THE TWO-BODY PROBLEM IN THE (TRUNCATED) PPN – THEORY *

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Abstract. The solution of the two-body problem in the (truncated) PPN theory is presented. It is given in two different analytical forms (the Wagoner-Will and Brumberg representation) and by the method of osculting elements.

1. Introduction

Analyzing gravitational experiments in the solar system is usually done in the socalled PPN - framework (e.g. *Will 1981)*, where a number of PPN - parameters designate the corresponding post - Newtonian limit of a certain metric theory of gravity. Now, the discovery of the binary pulsar PSR1913+16 (e.g. *Taylor & Weisberg 1982)* and subsequent extremely precise tracking of its orbital motion by analyzing pulse arrival times lead to the necessity to solve for the full two - body problem at least at the post - Newtonian level. For the Einstein post - Newtonian theory one solution to the two - body problem has been presented by *Wagoner & Will (1976), Epstein (1977)* and *Haugan (1985)*; a solution with osculting elements for this case was presented by *Damour & Deruelle (1985)*. In a series of papers *Barker & O'Connell (1975, 1976, 1981)* and *Barker et al. (1982, 1986)* dealt with the full post - Newtonian two - body problem even including spin and quadrupole moment effects. However, their main interest was lying in the precession and nutations of the spins and the *secular* motions of the classical angular momentum vector, the Runge - Lenz vector and the mean anomaly rather than solving for the detailed motions of the bodies.

This paper presents solutions to the full two - body problem in the (truncated) PPN - framework with parameters β and γ. Solutions are given in two different analytical forms (the Wagoner-Will and Brumberg representation) and by the method of osculting elements.

The Lagrangian for the two - body problem in the PPN - formalism truncated to the Eddington - Robertson parameters β and γ in standard post - Newtonian coordinates $(t, \mathbf{x})$ reads (e.g. *Will 1981)*:

$$
\begin{aligned}
\mathcal{L} &= -(m_1 + m_2)c^2 + \mathcal{L}_N + \mathcal{L}_{PN}/c^2 \qquad (1)\\
\mathcal{L}_N &= \frac{m_1}{2}\mathbf{v}_1^2 + \frac{m_2}{2}\mathbf{v}_2^2 + \frac{Gm_1m_2}{r}\\
\mathcal{L}_{PN} &= \frac{1}{8}m_1\mathbf{v}_1^4 + \frac{1}{8}m_2\mathbf{v}_2^4 + \frac{Gm_1m_2}{2r}\Big[(2\gamma+1)(\mathbf{v}_1^2+\mathbf{v}_2^2)\\
&\quad - (4\gamma+3)\mathbf{v}_1\cdot\mathbf{v}_2 - (\mathbf{v}_1\cdot\hat{\mathbf{n}})(\mathbf{v}_2\cdot\hat{\mathbf{n}}) - (2\beta-1)\frac{G(m_1+m_2)}{r}\Big]
\end{aligned}
$$

with

$$\hat{\mathbf{n}} = \frac{\mathbf{x}_1 - \mathbf{x}_2}{r} \quad ; \quad r = |\mathbf{x}_1 - \mathbf{x}_2|$$

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One finds that the total momentum $\mathbf{P}$ of the system can be obtained in the usual way from $\partial\mathcal{L}/\partial\mathbf{v}_1 + \partial\mathcal{L}/\partial\mathbf{v}_2$ and is given by:

$$\mathbf{P} = m_1\mathbf{v}_1 + m_2\mathbf{v}_2 + \frac{1}{2}m_1\mathbf{v}_1 v_1^2/c^2 + \frac{1}{2}m_2\mathbf{v}_2 v_2^2/c^2 + \frac{Gm_1m_2}{2c^2r}\left[2(2\gamma+1)(\mathbf{v}_1+\mathbf{v}_2) - (4\gamma+3)(\mathbf{v}_1+\mathbf{v}_2) - \hat{\mathbf{n}}[\hat{\mathbf{n}}\cdot(\mathbf{v}_1+\mathbf{v}_2)]\right] \tag{2}$$

The center of mass $\mathbf{X}$

$$\mathbf{X} = (m_1^*\mathbf{x}_1 + m_2^*\mathbf{x}_2)/(m_1^* + m_2^*) \tag{3}$$

with

$$m_a^* \equiv m_a + \frac{1}{2}m_a v_a^2/c^2 - \frac{1}{2}Gm_1m_2/r \tag{4}$$

then is not accelerated according to the equations of motion and the center of mass velocity is proportional to $\mathbf{P}$. We can then go to a post - Newtonian center of mass frame where $\mathbf{P} = \mathbf{X} = 0$ and

$$\mathbf{x}_1 = \left[\frac{m_2}{m} + \frac{\mu\,\delta m}{2m^2}(\mathbf{v}^2 - \frac{Gm}{r})\right]\mathbf{x} \tag{5a}$$

$$\mathbf{x}_2 = \left[-\frac{m_1}{m} + \frac{\mu\,\delta m}{2m^2}(\mathbf{v}^2 - \frac{Gm}{r})\right]\mathbf{x} \tag{5b}$$

with

$$\mathbf{x} = \mathbf{x}_1 - \mathbf{x}_2 \quad ; \quad \mathbf{v} = \mathbf{v}_1 - \mathbf{v}_2 \quad ; \quad m = m_1 + m_2$$

$$\delta m = m_1 - m_2 \quad ; \quad \mu = m_1m_2/m$$

For the relative motion one finds (*e.g. Barker et al. 1986)*:

$$\frac{d\mathbf{v}}{dt} = -\frac{Gm\hat{\mathbf{n}}}{r^2} + \frac{Gm\hat{\mathbf{n}}}{c^2r^2}\left\{\frac{Gm}{r}(2(\beta+\gamma)+2\nu) - \mathbf{v}^2(\gamma+3\nu) + \frac{3}{2}\nu(\hat{\mathbf{n}}\cdot\mathbf{v})^2\right\} + \frac{Gm}{c^2r^2}\,\mathbf{v}(\hat{\mathbf{n}}\cdot\mathbf{v})(2\gamma+2-2\nu) \tag{6}$$

$$\nu \equiv \frac{m_1m_2}{m^2} = \frac{\mu}{m}$$

and the corresponding Lagrangian takes the form:

$$\mathcal{L} = \frac{1}{2}\mathbf{v}^2 + \frac{Gm}{r} + \frac{1}{8}(1-3\nu)\frac{\mathbf{v}^4}{c^2} + \frac{Gm}{2c^2r}\left[(2\gamma+1+\nu)\mathbf{v}^2 + \nu(\hat{\mathbf{n}}\cdot\mathbf{v})^2 - (2\beta-1)\frac{Gm}{r}\right] \tag{7}$$

This Lagrangian is particularly useful in deriving first integrals of motion. For the (specific) post - Newtonian energy $\mathcal{E}$ and angular momentum $\mathcal{J}$ one finds:

$$\mathcal{E} = \mathbf{v}\frac{\partial\mathcal{L}}{\partial\mathbf{v}} - \mathcal{L} = \frac{1}{2}\mathbf{v}^2 - \frac{m}{r} + \frac{3}{8}(1-3\nu)\mathbf{v}^4 + \frac{m}{2r}\left[(2\gamma+1+\nu)\mathbf{v}^2 + \nu(\hat{\mathbf{n}}\cdot\mathbf{v})^2 + (2\beta-1)\frac{m}{r}\right] \tag{8}$$

and

$$\mathcal{J} = |\mathbf{x}\wedge\frac{\partial\mathcal{L}}{\partial\mathbf{v}}| = |\mathbf{x}\wedge\mathbf{v}|\left[1 + \frac{1}{2}(1-3\nu)\mathbf{v}^2 + (2\gamma+1+\nu)\frac{m}{r}\right] \tag{9}$$

2. The Wagoner - Will representation

In the first approach we will follow the route as taken by *Wagoner & Will (1976)* to derive an expression for the form of the post - Newtonian orbit. The time dependence is then obtained in analogy to the treatments by *Epstein (1977)* and *Haugan (1985).*

In the *Newtonian limit* the solution of (6) is given by †

$$\mathbf{x} = r(\cos\phi, \sin\phi, 0)$$

$$r = \frac{p}{1 + e\cos(\phi - \omega_0)} \tag{10}$$

$$r^2 \frac{d\phi}{dt} = \sqrt{mp} \tag{11}$$

The post - Newtonian solution can then be obtained with the ansatz:

$$r^2 \frac{d\phi}{dt} = |\mathbf{x} \wedge \mathbf{v}| = \sqrt{mp}\ (1 + \delta h) \tag{12}$$

$$\mathbf{v} = \frac{d\mathbf{x}}{dt} = \left(\frac{m}{p}\right)^{1/2}(-\sin\phi, e + \cos\phi, 0) + \mathbf{v}_{PN} = \mathbf{v}_N + \mathbf{v}_{PN} \tag{13}$$

We obtain ($\phi' = \phi - \omega_0$):

$$r^2 \frac{d\phi}{dt} = \sqrt{mp}\ [1 - \frac{me}{p}(2\gamma + 2 - 2\nu)\cos\phi'] \tag{14}$$

and

$$v^x_{PN} = \sqrt{\frac{m^3}{p^3}}\ [-(2\gamma + 2 - \beta)e\phi' + (2\beta + \gamma - \nu)\sin\phi' - (\gamma + \frac{21}{8}\nu)e^2\sin\phi' + \frac{1}{2}(\beta - 2\nu)e\sin 2\phi' - \frac{\nu}{8}e^2\sin 3\phi'] \tag{15a}$$

$$v^y_{PN} = \sqrt{\frac{m^3}{p^3}}\ [-(2\beta + \gamma - \nu)\cos\phi' - (\gamma + 2 - \frac{31}{8}\nu)e^2\cos\phi' - \frac{1}{2}(\beta - 2\nu)e\cos 2\phi' + \frac{\nu}{8}e^2\cos 3\phi'] \tag{15b}$$

$$v^z_{PN} = 0 \tag{15c}$$

An expression for $r(\phi)$ is obtained if the last two relations are substituted into the identity:

$$\frac{d}{d\phi}\frac{1}{r} \equiv -\frac{1}{r^2\dot{\phi}}(\mathbf{x}\cdot\mathbf{v}/r) \tag{16}$$

and one integrates w.r.t. ϕ. The integration constant is fixed by the requirement that the resulting formula for $\mathbf{x}$ yields expression (15) for the post - Newtonian velocity. One finds:

$$\frac{p}{r} = 1 + e\cos\phi' + \left(\frac{m}{p}\right)[-(2\beta + \gamma - \nu) + (\gamma + \frac{9}{4}\nu)e^2 + \frac{1}{2}(4\gamma + 4 - \beta - 2\nu)e\cos\phi' + (2\gamma + 2 - \beta)e\phi'\sin\phi' - \frac{\nu}{4}e^2\cos 2\phi'] \tag{17}$$

From this we see that the *secular* drift in the periastron motion is given by

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† We usually set $G = c = 1$ in the following

$$\Delta\phi = 2\pi\,(2\gamma + 2 - \beta)\frac{m}{p} \tag{18}$$

suggesting that we introduce as new angular variable the "true anomaly" η *(Epstein (1977), Haugan (1985)* with

$$\eta \equiv (1 - (2\gamma + 2 - \beta)\frac{m}{p})\phi - \omega_0 \tag{19}$$

With this eqs. (14) and (17) take the form:

$$r^2\frac{d\phi}{dt} = \sqrt{mp}\,[1 - \frac{me}{p}(2\gamma + 2 - 2\nu)\cos\eta] \tag{20}$$

$$\begin{aligned} \frac{p}{r} = 1 + e\cos\eta + (\frac{m}{p})[&-(2\beta + \gamma - \nu) + (\gamma + \frac{9}{4}\nu)e^2 \\ &+ \frac{1}{2}(4\gamma + 4 - \beta - 2\nu)e\cos\eta - \frac{\nu}{4}e^2\cos 2\eta] \end{aligned} \tag{21}$$

Introducing the eccentric anomaly E' instead of the "true anomaly η" in the usual way by

$$\sin\eta = \frac{(1 - e^2)^{1/2}\sin E'}{1 - e\cos E'} \quad ; \quad \cos\eta = \frac{\cos E' - e}{1 - e\cos E'} \tag{22}$$

we can rewrite the last expression in the form ($p = a(1 - e^2)$):

$$\begin{aligned} r = a(1 - e\cos E') \\ &- \frac{m}{c^2(1 - e^2)^2}\{-(2\beta + \gamma) + \frac{(10\gamma + \beta + 12)}{4}e^2 + \frac{1}{2}\gamma e^4 + (1 + \frac{17}{4}e^2 + \frac{3}{4}e^4)\nu \\ &+ e\cos E'[(\frac{8\gamma + 7\beta + 4}{2} + \frac{4 - \beta}{2}e^2) - (3 + 5e^2)\nu] \\ &+ e^2\cos 2E'[(-\frac{6\gamma + 3\beta + 4}{4} + \frac{1}{2}\gamma e^2) + (\frac{3}{4} + \frac{5}{4}e^2)\nu]\} \end{aligned} \tag{23}$$

The time dependence of the post - Newtonian relative two - body orbit can then be put into a generalized Kepler equation. Using (20) and the relations:

$$\dot\phi = (1 + (2\gamma + 2 - \beta)\frac{m}{p})\,\dot\eta \quad ; \quad \dot\eta = \frac{(1 - e^2)^{1/2}}{1 - e\cos E'}\,\dot E' \tag{24}$$

one finds that

$$(1 - (2\gamma + 2 - \beta)\frac{m}{p})\sqrt{mp} = [1 + \frac{me}{p}(2\gamma + 2 - 2\nu)(\frac{\cos E' - e}{1 + e\cos E'})]\,r^2(E')\dot\eta \tag{25}$$

where $r(E')$ is given by (23). Integrating this expression w.r.t. the time coordinate t finally gives the desired Kepler equation in the form:

$$t\,(\frac{2\pi}{T_{E'}}) + \sigma = E' - ge\sin E' - h\sin 2E' \tag{26}$$

where the E' period $T_{E'}$ is given by:

$$\begin{aligned} T_{E'} = 2\pi(\frac{a^3}{m})^{1/2}[1 + (2\gamma + 2 - \beta)\frac{m}{p} + \frac{m}{2a(1 - e^2)^2}\{&(8\beta + 4\gamma) \\ &+ (6\gamma + \beta + 8)e^2 + (2\gamma + 4)e^4 - (4 + 13e^2 + 7e^4)\nu\}] \end{aligned} \tag{27}$$

and

$$g = 1 + \frac{m}{2a(1-e^2)^2}[8\gamma + 6\beta + 4 - (2\gamma + 3\beta - 4)e^2 - (2\gamma + 4)e^4 - (4 + 11e^2 - 7e^4)\nu] \tag{28}$$

$$h = \frac{e^2 m}{4a(1-e^2)^2}[-(6\gamma + 3\beta + 4) + 2\gamma e^2 + (3 + 5e^2)\nu] \tag{29}$$

We finally note that in this representation e and p are related to $\mathcal{E}$ and $\mathcal{J}$ by

$$\mathcal{E} = -(\frac{m}{2p})\,\{(1-e^2) - (\frac{m}{4p})[(8\gamma + 8\beta + 3) - 5\nu + 2((4\gamma + 2\beta + 5) - 9\nu)e^2 + 3(1-3\nu)e^4]\} \tag{30}$$

$$\mathcal{J} = \sqrt{mp}\,\{1 + (\frac{m}{2p})[(4\gamma + 3 - \nu) + (1 - 3\nu)e^2]\} \tag{31}$$

3. The Brumberg representation

The expressions for the post - Newtonian specific energy $\mathcal{E}$ (8) and absolute value of the angular momentum $\mathcal{J}$ (9) can be written as:

$$\mathcal{E} = \frac{1}{2}(\dot{r}^2 + r^2\dot{\phi}^2) - \frac{m}{r} + \frac{3}{8}(1-3\nu)(\dot{r}^2 + r^2\dot{\phi}^2)^2 + \frac{m}{2r}[(2\gamma + 1 + 2\nu)\dot{r}^2 + (2\gamma + 1 + \nu)r^2\dot{\phi}^2 + (2\beta - 1)\frac{m}{r}] \tag{32}$$

$$\mathcal{J} = r^2\dot{\phi}\,[1 + \frac{1}{2}(1-3\nu)(\dot{r}^2 + r^2\dot{\phi}^2) + (2\gamma + 1 + \nu)\frac{m}{r}] \tag{33}$$

leading to first order eqs. of motion in the form:

$$r^2\dot{\phi} = \mathcal{J}[1 + (3\nu - 1)\mathcal{E} + \frac{m}{r}(2\nu - 2\gamma - 2)] \tag{34}$$

and

$$\dot{r}^2 = -r^2\dot{\phi}^2 + \frac{2m}{r} + 2\mathcal{E} + \frac{2}{c^2}\{-\frac{3}{2}(1-3\nu)\mathcal{E}^2 + \frac{\nu}{2}\frac{\mathcal{J}^2}{r^2}\frac{m}{r} - \frac{\mathcal{E}m}{r}(2\gamma + 4 - 7\nu) - \frac{m^2}{r^2}(2\gamma + \beta + 2 - \frac{5}{2}\nu)\} \tag{35}$$

Eliminating the $\dot{\phi}^2$ term the last equation can also be written as

$$\dot{r}^2 = A + \frac{B}{r} + \frac{C}{r^2} + \frac{D}{r^3} \tag{36}$$

with

$$A = 2\mathcal{E}\,(1 + \frac{3}{2}(3\nu - 1)\mathcal{E})$$
$$B = 2m\,(1 + (7\nu - 2\gamma - 4)\mathcal{E})$$
$$C = -\mathcal{J}^2\,(1 + 2(3\nu - 1)\mathcal{E}) + (5\nu - 4\gamma - 2\beta - 4)\frac{m^2}{r^2}$$
$$D = (4\gamma + 4 - 3\nu)\,\mathcal{J}^2 m$$

Using

$$\dot{r}^2 = \Big(\frac{dr(\phi(t))}{dt}\Big)^2 = \mathcal{J}^2\Big(\frac{d(1/r)}{d\phi}\Big)^2(1 - 2[(2\gamma + 2 - 2\nu)\frac{m}{r} + (1 - 3\nu)\mathcal{E}]) \tag{37}$$

the radial equation can be written in the form:

$$\Big(\frac{d(1/r)}{d\phi}\Big)^2 = A' + \frac{B'}{r} + \frac{C'}{r^2} + \frac{D'}{r^3} \tag{38}$$

with

$$\begin{aligned} A' &= \frac{2\mathcal{E}}{\mathcal{J}^2}(1 + \frac{1}{2}(1 - 3\nu)\mathcal{E}) \\ B' &= \frac{2m}{\mathcal{J}^2}(1 + (2\gamma + 2 - 3\nu)\mathcal{E}) \\ C' &= -1 + (4\gamma + 4 - 2\beta - 3\nu)\frac{m^2}{\mathcal{J}^2} \\ D' &= \nu m \end{aligned}$$

Notice that the right hand side of (38) is a third order polynomial in r^{-1}. This suggests to write (38) in the form:

$$\Big(\frac{d(1/r)}{d\phi}\Big)^2 = \Big(\frac{1}{r} - \frac{1}{a(1+e)}\Big)\Big(\frac{1}{a(1-e)} - \frac{1}{r}\Big)\Big(C_1 + \frac{C_2}{r}\Big) \tag{39}$$

where a comparison of coefficients yields:

$$\begin{aligned} C_1 &= 1 - (4\gamma + 4 - 2\beta - \nu)\frac{m}{a(1-e^2)} \\ C_2 &= -\nu m \end{aligned}$$

From the form of (39) we see that $r_\pm = a(1 \pm e)$ represent the minimal and maximal value for r and hence a and e play the role as semimajor axis and eccentricity of the post - Newtonian orbit. a and e can be considered as integration constants alternatively to $\mathcal{E}$ and $\mathcal{J}$. Solving for $\mathcal{E}$ and $\mathcal{J}$ in terms of a and e one finds:

$$\mathcal{E} = -\frac{m}{2a}\Big[1 - \Big(\frac{4\gamma + 3}{4} - \frac{\nu}{4}\Big)\frac{m}{a}\Big] \tag{40}$$

$$\mathcal{J}^2 = ma(1 - e^2)\Big[1 + \Big(-\gamma - 1 + \nu + \frac{4\gamma + 4 - 2\beta - \nu}{1 - e^2}\Big)\frac{m}{a}\Big] \tag{41}$$

The solution of (39) can then be written as:

$$r = \frac{a(1 - e^2)}{1 + e\cos f} \tag{42}$$

with the true anomaly f obeying

$$\Big(\frac{df}{d\phi}\Big)^2 = C_1 + \frac{C_2}{r} \tag{43}$$

or

$$\frac{df}{d\phi} = F \cdot \Big[1 - \frac{\nu}{2}\frac{m}{a(1 - e^2)}e\cos f\Big] \tag{44}$$

$$F = 1 - (2\gamma + 2 - \beta)\frac{m}{a(1 - e^2)}$$

Hence

$$f = F \cdot (\phi - \omega_0) - \frac{\nu}{2} \frac{m}{a(1-e^2)} e \sin[F \cdot (\phi - \omega_0)] \tag{45}$$

leading again to expression (18) for the secular drift in the perihelion motion. Eliminating $\mathcal{E}$ and $\mathcal{J}$ from (34) we get

$$r^2 \dot{\phi} = \sqrt{ma(1-e^2)} \{1 + [-\frac{1}{2}(\gamma + 2\nu) + \frac{(2\gamma + 2 - \beta - \nu/2)}{(1-e^2)} - 2(\gamma + 1 - \nu)\frac{a}{r}] \frac{m}{a}\}$$

or using (44)

$$\sqrt{ma(1-e^2)}\, dt = r^2 df\, [1 + (2\gamma + 2 - \frac{3}{2}\nu)\frac{m}{r} + \frac{1}{2}(\gamma + 2\nu)\frac{m}{a}] \tag{46}$$

Now, for a circular orbit $r = a$, $e = 0$ and

$$\dot{\phi}^2 \equiv n^2 = \frac{m}{a^3}[1 - (2\beta + \gamma - \nu)\frac{m}{a}] \tag{47}$$

defines the mean motion of the post - Newtonian orbit. Defining the mean anomaly M and eccentric anomaly E by relations (22) and

$$M = nt + M_0 \tag{48}$$

an integration of (46) leads to the corresponding Kepler equation in the form:

$$M = [1 + (2\gamma + 2 - \beta)\frac{m}{a}]E - (1 + (\frac{3}{2}\nu - \beta)\frac{m}{a})\, e \sin E \tag{49}$$

The *siderial period* T_ϕ of the orbit (ϕ changes by 2π) is finally found to be

$$\begin{aligned} T_\phi &= 2\pi\sqrt{\frac{a^3}{m}}\, [1 + \frac{m}{a}\{\frac{1}{2}(5\gamma + 4 - \nu) - \frac{(2\gamma + 2 - \beta)\sqrt{1-e^2}}{(1 + e\cos f_0)^2}\}] \\ &= T_f - 2\,\frac{(2\gamma + 2 - \beta)\sqrt{1-e^2}}{(1 + e\cos f_0)^2}\sqrt{\frac{a^3}{m}} \end{aligned} \tag{50}$$

where the *anomaleous* period

$$T_f = 2\pi\sqrt{\frac{a^3}{m}}\, [1 + \frac{1}{2}(5\gamma + 4 - \nu)\frac{m}{a}] \tag{51}$$

denotes the orbital period w.r.t. axes that precess with the secular perihelion motion.

4. The solution with osculting elements

Brumberg (1972) in his monography treats the restricted post - Newtonian ($\nu = 0$) two - body problem for a broad class of metric theories of gravity using parameters σ', β', α' and λ' † . It now turns out that his perturbing function is general enough to cover our case of the PPN two body problem. A comparison of his perturbing function with eq. (6) shows that for

† We added the primes to distinguish them from the usual PPN - parameters.

$$\begin{aligned}\sigma' &= \beta + \gamma + \nu \\ \beta' &= \frac{1}{2}(\gamma + 3\nu) \\ \alpha' &= \frac{1}{2}\nu \\ \lambda' &= \gamma + 1 - \nu\end{aligned} \tag{52}$$

Brumberg's results for the osculting elements apply for our PPN two body problem. Hence, the post Newtonian acceleration $\mathbf{a}_{PN}$ can be written as:

$$\mathbf{a}_{PN} = S\,\mathbf{n} + T\,(\mathbf{k} \wedge \mathbf{n}) + W\,\mathbf{k}$$

$$\mathbf{k} = \begin{pmatrix} \sin I \sin\Omega \\ -\sin I \cos\Omega \\ \cos I \end{pmatrix}$$

with

$$S = \frac{m}{r^2}\,[2(\beta + \gamma + \nu)\frac{m}{r} - (\gamma + 3\nu)\mathbf{v}^2 + (2\gamma + 2 - \frac{1}{2}\nu)\dot{r}^2] \tag{53a}$$

$$T = \frac{m}{r^2}\,(2\gamma + 2 - 2\nu)\frac{n^2 a^3}{r}\,e \sin f \tag{53b}$$

$$W = 0 \tag{53c}$$

The solution of Lagrange's planetary equations is then given by:

$$I = \text{const.} \quad ; \quad \Omega = \text{const.} \tag{54a}$$

$$\begin{aligned}\Delta a = {} & \frac{me}{c^2(1-e^2)^2}\,\{[(6\nu - (6\gamma + 4\beta + 4)) + e^2(\frac{31}{4}\nu - (4 + 2\gamma))]\,\cos f \\ & + (4\nu - (2\gamma + 2 + \beta))\,e\cos 2f + \frac{\nu}{4}e^2\cos 3f\}|_{t_0}^{t}\end{aligned} \tag{54b}$$

$$\begin{aligned}\Delta e = {} & \frac{m}{c^2 a(1-e^2)}\,\{[(\nu - 2\beta - \gamma) + e^2(\frac{47}{8}\nu - 4 - 3\gamma)]\,\cos f \\ & + (2\nu - \gamma - 1 - \frac{1}{2}\beta)e\cos 2f + \frac{\nu}{8}e^2\cos 3f\}|_{t_0}^{t}\end{aligned} \tag{54c}$$

$$\begin{aligned}\Delta\omega = {} & \frac{m}{c^2 a(1-e^2)}\,\{(2\gamma + 2 - \beta)f + [\frac{\nu - \gamma - 2\beta}{e} + (\gamma + \frac{21}{8}\nu)e]\,\sin f \\ & + (2\nu - \gamma - 1 - \frac{1}{2}\beta)\sin 2f + \frac{\nu}{8}e\sin 3f\}|_{t_0}^{t}\end{aligned} \tag{54d}$$

$$\begin{aligned}\Delta\epsilon = {} & (1 - \sqrt{1-e^2})\,\Delta\omega + \frac{m}{c^2 a\sqrt{1-e^2}}[(2\gamma + 4 - 7\nu)\sqrt{1-e^2}E \\ & + (-4\gamma - 4\beta - 4 + 9\nu)f + (4\gamma + 4 - \nu)\,e\sin f]|_{t_0}^{t}\end{aligned} \tag{54e}$$

$$\begin{aligned}\int_{t_0}^{t} \Delta n\,dt = {} & \frac{3m}{c^2 a}\{-(\gamma + 2 - \frac{7}{2}\nu)E + \frac{(2\gamma + \beta + 2 - 3\nu)}{\sqrt{1-e^2}}f - \frac{\nu}{2}\frac{e\sin f}{\sqrt{1-e^2}} \\ & + [\frac{\nu}{2}(1-e^2)(\frac{a}{r_0})^3 + (-2\gamma - \beta + 2 + \frac{5}{2}\nu)(\frac{a}{r_0})^2 + (\gamma + 2 - \frac{7}{2}\nu)\frac{a}{r_0}]M\}|_{t_0}^{t}\end{aligned} \tag{55}$$

$$\Delta M = \Delta\epsilon - \Delta\omega + \int_{t_0}^{t} \Delta n\,dt$$

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