Invent. math. 123, 1–34 (1996)

Inventiones mathematicae

Cellular algebras

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Oblatum 27-III-1995

Abstract. A class of associative algebras ("cellular") is defined by means of multiplicative properties of a basis. They are shown to have cell representations whose structure depends on certain invariant bilinear forms. One thus obtains a general description of their irreducible representations and block theory as well as criteria for semisimplicity. These concepts are used to discuss the Brauer centraliser algebras, whose irreducibles are described in full generality, the Ariki–Koike algebras, which include the Hecke algebras of type A and B and (a generalisation of) the Temperley–Lieb and Jones' recently defined "annular" algebras. In particular the latter are shown to be non-semisimple when the defining parameter δ satisfies $\gamma_{g(n)}(\frac{-\delta}{2}) = 1$, where γ_n is the n-th Tchebychev polynomial and $g(n)$ is a quadratic polynomial.

Introduction

The importance of the representation theory of Hecke (or "Hecke–Iwahori") algebras for the representation theory of reductive groups over finite fields is well understood and there is a large literature on the subject (see, e.g. [Lu1], [Cu], [HL1], [HL2], [Ca]). These are algebras A over the ring $R = \mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$ (q an indeterminate) which are generically semisimple, i.e. they are semisimple as algebras over the quotient field of R. However they have non-semisimple specialisations $A' = R' \otimes_R A$, which arise from ring homomorphisms: $R \to R'$. While most of the applications of their representation theory have hitherto been in the semisimple case, it is now clear that the representation theory of non-semisimple specialisations of Hecke algebras is intimately connected with the modular representation theory of reductive groups over finite fields (see [DJ3], [D], [G]). Furthermore, there are intimate connections with the representation theory of quantum groups (cf. [Ji], [D], [Lu3], [Dr]), statistical mechanics (cf. [J2], [J4]) and knot theory.

The principal motivation for this work is a systematic understanding of the non-semisimple specialisations of Hecke algebras and of a variety of other algebras with geometric connections. Kazhdan and Lusztig [KL1] have introduced bases for Hecke algebras A which are indexed by a partially ordered set (the Weyl group W). Multiplication of these basis elements reflects the ideal structure of A and as a result, Kazhdan and Lusztig were able to discuss the representation theory of A in the context of various orderings on W. In particular, they introduced the terms *cells* and *cell representations* in this context.

In this work, we define in Sect. 1 a class of associative algebras over a commutative ring R by stipulating the existence of a basis with certain combinatorial properties which reflect those of the "Robinson Schensted correspondence" in Hecke algebras of type A_n. These are the "cellular algebras" of the title. For these, we shall define (in Sect. 1) "cell representations" and give a complete parametrisation (in Sect. 3) of their irreducible modules (up to equivalence) in terms of the properties of certain invariant bilinear forms on the cell representations. A key property of these algebras is that the "cell datum" (see (1.1) below) which defines them and the cell representations is unchanged by specialisation. Thus questions about their specialisations are distilled into the study of these forms. In addition, we shall describe in Sect. 2 "canonical filtrations" for projective modules, classify the projective indecomposables over a field and show that if D is the "decomposition matrix" relating the cell modules to the irreducibles and C is the Cartan matrix, one has the usual relation $C = D^t D$ (see Theorem (3.7)). The block theory of A is implicitly contained in this relation.

After developing the general theory in Sects. 1, 2 and 3, we apply it in Sects. 4, 5 and 6 respectively to the Brauer centralizer algebras, the Ariki–Koike Hecke algebras and the Temperley–Lieb and Jones algebras, which are subalgebras of the Brauer algebra, defined by topological conditions (we actually treat a slightly more general class of algebras than the latter). In each case the general theory is immediately applicable, and provides more information than was hitherto available. In particular, we obtain an explicit parameterisation of the irreducible representations of the Brauer centralizer algebra over *any* field (regardless of whether it is semisimple). We also obtain new information about the semisimplicity of Jones' algebra $J_n(R)$ by studying one of its cell representations carefully (see (6.20) below).

In somewhat more detail, the contents of this paper are as follows. In Sect. 1 we define cellular algebras, give some examples and prove some easy consequences of the definitions. In Sect. 2 we begin the study of the representation theory with the construction of cell representations and their invariant forms (2.4). Projective modules and homomorphisms are also discussed here. In Sect. 3 a complete description of the representation theory of a cellular algebra is given in the case when R is a field. The principal results are Theorem (3.4), which gives the irreducibles, Theorem (3.7), which describes the projective indecomposables and block theory in terms of the cell representations and Theorem (3.8), which gives a criterion for semi-simplicity.

In Sect. 4 we analyse the Brauer centraliser algebra in our context, showing that it is cellular by developing a calculus of pairs of involutions in the symmetric group $\mathrm{Sym}\{1,2,\dots,n\}$. The result is an explicit description of the irreducible modules which is "characteristic free". In principle, the blocks are describable in our setup and the problem of determining the dimensions of the irreducibles (which of course are known in the semisimple case) is reduced to questions concerning the rank of certain explicitly defined bilinear forms. The ideas in this chapter have something in common with those of [FG].

In Sect. 5 we show that the "Hecke algebra" defined by Ariki and Koike ([AK]) for the unitary reflection group $G(r,1,n) \cong (\mathbb{Z}/r\mathbb{Z}) \wr \mathrm{Sym}(\mathbf{n})$ (where $\mathrm{Sym}(S)$ denotes the symmetric group on the set S and $\mathrm{Sym}(\mathbf{n}) := \mathrm{Sym}(\{1,2,\dots,n\})$) of order $r^n n!$ is a cellular algebra and we describe the combinatorial data necessary for the discussion of its representation theory in general. This includes the case of Hecke algebras of type A_{n-1} (the case $r = 1$) and B_n (the case $r = 2$) where there are two independent parameters. Our work therefore systematises that of Dipper and James in the latter cases. We also give some results about the block theory of this algebra in the general case. Our cell structure is different from the "ϕ-cells" introduced by Lusztig in the case $r = 2$ [Lu2] for certain parameter values.

In Sect. 6 we use the calculus of Sect. 4 to define an algebra $\widetilde{\mathrm{TL}}_n(R)$ which is, roughly, "double" the Temperley–Lieb algebra. We describe both its cellular structure and that of $\mathrm{TL}_n(R)$ and point out easy consequences for their representation theory (well understood in the case of $\mathrm{TL}_n(R)$ by [GW]). Jones' "annular" algebra $J_n(R)$ (cf. [J3]) is treated similarly in this section, and by studying one of the cell representations (the analogue of the "reflection representation" of a Hecke algebra) we are able to deduce results about the non-semisimplicity of $J_n(R)$ for certain parameter values.

In [Gr], the first author proves that a wide class of algebras, called "projection algebras" are cellular. They are, roughly, the algebra analogues of Coxeter groups and they include certain infinite families of "generalised Temperley–Lieb" algebras. The latter are defined as quotients of Hecke algebras and may be finite dimensional, even when the Hecke algebra is infinite dimensional. Examples are the infinite series $T(E_n)$ $(n \geqq 6)$, $T(F_n)$ $(n \geqq 5)$ and $T(H_n)$ $(n \geqq 5)$.

We shall adopt the following notation. If X is any (finite) set, $\mathrm{Sym}(X)$ will be the symmetric group on X. For any positive integer n, we write $\mathbf{n} = \{1,\dots,n\}$, so that $\mathrm{Sym}(\mathbf{n})$ denotes the symmetric group on $\{1,\dots,n\}$.

1 Cellular algebras

Let R be a commutative ring with identity.

(1.1) Definition. *A* **cellular algebra** *over R is an associative (unital) algebra A, together with* **cell datum** $(\Lambda, M, C, *)$ *where*

(C1) *Λ is a partially ordered set (poset) and for each $\lambda \in \Lambda, M(\lambda)$ is a finite set (the set of "tableaux of type λ") such that $C : \coprod_{\lambda\in\Lambda} M(\lambda)\times M(\lambda) \to A$ is an injective map with image an R-basis of A.*

(C2) *If $\lambda \in \Lambda$ and $S,T \in M(\lambda)$, write $C(S,T) = C^{\lambda}_{S,T} \in A$. Then $*$ is an R-linear anti-involution of A such that $(C^{\lambda}_{S,T})^* = C^{\lambda}_{T,S}$.*

(C3) *If $\lambda \in \Lambda$ and $S,T \in M(\lambda)$ then for any element $a \in A$ we have*

$$aC^{\lambda}_{S,T} \equiv \sum_{S'\in M(\lambda)} r_a(S',S)C^{\lambda}_{S',T} \pmod{A(<\lambda)}$$

where $r_a(S',S) \in R$ is independent of T and where $A(<\lambda)$ is the R-submodule of A generated by $\{C^{\mu}_{S'',T''} | \mu < \lambda; S'',T'' \in M(\mu)\}$.

Note that if we apply $*$ to the equation in (C3), we obtain

(C3)′

$$C^{\lambda}_{T,S}a^* \equiv \sum_{S'\in M(\lambda)} r_a(S',S)C^{\lambda}_{T,S'} \pmod{A(<\lambda)}\ (\forall a \in A \quad \text{and} \quad S,T \in M(\lambda))\,.$$

The axioms above are modelled upon the Robinson–Schensted correspondence for Sym(**n**), under which a permutation $w \in$ Sym(**n**) corresponds to a pair of standard tableaux of the same shape λ (λ being a partition of n).

(1.2) Example. The Hecke algebra of type A_{n-1}. Let $R = \mathbb{Z}[q]$ (q an indeterminate) and let A be the Hecke algebra of type A_{n-1} over R. Then A has a basis $\{C_w | w \in \text{Sym}(\mathbf{n})\}$ defined in [KL1]. If w corresponds to the pair (S,T) (let us write $w \sim (S,T)$) of standard tableaux then it is known [KL1, (1.4)], that T corresponds to the left cell of w and S to the right cell. The property $w^{-1} \sim (T,S)$ of the Robinson–Schensted correspondence [Kn] shows that the anti-involution defined by $* : T_w \mapsto T_{w^{-1}}$ (where $\{T_w | w \in \text{Sym}(\mathbf{n})\}$ is the usual (defining) basis of A) satisfies (C2). For $*$ is easily seen to be an anti-involution and clearly $C^*_w = C_{w^{-1}}$. Hence if $w \sim (S,T)$, write $C_w = C^{\lambda}_{S,T}$ where λ is the appropriate partition of n. The relation (C3) is implicit in [KL1] (see also [BV] and [V]). It asserts the isomorphism of the left cell representations in a given two-sided cell in the sense of [KL1].

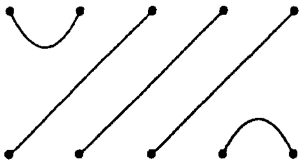
(1.3) Example. We give the following "banal" example to illustrate the results we prove below. It also comes into the cellular structure of Jones' algebra. Let σ be a function from a finite set Λ to a ring R as in (1.1). Let $A = R[X]/f(X)$ where $f(X) = \prod_{\lambda\in\Lambda}(X - \sigma(\lambda))$. Choose a partial order Λ such that for each pair $\mu, \lambda \in \Lambda$, we have $\mu \leqq \lambda, \lambda \leqq \mu$ or $\sigma(\mu) - \sigma(\lambda)$ is invertible in R (e.g. a total order). For $\lambda \in \Lambda$, let $M(\lambda) = \{\lambda\}$ and write

$$C^{\lambda}_{\lambda,\lambda} = C^{\lambda} = \left[\prod_{\mu\not\leqq\lambda}(X - \sigma(\mu))\right],$$

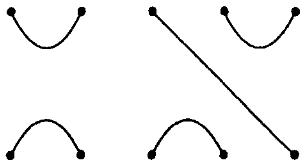
where $[g(X)]$ denotes the image of $g(X)$ in A under the natural map. To satisfy (C2), take $*$ to be id_A (an anti-involution since A is commutative). Observe that for $g(X) \in R[X]$ we have $[g(X)]C^\lambda \equiv g(\sigma(\lambda))C^\lambda \pmod{A(<\lambda)}$, which proves (C3). Thus A is a cellular algebra.

(1.4) Example. The Temperley–Lieb algebra (of type A). Let R be a commutative ring with identity and let $\delta \in R$ (possibly $\delta = 0$). Define the algebra TL_n ($= \mathrm{TL}_n(R,\delta)$) as follows: TL_n has an R-basis consisting of "planar Brauer diagrams" on $2n$ points (cf. [GW] or [J4]). These consist of two rows of n dots in which each dot is joined to just one other dot and none of the joins intersect when drawn in the rectangle defined by the two rows of n dots. "Planar" refers to the intersection condition.

We illustrate two such diagrams when $n = 5$.

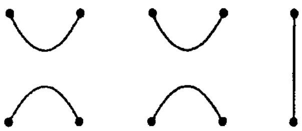


D_1

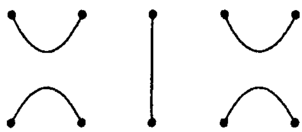


D_2

If D_1 and D_2 are two diagrams, $D_1 \circ D_2$ is their concatenation, with interior circuits removed. This is illustrated below



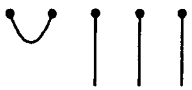
$D_1 \circ D_2$



$D_2 \circ D_1$

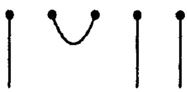
If $n(D_1, D_2)$ is the number of deleted interior circuits, the multiplication in A is defined by $D_1 D_2 = (\delta)^{n(D_1,D_2)} D_1 \circ D_2$.

Now let $\Lambda = \{t \in \{0,1,\ldots,n\} | n - t \in 2\mathbb{Z}\}$. For $\lambda \in \Lambda$, define a *planar* involution with λ fixed points as a diagram consisting of n dots arranged linearly with $n - \lambda$ of them joined in pairs and λ of them with an "end" attached. The pairs correspond to interchanges of the involution, while an "end" is a fixed point. The condition that the involution be planar is that it can be drawn in a $\frac{1}{2}$-plane defined by the line of n points with no intersections, if the ends are extended arbitrarily. We illustrate some planar involutions below:



S_1

S_2



S_3

Now for $\lambda \in \Lambda$, let $M(\lambda)$ be the set of planar involutions with λ fixed points. If S_1 and $S_2 \in M(\lambda)$, let $C^{\lambda}_{S_1,S_2}$ be the diagram with S_1 on top and S_2 on the bottom, the ends being joined in the unique way which creates a planar diagram. Reversing S_1 and S_2 defines an anti-involution of TL_n which clearly satisfies (C2) and the relation in (C3) is clear from the geometry. Note that the elements of $M(\lambda)$ are in bijection with standard tableaux with 2 rows of size $\frac{n+\lambda}{2}, \frac{n-\lambda}{2}$. One takes the right hand dots of the horizontal joins to be the second row of a tableau.

We shall meet further examples below.

To conclude this section, we record some elementary observations about the structure of A. If A is as in (1.1) and Λ' is any subset of Λ, define $A(\Lambda') = \langle C^{\lambda}_{S,T} | \lambda \in \Lambda' \rangle_R$.

(1.5) Lemma. *Suppose Φ is an ideal of Λ (i.e. $\phi \in \Phi, \lambda \in \Lambda, \lambda \leqq \phi \Rightarrow \lambda \in \Phi$). Then $A(\Phi) = \langle C^{\lambda}_{S,T} | \lambda \in \Phi \rangle_R$ is a two-sided ideal of A.*

This is clear from (C3) and (C3)$'$.

(1.6) Definition. *If $\Phi' \subseteq \Phi$ are two ideals of Λ, we define $Q(\Phi \backslash \Phi')$ as the (A,A) bimodule $A(\Phi)/A(\Phi')$.*

(1.6)$'$ *Remark.* Clearly $Q(\Phi\backslash\Phi')$ depends only on the set $\Phi\backslash\Phi'$, and not on Φ and Φ'. This accounts for the notation.

Observe that we have an obvious R-module monomorphism: $Q(\Phi\backslash\Phi') \longrightarrow A$, whose image is $A(\Phi\backslash\Phi')$ in the notation above. We shall be making particular use of $A(\{\lambda\})$ and $Q(\{\lambda\})$ where $\lambda \in \Lambda$.

(1.7) Lemma. *Let $\lambda \in \Lambda$ and $a \in A$. Then for any elements $S_1, S_2, T_1, T_2 \in M(\lambda)$, we have*

$$C^{\lambda}_{S_1,T_1} a C^{\lambda}_{S_2,T_2} \equiv \phi_a(T_1, S_2) C^{\lambda}_{S_1,T_2} \pmod{A(<\lambda)}$$

where $\phi_a(T_1,S_2) \in R$ depends only on a, T_1 and S_2 (i.e. is independent of T_2, S_1).

Proof. By (C3) and (C3)$'$, the left side of (1.7) is equal to $rC^{\lambda}_{S_1,T_2} \bmod A(<\lambda)$ (some $r \in R$). By (C3), r is independent of T_2 and by (C3)$'$, r is independent of S_1. □

(1.8) Specialisation. *If $\sigma : R \rightarrow R'$ is a homomorphism of commutative rings, then R' becomes an R-module in the obvious way. The R'-algebra $A^{\sigma} := R' \otimes_R A$ is called the* **specialisation** *of A at σ. If A is a cellular algebra with cell datum $(\Lambda, M, *, C)$ then A^{σ} is a cellular algebra with essentially the same cell datum, with $*$ and C being modified in the obvious way ($(r' \otimes a)^* := r' \otimes a^*$ and $C(S,T) = 1_{R'} \otimes C^{\lambda}_{S,T}$ for $S, T \in M(\lambda)$).*

2 Cell representations and projective modules

Let A be a cellular algebra with cell datum $(\Lambda, M, *, C)$ as defined in (1.1). In this section we define (and study) a set of representations of A whose existence is a natural consequence of the axioms.

(2.1) Definition. *For each $\lambda \in \Lambda$ define the (left) A-module $W(\lambda)$ as follows: $W(\lambda)$ is a free R-module with basis $\{C_S | S \in M(\lambda)\}$ and A-action defined by*

$$aC_S = \sum_{S' \in M(\lambda)} r_a(S', S) C_{S'} \quad (a \in A, S \in M(\lambda)) \tag{2.1.1}$$

where $r_a(S', S)$ is the element of R defined in (1.1)(C3). *It is called the* **cell representation** *of A corresponding to $\lambda \in \Lambda$.*

It is a consequence of (1.1)(C3) that (2.1.1) does define an action of A on $W(\lambda)$. Now $W(\lambda)$ may be thought of as a *right* A-module via

$$C_S a = \sum_{S' \in M(\lambda)} r_{a^*}(S', S) C_{S'} . \tag{2.1.2}$$

Once again, (1.1)(C3)$'$ shows that (2.1.2) defines a right action on $W(\lambda)$. Of course, the left and right actions do not generally commute. We use the notation $W(\lambda)$ for the *left* A-module and $W(\lambda)^*$ for the right A-module; we shall be making essential use of the following elementary observations later.

(2.2) Lemma.

(i) *There is a natural isomorphism of R-modules $C^\lambda : W(\lambda) \otimes_R W(\lambda)^* \to A(\{\lambda\})$, defined by $(C_S, C_T) \to C^\lambda_{S,T} (S, T \in M(\lambda))$. If $A(\{\lambda\})$ is identified with the (A, A) bimodule $Q(\{\lambda\})$ (see the remarks following* (1.6)) *then C^λ becomes an isomorphism of (A, A) bimodules.*

(ii) $A = \bigoplus_{\lambda \in \Lambda} A(\{\lambda\})$ *(as R-module).*

(iii) *If $a \in A(\{\lambda\})$ and $S, T \in M(\mu), (\lambda, \mu \in \Lambda)$ then $r_a(S, T) = 0$ unless $\lambda \geqq \mu$.*

The proof is easy.

A key to understanding the structure of $W(\lambda)$ is the R-bilinear form ϕ_λ which we now define.

(2.3) Definition. *For $\lambda \in \Lambda$, define $\phi_\lambda : W(\lambda) \times W(\lambda) \to R$ by $\phi_\lambda(C_S, C_T) = \phi_1(S, T)$, $S, T \in M(\lambda)$ (in the notation of* (1.7)), *extended bilinearly.*

(2.4) Proposition. *Keep the notation above and let $\lambda \in \Lambda$. Then*

(i) *The form ϕ_λ is symmetric; i.e. for $x, y \in W(\lambda)$, $\phi_\lambda(x, y) = \phi_\lambda(y, x)$.*

(ii) *For $x, y \in W(\lambda)$ and $a \in A$, we have*

$$\phi_\lambda(a^* x, y) = \phi_\lambda(x, ay) .$$

(iii) *For* $x, y, z \in W(\lambda)$ *we have*

$$C^{\lambda}(x \otimes y)z = \phi_{\lambda}(y, z)x$$

(*where* $C^{\lambda} : W(\lambda) \otimes_R W(\lambda)^* \to A(\{\lambda\})$ *is as in* (2.2).)

Proof. The form ϕ_{λ} is defined by the equation

$$C^{\lambda}_{S_1,T_1} C^{\lambda}_{S_2,T_2} \equiv \phi_{\lambda}(C_{T_1}, C_{S_2}) C^{\lambda}_{S_1,T_2} \bmod A(<\lambda)(S_i, T_i \in M(\lambda)) . \tag{2.4.1}$$

(i) This follows immediately by applying $*$ to (2.4.1).

(ii) We show that the matrices $\phi_{\lambda} = (\phi_{\lambda}(C_S, C_T))$ and $r_a = (r_a(S, T))$, $(a \in A)$ satisfy the relation $(r_{a^*})^t \phi_{\lambda} = \phi_{\lambda} r_a$. Take $S, T \in M(\lambda)$ and $a \in A$; then

$$\begin{aligned} C^{\lambda}_{S,S} a C^{\lambda}_{T,T} &\equiv C^{\lambda}_{S,S} \sum_{T' \in M(\lambda)} r_a(T', T) C^{\lambda}_{T',T} \\ &\equiv \sum_{T' \in M(\lambda)} r_a(T', T) \phi_{\lambda}(C_S, C_{T'}) C^{\lambda}_{S,T} . \end{aligned}$$

Thus the coefficient of $C^{\lambda}_{S,T}$ is the (S, T) entry in the matrix $\phi_{\lambda} r_a$. But evaluating $C^{\lambda}_{S,S} a C^{\lambda}_{T,T}$ as $(C^{\lambda}_{S,S} a) C^{\lambda}_{T,T}$, we find the relation asserted above.

The statement in (ii) is simply the matrix form of $\phi_{\lambda}(a^* x, y) = \phi_{\lambda}(x, ay)$ with respect to the basis $\{C_S | S \in M(\lambda)\}$ of $W(\lambda)$.

(iii) Since both sides are linear in each variable (i.e. in x, y, z), it clearly suffices to prove (iii) for $x = C_S$, $y = C_T$, $z = C_U (S, T, U \in M(\lambda))$. In this case, we have

$$\begin{aligned} C^{\lambda}(C_S \otimes C_T) C_U &= C^{\lambda}_{S,T} C_U \\ &= \sum_{V \in M(\lambda)} r_V C_V \end{aligned}$$

where r_V is the coefficient of $C^{\lambda}_{V,U'}$ in $C^{\lambda}_{S,T} C^{\lambda}_{U,U'}$ (any $U' \in M(\lambda)$). By (1.7) and (2.3), $r_V = \delta_{S,V} \phi_{\lambda}(C_T, C_U)$ (δ = Kronecker delta) whence we have

$$C^{\lambda}(C_S \otimes C_T) C_U = \phi_{\lambda}(C_T, C_U) C_S$$

as required. □

(2.5) Corollary. *For* $z \in W(\lambda)$, *let* R_z *be the ideal of* R *defined by*

$$R_z = \{\phi_{\lambda}(y, z) | y \in W(\lambda)\} .$$

Then

(i) *If* $a \in A$, $R_{az} \subseteq R_z$.

(ii) *If* $z \in W(\lambda)$, *we have* $Az \supseteqq R_z W(\lambda) = A(\{\lambda\})z$. *In particular if* $R_z = R$, $W(\lambda) = Az$.

Proof.

(i) Since $\phi_{\lambda}(y, az) = \phi_{\lambda}(a^* y, z)$ for $y \in W(\lambda)$, $a \in A$, clearly the image of $\phi_{\lambda}(-, az)$ is contained in R_z.

(ii) By (2.4)(iii), $A(\{\lambda\})z$ consists of all elements of the form $\phi_\lambda(y,z)x$ $(y,x \in W(\lambda))$, which is the first assertion. If $R_z = R$, then $RW(\lambda) = W(\lambda)$, which is the second. □

(2.6) Proposition. *Let $\lambda,\mu \in \Lambda$ and suppose $\theta : W(\lambda) \to W(\mu)/W'$ is a homomorphism of A-modules where W' is an A-submodule of $W(\mu)$. Assume that $W(\mu)/W'$ is free as R-module and suppose $\phi_\lambda \neq 0$. Then*

(i) *The function $\theta = 0$ unless $\lambda \geqq \mu$.*

(ii) *If $\lambda = \mu$, then there are elements $r_0 \neq 0$ and r_1 in R such that for all $x \in W(\lambda)$, we have $r_0\theta(x) = r_1x + W'$.*

Proof.

(i) It follows from (2.2)(iii) that if $\lambda \ngeqq \mu$ then $\theta(A(\{\lambda\})W(\lambda)) = 0$, because if $a \in A(\{\lambda\})$ and $z \in W(\lambda)$, then $\theta(az) = a\theta(z) = 0$ by (2.2)(iii). Now if $\phi_\lambda \neq 0$, there is an element $z \in W(\lambda)$ such that $R_z \neq 0$ (see (2.5)). Then by (2.5)(ii), $A(\{\lambda\})z = R_zW(\lambda)$; hence if $\lambda \ngeqq \mu$ then $\theta(R_zW(\lambda)) = 0$. But $W(\mu)/W'$ is free as R-module by hypothesis, whence $\theta = 0$.

(ii) Take $\lambda = \mu$. Since $\phi_\lambda \neq 0$, there are elements $y,z \in W(\lambda)$ such that $\phi_\lambda(y,z) = r_0 \neq 0$. Then for any $x \in W(\lambda)$, we have $C^\lambda(x \otimes y)z = r_0x$. Write $\theta(z) = z' + W'(z' \in W(\lambda))$. Then $\theta(r_0x) = C^\lambda(x \otimes y)\theta(z) = C^\lambda(x \otimes y)z' + W' = \phi_\lambda(y,z')x + W'$. If we write $r_1 = \phi_\lambda(y,z')$, this shows that $\theta(r_0x) = r_1x + W'$ for all $x \in W(\lambda)$ as stated. □

(2.6)′ Corollary. *Let $\lambda \in \Lambda$ and suppose $\phi_\lambda \neq 0$. If R is an integral domain, then* $\mathrm{Hom}_A(W(\lambda), W(\lambda)) \cong R$.

Proof. It follows from the proof of (2.6)(ii) that there is an element $r_0 \neq 0$ $(r_0 \in R)$ such that for any $\theta \in \mathrm{Hom}_A(W(\lambda), W(\lambda))$, there is an element $r_1(\theta) \in R$ such that

$$r_0\theta(x) = r_1(\theta)x \quad (x \in W(\lambda))\,. \tag{2.6.1}$$

Recall that $W(\lambda)$ has R-basis $\{C_S \mid S \in M(\lambda)\}$. If $\theta(C_S) = \sum_{S \in M(\lambda)} r_S C_S$, we see from (2.6.1) (with $x = C_S$) that $r_0r_T = 0\,(T \neq S)$, whence $r_T = 0$ since R is an integral domain. Thus we see

$$\theta(C_S) = r_SC_S \quad (\text{some } r_S \in R,\ \text{any } S \in M(\lambda))\,. \tag{2.6.2}$$

Using the same argument, we see that

$$r_S = r_T \quad (S,T \in M(\lambda))\,. \tag{2.6.3}$$

It follows that θ is of the form $\theta(x) = r(\theta)x$ (some $r(\theta) \in R$) and the map $\theta \mapsto r(\theta)$ (well defined by the R-free nature of $W(\lambda)$) yields the desired isomorphism. □

It is apparent from (2.6) that the axioms in (1.1) provide the combinatorial framework for a detailed study of the set of (equivalence classes of) irreducible A-modules and we shall carry this out in the case where R is a field in the next section. In addition, one obtains natural filtrations of projective A-modules. Recall that if Φ, Φ' are ideals of Λ such that $\Phi' \subset \Phi$, then we have an (A,A) bimodule $Q(\Phi\backslash\Phi')$ as defined in (1.6).

(2.7) Definition. *Let P be any A-module, with Φ, Φ' as above. Then $P(\Phi\backslash\Phi')$ is the A-module $Q(\Phi\backslash\Phi')\otimes_A P$.*

In particular, if Φ' is empty, we have

$$P(\Phi) = Q(\Phi)\otimes_A P = A(\Phi)\otimes_A P\ .$$

(2.8) Lemma. *Let Φ be an ideal of Λ.*

(i) *If P is any projective A-module, there is a natural isomorphism: $P(\Phi) \to A(\Phi)P$ defined by $a \otimes p \mapsto ap$ ($a \in A(\Phi)$, $p \in P$).*

(ii) *If e is an idempotent of A, then*

$$A(\Phi)Ae = A(\Phi)e = A(\Phi)\cap Ae\ .$$

Proof.

(i) The map $a\otimes p \mapsto ap : A\otimes_A P \to P$ is an isomorphism for any A-module P. If P is projective, $A(\Phi)\otimes_A P$ is naturally a submodule of $A\otimes_A P$ and the above map clearly takes $A(\Phi)\otimes_A P$ into $A(\Phi)P$.

(ii) This is clear, since $A(\Phi)$ is a two-sided ideal of A. □

We complete this section with two technical results.

(2.9) Lemma.

(i) *If, in* (2.7) *and* (2.8) *P is projective (as A-module), then for any two ideals $\Phi \subseteq \Phi'$ of Λ, we have an exact sequence $0 \to P(\Phi) \to P(\Phi') \to P(\Phi'\backslash\Phi) \to 0$.*

(ii) *For any finitely generated projective A-module P, there is a filtration $0 = P_0 \leqq P_1 \leqq P_2 \cdots \leqq P_d = P$ of P by projective modules P_i, such that $P_i/P_{i-1} \cong P(\{\lambda\})$ for some $\lambda \in \Lambda$.*

Proof.

(i) We clearly have an exact sequence of (A,A) bimodules: $0 \to A(\Phi') \to A(\Phi) \to Q(\Phi\backslash\Phi') \to 0$. Since P is projective, the functor $-\otimes_A P$ is exact, whence (i).

(ii) Let $\emptyset = \Phi_0 \subset \Phi_1 \subset \Phi_2 \subset \cdots \subset \Phi_d = \Lambda$ be a maximal chain of ideals of Λ. It is elementary that by maximality, $\Phi_i\backslash\Phi_{i-1}$ is a single element for $i \in \{1,\ldots,d\}$. The result now follows from (i). □

(2.10) Lemma. *Let P be any A-module and let $\lambda \in \Lambda$. Define the R-module P^λ by $P^\lambda := W(\lambda)^* \otimes_A P$.*

(i) *In the notation of* (2.7), *$P(\{\lambda\}) \cong W(\lambda)\otimes_R P^\lambda$.*

(ii) *If $\phi_\lambda \neq 0$ and R is an integral domain, then* $\mathrm{Hom}_A(P(\{\lambda\}), W(\lambda)) \cong \mathrm{Hom}_R(P^\lambda, R)$ *(as R-modules).*

Proof.

(i) By (2.7), $P(\{\lambda\}) = Q(\{\lambda\})\otimes_A P$. But by (2.2), $Q(\{\lambda\}) \cong W(\lambda)\otimes_R W(\lambda)^*$. It follows that $P(\{\lambda\}) \cong (W(\lambda)\otimes_R W(\lambda)^*)\otimes_A P$ and (i) follows from the associativity of tensor products.

(ii) We have

$$\begin{aligned}\mathrm{Hom}_A(P(\{\lambda\}), W(\lambda)) &\cong \mathrm{Hom}_A(W(\lambda)\otimes_R P^\lambda, W(\lambda))\\ &\cong \mathrm{Hom}_R(P^\lambda, \mathrm{Hom}_A(W(\lambda), W(\lambda))\\ &\cong \mathrm{Hom}_R(P^\lambda, R)\end{aligned}$$

since $\mathrm{Hom}_A(W(\lambda), W(\lambda)) \cong R$ if R is an integral domain and if $\phi_\lambda \neq 0$, by Corollary (2.6)′. □

3 Representation theory over a field

In this section we assume without further comment that R *is a field* and that all modules are finite dimensional over R. We maintain the notation of Sects. 1 and 2.

(3.1) Definition. *Let* $(\Lambda, M, C, *)$ *be a cell datum* (see (1.1)). *For* $\lambda \in \Lambda$, *define* $\mathrm{rad}\,(\lambda) := \{x \in W(\lambda) \mid \phi_\lambda(x, y) = 0$ *for all* $y \in W(\lambda)\}$.

(3.2) Proposition. *Let* $\lambda \in \Lambda$ *as above. Then*

(i) $\mathrm{rad}(\lambda)$ *is an* A*-submodule of* $W(\lambda)$.

(ii) *If* $\phi_\lambda \neq 0$, *the quotient* $W(\lambda)/\mathrm{rad}(\lambda)$ *is absolutely irreducible.*

(iii) *If* $\phi_\lambda \neq 0$, $\mathrm{rad}(\lambda)$ *is the radical of the* A*-module* $W(\lambda)$ (*i.e. the minimal submodule with semisimple quotient*).

Proof.

(i) If $x \in \mathrm{rad}(\lambda)$ and $a \in A$, then by (2.4)(ii), $\phi(ax, y) = \phi(x, a^* y) = 0$ (all $y \in W(\lambda)$), whence $ax \in \mathrm{rad}(\lambda)$.

(ii) If $z \in W(\lambda)$, $z \notin \mathrm{rad}(\lambda)$ then (2.5)(ii) shows that $W(\lambda) = Az$. It follows that $W(\lambda)/\mathrm{rad}(\lambda)$ is irreducible. Moreover it follows from (2.6)(ii) that (as R-modules) $\mathrm{End}_A(W(\lambda)/\mathrm{rad}(\lambda)) \cong R$. Hence $W(\lambda)/\mathrm{rad}(\lambda)$ is absolutely irreducible.

(iii) Let $\mathrm{Rad}(\lambda)$ be the radical of $W(\lambda)$. Since $W(\lambda)/\mathrm{rad}(\lambda)$ is semisimple (by (ii)), $\mathrm{Rad}(\lambda) \leqq \mathrm{rad}(\lambda)$. Consider the short exact sequence

$$0 \to \ker(\theta) \to W(\lambda)/\mathrm{Rad}(\lambda) \to W(\lambda)/\mathrm{rad}(\lambda) \to 0$$

where $\theta : W(\lambda)/\mathrm{Rad}(\lambda) \to W(\lambda)/\mathrm{rad}(\lambda)$ is the natural map. Choose a splitting s of θ (as linear map) and let m be a non-zero element of $W(\lambda)/\mathrm{rad}(\lambda)$. If $z \in W(\lambda)$ is such that $s(m) = z + \mathrm{Rad}(\lambda)$, then $W(\lambda) = Az$ whence $W(\lambda)/\mathrm{Rad}(\lambda) = As(m)$. Moreover if $\theta(as(m)) = 0$ for some $a \in A$ then $a = 0$. It follows that $\ker(\theta) = 0$, whence the result. □

(3.3) Definition. *Denote the* (*absolutely irreducible*) A*-module* $W(\lambda)/\mathrm{rad}(\lambda)$ $(\lambda \in \Lambda, \phi_\lambda \neq 0)$ *by* L_λ.

(3.4) Theorem. *Let* R *be a field and let* $(\Lambda, M, C, *)$ *be a cell datum* (*see* (1.1)) *for the* R*-algebra* A. *For each* $\lambda \in \Lambda$, *define the* (*left*) A*-module*

$W(\lambda)$ and bilinear form ϕ_λ on $W(\lambda)$ as in (2.1) *and* (2.3) *respectively. Let $\Lambda_0 = \{\lambda \in \Lambda \mid \phi_\lambda \neq 0\}$.*

(i) *The set $\{L_\lambda | \lambda \in \Lambda_0\}$ is a complete set of* (*representatives of equivalence classes of*) *absolutely irreducible A-modules.*

(ii) *If P_λ is the principal indecomposable A-module with head isomorphic to L_λ then $P_\lambda \cong P_\lambda(\leqq \lambda)$ in the notation of* (2.7) ($\lambda \in \Lambda_0$).

Proof. Observe first that if $\theta : L_\lambda \to L_\mu$ is an isomorphism ($\lambda, \mu \in \Lambda_0$) then by (2.6), $\lambda \geqq \mu \geqq \lambda$, whence $\lambda = \mu$. Thus from (3.2) we see that $\{L_\lambda | \lambda \in \Lambda_0\}$ is a set of non-isomorphic absolutely irreducible A-modules. We complete the proof of (i) by showing that any principal indecomposable A-module is of the form (ii).

Let $P = Ae$ be a principal indecomposable A-module, where e is a primitive idempotent in A. Let Φ be the ideal of Λ generated by $\{\lambda \in \Lambda | P(\{\lambda\}) \neq 0\}$. Clearly $P \cong P(\Phi)$ in the notation of (2.7), because the exact sequence of (2.9) provides a filtration of $P(\Lambda \backslash \Phi)$ with zero quotients. Thus $Ae = P = P(\Phi) = A(\Phi)e = A(\Phi) \cap Ae$, whence $e \in A(\Phi)$ (cf. (2.8)).

Now let λ_0 be any maximal element of Φ. Then clearly $P(\{\lambda_0\}) \neq 0$ and we claim $\phi_{\lambda_0} \neq 0$. To prove the latter statement, observe that if $\phi_{\lambda_0} = 0$, then a short computation shows that $A(\Phi)$ annihilates $W(\lambda_0)$. But by (2.10), $P(\{\lambda_0\}) \cong W(\lambda_0) \otimes_R P^{\lambda_0}$, where $P^{\lambda_0} = W(\lambda_0)^* \otimes_A P \cong W(\lambda_0)^* e \cong e^* W(\lambda_0)$ (as R-modules). Moreover $e^* \in A(\Phi)$, whence $P^{\lambda_0} = 0$, a contradiction. Thus $\phi_{\lambda_0} \neq 0$. Hence by (2.10), $\mathrm{Hom}_A(P(\{\lambda_0\}), W(\lambda_0)) \cong \mathrm{Hom}_R(P^{\lambda_0}, R) \neq 0$, whence it follows that $P(\{\lambda_0\})$ has a quotient isomorphic to L_{λ_0}. But $P(\{\lambda_0\})$ is itself a quotient of P, whence P has head L_{λ_0}; it follows that λ_0 is the unique maximal element of Φ and that $P \cong P(\leqq \lambda_0)$ as stated in (ii). It follows also that any irreducible quotient of a principal indecomposable A-module is isomorphic to L_λ, for some $\lambda \in \Lambda_0$, proving (i). □

As a consequence of Theorem (3.4), each A-module $W(\lambda)$ ($\lambda \in \Lambda$) has a composition series with quotients isomorphic to L_μ (some $\mu \in \Lambda_0$). Since the Jordan–Hölder theorem applies here, we may speak of the *multiplicity* of L_μ in $W(\lambda)$.

(3.5) Definition. *For $\lambda \in \Lambda$ and $\mu \in \Lambda_0$, write $d_{\lambda\mu}$ for the multiplicity of L_μ in $W(\lambda)$. The matrix $(d_{\lambda\mu})_{\lambda \in \Lambda, \mu \in \Lambda_0}$ will be denoted D; it is called the* **decomposition matrix** *of A.*

(3.6) Proposition. *The matrix D is upper unitriangular, i.e. $d_{\lambda\mu} = 0$ unless $\lambda \leqq \mu$ and $d_{\lambda\lambda} = 1$.*

Proof. If $d_{\lambda\mu} \neq 0$, there is a nontrivial homomorphism $\theta : W(\mu) \to W(\lambda)/W'$, where W' is a submodule of $W(\lambda)$. Thus $d_{\lambda\mu} = 0$ unless $\mu \geqq \lambda$ by (2.6)(i). If $\lambda = \mu \in \Lambda_0$, then by (2.6)(ii), any homomorphism θ (as above) is of the form $\theta(x) = rx + W'$ for some $r \in R$. Hence if $\theta \neq 0$, $\mathrm{im}(\theta) = W(\lambda)/W'$ and it follows from (3.2) that $W' = \mathrm{rad}(\lambda)$. Thus $W(\lambda)$ has just one subquotient (viz. $W(\lambda)/\mathrm{rad}(\lambda)$) isomorphic to L_λ. □

(3.7) Theorem. *Let P_λ be the projective indecomposable A-module corresponding to $\lambda \in \Lambda_0$. Then*

(i) *$P_\lambda \cong Ae \cong Ae^*$ for some (primitive) idempotent e of A such that $eA(\{\lambda\}) \neq 0$.*

(ii) *If $\lambda \geqq \mu$, then $\dim_R(P_\lambda)^\mu = d_{\mu\lambda} (\lambda \in \Lambda_0, \mu \in \Lambda)$, where $P^\mu = W(\mu)^* \otimes_A P$ for any A-module P (see (2.10)).*

(iii) *If $c_{\lambda\mu}$ is the multiplicity of L_μ in $P_\lambda (\lambda, \mu \in \Lambda_0)$ then writing $C = (c_{\lambda\mu})_{\lambda,\mu \in \Lambda_0}$ we have $C = D^t D$.*

Proof.

(i) If $e \in A$ is a primitive idempotent, then by the proof of (3.4)(ii) we have $P_\lambda \cong Ae$ if and only if $\Phi = \{\mu \in \Lambda | \mu \leqq \lambda\}$ is the smallest ideal of Λ with $e \in A(\Phi)$. Clearly if e has this property, so does e^*, whence $P_\lambda \cong Ae^*$.

(ii) The multiplicity $d_{\mu\lambda}$ is the dimension of $\operatorname{Hom}_A(P_\lambda, W(\mu))$. We have

$$\operatorname{Hom}_A(P_\lambda, W(\mu)) \cong \operatorname{Hom}_A(Ae^*, W(\mu)) \cong e^* W(\mu) \cong W(\mu)^* e$$
$$\cong W(\mu)^* \otimes_A Ae = (P_\lambda)^\mu .$$

(iii) Following the proof of (3.4)(i), let Φ be the ideal of Λ generated by $\{\lambda \in \Lambda | P(\{\lambda\}) \neq 0\}$. Take a maximal chain $\emptyset = \Phi_0 \subset \Phi_1 \subset \cdots \subset \Phi_m = \Phi$ of ideals of Λ. By (2.9)(ii) the set of subquotients of the filtration $0 = P(\Phi_0) \subset P(\Phi_1) \subset \cdots \subset P(\Phi_m) = P(\Phi) \cong P$ is precisely $\{P_\lambda(\{\nu\}) | \nu \leqq \lambda\}$. Now by (2.10)(i), $P_\lambda(\{\nu\}) \cong W(\nu) \otimes_R (P_\lambda)^\nu$, whence the multiplicity of L_μ in $P_\lambda(\{\nu\})$ (for any $\mu \in \Lambda_0$) is $d_{\nu\mu} \cdot (\dim_R(P_\lambda)^\nu) = d_{\nu\mu} d_{\nu\lambda}$ (by (ii) above). It follows that $c_{\lambda\mu} = \sum_{\nu \leqq \mu, \lambda} d_{\nu\mu} d_{\nu\lambda}$, which is the statement (iii). □

Next we show how the issue of semisimplicity is dealt with in this context.

(3.8) Theorem. *Let A be an R-algebra (R a field) with cell datum $(\Lambda, M, C, *)$. Then the following are equivalent.*

(i) *The algebra A is semisimple.*

(ii) *The nonzero cell representations $W(\lambda)$ are irreducible and pairwise inequivalent.*

(iii) *The form ϕ_λ (cf. (2.3)) is nondegenerate (i.e. $\operatorname{rad}(\lambda) = 0$) for each $\lambda \in \Lambda$.*

Proof. We show that each statement is equivalent to

(3.8.1) $$\text{For } \mu \in \Lambda \text{ and } \lambda \in \Lambda_0, \quad d_{\mu,\lambda} = \begin{cases} 1 & \text{if } \mu = \lambda , \\ 0 & \text{if } \mu \neq \lambda . \end{cases}$$

First assume (3.8.1) holds. Then by (3.7)(iii), if $\mu \in \Lambda_0$, we have $P_\mu \cong W(\mu) \cong L_\mu$, while if $\mu \notin \Lambda_0$, $W(\mu) = 0$, Statements (ii) and (iii) follow immediately; for (i), recall that A is a sum of principal indecomposable modules, each of which is isomorphic to P_μ for some $\mu \in \Lambda$ (of course for a given $\mu \in \Lambda$, P_μ "occurs" with a multiplicity, which may be zero). By (3.8.1), each P_μ is either zero or irreducible, whence A is semisimple.

Conversely, if A is semisimple and $\lambda \in \Lambda_0$, then $P_\lambda = L_\lambda$, whence (3.8.1) follows from (3.7)(iii). Thus (i) implies (3.8.1). It is clear that (3.8.1) is also a consequence of (ii). Finally, assume $\mathrm{rad}(\mu) = 0$ for any $\mu \in \Lambda$. If $\mu \in \Lambda_0$, then by (3.2), $L_\mu = W(\mu)$ is irreducible; if $\mu \notin \Lambda_0$, then $W(\mu) = 0$, whence (3.8.1). This completes the proof of (3.8). □

(3.9) Remarks. Consider the following questions concerning the R-algebra A.

(3.9.1) Determine the (equivalence classes of) irreducible left A-modules.

(3.9.2) Determine the dimensions of the irreducible left A-modules.

(3.9.3) Two irreducible A-modules L and M are said to be *linked* if M is a composition factor of the projective indecomposable module corresponding to L. Determine the classes of the equivalence relation generated by linkage, i.e. the *blocks* of A.

(3.9.4) Determine the Cartan matrix of A (in the usual sense of representation theory).

(3.9.5) Is A semisimple?

When A is a cellular algebra and R is a field, these equations are reduced by the foregoing development to standard questions in linear algebra. In somewhat more detail we have:

(3.9.6) The irreducible modules are parametrised by

$$\Lambda_0 := \{\lambda \in \Lambda \mid \phi_\lambda \neq 0\} .$$

(3.9.7) The dimensions are given by

$$\dim_R(L_\lambda) = |M(\lambda)| - \dim_R(\mathrm{rad}(\lambda)) .$$

(3.9.8) Let $\lambda, \mu \in \Lambda$. Say λ, μ are *cell-linked* if $\lambda \in \Lambda_0$ and L_λ is a composition factor of $W(\mu)$. The classes of the equivalence relation of Λ generated by this relation are called *cell-blocks*. The intersection of a cell-block with Λ_0 corresponds to a *block* in the sense of (3.9.3). Thus the solution of (3.9.3) reduces to

(3.9.9) Determine the set $\{(\mu, \lambda) \mid \mu \in \Lambda,\ \lambda \in \Lambda_0,\ d_{\mu,\lambda} \neq 0\}$.

(3.9.10) The Cartan matrix C is given by (3.7)(iii) and D is computed as indicated in the proof of (3.7)(ii).

(3.9.11) By (3.8), the problem of semisimplicity reduces to the computation of the discriminants Δ_λ of the forms $\{\phi_\lambda \mid \lambda \in \Lambda\}$.

In the examples below we generally have only incomplete answers to the first three questions above.

(3.10) Remark. It is clear from the definition (see, e.g. [CPS]) that A is quasi-hereditary if ϕ_λ is nonzero for each $\lambda \in \Lambda$ (i.e. if in the notation of (3.4), we have $\Lambda = \Lambda_0$).

4 The Brauer algebra

In this section we show that Brauer's centraliser algebra has the structure of a cellular algebra and we give a complete parametrisation of its irreducible representations over any field. Let R be a commutative ring with identity and let $\delta \in R$. The Brauer algebra $B(n) = B(n, \delta)$ (cf. [B], [HW], [W]) has R-basis consisting of diagrams D, which consist of two rows of n points, labelled $\{1, \ldots, n\}$, with each dot joined to precisely one other dot (distinct from itself) (see, e.g. Fig. (4.1)).

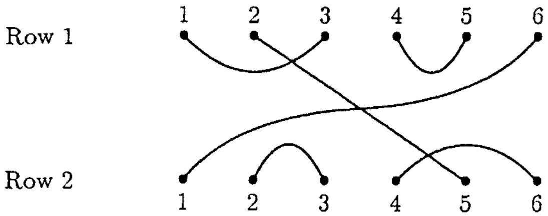


Fig. 4.1

Two diagrams D_1, D_2 may be "composed" as in example (1.4) above, to get $D_1 \circ D_2$ by placing D_1 above D_2 and joining corresponding points; interior loops are deleted. The multiplication in $B(n)$ is then defined by

(4.2) $$D_1 \circ D_2 = \delta^{n(D_1, D_2)} D_1 \circ D_2$$

where $n(D_1, D_2)$ is the number of deleted interior loops.

(4.3) Now to discuss the cellular structure of $B(n)$, observe that given a diagram D, we may associate with it the following data

(4.3.1) $t(D) :=$ the number of "through strings", where a through string is a join between dots in different rows.

(4.3.2) Two involutions $S_1(D), S_2(D)$ in $\mathrm{Sym}(\mathbf{n})$, where $S_i(D)$ is the involution interchanging the ends of the joins between points in row $i (i = 1, 2)$.

(4.3.3) Sets $\mathrm{Fix}(S_i(D)) \subseteq \mathbf{n}$, which are the fixed points of the involutions $S_i(D)$.

(4.3.4) $w(D) \in \mathrm{Sym}(\mathbf{t}), t = t(D)$; this is the permutation of $\mathrm{Fix}(S_1(D))$ determined by taking the end points of the through strings (regarded as joining from row 2 to row 1) in the order determined by taking their starting points in row 2 in increasing order.

The permutation $w(D)$ may be thought of as an "attaching map": $\mathrm{Fix}(S_2(D)) \to \mathrm{Fix}(S_1(D))$, which corresponds to the through strings of D. It is clear that conversely, D is determined by the triple $[S_1(D), S_2(D), w(D)]$. We have therefore shown

(4.4) Lemma. *Given an integer $n \geqq 2$, let $\mathscr{T}(n) = \{t \in \mathbf{n} \cup \{0\} | n - t \in 2\mathbb{Z}\}$. For any $t \in \mathscr{T}(n)$, let $I(t)$ be the set of involutions S in* $\mathrm{Sym}(\mathbf{n})$ *such that* $|\mathrm{Fix}(S)| = t$. *($I(n) = \{\mathrm{id}\}$). Then $B(n)$ has basis $\coprod_{t \in \mathscr{T}(n)} \{[S_1, S_2, w] | S_i \in I(t)$, $w \in$* $\mathrm{Sym}(\mathbf{t})\}$.

The proof is clear from (4.3).

(4.4.1) *Remark.* We shall abuse notation by using w to denote the element of $\mathrm{Sym}(\mathbf{t})$ which corresponds to a bijection: $\mathrm{Fix}(S_2) \to \mathrm{Fix}(S_1)$, taking the canonical order on $\mathrm{Fix}(S_i)$ to be the one inherited from $\mathbf{n}$.

We now describe the multiplication in $B(n)$ in terms of this basis. To compute the product $[S_1, S_2, w][S_1', S_2', w']$ the key is to investigate what happens when the second row of the first diagram is identified with the first row of the second. This turns out to be equivalent to analysing the orbits of the group H generated by S_2, S_1' on $\mathbf{n}$.

(4.5) Proposition. *Let S, S' be involutions in* $\mathrm{Sym}(\mathbf{n})$ *and write $H = \langle S, S' \rangle$. The H-orbits $\mathcal{O}$ on* $\mathbf{n}$ *fall into the following mutually exclusive classes.*

(i) *$\mathcal{O}$ contains no point of* $\mathrm{Fix}(S) \cup \mathrm{Fix}(S')$. *In this case $\mathcal{O}$ is called a* **loop**.

(ii) *$\mathcal{O}$ contains 2 points of* $\mathrm{Fix}(S)$ *(resp.* $\mathrm{Fix}(S')$*) and no points of* $\mathrm{Fix}(S')$ *(resp.* $\mathrm{Fix}(S)$*). In this case we say $\mathcal{O}$ is an S-***arc** *(resp. S'-***arc***)*.

(iii) *$\mathcal{O}$ contains precisely one point of* $\mathrm{Fix}(S)$ *and one point of* $\mathrm{Fix}(S')$. *In this case we say that $\mathcal{O}$ is a* **through arc**.

Proof. If $g, h \in H$ and $i = 0, 1, 2, \ldots$ write $(gh)_i = \cdots hgh$ (i factors). Thus if $i (> 0)$ is even $(gh)_i$ commences with g, otherwise with h; $(gh)_0 = 1$ by definition.

Suppose $\mathcal{O}$ contains a point fixed by S, viz. $i (\in \mathbf{n})$. Then clearly $\mathcal{O} = \{(SS')_j i | j = 0, 1, \ldots\}$ since the latter set is clearly invariant under S and S'. Let j be minimal such that $(SS')_j i = (SS')_{j'} i$ (some $j' < j$). Since $(SS')_{j-1} i = (SS')_{j'+1} i$, we must have (by minimality) $j - 1 \leqq j' + 1$, i.e. $j \leqq j' + 2 (< j + 2)$, so that $j = j' + 1$. Thus $\mathcal{O} = \{(SS')_k i, k = 0, 1, \ldots, j'\}$ and the elements $(SS')_k i$ are distinct for $k = 0, 1, \ldots, j'$. If j' is odd, the above argument shows that $(SS')_{j'} i$ is fixed by S, so that we are in case (ii). Otherwise j' is even and (iii) applies. □

(4.6) Remarks.

(i) Note that in (4.5) the cardinality of $\mathcal{O}$ in cases (i), (ii) and (iii) is (respectively) even, even and odd.

(ii) In case $\mathcal{O}$ is an arc or through arc we may speak of its *end points* (its intersection with $\mathrm{Fix}(S) \cup \mathrm{Fix}(S')$). More precisely, we may speak of S-ends and S'-ends of arcs. A through arc has one of each, while an S arc has two S-ends and no S'-ends, etc. Write $T_S(S, S')$ for the S-ends of the through arcs and $T_S'(S, S')$ for the S'-ends of the through arcs.

(iii) We denote by $t(S, S')$ the number of through arcs of $\langle S, S' \rangle$. If $t = t(S, S')$ we have a map: $\{S'$-ends of through arcs$\} \to \{S$-ends of through arcs} defined by following arcs. This defines an element $w(S,S') \in \mathrm{Sym}(\mathbf{t})$ in analogy with (4.3.4). Observe that $w(S,S')$ is not symmetric in S,S', in fact $w(S,S') = w(S,S')^{-1}$.

(iv) Define the involution $i_S(S,S')$ in $\mathrm{Sym}(\mathrm{Fix}(S))$ as the permutation of $\mathrm{Fix}(S)$ whose cycles are the end points of the S-arcs. Similarly, define $i_{S'}(S,S') \in \mathrm{Sym}(\mathrm{Fix}(S'))$.

Clearly we have

$$|\mathrm{Fix}(i_S(S,S'))| = t(S,S') = |\mathrm{Fix}(i_{S'}(S,S'))| \tag{4.6.1}$$

With these notations, we now have

(4.7) Proposition. *Let* $[S_1,S_2,w],[S_1',S_2',w']$ *be basis elements of* $B(n)$ *as in* (4.4). *Then* $[S_1,S_2,w][S_1',S_2',w'] = \delta^{n(S_2,S_1')}[S_1'',S_2'',w'']$ *where* $n(S_2,S_1')$ *is the number of* $\langle S_2,S_1'\rangle$*-loops in* $\mathbf{n}$ (*see* (4.5)(*i*)) *and*

$$S_1'' = S_1 w(i_{S_2}(S_2,S_1'))w^{-1}$$

$$S_2'' = S_2'(w')^{-1} i_{S_1'}(S_2,S_1')w'$$

$$w'' = w|_{T_{S_2}(S_2,S_1')} w(S_2,S_1')w'|_{(w')^{-1}T_{S_1'}(S_2,S_1')}$$

and $w|_S$ (S *a subset of* $\mathrm{Fix}(S_2)$) *denotes the element of* $\mathrm{Sym}(S)$ *obtained by restricting* w *to* S *in a way analogous to* (4.3.4). (*Recall that* $T_{S_2}(S_2,S_1')$ *denotes the set of* S_2*-ends of through arcs etc*).

Proof. It is straightforward to check, using (4.3), (4.4), (4.5) and (4.6) that composition of diagrams amounts to the statement of (4.7). Observe that since $i_{S_2}(S_2,S_1')$ is an involution in $\mathrm{Sym}(\mathrm{Fix}(S_2)), w i_{S_2}(S_2,S_1')w^{-1}$ is an involution in $\mathrm{Sym}(\mathrm{Fix}(S_1))$, so that $S_1 w(i_{S_2}(S_2,S_1'))w^{-1}$ is an involution in $\mathrm{Sym}(\mathbf{n})$. □

We are now in a position to describe the cellular structure of $B(n)$. Recall (cf. example (1.2) above) that $\mathbb{Z}\,\mathrm{Sym}(\mathbf{t})$ has a cellular basis $\{C_w | w \in \mathrm{Sym}(\mathbf{t})\}$ (the Kazhdan–Lusztig basis), where, in the notation of the cell datum, $C_w = C^\lambda_{T_1,T_2}$, where $\lambda \in \mathscr{P}(t)$ (the poset of partitions of t) and the pair (T_1,T_2) ($\in M(\lambda)$, the set of standard tableaux of shape λ) corresponds to w under the Robinson Schensted correspondence.

(4.8) Definition. *Let* $t \in \mathbb{Z}_{\geqq 0}$. *For* $\lambda \in \mathscr{P}(t)$ *and tableaux* $T_1,T_2 \in M(\lambda)$, *define* $p^\lambda_{T_1,T_2}(w) \in \mathbb{Z}$ *for each* $w \in \mathrm{Sym}(\mathbf{t})$ *by*

$$C^\lambda_{T_1,T_2} = \sum_{w \in \mathrm{Sym}(\mathbf{t})} p^\lambda_{T_1,T_2}(w)w\,.$$

Next, given $n \geqq 2$ define the poset Λ as follows. For $t \in \mathscr{T}(n) = \{0 \leqq t \leqq n | n - t \in 2\mathbb{Z}\}$, let $\mathscr{P}(t)$ be the set of partitions of t, ordered by dominance. Then $\Lambda = \{(t,\lambda) | t \in \mathscr{T}(n), \lambda \in \mathscr{P}(t)\}$ with lexicographic ordering: $(t,\lambda) \leqq (t',\lambda')$ if $t < t'$ or if $t = t'$ and $\lambda \leqq \lambda'$.

(4.9) Definition. *For $(t,\lambda) \in \Lambda$ (see above), define $M(t,\lambda) := \{(S,T) | S \in I(t), T \in M(\lambda)\}$ where $I(t)$ is the set of involutions S of* Sym(**n**) *such that $|\mathrm{Fix}(S)| = t$ and $M(\lambda)$ is the set of standard tableaux of shape λ.*

(4.10) Theorem. *Let $(t,\lambda) \in \Lambda$ and take $(S_i,T_i) \in M(t,\lambda)(i = 1,2)$ (see (4.9)). Define $C^{t,\lambda}_{(S_1,T_1),(S_2,T_2)} := \sum_{w\in \mathrm{Sym}(\mathbf{t})} p^{\lambda}_{T_1,T_2}(w)[S_1,S_2,w]$ (where $[S_1,S_2,w]$ is as in (4.4) and $p^{\lambda}_{T_1,T_2}(w)$ is defined in (4.8), interpreted as an element of $\mathbb{Z}1 \subseteq R$). Define $* : B(n) \to B(n)$ by the linear extension of $[S_1,S_2,w]^* = [S_2,S_1,w^{-1}]$ (this corresponds to reflecting diagrams in a horizontal axis). Then $(\Lambda, M, C, *)$ is a cell datum for $B(n)$.*

Proof. We check the axioms (C1), (C2) and (C3) of (1.1). First, note that $\{C'_w | w \in \mathrm{Sym}(\mathbf{t})\}$ is a basis of $R\,\mathrm{Sym}(\mathbf{t})$ for any R, since the matrix $(p_{T_1,T_2}(w))$ is upper unitriangular with respect to the Bruhat order on $\mathrm{Sym}(\mathbf{t})$. Hence $\{C^{t,\lambda}_{(S_1,T_1),(S_2,T_2)}\}$ has the same R-linear span as $\{[S_1,S_2,w]\}$, proving (C1).

Next, observe that

$$
\begin{aligned}
(C^{t,\lambda}_{(S_1,T_1),(S_2,T_2)})^* &= \sum_{w\in \mathrm{Sym}(\mathbf{t})} p^{\lambda}_{T_1,T_2}(w)[S_2,S_1,w^{-1}] \\
&= \sum_{w\in \mathrm{Sym}(\mathbf{t})} p^{\lambda}_{T_2,T_1}(w)[S_2,S_1,w]\,,
\end{aligned}
$$

since $p^{\lambda}_{T_1,T_2}(w^{-1}) = p^{\lambda}_{T_2,T_1}(w)$ (cf. (1.2)). Thus $(C^{t,\lambda}_{(S_1,T_1),(S_2,T_2)})^* = C^{t,\lambda}_{(S_2,T_2)(S_1,T_1)}$, proving (C2).

Finally, to prove (C3), it suffices to prove that for any $[U_1,U_2,w_0]$ with $U_i \in I(t_0), w_0 \in \mathrm{Sym}(\mathbf{t}_0)$, the product $[U_1,U_2,w_0]C^{t,\lambda}_{(S_1,T_1)(S_2,T_2)}$ is a sum of $r(S_1',T_1')C^{t,\lambda}_{(S_1',T_1'),(S_2,T_2)}$ (modulo terms with smaller (t,λ)) where $r(S_1',T_1')$ depends only on $[U_1,U_2,w_0]$ and (S_1,T_1).

We have

$$
(4.10.1)\quad [U_1,U_2,w_0]C^{t,\lambda}_{(S_1,T_1),(S_2,T_2)} = \sum_{w\in \mathrm{Sym}(\mathbf{t})} p^{\lambda}_{T_1,T_2}(w)[U_1,U_2,w_0][S_1,S_2,w]\,.
$$

Now $[U_1,U_2,w_0][S_1,S_2,w] = \delta^{n(U_2,S_1)}[S_1',S_2',w']$, where, by (4.7), if $|\mathrm{Fix}(S_1')| = t$ (these are the terms which are of interest in the product), we have $S_2' = S_2, S_1' = U_1 w_0(i_{U_2}(U_2,S_1))w_0^{-1}$, and $w' = x(U_2,S_1,w_0)w$ for some element $x(U_2,S_1,w_0)$ of $\mathrm{Sym}(\mathbf{t})$, which depends only on w_0, U_2 and S_1. This is because if $|\mathrm{Fix}(S_1')| = t$, then $i_{S_1}(U_2,S_1) = 1$ (in the notation of (4.7)), and by (4.7)

$$
\begin{aligned}
w' &= w_0|_{T_2(U_2,S_1)}w(U_2,S_1)w \\
&= x(U_2,S_1,w_0)w\,,
\end{aligned}
$$

where $x(U_2,S_1,w_0) \in \mathrm{Sym}(\mathbf{t})$ depends only on U_2, S_1 and w_0.

Hence the right side of (4.10.1) becomes

$$
(4.10.2)\qquad \sum_{w\in \mathrm{Sym}(\mathbf{t})} p^{\lambda}_{T_1,T_2}(w)\delta^{n(U_2,S_1)}[S_1',S_2,x(U_2,S_1,w_0)w]
$$

(modulo terms with smaller t).

But the cellular property for the basis $\{C'_w | w \in \mathrm{Sym}(\mathbf{t})\}$ may easily be shown to imply that for any $w, w_1 \in \mathrm{Sym}(\mathbf{t})$, we have

$$(4.10.3) \qquad p^{\lambda}_{T_1,T_2}(w_1^{-1}w) = \sum_{T'_1 \in M(\lambda)} r_{w_1}(T'_1, T_1) p^{\lambda}_{T'_1,T_2}(w) + \sum_{\substack{T''_1, T''_2 \in M(\lambda'') \\ \lambda'' < \lambda}} r(T''_1, T''_2) p^{\lambda''}_{T''_1,T''_2}(w)$$

where $r_{w_1}(T'_1, T_1), r(T''_1, T''_2) \in R$.

Substituting (4.10.3) into (4.10.2), we see that modulo the R-linear span of $\{C^{t_1,\lambda_1}_{(S,T),(S',T')} | (t_1, \lambda_1) < (t, \lambda)\}$, the product in (4.10.1) is equal to (writing $w_1 = w(U_2, S_1, w_0)$)

$$(4.10.4) \qquad \sum_{T'_1 \in M(\lambda)} r_{w_1}(T'_1, T_1) \delta^{n(U_2,S_1)} C^{t,\lambda}_{(S'_1,T'_1),(S_2,T_2)} .$$

This completes the proof of (C3), and hence of (4.10). □

It now follows that all the results of Sects 2,3 above apply to $B(n)$. We shall give a parametrisation of its irreducible modules when R is a field using a result of Dipper and James. We begin by computing the bilinear form $\phi_{(t,\lambda)}((t,\lambda) \in \Lambda)$, (see (2.3) above).

In order to formulate our expression for $\phi_{(t,\lambda)}$, we require the following notation from the cellular structure of the group ring $R\ \mathrm{Sym}(\mathbf{t})$ $(t \in \mathbb{Z}_{\geqq 0})$. If $\lambda \in \mathscr{P}(t)$ and $T_1, T_2 \in M(\lambda)$ (i.e. the T_i are standard tableaux of shape λ) then for $w \in \mathrm{Sym}(\mathbf{t})$ we have

$$(4.11) \qquad C^{\lambda}_{T_1,T_2} w C^{\lambda}_{T_1,T_2} = \psi(w, T_1, T_2) C^{\lambda}_{T_1,T_2}$$

+ lower terms (for some $\psi(w, T_1, T_2) \in R$).

Note that $\psi(1, T_1, T_2) = \phi_\lambda(C_{T_1}, C_{T_2})$ where ϕ_λ is the bilinear form of (2.3) on the cell module $W(\lambda)$ for $R\ \mathrm{Sym}(\mathbf{t})$.

(4.12) Lemma. *With the above notation,* $\phi_\lambda = 0$ *if and only if* $\psi(w, T_1, T_2) = 0$ *for all* $w \in \mathrm{Sym}(\mathbf{t})$ *and* $T_1, T_2 \in M(\lambda)$.

Proof. The "if" statement is trivial. Conversely, if $\phi_\lambda = 0$ then $C^{\lambda}_{T_1,T_2} C^{\lambda}_{T'_1,T'_2} \equiv 0$ (mod lower terms) for all $T_1, T_2, T'_1, T'_2 \in M(\lambda)$. Hence

$$C^{\lambda}_{T_1,T_2} w C^{\lambda}_{T_1,T_2} \equiv C^{\lambda}_{T_1,T_2} \left(\sum_{T'_1} r_{T'_1} C^{\lambda}_{T'_1,T_2} \right) \equiv \sum_{T'_1} r_{T'_1} C^{\lambda}_{T_1,T_2} C^{\lambda}_{T'_1,T_2} \equiv 0$$

(mod lower terms). Hence $\psi(w, T_1, T_2) = 0$. □

(4.13) Proposition. *Let* $(t, \lambda) \in \Lambda$ *(i.e.* $t \in \mathscr{T}(n), \lambda \in \mathscr{P}(t)$*) and let* $\sigma_1 = (S_1, T_1)$ *and* $\sigma_2 = (S_2, T_2) \in M(t, \lambda)$ *(so that* $S_i \in I(t), T_i \in M(\lambda), i = 1, 2$*). Then the* *bilinear form* $\phi_{(t,\lambda)}$ *on* $W(t,\lambda)$ *is given by*

$$\phi_{(t,\lambda)}(C_{\sigma_1}, C_{\sigma_2}) = \begin{cases} \delta^{n(S_1,S_2)}\psi(w(S_1,S_2),T_1,T_2) & \text{if } \langle S_1,S_2\rangle \text{ has } t \text{ through arcs} \\ 0 & \text{otherwise} \end{cases}$$

where ψ *is as in* (4.11).

Proof. We have $\left(C^{t,\lambda}_{(S_1,T_1),(S_2,T_2)}\right)^2 \equiv \phi_{t,\lambda}(C_{\sigma_1},C_{\sigma_2})C^{t,\lambda}_{(S_1,T_1),(S_2,T_2)}$ (mod lower terms). Now $C^{t,\lambda}_{(S_1,T_1),(S_2,T_2)} = \sum_{w\in \mathrm{Sym}(\mathbf{t})} p^{\lambda}_{T_1,T_2}(w)[S_1,S_2,w]$. Moreover if $w_1,w_2 \in \mathrm{Sym}(\mathbf{t})$ we have by (4.7),

$$[S_1,S_2,w_1][S_1,S_2,w_2] = \delta^{n(S_1,S_2)}[S_1',S_2',w'] \tag{4.13.1}$$

where S_1',S_2',w' are as given in (4.7).

From the formulae in (4.7), if $|\mathrm{Fix}(S_1')| = t$, then $i_{S_1}(S_1,S_2) = i_{S_2}(S_1,S_2) = 1$, whence $S_1' = S_1$ and $S_2' = S_2$; moreover $w' = w_1w(S_1,S_2)w_2$. Hence

$$(C^{t,\lambda}_{(S_1,T_1),(S_2,T_2)})^2 \equiv \delta^{n(S_1,S_2)} \sum_{w_1,w_2\in\mathrm{Sym}(\mathbf{t})} p^{\lambda}_{T_1,T_2}(w_1)p^{\lambda}_{T_1,T_2}(w_2) \times [S_1,S_2,w_1w(S_1,S_2)w_2]\,.$$

But by (4.11),

$$\sum_{\substack{w_1,w_2\\ w_1ww_2=x}} p^{\lambda}_{T_1,T_2}(w_1)p^{\lambda}_{T_1,T_2}(w_2) = \psi(w,T_1,T_2)p^{\lambda}_{T_1,T_2}(x) + \sum_{\substack{\lambda'<\lambda\\ T_1',T_2'\in M(\lambda')}} r_{\lambda'}(T_1',T_2')p^{\lambda'}_{T_1',T_2'}(x)$$

(for some coefficients $r_{\lambda'}(T_1',T_2') \in M(\lambda')$). Therefore

$$(C^{t,\lambda}_{(S_1,T_1),(S_2,T_2)})^2 \equiv \delta^{n(S_1,S_2)}\psi(w(S_1,S_2),T_1,T_2)C^{t,\lambda}_{(S_1,T_1),(S_2,T_2)}$$

(mod lower terms).

This completes the proof of (4.13). □

(4.14) Corollary. *If* $\delta \neq 0$ *then* $\phi_{t,\lambda} = 0$ *if and only if* $\phi_\lambda = 0$.

Proof. For any $t \in \mathcal{T}(n)$, if we take $S_1 = S_2$ then $\langle S_1,S_2\rangle$ has t through strings and $w(S_1,S_2) = 1$, so that the matrix of $\phi_{t,\lambda}$ has a submatrix equal to $\delta^{n(S_1,S_1)}$ times the matrix of ϕ_λ. Thus if $\phi_\lambda \neq 0$, $\phi_{(t,\lambda)} \neq 0$. Conversely, if $\phi_\lambda = 0$, then $\psi(w,T_1,T_2) = 0$ for all $w \in \mathrm{Sym}(\mathbf{t})$ and $T_1,T_2 \in M(\lambda)$ (by (4.12)). Hence $\phi_{(t,\lambda)} = 0$. □

To deal with the case $\delta = 0$, we require

(4.15) Lemma. *For any $t \in \mathscr{T}(n)\backslash\{0\}$, there are involutions $S_1, S_2 \in I(t)$ such that $w(S_1, S_2) = 1$ and $n(S_1, S_2) = 0$. If $t = 0$, $n(S_1, S_2) > 0$ for any $S_1, S_2 \in I(t)$, provided $n \geqq 2$.*

Proof. Since all elements of $I(t)$ are conjugate in $\mathrm{Sym}(\mathbf{n})$, we may (w.l.o.g.) assume that $S_1 = (12)(34)\cdots(2k-1, 2k)$, where $t = n - 2k$. Let $S_2 = (23)(45)\cdots(2k, 2k+1)$ (here we use $t > 0$). Then $n(S_1, S_2) = 0$ and $w(S_1, S_2) = 1$. If $t = 0$, then clearly each $\langle S_1, S_2\rangle$ orbit on $\mathbf{n}$ is a loop, since there are no fixed points. Hence $n(S_1, S_2) > 0$. □

(4.16) Corollary. *Suppose $\delta = 0$*

(i) *If $t > 0$, then $\phi_{t,\lambda} = 0$ if and only if $\phi_\lambda = 0$.*

(ii) *We have $\phi_{0,\lambda} = 0$.*

Proof.

(i) By (4.15) and (4.13) all values of ϕ_λ occur as values of $\phi_{t,\lambda}$, so if the latter is 0, so is the former. Conversely, if $\phi_\lambda = 0$, (4.12) and (4.13) show that $\phi_{(t,\lambda)} = 0$.

(ii) is clear from (4.15). □

We now have complete information about the set of irreducible $B(n)$ modules in case R is a field.

(4.17) Theorem. *Let $B(n)$ ($n \geqq 2$) be the Brauer algebra over R, assumed to be a field of characteristic p (possibly $p = 0$), corresponding to the parameter $\delta \in R$. The set of (equivalence classes of) irreducible $B(n)$-modules is parametrised by $\{(t, \lambda) | t \in \mathscr{T}(n),\ \lambda \in \mathscr{P}(t),\ \lambda$ is p-regular$\}$ (a partition is p-regular if it does not have p equal parts ($p \neq 0$); if $p = 0$ all partitions are p-regular) where $\mathscr{T}(n) = \{t \in \mathbf{n} \cup \{0\} | n - t \in 2\mathbb{Z}\}$ and $\mathscr{P}(t)$ is the set of partitions of t, except if $\delta = 0$, in which case one removes from the parameter set above the element $(0, \lambda)$, where λ is the empty partition.*

Proof. It follows from Theorems (3.4) and (4.10) that the irreducible $B(n)$-modules are parametrised by $\{(t, \lambda) \in \Lambda | \phi_{(t,\lambda)} \neq 0\}$. But for $t \neq 0$, it follows from (4.14) and (4.16) that $\phi_{(t,\lambda)} \neq 0$ if and only if $\phi_\lambda \neq 0$. Moreover it is a result of Dipper and James [DJ1, (7.6)] that $\phi_\lambda \neq 0$ if and only if λ is p-regular. Finally $\phi_{0,\lambda}$ is nonzero precisely when $\delta \neq 0$. This completes the proof of the Theorem. □

5 The Ariki–Koike Hecke algebras

In [AK], Ariki and Koike defined a "Hecke algebra" corresponding to the group $G(r, 1, n) = (\mathbb{Z}/r\mathbb{Z}) \wr \mathrm{Sym}(\mathbf{n})$. This is an associative algebra over the ring $R := \mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}, u_1, u_2, \ldots, u_r]$ where $q^{\frac{1}{2}}$ and the u_i are indeterminants over $\mathbb{Z}$. This algebra, which we denote (following [AK]) by $H_{n,r}(R)$ includes the usual Hecke algebras of type A_{n-1} (the case $r = 1$) and B_n and C_n (the case $r = 2$).

In this section we shall show that $H_{n,r}(R)$ has a cellular structure (as in (1.1)) and its representations may therefore be studied as in Sect. 3. We follow the notation of [AK] as far as possible.

(5.1) Definition. Fix $R = \mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}, u_1, u_2, \ldots, u_r]$ *as above and define the associative R-algebra* $H_{n,r} = H_{n,r}(R)$ *as follows:* $H_{n,r}$ *is generated by* $\{t = a_1, a_2, \ldots, a_n\}$ *subject to relations.*

(H1) $(t - u_1)(t - u_2)\ldots(t - u_r) = 0$

(H2) $a_i^2 = (q-1)a_i + q \quad (i = 2, \ldots, n)$

(H3) $ta_2ta_2 = a_2ta_2t$

(H4) $a_ia_j = a_ja_i \quad (|i-j| \geqq 2)$

(H5) $a_ia_{i+1}a_i = a_{i+1}a_ia_{i+1} \quad (i = 2, \ldots, n-1)$

Note that these are deformations of a well known set of defining relations for the unitary reflection group $G(r, 1, n)$.

Following [AK], define elements $s_1, s_2, \ldots, s_n \in H_{n,r}$ inductively by $s_1 = t$ and $s_i = q^{-1}a_is_{i-1}a_i$ if $i > 1$. These elements generate an abelian subalgebra $U_{n,r}$ [AK, Lemma (3.3)]. If $\tau : \mathbf{n} \to \mathbf{r}$ is any function, define

$$s^\tau := \prod_{i \in \mathbf{n}} s_i^{\tau(i)-1} . \tag{5.11}$$

Next, observe that $\{a_2, a_3, \ldots, a_n\}$ generates a quotient $H_{n,r}^0$ of the Hecke algebra of type A_{n-1}. Hence for w in Sym($\mathbf{n}$), we may speak of the corresponding element $a_w \in H_{n,r}$ (defined in terms of a reduced expression for w as a product of simple reflections in Sym($\mathbf{n}$)).

(5.2) Theorem. [AK, Theorem (3.10)].

The algebra $H_{n,r}$ *is R-free with basis*

$$\{s^\tau a_w | \tau \textit{ is a function: } \mathbf{n} \to \mathbf{r}, w \in \mathrm{Sym}(\mathbf{n})\} .$$

We may therefore identify $H_{n,r}^0$ with the Hecke algebra of type A_{n-1}.

We shall define a cellular structure for $H_{n,r}$ with only sketches of the computations necessary to justify the statements because these are straightforward and in any case similar ones may be found in [AK].

The symmetric group Sym($\mathbf{n}$) acts on the set of functions $\tau : \mathbf{n} \to \mathbf{r}$ by composition: $\tau w := \tau \circ w$. Each orbit of this action contains a unique function $\tau : \mathbf{n} \to \mathbf{r}$ which is nonincreasing; that is, if $i \leqq j$, $\tau(i) \geqq \tau(j)$. The set $\Lambda_{n,r}^+$ of such orbit representatives is partially ordered as follows: $\tau < \tau'$ iff

(i) $\quad \sum_i \tau(i) < \sum_i \tau'(i)$ or

(ii) $\quad \sum_i \tau(i) = \sum_i \tau'(i)$ and $\sum_{i<j} (\tau(i) - \tau(j))^2 < \sum_{i<j} (\tau'(i) - \tau'(j))^2$.

We then say $\tau \leqq \tau'$ iff $\tau < \tau'$ or $\tau = \tau'$.

Let $\tau \in \Lambda_{n,r}^+$. The stabiliser $S(\tau) := \{w \in \mathrm{Sym}(\mathbf{n}) | \tau w = \tau\}$ is a standard parabolic subgroup of Sym($\mathbf{n}$). Noting that $S(\tau) = \mathrm{Sym}(\tau^{-1}(1)) \times \mathrm{Sym}(\tau^{-1}(2)) \times \cdots \times \mathrm{Sym}(\tau^{-1}(r))$, we define the set Λ^τ of r-tuples $\alpha = (\alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(r)})$

where $\alpha^{(i)}$ is a partition of $|\tau^{-1}(i)|$. This set is partially ordered by the componentwise dominance order, that is, $\alpha \leqq \beta$ iff $\alpha^{(i)} \leqq \beta^{(i)}$ for i in $\mathbf{r}$.

(5.3) Definition. *The partially ordered set Λ is defined by $\Lambda := \{(\tau,\alpha)| \tau \in \Lambda^+_{n,r},\ \alpha \in \Lambda^\tau\}$ with the order given by $(\tau,\alpha) \geqq (\sigma,\beta)$ iff* (i) $\tau < \sigma$ *(see above) or (ii) $\tau = \sigma$ and $\alpha \geqq \beta$.*

Next we construct the sets $M(\tau,\alpha)$ required by (1.1). Let $(\tau,\alpha) \in \lambda$. Recall that there exists a set $D(\tau)$ of distinguished (shortest, left) coset representatives for $S(\tau)$ in $\mathrm{Sym}(\mathbf{n})$. Explicitly, $D(\tau) = \{w \in \mathrm{Sym}(\mathbf{n})|w(i) < w(j)$ for i,j in $\mathbf{n}$ such that $i < j$ and $\alpha(i) = \alpha(j)\}$. A *multi-tableau* $\mathbf{S}$ of shape α is an r-tuple $(S^{(1)},S^{(2)},\ldots,S^{(r)})$ of standard tableaux, where $S^{(i)}$ has shape $\alpha^{(i)}$ and entries $\tau^{-1}(i)$.

(5.4) Definition. *For* $(\tau,\alpha) \in \Lambda$ (see (5.3)), *the set $M(\tau,\alpha)$ is defined to be $\{(d,\mathbf{S})|d \in D(\tau),\mathbf{S}$ is a multi-tableau of shape $\alpha\}$.*

Let $(\tau,\alpha) \in \Lambda$. If $(d_1,\mathbf{S}_1), (d_2,\mathbf{S}_2) \in M(\tau,\alpha)$, definition (1.1) calls for a basis element $C^{(\tau,\alpha)}_{(d_1,\mathbf{S}_1)(d_2,\mathbf{S}_2)}$ of $H_{n,r}$. Define

$$p^\tau := \prod_{i\in\mathbf{n}} \prod_{1\leqq j<\tau(i)} (s_i - u_j)\,.$$

By [AK, Lemma (3.3)], this element commutes with a_w if $w \in S(\tau)$. The R-span H^τ of $\{a_w|w \in S(\tau)\}$ is a subalgebra of $H_{n,r}$ which is isomorphic to the Hecke algebra of $S(\tau)$ over R. We therefore have the Kazhdan–Lusztig basis elements $C'_w(w \in S(\tau))$ of this subalgebra. Clearly w in $S(\tau)$ corresponds to a pair $(\mathbf{S}_1, \mathbf{S}_2)$ of multi-tableaux of the same shape (α, say). Thus we may write $C^\alpha_{\mathbf{S}_1,\mathbf{S}_2} := C'_w$ (for $w \in S(\tau)$).

(5.5) Theorem. *Let $H_{n,r}$ be the R-algebra defined in* (5.1). *Then $H_{n,r}$ has a cell datum $(\Gamma,M,C,*)$ given as follows: Λ is the partially ordered set defined in* (5.3); *for $(\tau,\alpha) \in \Lambda$, the set $M(\tau,\alpha)$ is defined in* (5.4); *if $(\tau,\alpha) \in \Lambda$ and $(d_1,\mathbf{S}_1)$, $(d_2,\mathbf{S}_2) \in M(\tau,\alpha)$, then $C^{(\tau,\alpha)}_{(d_1,\mathbf{S}_1),(d_2,\mathbf{S}_2)} := a_{d_1^{-1}} p^\tau C^\alpha_{\mathbf{S}_1,\mathbf{S}_2} a_{d_2}$. Finally $*$ is the unique anti-involution of $H_{n,r}$ such that $a_i^* = a_i$ if $i \in \mathbf{n}$.*

Proof (Sketch). The partial order defined above on $\Lambda^*_{n,r}$ extends readily to the set of all functions $\tau\colon \mathbf{n} \to \mathbf{r}$. If $\tau \in \Lambda^+_{n,r}$ and $d \in D(\tau)$, elaborating [AK, Lemma (3.3)] yields

$$a_d^* p^\tau = a_d^* s^\tau = q^{l(d)} s^{\tau d} (a_d)^{-1} \tag{5.5.1}$$

modulo the R-span of $\{s^\sigma a_w|\sigma\colon \mathbf{n} \to \mathbf{r}, \sigma < \tau,\ w \in \mathrm{Sym}(\mathbf{n})\}$.

It follows Theorem (5.2), that the set

$$\{C^{(\tau,\alpha)}_{(d_1,\mathbf{S}_1),(d_2,\mathbf{S}_2)}|(\tau,\alpha) \in \Lambda;(d_2,\mathbf{S}_1),(d_2,\mathbf{S}_2) \in M(\tau,\alpha)\}$$

is a basis of $H_{n,r}$, whence (C1) of (1.1). (C3) is a consequence of the following computation: if $i \in \mathbf{n}$ and $\tau \in \Lambda^+_{n,r}$, we have

$$s_i p^\tau = u_{\tau(i)} L_i^\tau p^\tau \tag{5.5.2}$$

modulo the R-linear span of $\{H_{n,r}^0 p^\sigma H_{n,r}^0 | \sigma \in A_{n,r}^+, \sigma > \tau\}$, where $L_i^\tau := q^{-\ell(x)} a_x^* a_x$, $x = (j, j+1, \ldots, i)$ and j is the least element of $\mathbf{n}$ such that $\alpha(j) = \alpha(i)$. (We use the notation L_i^τ to indicate the close relationship between this element and the Murphy operators of [M].) □

To study the irreducible representations and block theory of $H_{n,r}$, we consider the action of the abelian subalgebra $U_{n,r}$ on the cell representation $W(\tau, \alpha)$. In order to describe this it is convenient to order the indexing set $M(\tau, \alpha)$. Let $(\tau, \alpha) \in \Lambda$. If $i \in \mathbf{n}$ and $\mathbf{S}$ is a multi-tableau of shape α, then those nodes of $s^{\tau(i)}$ which are labelled by $\{j \in \mathbf{n} | \tau(i) = \tau(j), j \leqq i\}$ form a tableau whose shape defines a partition which we call $\mathrm{Sh}_i(\mathbf{S})$. Restrict the Bruhat order on $\mathrm{Sym}(\mathbf{n})$ to $D(\tau)$.

(5.6) Definition. *If* $(d_1, \mathbf{S}_1)$, $(d_2, \mathbf{S}_2) \in M(\tau, \alpha)$, *we say that* $(d_1, \mathbf{S}_1) \leqq (d_2, \mathbf{S}_2)$, *iff*

(i) $d_1 < d_2$ *or*
(ii) $d_1 = d_2$ *and* $\mathrm{Sh}_i(\mathbf{S}_1) \leqq \mathrm{Sh}_i(\mathbf{S}_2)$ *for each* i *in* $\mathbf{n}$.

Next, if $i \in \mathbf{n}$ and $\mathbf{S}$ is a multi-tableau of shape α, recall that the *content* $c(\mathbf{S}, i)$ is $m - \ell$ where the node of $S^{(\tau(i))}$ labelled by i is located in the mth column and the ℓth row. If $(d, \mathbf{S}) \in M(\tau, \alpha)$, define the function

$$z_{d,\mathbf{S}} : \mathbf{n} \to R : i \mapsto u_{rd(i)} q^{c(\mathbf{S}, d(i))} .$$

Note that the multiset of values $z(\tau, \alpha) := \{z_{d,\mathbf{S}}(i) | i \in \mathbf{n}\}$ depends only on (τ, α) since $z_{d,\mathbf{S}}(i)$ depends only on the position which $d(i)$ occupies in $\mathbf{S}$.

(5.7) Lemma. *If* $(\tau, \alpha) \in \Lambda$ *and* $M(\tau, \alpha)$ *is ordered as above, the abelian subalgebra* $U_{n,r}$ *acts on the cell representation* $W(\tau, \alpha)$ *in triangular fashion: if* $i \in \mathbf{n}$ *and* $(d, \mathbf{S}) \in M(\tau, \alpha)$ *we have* $s_i C^{(\tau,\alpha)}(d, \mathbf{S}) = z_{d,\mathbf{S}}(i) C^{(\tau,\alpha)}(d, \mathbf{S})$ *modulo the span of* $\{C^{(\tau,\alpha)}(x, \mathbf{T}) | (x, \mathbf{T}) \in M(\tau, \alpha)$ *and* $(x, \mathbf{T}) < (d, \mathbf{S})\}$.

Proof (Sketch). Recall the cellular structure of the Hecke algebra of type A_{n-1} defined in example (1.2). If λ is a partition of n, the cell representation $W(\lambda)$ is isomorphic to the Specht module S_λ defined in [M]. Furthermore, with the notation of [M, (3.5)], the basis $\{q^{\frac{1}{2}\ell(d(\mathbf{S}))} C^\lambda(S) | S \in M(\lambda)\}$ is related to the "standard basis" by a unitriangular matrix by [KL1, (2.3)]. Hence [M, Theorem 4.6] may be reformulated in the notation of the statement as follows: if $\mathbf{S}$ is a multi-tableau of shape α and $i \in \mathbf{n}$, we have $L_i^\tau C^\alpha(\mathbf{S}) = q^{c(\mathbf{S},i)} C^\alpha(\mathbf{S})$ modulo the R-span of $\{C^\alpha(\mathbf{T}) | \mathbf{T} < \mathbf{S}\}$. On the other hand, [AK, Lemma (3.3)] ensures that if $w \in \mathrm{Sym}(\mathbf{n})$, we have $a_w s_i = s_{w(i)} a_w$ modulo the R-span of

$$\{s_j a_x | j \in \mathbf{n},\ x \in \mathrm{Sym}(\mathbf{n}),\ x < w\} .$$

Hence the computation in the proof of (5.5) may be sharpened to yield

$$s_i a_d^* p^\tau = a_d^* s_{d(i)} p^\tau = u_{\tau d(i)} a_d^* p^\tau L_{d(i)}^\tau \tag{5.7.1}$$

modulo the sum of $H_{n,r} p^\sigma H_{n,r}$ for $\sigma > \tau$ and the R-span of $\{a_x^* p^\tau H^\tau | x \in D^\tau, x < d\}$. Combining these facts yields the lemma. □

Comparing this lemma with [AK, Proposition (3.16)] we obtain

(5.8) Corollary. *In the notation of* [AK], *the irreducible module V_α of $H_{n,r}(K)$ is isomorphic to the cell representation $W(\tau,\alpha)$ where K is the field of fractions of R.*

(5.9) Proposition. *Let F be a field, $\phi: R \to F$ be a ring homomorphism, and consider the specialisation $H_{n,r}(F) = F \otimes_R H_{n,r}(R)$.*

(i) *Let $(\tau,\alpha) \in \Lambda$. Suppose there exists $(d,\mathbf{S}) \in M(\tau,\alpha)$ such that the function $z_{d,\mathbf{S}} \neq z_{x,\mathbf{T}}$ for any pair $(x,\mathbf{T}) \in M(\sigma,\beta)$ such that $(\sigma,\beta) > (\tau,\alpha)$. Then $L_{(\tau,\alpha)}$ is nonzero.*

(ii) *If the irreducible modules $L_{(\tau,\alpha)}$ and $L_{(\sigma,\beta)}$ lie in the same block, then the multisets $z(\tau,\alpha)$ and $z(\sigma,\beta)$ are equal.*

Proof. (i) Lemma (5.7) provides a filtration of $W(\tau,\alpha)$ as a $U_{n,r}$-module with irreducible quotients indexed by $M(\tau,\alpha)$. By way of contradiction, suppose $L_{(\tau,\alpha)} = 0$.

In the Grothendieck group spanned by irreducible $H_{n,r}$ modules, we then have, using (3.6) that

$$W(\tau,\alpha) = \sum_{(\sigma,\beta)>(\tau,\alpha)} n_{\beta,\alpha} W(\sigma,\beta) \tag{5.9.1}$$

where $n_{\beta,\alpha} \in \mathbb{Z}$. Restricting the action to $U_{n,r}$ the assumption in the statement of part (i) implies that the irreducible $U_{n,r}$-module R with action $s_i r = z_{d,\mathbf{S}}(i) r$ ($r \in R$) appears in $W(\tau,\alpha)$, but not in any term on the right hand side. This contradiction proves (i).

(ii) By [AK, Lemma (3.3)], the elementary symmetric polynomial

$$e_j = \sum_{\substack{J \subseteq \mathbf{n} \\ |J|=j}} \prod_{i \in J} s_i$$

is central in $H_{n,r}(R)$. The algebra $H_{n,r}$ is generically (i.e. over K) semisimple and $W(\tau,\alpha)$ is irreducible [AK, Theorem (3.10)]. Therefore e_j acts via a scalar $\lambda_j(\tau,\alpha)$ on this module. The same thing applies over R via restriction, and over F via specialisation.

Using (5.7), it is apparent that if $(d,\mathbf{S}) \in M(\tau,\alpha)$ the scalar $\lambda_j(\tau,\alpha)$ is the coefficient of X^j in the polynomial $\prod_{i\in\mathbf{n}}(X - z_{d,\mathbf{S}}(i))$, which we shall call $f_{(\tau,\alpha)}(X)$ (cf. remarks following (5.6)). Now clearly if $L_{(\tau,\alpha)}$ and $L_{(\sigma,\beta)}$ are in the same block, e_j acts via the same scalar on both of these modules (for all $j \in \mathbf{n}$). Thus $f_{(\tau,\alpha)}(X) = f_{(\sigma,\beta)}(X)$, as stated. □

When $r = 1$ and $q \neq 1$, Dipper and James have obtained both (5.9) and its converse [DJ1], [DJ2]. When $r = 2$ and $q \neq 1$, Dipper, James and Murphy have proved (5.9) and conjectured its converse [DJM]. Hence it seems reasonable to ask

(5.10) Question. With notation as above, suppose $q \neq 1$.

(i) Is the condition of (5.9) (i) necessary in order for $L_{(\tau,\alpha)}$ to be nonzero?

(ii) Is the condition $z(\tau, \alpha) = z(\sigma, \beta)$ sufficient in order that $L_{(\tau,\alpha)}$ and $L_{(\sigma,\beta)}$ lie in the same block?

6. The Temperley–Lieb and Jones algebras

These are subalgebras of $B_n(R, \delta)$ (see Sect. 4) defined as follows.

(6.1) Definitions.

(i) *The* **Temperley–Lieb algebra** $\mathrm{TL}_n(R)$ *is the subalgebra of* $B_n(R)$ *spanned by the planar diagrams of the form* (1.4) (*i.e. those which may be drawn in the plane without any intersections* (*in the convex hull of the* $2n$ *points*).

(ii) *The* **Jones algebra** $J_n(R)$ (cf. [J3]) *is the subalgebra of* $B_n(R)$ *spanned by diagrams which are planar when the* $2n$ *points are on the inner and outer circles of a plane annulus.*

The key to understanding the cellular structure of these algebras is to describe them according to the setup of Sect. 4.

Let us call an involution S of $\mathrm{Sym}(\mathbf{n})$ *planar* if $[S, T, w]$ is a basis element of TL_n for some T, w (see (4.4)).

(6.2) Lemma.

(i) *An involution* $S \in \mathrm{Sym}(\mathbf{n})$ *is planar if and only if, for any pair* i, j *interchanged by* $S(i < j)$, *we have*

(a) $S[i, j] = [i, j]$ *and*

(b) $[i, j] \cap \mathrm{Fix}\, S = \emptyset$

(*where* $[i, j] = \{k \in \mathbf{n} | i \leqq k \leqq j\}$).

(ii) *If* S *is a planar involution of* $\mathbf{n}$ *and* T *is a planar involution of* $\mathrm{Fix}\,\mathbf{S}$ (*naturally linearly ordered*), *then* ST *is a planar involution of* $\mathbf{n}$.

The proof is easy.

For any linearly ordered set X, denote by $\mathrm{Plan}(X)$ the set of planar involutions in $\mathrm{Sym}(X)$.

(6.3) Lemma. *There is a canonical bijection between* $\mathrm{Plan}(\mathbf{n})$ *and standard tableaux of size* n, *which have* 1 *or* 2 *rows.*

Proof. Given $S \in \mathrm{Plan}(\mathbf{n})$, we take the second row of a standard tableau to be $\{j | Si = j, i < j\}$. A little reflection shows this is a bijection. □

(6.4) Lemma. *Suppose* S, S' *are planar involutions of* $\mathbf{n}$. *Then* (*in the notation of* (4.6))

(i) $i_S(S, S')$ *and* $i_{S'}(S, S')$ *are planar involutions of* $\mathrm{Fix}\, S$ *and* $\mathrm{Fix}\, S'$ *respectively.*

(ii) $w(S, S') = \mathrm{id}$.

The proof is straightforward, given the characterisation of planar involutions in (6.2).

Recall (4.9) that $\mathscr{T}(n) = \{t \in \boldsymbol{n} \cup \{0\} | n - t \in 2\mathbb{Z}\}$. For any $t \in \mathbb{Z}_{\geq 0}$, define $w(t)$ as the "longest element" in Sym(t), i.e. $w(t)i = t - i + 1$ (if $t = 0$ or 1, $w(t) = \mathrm{id}$). For any $X \subseteq t$, write $w(X)$ for the corresponding element of $\mathrm{Sym}(X)$, X being naturally linearly ordered as a subset of $\mathbf{t}$.

If X is any subset of $\mathbf{t}$, then in the notation of (4.7), we have $w(t)|_X = w(X)$. Moreover, if $r \in \mathrm{Plan}(\mathbf{t})$ then $w(t)r(w(t))^{-1} \in \mathrm{Plan}(\mathbf{t})$.

(6.5) Proposition.

(i) *The set* $\{[S_1, S_2, w] | S_t \in \mathrm{Plan}(\mathbf{t}),\ w = w(t)$ *or* $\mathrm{id},\ t \in \mathscr{T}(n)\}$ *is an* R*-basis of a subalgebra* $\widetilde{\mathrm{TL}}_n(R)$ *of* $B_n(R)$.

(ii) *The subset of the basis in* (i) *consisting of elements with* $w = \mathrm{id}$ *is a basis of the Temperley–Lieb algebra* $\mathrm{TL}_n(R)$.

Proof. Both statements follow from the multiplication formula (4.7) and the results above, given the restriction and conjugation properties of $w(t)$ mentioned above. □

Notice that $\dim_R \widetilde{\mathrm{TL}}_n(R) = 2 \dim \mathrm{TL}_n(R) - x_n^2$, where x_n is the number of planar involutions with at most 1 fixed point. Thus $x_n = \frac{1}{k+1}\binom{2k}{k}$ if $n = 2k$ is even, and $x_n = x_{n+1}$ if n is odd; i.e. x_n is the kth Catalan number, where $k = [\frac{n+1}{2}]$. This is because for each $t \in \mathscr{T}(n)$, there are twice as many basis elements as there are for $\mathrm{TL}_n(R)$, except when $t = 0$ or 1. In the latter case, the number of basis elements of $\mathrm{TL}_n(R)$ is x_n (cf. (4.2)).

(6.6) Corollary. *Multiplication of basis elements in* $\widetilde{\mathrm{TL}}_n(R)$ *is given by*

$$[S_1, S_2, w][S_1', S_2', w'] = \delta^{n(S_2, S_1')}[S_1'', S_2'', w'']$$

where $S_i \in \mathrm{Plan}(\mathbf{n}) \cap I(t)$, $S_i' \in \mathrm{Plan}(\mathbf{n}) \cap I(t')$ *and* $S_i'', (i = 1, 2)$ *are given by* (4.7) *and* w'' *is given by the following table*

$w' \backslash w$	id	$w(t)$
id	id	$w(t'')$
$w(t')$	$w(t'')$	id

This follows immediately from (6.3) and (6.4). A cellular structure may now be identified in $\widetilde{\mathrm{TL}}_n(R)$ using the "banal" example (1.3) with $f(x) = x^2 - 1$. Let $\Lambda_2 = \{(t, \sigma) | t \in \mathscr{T}(n), \sigma \in \mathscr{D}(t)\}$, where $\mathscr{T}(n) = \{t \in \{0, 1, \ldots, n\} | n - t \in 2\mathbb{Z}\}$ and for $t \in \mathscr{T}(n)$,

$$\mathscr{D}(t) = \begin{cases} \{1\} & \text{if } t = 0 \text{ or } 1 \\ \{0, 1\} & \text{if } t > 1\,. \end{cases}$$

The partial order on Λ_2 is given by $(t, \sigma) \leqq (t', \sigma')$ if $t < t'$ or $t = t'$ and $\sigma \leqq \sigma'$. For $(t, \sigma) \in \Lambda_2$, let $M_2(t, \sigma) := \mathrm{Plan}(\mathbf{n}) \cap I(t)$.

(6.7) Theorem. *With the above notation, the algebras* $\mathrm{TL}_n(R)$ *and* $\widetilde{\mathrm{TL}}_n(R)$ *have cell data* $(\Lambda_1, M_1, C, *)$ *and* $(\Lambda_2, M_2, C, *)$ *respectively, where* $\Lambda_1 = \mathscr{T}(n), M_1(t) = \mathrm{Plan}(\mathbf{n}) \cap I(t)$, $C^t_{S_1, S_2} = [S_1, S_2, 1]$ *and* $[S_1, S_2, 1]^* = [S_2, S_1, 1]$

and where Λ_2 and M_2 are as above and

$$C^{t,\sigma}_{S_1,S_2} := \begin{cases} [S_1,S_2,1]+[S_1,S_2,w(t)] & (\sigma=0) \\ [S_1,S_2,1] & (\sigma=1) \end{cases}$$

(where $S_1,S_2 \in \mathrm{Plan}(\mathbf{n}) \cap I(t)$). The map $*$ on $\widetilde{\mathrm{TL}}_n(R)$ is given by $(C^{t,\sigma}_{S_1,S_2})^* = C^{t,\sigma}_{S_2,S_1}$.

Proof. The result for TL_n is easy. For $\widetilde{\mathrm{TL}}_n(R)$, the statements (C1), (C2) and (C3) of (1.1) need to be checked. Since (C1) and (C2) are clear, we check only (C3). It is sufficient to take any basis element $C^{t,\sigma}_{S_1,S_2}(S_i \in \mathrm{Plan}(\mathbf{n})\cap I(t))$ and $a=[S'_1,S'_2,w']$, ($S'_i \in I(t')$, $w'=\mathrm{id}$ or $w(t')$) and prove the relation (C3).

Now by (4.7), $[S'_1,S'_2,w'][S_1,S_2,w]$ is an R-linear combination of lower terms unless $i_{S_1}(S_1,S'_2)=\mathrm{id}$, in which case $[S'_1,S'_2,w'][S_1,S_2,w]=[S''_1,S_2,w'']$ where S''_1 depends on S'_1,S'_2 and S_1. Moreover w'' is given by the table in (6.5). One therefore sees easily that in the above notation,

$$[S'_1,S'_2,w']C^{t,\sigma}_{S_1,S_2} \equiv \begin{cases} \delta^{n(S_1,S'_2)}C^{t,\sigma}_{S''_1,S_2} & (\sigma=1 \text{ or } w'=\mathrm{id}) \\ -\delta^{n(S_1,S'_2)}C^{t,\sigma}_{S''_1,S_2} & (\sigma=0,\ w'\neq\mathrm{id}) \end{cases}$$

where $\equiv$ denotes equivalence modulo terms $C^{t',\sigma'}_{S_3,S_4}$ with $(t',\sigma')<(t,\sigma)$.

This proves the desired relation. □

(6.8) Corollary. *If R is any field, the (equivalence classes of) irreducible representations of TL_n are parametrised by $\mathscr{T}(n)$ unless n is even and $\delta=0$, in which case the set of irreducibles is parametrised by $\mathscr{T}(n)\backslash\{0\}$. (We refer to this set as $\Lambda_{1,0}(\delta)$).*

Proof. The value of the bilinear form ϕ_t may be computed using the equation $(C^t_{S_1,S_2})^2 \equiv \phi_t(S_1,S_2)C^t_{S_1,S_2}$ (modulo lower terms). This shows that $\phi_t(S_1,S_2) = \delta^{n(S_1,S_2)}$ for all $S_1,S_2 \in I(t)\cap\mathrm{Plan}(\mathbf{n})$. But for $t>0$, it is possible to find $S_1,S_2 \in \mathrm{Plan}(\mathbf{n})\cap I(t)$ such that $n(S_1,S_2)=0$ (see Sect. 4), whence the result. □

(6.9) Corollary. *The dimension $d(t)$ of the irreducible representation of $\mathrm{TL}_n(R)$ (R a field) corresponding to $t\in\mathscr{T}(n)$ is equal to the rank of the matrix $(\delta^{n(S_1,S_2)})$ ($S_i \in \mathrm{Plan}(\mathbf{n})\cap I(t)$), where $n(S_1,S_2)$ = number of loop orbits of (S_1,S_2) on $\mathbf{n}$ and the nonzero entries of the matrix occur only for pairs (S_1,S_2) such that $t(S_1,S_2)=t$.*

(6.10) Corollary. *The irreducible representations of $\widetilde{\mathrm{TL}}_n(R)$ are parametrised as follows. If the characteristic of R is 2, then the representations are parametrised by $\Lambda_{1,0}(\delta)$ (i.e. by the same set as those of $\mathrm{TL}_n(R)$). If the characteristic of R is not 2, then a parameter set is $\Lambda_{2,0}(\delta)=\{(t,\sigma)\in\Lambda_2 \mid t\in\Lambda_{1,0}(\delta)\}$*

Proof. It is straightforward to show that

$$(6.10.1) \qquad (C^{t,\sigma}_{S_1,S_2})^2 \equiv \begin{cases} \delta^{n(S_1,S_2)} C^{t,\sigma}_{S_1,S_2} & \text{if } \sigma = 1 \\ 2\delta^{n(S_1,S_2)} C^{t,\sigma}_{S_1,S_2} & \text{if } \sigma = 0\,. \end{cases}$$

The result follows. It is furthermore clear that the dimension of the irreducible representation corresponding to (t_0, σ_0) is equal to that of TL_n which corresponds to t_0. □

To discuss the Jones algebra $J_n(R)$ we use the notation of an *annular* involution. For a definition of the algebra see [J3].

(6.11) Definition. *An involution $S \in \mathrm{Sym}(\mathbf{n})$ is* **annular** *if there exists an element $[S, S', w]$ in the Jones algebra $J_n(R)$.*

(6.12) Lemma.

(i) *The involution $S \in \mathrm{Sym}(\mathbf{n})$ is annular if and only if, for each pair i, j interchanged by $S(i < j)$, we have* (cf. (6.2)).
  - (a) $S[i,j] = [i,j]$
  - (b) $[i,j] \cap \mathrm{Fix}\, S = \emptyset$ or $\mathrm{Fix}\, S \subseteq [i,j]$

(ii) *For $k \in \mathbf{n}$, let τ_k be the permutation of $\mathbf{n}$ given by $\tau_i = i + k$* (mod n). *Then the set* $\mathrm{Ann}(\mathbf{n})$ *is invariant under conjugation by τ_k.*

Proof. The proof of (i) is straightforward and is omitted. If $\tau = \tau_k$ (some $k \in \mathbf{n}$) and $i, j \in \mathbf{n}(i < j)$ then $\tau[i,j] = [\tau i, \tau j]$ if $\tau i < \tau j$ or $\tau[i,j] = \mathbf{n} \backslash [\tau j, \tau i] \cup \{\tau i, \tau j\}$ if $\tau i > \tau j$. Since both conditions (a) and (b) of (i) are symmetric with respect to $[i,j]$ and its complement, (ii) follows. □

As in the case of planar involutions, if X is any totally ordered set, we may speak of $\mathrm{Ann}(X)$ as the set of involutions in $\mathrm{Sym}(X)$ which satisfy the conditions of (6.12)(i).

(6.13) Lemma.

(i) *If $S \in \mathrm{Ann}(\mathbf{n})$ and $w \in \mathrm{Ann}(\mathrm{Fix}\, S)$, then $Sw \in \mathrm{Ann}(\mathbf{n})$.*

(ii) *$S, S' \in \mathrm{Ann}(\mathbf{n})$ then (cf.* (4.6))
  - (a) $i_S\ (S, S') \in \mathrm{Ann}(\mathrm{Fix}\, S)$
  - (b) $w(S, S') \in \mathrm{Sym}(t(S, S')) = \tau_k$ *for some k.*

(iii) *Suppose $k \in \mathbf{n}$. If $X = \mathrm{Fix}\, S \subseteq \mathbf{n}$, for some $S \in \mathrm{Ann}(\mathbf{n})$, then $\tau_k|_X = \tau_{k'} \in \mathrm{Sym}(X)$ for some k' (see* (4.7)).

Proof. (i) and (ii) are clear from the diagrammatic viewpoint, since the annular composition process leads to no self intersections. Alternatively, they may be proved directly from the definitions, using (6.12).

For (iii), observe that by downward induction on the number of fixed points of S, it is sufficient to take $X = \mathbf{n} \backslash \{i, i+1\}$, some $i \in \mathbf{n}$ (if $i = n$, then $\{i, i+1\} = \{1, n\}$). In this case, the statement is clear. □

(6.14) Proposition. *The set $\{[S_1, S_2, w] | S_i \in \mathrm{Ann}(\mathbf{n}) \cap I(t), t \in \mathscr{T}(n), w = \tau_k$ for $k \in \mathbf{t}\}$ is the basis of a subalgebra $J_n(R)$ of $B_n(R)$. This algebra coincides with the algebra defined by Jones in* [J3].

Proof. We need to show only that the R-linear span of the set of basis elements mentioned is closed under multiplication. Recall the formula of (4.7):

$$[S_1, S_2, w][S_1', S_2', w'] = \delta^{n(S_2,S_1')}[S_1'', S_2'', w''] \tag{6.14.1}$$

where S_1'', S_2'', w'' are as given in (4.7). Thus $S_1'' = S_1 w(i_{S_2}(S_2, S_1'))w^{-1}$. But by (6.13) (ii), $i_{S_2}(S_2, S_1') \in \mathrm{Ann}(\mathrm{Fix}\, S_2)$ and by (6.12) (ii), since $w = \tau_k$ (some k), $w(i_{S_2}(S_2, S_1'))w^{-1} \in \mathrm{Ann}(\mathrm{Fix}\, S)$. Hence by (6.13) (i), $S_1'' \in \mathrm{Ann}(\mathbf{n})$ and similarly $S_2'' \in \mathrm{Ann}(\mathbf{n})$. Since restriction to $\mathrm{Fix}(S)$ (S annular) and composition leave the set $\{\tau_k | k = 1, 2, \ldots\}$ invariant, $w'' = \tau_{k''}$, some k'' and the proof is complete. □

To state our next result, we introduce the following notation. Recall that $w(n)$ is the "longest" element of $\mathrm{Sym}(\mathbf{n})$; for any involution $S \in I(n)$, write $w(n)^{-1} S w(n) = S^*$. If $t = 0$, write $\mathbf{t} = \{0\}$.

(6.15) Theorem. *Fix a positive integer $n \geqq 1$ and assume that for each $t \in \mathscr{T}(n)$ $(= \{i \in \mathbf{n} \cup \{0\} | n - i \in 2\mathbb{Z}\})$, the polynomial $x^t - 1 \in R[x]$ splits over the field R (into not necessarily distinct linear factors). For $t \in \mathscr{T}(n)$, $t \neq 0$, write $x^t - 1 = \prod_{l=1}^{t}(x - r_l(t))$ and $f_j(x) = \prod_{i>j}(x - r_i(t)) = \sum_{i=0}^{t-j} r_{ij}(t) x^i$ $(j = 1, \ldots, t)$. Write $f_0(x) = 1$, so that $r_{00}(0) = 1$. Then $J_n(R)$ has a cell datum $(\Lambda, M, C, *)$ (see (1.1)) defined as follows: $\Lambda = \{(t,j) | t \in \mathscr{T}(n), j \in \mathbf{t}\}$ ordered lexicographically; for $(t,j) \in \Lambda, M(t,j) = \mathrm{Ann}(\mathbf{n}) \cap I(t)$; if $S_1, S_2 \in M(t,j)$ then $C^{t,j}_{S_1,S_2} = \sum_{l=0}^{t-j} r_{ij}(t)[S_1, S_2^*, \tau_l]$. Finally, $[S_1, S_2, w]^* = [S_2^*, S_1^*, w]$.*

Proof. Since (C1) and (C2) are clear, we have only to verify the relation (C3) of (1.1). Take $a = [S_1', S_2', \tau_{k'}] \in J_n(R)$, $S_i' \in \mathrm{Ann}(\mathbf{n}) \cap I(t'), k' \in \mathbf{t}'$ and compute $aC^{t,j}_{S_1,S_2}$. The product $[S_1', S_2', \tau_{k'}][S_1, S_2^*, \tau_l] = \delta^{n(S_2', S_1)}[S_1'', S_2'', \tau_{k''}]$ where S_1'', S_2'', τ_k'' are given by (6.14) (see also (4.7)).

If $|\mathrm{Fix}\, S_2''| = t$, then $S_2'' = S_2^*$; moreover $\tau_{k''} = \tau_{k'}|_{\mathrm{Fix}\, i_{S_2'}(S_2', S_1)} w(S_2', S_1)\tau_i = w'\tau_i$, where $w' = \tau_\ell$ for some $\ell \in \mathbf{t}$ which depends only on S_2', S_1 and k'. We therefore see that if $i_{S_1}(S_2', S_1) = \mathrm{id}$, we have

$$[S_1', S_2', \tau_{k'}]C^{t,j}_{S_1,S_2} = \delta^{n(S_2',S_1)} \sum_{i=0}^{t-j} r_{ij}(t)[S_1'', S_2^*, \tau_{\ell+i}]\,. \tag{6.15.1}$$

But by construction, $\sum_i r_{ij}(t) x^{i+1} \equiv r_j(t) \sum_i r_{ij}(t) x^i \mod \langle f_1(x), \ldots, f_{j-1}(x) \rangle_R$. Hence $\sum_i r_{ij}(t) x^{i+\ell} = r_j(t)^\ell \sum_i r_{ij}(t) x^i + \sum_{k<j} u_k f_k(x)$ (for appropriate coefficients $u_k \in R$). It follows that

$$[S_1', S_2', \tau_{k'}]C^{t,j}_{S_1,S_2} = \delta^{n(S_2',S_1)} r_j(t)^\ell C^{t,j}_{S_1'',S_2} + \delta^{n(S_2',S_1)} \sum_{k<j} u_k C^{t,k}_{S_1'',S_2} \tag{6.15.2}$$

where ℓ depends only on S_2', S_1 and k' and S_1'' depends only on S_1', S_2' and S_1. This proves the relation (C3) and completes the proof of (6.15). □

We may now apply (3.4) to describe the irreducible representations of $J_n(R)$.

(6.16) Scholium. *The bilinear form $\phi_{t,j}$ on the cell representation $W(t,j)$ is given by $\phi_{t,j}(C_{S_1},C_{S_2}) = \delta^{n(S_1^*,S_2)}r_j(t)^{\ell(S_1^*,S_2)}\prod_{i<j}(r_j(t)-r_i(t))$ (for S_1, $S_2 \in \mathrm{Ann}(\mathbf{n})\cap I(t)$), where $\ell = \ell(S_1^*,S_2)$ is defined by $w(S_1^*,S_2) = \tau_\ell$, provided $t(S_1^*,S_2) = t$. Otherwise $\phi_{t,j}(C_{S_1},C_{S_2}) = 0$.*

Proof. The desired value is the coefficient of $C^{t,j}_{S_1,S_2}$ in $(C^{t,j}_{S_1,S_2})^2$. This may be calculated using the relation (6.15.2) and the result is the stated formula. □

(6.17) Corollary. *If R is a field, then the irreducible representations of $J_n(R)$ are parametrised by the set $\Lambda_0(R)$, defined as follows.*

(a) *If $\delta \neq 0$, $\Lambda_0(R) = \{(t,\omega) \mid t \in \mathcal{T}(n), \omega \in R, \omega^t = 1\}$*

(b) *If $\delta = 0$, $\Lambda_0(R) = \{(t,\omega) \mid t \in \mathcal{T}(n)\backslash\{0\},\ \omega \in R,\ \omega^t = 1\}$.*

Proof. The required parameter set is $\{(t,j) \mid \phi_{t,j} \neq 0\}$ by (3.4). From (6.16) it is easily checked that $\Lambda_0(R)$ is this set. □

Although it is beyond the scope of this work to enter the details of the representations of $J_n(R)$ or the other algebras treated here, we show by means of an example how our approach may be used to discuss the issue of semisimplicity.

(6.18) Proposition. *Let R be a field satisfying the conditions of (6.15) and suppose $n > 2$ and $\omega \in R$ satisfies $\omega^{n-2} = 1$. Then the discriminant of the form $\phi_{n-2,\omega}$ is given by $\Delta_{n-2,\omega} = c\{u^n + u^{-n} + (-1)^{n-1}(\omega^2+\omega^{-2})\}$ where the element $u \in R$ is related to δ by $\delta = u + u^{-1}$ and where the constant c is non-zero, provided that the $(n-2)$-nd roots of 1 are distinct in R.*

Proof. The cell representation $W(n-2,\omega)$ has basis $\{C_S \mid S \in \mathrm{Ann}(\mathbf{n})\cap I(n-2)\}$. Now $\mathrm{Ann}(\mathbf{n}) \cap I(n-2) = \{(i,i+1),(1,n) \mid i = 1,2,\ldots,n-1\}$, and the value of $\phi_{n-2,\omega}(C_S,C_{S'})$ is given by (6.16). Using this, it becomes clear that $\phi_{n-2,\omega}$ has matrix (with respect to the bases $\{C_S\}$ and $\{C_{S^*}\}$) equal to a nonzero constant times $M_{n-2}(\omega)$, where

$$M_{n-2}(\omega) = \begin{pmatrix} \delta & \omega & 0 & \cdots & \cdots & \cdots & \cdots 0 & \omega^{-1} \\ \omega^{-1} & \delta & 1 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & 1 & \delta & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \delta & 1 & 0 & \cdots & 0 \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & 1 \\ \omega & 0 & 0 & \cdots & \cdots & \cdots & 1 & \delta \end{pmatrix}.$$

To compute the determinant of $M_{n-2}(\omega)$ one uses the formula

$$\det M_{n-2}(\omega) = \delta d_{n-1} - (-1)^n(\omega^2+\omega^{-2}) - 2d_{n-2} \tag{6.18.1}$$

where d_n is the determinant of the $n \times n$ matrix

$$D_n = \begin{pmatrix} \delta & 1 & 0 & \cdots & \cdots \\ 1 & \delta & 1 & \cdots & 0 \\ & 1 & \ddots & \ddots & \\ & \ddots & \ddots & \ddots & 1 \\ 0 & & & 1 & \delta \end{pmatrix}.$$

Further, since $d_n = \delta d_{n-1} - d_{n-2}$, it follows easily by induction that

(6.18.2) $$d_n = u^n + u^{n-2} + \cdots + u^{2-n} + u^{-n}.$$

The stated formula for $\Delta_{n-2,\omega}$ follows directly. □

(6.19) Corollary. *$J_n(R)$ is not semisimple if $\delta = u + u^{-1}$, where u is a zero of the Laurent polynomial $\Delta_{n-2,\omega}(u)$ of* (6.18).

Proof. This is a straightforward consequence of (3.8), which implies that if $J_n(R)$ is semisimple then $\phi_{n-2,\omega}$ is nondegenerate, whence $\Delta_{n-2,\omega} \neq 0$. □

This may be used to determine cases where $J_n(R)$ is not semisimple.

(6.20) Corollary. *The algebra $J_n(\mathbb{C})$ is not semisimple in the following cases*

(i) *If n is even and $\delta = 2\cos\frac{4k\pi}{n(n-2)}(k \in \mathbb{Z})$.*

(ii) *If n is odd and $\delta = -2\cos\frac{2k\pi}{n(n-2)}(k \in \mathbb{Z})$.*

Proof. By (6.18), the roots of the polynomial equation $\Delta_{n-2,\omega} = 0$ are given by $(-u)^n = w^{\pm 2}$. By (3.8), these values of u correspond to parameter values δ at which $J_n(R)$ is non-semisimple. The statement follows easily. □

(6.21) Remarks.

(i) Jones [J3] showed that if $\delta = -2$ then $J_n(\mathbb{C})$ is not semisimple, which is an immediate consequence of (6.20) (take $k = 0$ if n odd, and $k = n(n-2)/4$ if n even).

(ii) It is an easy consequence of (6.20) that if a is any *odd* divisor of n then $J_n(\mathbb{C})$ is not semisimple if $\delta = 2\cos\frac{\pi}{a}$. Thus (e.g.) if $3|n$, then $\delta = 1$ is such a value.

(iii) Since $\cos(\pi - \frac{2k\pi}{n}) = \cos(\frac{n-2k}{n}\pi) = -\cos\frac{2k\pi}{n}$, it follows that if n is even and $J_n(\mathbb{C}, \delta)$ is not semisimple, the same is true of $J_n(\mathbb{C}, -\delta)$. Thus (e.g) from (i) and (ii), if n is even, $J_n(\mathbb{C}, 2)$ is not semisimple, while if $6|n$ then $J_n(\mathbb{C}, -1)$ is not semisimple.

(iv) The "change of variable" $\delta = u + u^{-1}$ is suggested by the Hecke algebra context, where the Kazhdan–Lusztig elements C'_r (r a simple reflection) satisfy $C'^2_r = (q^{\frac{1}{2}} + q^{-\frac{1}{2}})C'_r$. The same change of variables simplifies Jones' discussion of his trace in terms of Tchebychev polynomials.

(6.20)′ Corollary. *The algebra $J_n(\mathbb{C}, \delta)$ is not semisimple if δ satisfies $\gamma_{g(n)}(-\frac{\delta}{2}) = 1$, where γ_n is the n-th Tchebychev polynomial and $g(n) = n(n-2)$ if n is odd, and $g(n) = n(n-2)/2$ if n is even.*

If R has characteristic zero, our techniques provide a method for the determination of the decomposition matrix D for the algebras $\mathrm{TL}_n(R)$, $\widetilde{\mathrm{TL}}_n(R)$ and $J_n(R)$. In the first case, the results of [GW] may be recovered.

We conclude by mentioning that one also has an algebra $\tilde{J}_n(R)$ analogous to $\widetilde{\mathrm{TL}}_n(R)$. This has basis $\{[S_1, S_2, w] | S_i \in \mathrm{Ann}(n) \cap I(t),\ t \in \mathscr{T}(n),\ w \in \langle \tau_1, w(t) \rangle\}$ where τ_1 is the cyclic permutation $i \mapsto i+1$ of $\mathbf{t}$ and $w(t)$ is the permutation $i \mapsto t+1-i$. This also has a cellular structure, which arises from the cellular structure of the group ring of a dihedral group.

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