mathrm{GL}_2(\mathbb{A}_K)$ *such that π_∞ has Langlands parameter*

$$W_{\mathbb{C}} = \mathbb{C}^\times \to \mathrm{GL}_2(\mathbb{C})$$

$$z \mapsto \begin{pmatrix} z^{1-k} & 0 \\ 0 & {}^c z^{1-k} \end{pmatrix},$$

where $k \in \mathbb{Z}_{\geqq 2}$ (i.e. π is any regular algebraic cuspidal automorphic representation up to twist, i.e. any cuspidal automorphic representation contributing to cohomology of the standard local systems on the corresponding three-manifolds). Let χ denote the central character of π, let S denote the set of places of K where $K/\mathbb{Q}$ is ramified or π or π^c is ramified, and for $v \notin S$ let $\{\alpha_v, \beta_v\}$ denote the Langlands parameters of π_v. Let F_π denote the field generated by the $\alpha_v + \beta_v$ and $\alpha_v \beta_v$ for $v \notin S$, it is a number field.

Assume moreover

1. *that $\chi^c = \chi$,*
2. *that k is even,*
3. *that $\chi = \phi^2$ for some grossencharacter ϕ with $\phi^c = \phi$ and that either $\varepsilon(\pi \otimes \phi^{-1}, 1/2) = 1$ or π_v is special for some finite place v.*

Then there is an extension E/F_π of degree at most four and for each prime λ of F_π there is a continuous irreducible representation

$$\rho\colon \mathrm{Gal}(\bar{K}/K) \to \mathrm{GL}_2(E_{\lambda'})$$

(λ' a prime of E above λ) such that if v is a prime of K which is outside S_π and does not divide the residue characteristic l of λ, then ρ is unramified at v and either $\rho(\mathrm{Frob}_v)$ has characteristic polynomial

$$(X - \alpha_v)(X - \beta_v)$$

or $\alpha_{c_v} + \beta_{c_v} = 0$ and $\rho(\mathrm{Frob}_v)$ has characteristic polynomial $(X^2 + \alpha_v \beta_v) = (X - \alpha_{c_v})(X - \beta_{c_v})$. The first possibility occurs outside a set of Dirichlet density zero.

The assumption that $\chi = \chi^c$ is essential to the method. However the assumption that k be even is of a technical nature and one might hope to remove it. The third assumption is only needed to ensure the non-vanishing of the theta lift of this paper. If we assume the result of Bump, Friedberg and Hoffstein (Conjecture/Theorem 1) then this condition is not needed.

A weakness of this theorem is that one can only calculate the trace of Frobenius outside a set of Dirichlet density zero, though this set does have an explicit description. This is enough to determine the l-adic representations completely.

The theorem covers most of the cases where explicit computations have been made. These calculations compute cohomology classes on certain hyperbolic three-manifolds which are known to correspond to the π considered here. When the elliptic curve corresponding to a suitable π (at least $k=2$, χ trivial and $F_\pi = \mathbb{Q}$, but such a π might correspond to an abelian surface with quaternionic multiplication) can be found by some method, one can use the Faltings-Serre method to check that the dual of the Tate module is isomorphic to the l-adic representations we attach to π. This allows one to check the Ramanujan conjecture for π and to prove results about the L-function of the elliptic curve. In [Ta3] we carry this out for a specific example. In fact we prove the following.

Theorem 3 *Let $K=\mathbb{Q}(\sqrt{-3})$ and let $\mathfrak{n}$ denote the prime ideal generated by $(17+\sqrt{-3})/2$. Let $\mathscr{X}$ denote hyperbolic three space, let $\tilde{\Gamma}_0(\mathfrak{n})$ denote the matrices in $\mathrm{GL}_2(\mathcal{O}_K)$ which are congruent to an upper triangular matrix modulo $\mathfrak{n}$ and let $X_\mathfrak{n}^*$ denote $\Gamma_0(\mathfrak{n})\backslash(\mathscr{X}\cup\mathbb{P}^1(K))$. Then $H_1(X_\mathfrak{n}^*, \mathbb{Q})$ is one dimensional. For v a prime of K let $\theta(T_v)$ denote the eigenvalue of the Hecke operator T_v on $H_1(X_\mathfrak{n}^*, \mathbb{Q})$ (see [Cr] or [Ta3] for definitions). Let A denote the elliptic curve $y^2+xy=x^3+(3+\sqrt{-3})x^2/2+(1+\sqrt{-3})x/2$.*

1. *For all places v of K outside a set of Dirichlet density zero we have that $|\theta(T_v)|\leqq 2\sqrt{Nv}$. If this inequality fails (for $v\nmid 6\mathfrak{n}$) then v is split and $\theta(T_{c_v})=0$.*
2. *There is an L-function $L(\pi,s)=\prod_v L(\pi_v,s)$ with analytic continuation to all of $\mathbb{C}$ such that for all v outside a set of Dirichlet density zero we have $L(\pi_v,s)=(1-(1+Nv-\#A(\mathbb{F}_v))(Nv)^{-s}+(Nv)^{1-2s})^{-1}$ (and for all v, $L(\pi_v,s)$ is the inverse of a polynomial in $(Nv)^{-s}$). Moreover if we set $\Lambda(s)=\pi^{-s}(73)^{s/2}\,\Gamma(s)^2\,L(\pi,s)$ then $\Lambda(s)=\Lambda(2-s)$.*

Such results have been conjectured and much numerical evidence obtained for them by many authors. We mention [EGM] and [Cr] as examples. The example above is based on computations of Cremona.

6 Philosophy

Finally it might be useful to try to explain in a more conceptual way some of our results.

Let π be a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_K)$. Then π should correspond to a two dimensional representation of the Langlands group of K. For simplicity assume that π is algebraic so that this representation of the Langlands group should give rise to a system of l-adic representations of $\mathrm{Gal}(\bar{K}/K)$. Fix a prime λ of the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$. Let ρ be the λ-adic representation corresponding to π. Let χ denote the central character of π and assume that $\chi^c=\chi$. Thus there are two grossencharacters over $\mathbb{Q}$, $\tilde{\chi}_\pm$ such that $\chi=\tilde{\chi}_\pm\circ N$. Let $\tilde{\chi}_\pm^G$ be the corresponding λ-adic characters. Choose the notation so that $\tilde{\chi}_\pm^G(c)=\pm 1$. Note that $\tilde{\chi}_+/\tilde{\chi}_-$ is the quadratic character ε corresponding to $K/\mathbb{Q}$.

We can form the four dimensional representation R of $\mathrm{Gal}(\bar{K}/\mathbb{Q})$ induced from ρ. Then $\wedge^2 R$ is the sum of a four dimensional representation, $\tilde{\chi}_+^G$ and $\tilde{\chi}_-^G$. The four dimensional representation will be irreducible unless ρ is in some way degenerate. Thus one obtains (at least) two representations $R_\pm$: $\mathrm{Gal}(\bar{K}/\mathbb{Q})\to GSp_4((\bar{\mathbb{Q}}_l))$ such that $\mu\circ R_\pm=\tilde{\chi}_\pm^G$ and $i\circ R_\pm=R$, where i denotes the natural embedding of GSp_4 into GL_4. Explicitly, if V_ρ denotes the underlying space of ρ and $\langle\,,\rangle_\rho$ denotes its usual alternating form, then R is realised on $V_\rho\oplus V_\rho$

where $R(c)$: $(v, w) \mapsto (w, v)$ and if $\sigma \in \mathrm{Gal}(\bar{K}/K)$ then $R(\sigma)$: $(v, w) \mapsto (\rho(\sigma)v, \rho(c\sigma c)w)$. $R_\pm$ are obtained by taking the alternating forms $\langle (v, w), (v', w') \rangle = \langle v, v' \rangle_\rho \pm \langle w, w' \rangle_\rho$. In terms of matrices, if ρ is given by:

$$\mathrm{Gal}(\bar{K}/K) \to \mathrm{GL}_2(\mathbb{Q}_l)$$

$$\sigma \mapsto \begin{pmatrix} a_\sigma & b_\sigma \\ c_\sigma & d_\sigma \end{pmatrix}$$

then R_+ is given by:

$$\mathrm{Gal}(\bar{K}/K) \to GSp_2(\mathbb{Q}_l)$$

$$\sigma \mapsto \begin{pmatrix} a_\sigma & 0 & b_\sigma & 0 \\ 0 & a_{c\sigma c} & 0 & b_{c\sigma c} \\ c_\sigma & 0 & d_\sigma & \\ 0 & c_{c\sigma c} & 0 & d_{c\sigma c} \end{pmatrix}$$

$$c \mapsto \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

and R_- is given by:

$$\mathrm{Gal}(\bar{K}/K) \to GSp_2(\mathbb{Q}_l)$$

$$\sigma \mapsto \begin{pmatrix} a_\sigma & 0 & b_\sigma & 0 \\ 0 & a_{c\sigma c} & 0 & -b_{c\sigma c} \\ c_\sigma & 0 & d_\sigma & \\ 0 & -c_{c\sigma c} & 0 & d_{c\sigma c} \end{pmatrix}$$

$$c \mapsto \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

As homomorphisms into GSp_4, $R_\pm$ are not related in any simple way. They have the same degree four L-functions but different degree one L-functions. They are not (in general) twists of each other by a character. If π corresponds to an elliptic curve A (i.e. ρ is the dual of the l-adic Tate module of A) then R_- corresponds to the dual of the Tate module of the abelian surface $A \oplus {}^cA/\mathbb{Q}$ with its natural polarisation.

Suppose that π is a base change from $\mathbb{Q}$. Then ρ is the restriction of two dimensional l-adic representations, ρ_1 and ρ_2, of $\mathrm{Gal}(\bar{K}/\mathbb{Q})$. In this case R_+ is conjugate in GSp_4 to the representation:

$$\sigma \mapsto \begin{pmatrix} \rho_1(\sigma) & 0 \\ 0 & \rho_2(\sigma) \end{pmatrix}.$$

In particular it lands inside a Levi subgroup of a Siegel parabolic. One can also see this by noting that the vectors of the form $(v, \rho_1(c)v)$ form an invariant isotropic sub-plane. However R_- does not map to any parabolic subgroup. (If it did it would have to have an invariant proper subspace. This would have to be a plane as ρ is irreducible. Thus it would have to land in a Siegel parabolic and have an invariant isotropic plane. This would consist of all vectors of the form $(v, \alpha v)$ for some $\alpha \in \mathrm{End}(V_\rho)$ with $\det \alpha = 1$, $\alpha^2 = 1_2$ and $\rho(c\sigma c)\alpha = \alpha\rho(\sigma)$ for all $\sigma \in \mathrm{Gal}(\bar{K}/K)$. Then $\alpha = \lambda\rho_1(c)$ for some scalar λ with $\det \lambda 1_2 = -1$ and $\lambda^2 = 1$, a contradiction.) In fact R_- factors through the cuspidal subgroup isomorphic to $\{(a, b) \in \mathrm{GL}_2^2 \mid \det a = \det b\}$, and is given explicitly as $\rho_1 \oplus \rho_2$.

One would expect some sort of packets $\Pi^\pm$ of automorphic representations of $GSp_4(\mathbb{A})$ corresponding to $R_\pm$. Then for any δ we can form $\tilde{\chi}_\pm\, \Theta((\pi, \tilde{\chi}_\pm, \delta))$. The irreducible quotients of this representation will correspond to $R_\pm$ by the calculations of section two. We would thus expect $\tilde{\chi}_-\, \Theta((\pi, \tilde{\chi}_-, \delta))$ to be always cuspidal, which indeed we checked (in many cases) in section three. On the other hand we would expect $\tilde{\chi}_+\, \Theta((\pi, \tilde{\chi}_+, \delta))$ to be cuspidal except when π is a base change from $\mathbb{Q}$. If π is a base change we would expect $\tilde{\chi}_+\, \Theta((\pi, \tilde{\chi}_+, \delta))$ not to meet the space of cusp forms (because of the results of [PS] and [So] on CAP representations).

If for some choices $\delta^\pm$ neither $\tilde{\chi}_+\, \Theta((\pi, \tilde{\chi}_+, \delta^+))$ nor $\tilde{\chi}_-\, \Theta((\pi, \tilde{\chi}_-, \delta^-))$ vanish then we get two automorphic representations in different near equivalence classes, but with the same degree four L-function, i.e. the same lift to GL_4. If π is not a base change this lift will be cuspidal. If however π is a base change then we would get an Eisenstein and a non-CAP cuspidal representation with the same degree four L-function. We have not proved that for some choices $\delta^\pm$ neither $\tilde{\chi}_+\, \Theta((\pi, \tilde{\chi}_+, \delta^+))$ nor $\tilde{\chi}_-\, \Theta((\pi, \tilde{\chi}_-, \delta^-))$ vanish.

Similar comments apply locally. Let v be a place of $\mathbb{Q}$ which does not split in K. Let σ be a supercuspidal representation of $\mathrm{GL}_2(\mathbb{Q}_v)$ with central character $\tilde{\chi}$. Let π_v be its base change to K_v. Suppose that $v \nmid 2$. Then we have the following result.

Lemma 15 *Assume Conjecture/Theorem* 1. *Then there are admissible representations* $\theta(\pi_v, \tilde{\chi}_v, \pm)$ *and* $\theta(\pi_v, \tilde{\chi}_v\varepsilon_v, +)$ *of* $GSp_4(\mathbb{Q}_v)$ *which are associated to the representations* $(\pi_v, \tilde{\chi}_v, \pm)$ *and* $(\pi_v, \tilde{\chi}_v\varepsilon_v, +)$ *of* $GO(\mathbb{A})$.

Proof. There is a cuspidal automorphic representation σ of $\mathrm{GL}_2(\mathbb{A})$ whose local component at v is σ_v and whose base change to K remains cuspidal (either use [Cl1] or explicitly construct σ as an automorphic induction from a quadratic extension). Let π be the base change of σ to K and let $\tilde{\chi}$ be the central character of σ. Define $\delta^\pm$ by $\delta_w^\pm = 1$ if $v \neq w$ and $\delta_v^\pm = \pm 1$.

By Proposition 5 and Lemma 14 we can find a quadratic character $\eta^\pm$ of $\mathbb{A}_K^\times/K^\times$ such that $\eta_v^\pm = 1$ and $\Theta(\pi \otimes \eta^\pm, \tilde{\chi}, \delta^\pm) \neq (0)$. Similarly we can find a quadratic character η' with $\eta'_v = 1$ and $\Theta(\pi \otimes \eta', \tilde{\chi}\varepsilon, \delta^+) \neq (0)$. The lemma follows.

The restriction $v \nmid 2$ is not needed for the existence of $\theta(\pi_v, \tilde{\chi}_v, \pm)$ (see Lemma 14). The existence of the (non-supercuspidal) representation $\theta(\pi_v, \tilde{\chi}_v\varepsilon_v, +)$ probably follows from the methods of Kudla [K], even if $v \mid 2$.

Keeping the notation of the theorem we would expect $\theta(\pi_v, \tilde{\chi}_v\varepsilon_v, +)$ not to be supercuspidal (this presumably follows from the work of Cognet [Co]). On the other hand we would expect that $\theta(\pi_v, \tilde{\chi}_v, \pm)$ are both supercuspidal and form the two elements of a non-stable local L-packet on $GSp_4(\mathbb{Q}_v)$ (this presumably follows from Kudla's method, cf. Lemma 4). The centraliser of the image of this representation modulo $\mathbb{C}^\times$ has two elements, which is consistent with Langlands conjectures. These representations appear to have been missed by Vigneras [V] as have the corresponding representations $W_{\mathbb{Q}_v} \to GSp_4(\mathbb{C})$.

Returning to the global situation one might expect that the map from cuspidal automorphic representations $(\pi, \tilde{\chi})$ of $GO^\circ(\mathbb{A})$ to near equivalence classes of automorphic representations of $GSp_4(\mathbb{A})$, $(\pi, \tilde{\chi}) \mapsto [\tilde{\chi}\,\Theta(\pi, \tilde{\chi}, \delta)]$ is an injection, and that as δ varies the near equivalence class is exactly exhausted. This would appear to be suggested by the results of [KRS]. Indeed it was their theorem that convinced us that we must be able to get non-generic representations of $GSp_4(\mathbb{A})$ from this theta lift and led to this paper. Taking $\delta \equiv 1$ would presumably give the unique globally generic element of this near equivalence class. If $(\pi_v, \tilde{\chi}_v)$ does not admit -1 then presumably $\{\tilde{\chi}_v\,\theta(\pi_v, \tilde{\chi}_v, +)\}$ is a local L-packet. If it does admit -1 then presumably $\{\tilde{\chi}_v\,\theta(\pi_v, \tilde{\chi}_v, \pm)\}$ is a two element local L-packet. If π is not a base change then $[\tilde{\chi}\,\Theta(\pi, \tilde{\chi}, \delta)]$ is stable and one expects no global obstruction to products of elements of the local L-packets being stable. That this is the case is suggested by our results. If however π is a base change from $\mathbb{Q}$ from a cusp form of central character $\tilde{\chi}$ then $[\tilde{\chi}\,\Theta(\pi, \tilde{\chi}, \delta)]$ is not stable and one would expect a sign condition for products of elements of the local L-packets to be automorphic. This plausibly corresponds to the condition that $\prod_v \delta_v = 1$ which in this case we have for $(\pi, \tilde{\chi}, \delta)$ to be a cuspidal automorphic representation of $GO(\mathbb{A})$. If π is a base change from $\mathbb{Q}$ from a cusp form of central character other than $\tilde{\chi}$ then presumably $\tilde{\chi}\,\Theta(\pi, \tilde{\chi}, \delta)$ is not cuspidal. It would be interesting to check some of these assertions. Some are probably not very difficult, others may be more subtle.

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