EXPONENTIAL INSTABILITY OF COLLISION ORBIT IN THE ANISOTROPIC KEPLER PROBLEM

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Abstract. The straight-line collision solution in the anisotropic Kepler problem is extended to a periodic solution by means of Sundman's analytic continuation. It is shown that this collision periodic solution is always exponentially unstable.

1. Introduction

The anisotropic Kepler problem (in two dimension) is a two degrees of freedom Hamiltonian system

$$\frac{\mathrm{d}q}{\mathrm{d}t} = \frac{\partial H}{\partial p}, \quad \frac{\mathrm{d}p}{\mathrm{d}t} = -\frac{\partial H}{\partial q}, \tag{1.1}$$

with the Hamiltonian

$$H = \frac{1}{2}(p_1^2 + p_2^2) - \frac{1}{\sqrt{\mu_1 q_1^2 + \mu_2 q_2^2}}. \tag{1.2}$$

This system was first introduced by Gutzwiller [7]. The essential parameter of this system is the ratio μ_1/μ_2, and when $\mu_1 = \mu_2$, this system reduces to usual integrable Kepler problem.

Among the topics studied on this system previously are, for example,

(i) Non-regularizability of collision orbit (Devaney [3], [4]) and
(ii) Heteroclinic and chaotic behaviors. (Gutzwiller [7], [8]).

Almost complete lists of references of the investigations on this system are found in Casasayas and Llibre [2] and a review paper by Devaney [5].

In the present paper, we adopt a complex-analytic approach, and we regard the solution of Equation (1.1) as a complex analytic function defined in the complex t-plane. The straight-line orbit on the q_1-axis or q_2-axis is obviously a solution of (1.1), which begins and ends at collision $q_1 = q_2 = 0$, when the value of energy is negative. By the so-called Sundman's analytic continuation ([9], [11], [14], [15]), this solution can be extended beyond the instance of collision and can be considered a periodic solution defined for entire real t, $-\infty < t < +\infty$. Stability of this periodic solution is determined by the momodromy matrix $M[T]$ of the associated variational equations (Hill's equation [10], [12]). The solution is exponentially unstable if and only if $|\text{trace } M[T]| > 2$. Our main result of this paper is

THEOREM

In the anisotropic Kepler problem (1.1), *the analytically continued collisional periodic solution is exponentially unstable whenever* $\mu_1 \neq \mu_2$.

The crucial point which makes the Theorem hold, is the fact that the variational equation can be transformed, by a change of independent variable, to Gauss hypergeometric equation, as in Yoshida [17] and [19]. There must be a close relation between our complex-analytic instability and previous studies on real instability of collision solution or its heteroclinic behaviors, which entirely rely on the blowing up of the so-called collision manifold. ([2], [3], [4], [5], [11]). To make this relation clear, is beyond the reach of present paper.

2. Collision Orbit and its Analytic Continuation

By the form of equations for q_i,

$$d^2 q_i/\mathrm{d}t^2 = -\mu_i q_i (\mu_1 q_1^2 + \mu_2 q_2^2)^{-3/2}, \quad (i = 1, 2), \tag{2.1}$$

it is obvious that straight-line orbit $q_1 = 0$ and $q_2 = 0$ are solutions of (1.1) or (2.1). More precisely, let

$$q_1 = c_1 \phi(t), \quad q_2 = c_2 \phi(t), \tag{2.2}$$

be a particular solution of (2.1). This is possible when constants c_1 and c_2 satisfy the algebraic equations

$$c_i = \mu_i c_i (\mu_1 c_1^2 + \mu_2 c_2^2)^{-3/2}, \quad (i = 1, 2), \tag{2.3}$$

and the function $\phi(t)$ satisfies the differential equation

$$d^2 \phi/\mathrm{d}t^2 + \phi^{-2} = 0. \tag{2.4}$$

When $\mu_1 \neq \mu_2$, we have two particular solutions (2.2) with

(i) $\quad c_1 = \mu_1^{-1/6}, \quad c_2 = 0.$ (2.5)

(ii) $\quad c_1 = 0, \quad c_2 = \mu_2^{-1/6}.$ (2.6)

Particular solution (2.5) is a straight-line solution on the q_1-axis, and (2.6) on the q_2-axis.

The function $\phi(t)$, which is a solution of Equation (2.4) represents a solution of one-dimensional Kepler problem. If we fix the initial condition at $t = 0$ as

$$\phi = 1, \quad \mathrm{d}\phi/\mathrm{d}t = 0, \tag{2.7}$$

then, integrating (2.4) once, we find that $\phi(t)$ is the inverse function of

$$t = \frac{1}{\sqrt{2}} \int_{\phi}^{1} \sqrt{u/(1-u)}\,\mathrm{d}u$$

$$= \frac{1}{\sqrt{2}} \left\{ \frac{\pi}{2} + \sqrt{\phi(1-\phi)} - \arctan[\sqrt{\phi/(1-\phi)}] \right\}. \tag{2.8}$$

Figure 1 shows the function $\phi(t)$, which begins at $t = -t_0$ and terminates at $t = t_0$, where

$$t_0 = \frac{1}{\sqrt{2}} \int_{0}^{1} \sqrt{u/(1-u)}\,\mathrm{d}u = \frac{\pi}{2\sqrt{2}}. \tag{2.9}$$

We now extend the function $\phi(t)$, or the straight-line solution (2.2), beyond the instance of collision at $t = t_0(-t_0)$ so that the solution can be defined in the interval $-\infty < t < +\infty$.

One can easily see that near the instance of collision, say $t = t_0$, $\phi(t)$ has a series expansion of the form

$$\phi(t) = (t_0 - t)^{2/3} \sum_{n=0}^{\infty} a_n (t_0 - t)^{2n/3}, \tag{2.10}$$

with real expansion coefficients a_n, which are successively determined by a recursion formula from $a_0 = (9/2)^{1/2}$. In the expression (2.10), $(t_0 - t)^{2/3}$ is real when $t < t_0$, so that $\phi(t)$ represents the real function in Figure 1 for $t < t_0$. Although the point $t = t_0$

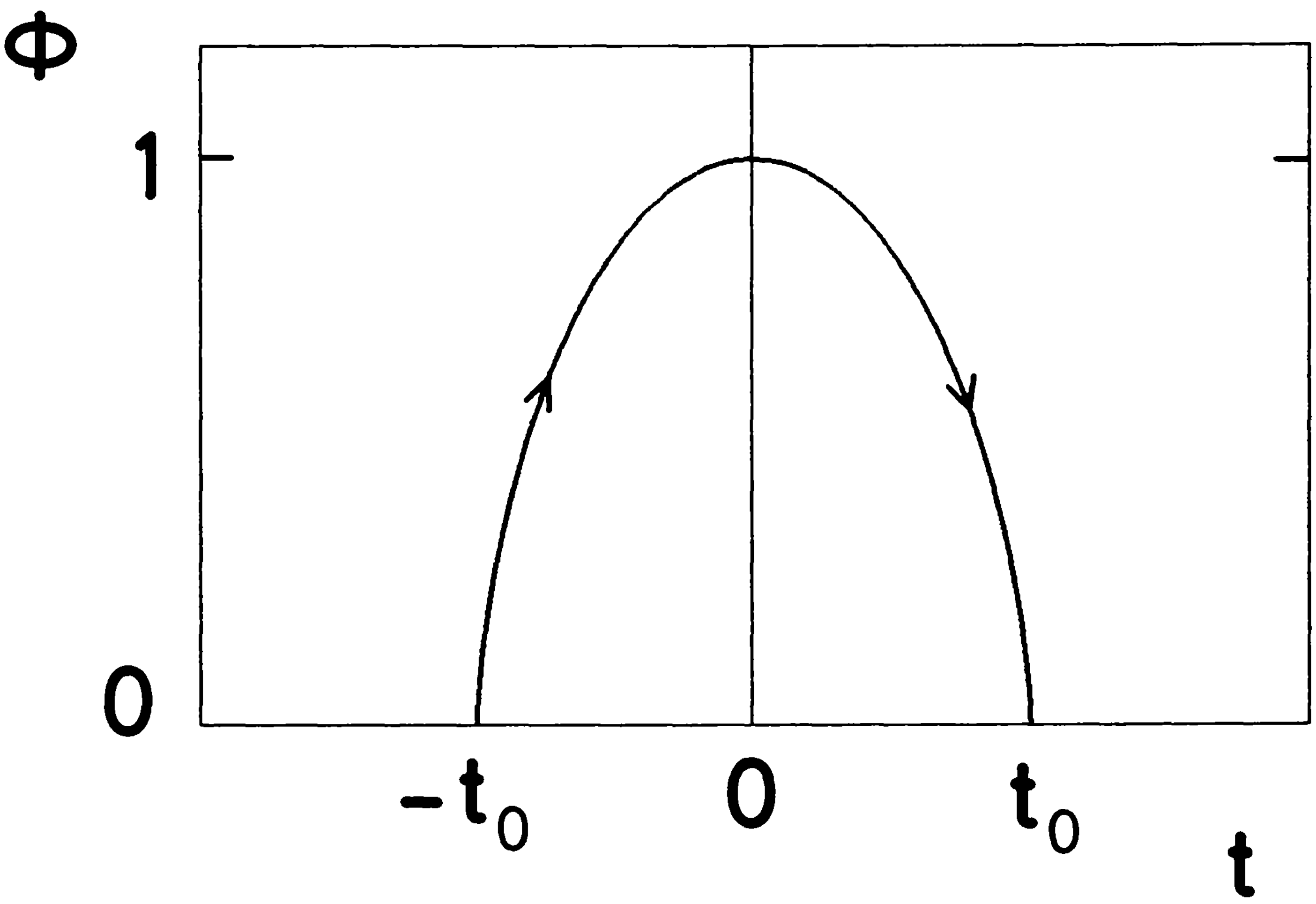

Fig. 1. Graph of function $\phi(t)$ in the interval $-t_0 \leqslant t \leqslant t_0$.

is a singularity (algebraic branch point), we shall continue the function $\phi(t)$ for $t < t_0$ across the point t_0 and make its analytic continuation to $t_0 < t$. For this purpose, we define the following path, $T_{3\pi}$ (or $T_{-3\pi}$) in the complex t-plane. See Figure 2.

(i) From $t = 0$ to $t = t_0 - \delta$ (δ is a sufficiently small real positive number), on the real t-axis.
(ii) From $t = t_0 - \delta$ to $t = t_0 + \delta$, on a circle C with radius δ and with center at t_0, where the argument of $(t_0 - t)$ is increased by $+3\pi$ (or -3π).
(iii) From $t = t_0 + \delta$ to $t = 2t_0$, on the real t-axis.

From expression (2.10) we see that when $\arg(t_0 - t)$ is increased $\pm 3\pi$ on the circle C, the function $\phi(t)$ again becomes real, since $[(t_0 - t)e^{\pm i3\pi}]^{2/3} = (t_0 - t)^{2/3} e^{\pm i2\pi} = (t_0 - t)^{2/3}$. Figure 2 and Figure 3 show the path of continuation in the complex t-plane and the graph of function $\phi(t)$, thus obtained, in the interval $[0, 2t_0]$. This continuation of solution physically means the elastic bounce of a particle in the one-dimensional Kepler problem, and this is also obtained in the limit of elliptic motion, e (eccentricity) $\to 1$, in the planer Kepler motion. This real-to-real continuation beyond a singularity, on the basis of series expansion, was first employed by Sundman in his research of collision in the restricted three body problem ([15], [9], [11], [14]), and is now called the Sundman's analytic continuation.

The path $T_{3\pi}$, or $T_{-3\pi}$, is, strictly speaking, not a period of the function $\phi(t)$, since $\arg(\phi)$ is changed by $\pm 2\pi$. We shall constitute a periodic function on its Riemann surface on which $\phi(t)$ is single valued, so that the function is periodic also in its argument. The simplest way to obtain periodic solution is to define the paths (periods) $T^{\langle 1 \rangle}$ and $T^{\langle 2 \rangle}$ by the succession of $T_{3\pi}$ and $T_{-3\pi}$ as

$$T^{\langle 1 \rangle} = T_{3\pi} \cdot T_{-3\pi}, \quad \text{and} \quad T^{\langle 2 \rangle} = T_{-3\pi} \cdot T_{3\pi}. \tag{2.11}$$

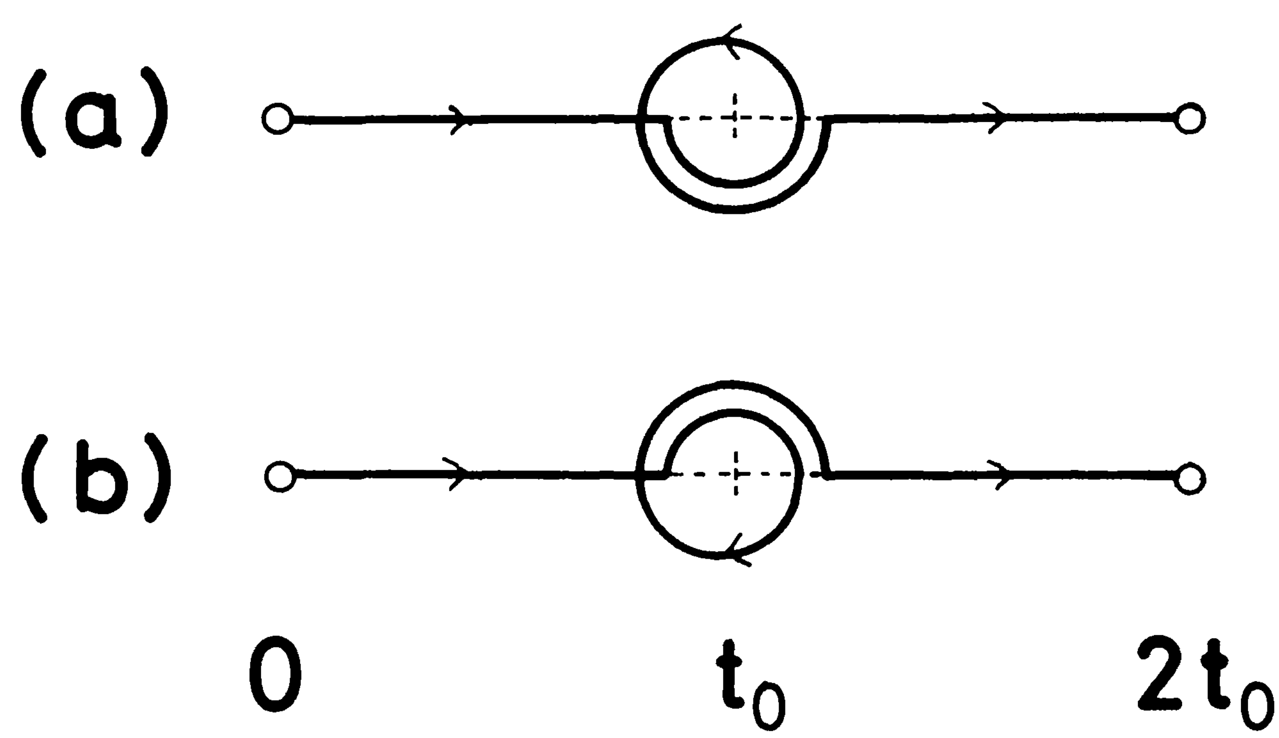

Fig. 2. Paths (a): $T_{3\pi}$ and (b): $T_{-3\pi}$ in the complex t-plane.

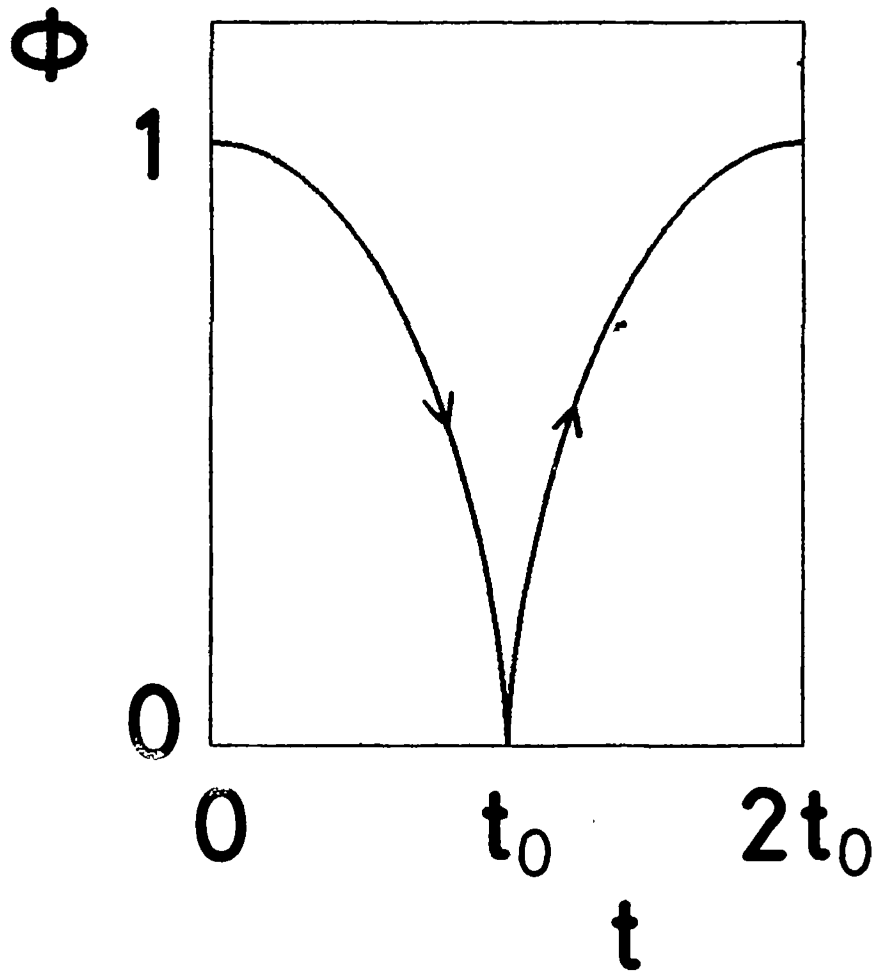

Fig. 3. Graph of function $\phi(t)$ after collision.

With these periods, $\phi(t)$ becomes completely periodic, cancelling the change of argument. We denote the periodic function with period $T^{\langle 1\rangle}$ by $\phi^{\langle 1\rangle}(t)$, and with period $T^{\langle 2\rangle}$ by $\phi^{\langle 2\rangle}(t)$. Both $\phi^{\langle 1\rangle}$ and $\phi^{\langle 2\rangle}$ have the same value $4t_0$ as the 'length' of period, and are identical in the interval $[0, t_0]$ and $[3t_0, 4t_0]$. In the interval $[t_0, 3t_0]$, $\arg\phi^{\langle 1\rangle} - \arg\phi^{\langle 2\rangle} = 4\pi$, and this means that $\phi^{\langle 1\rangle}$ and $\phi^{\langle 2\rangle}$ are defined on different Riemann sheets. Figure 4 shows the two periods (paths) $T^{\langle 1\rangle}$ and $T^{\langle 2\rangle}$. Figure 5 shows the graph of absolute value of periodic functions $\phi^{\langle 1\rangle}(t)$ and $\phi^{\langle 2\rangle}(t)$, both of which is identical. Figure 6 represents the change of arguments.

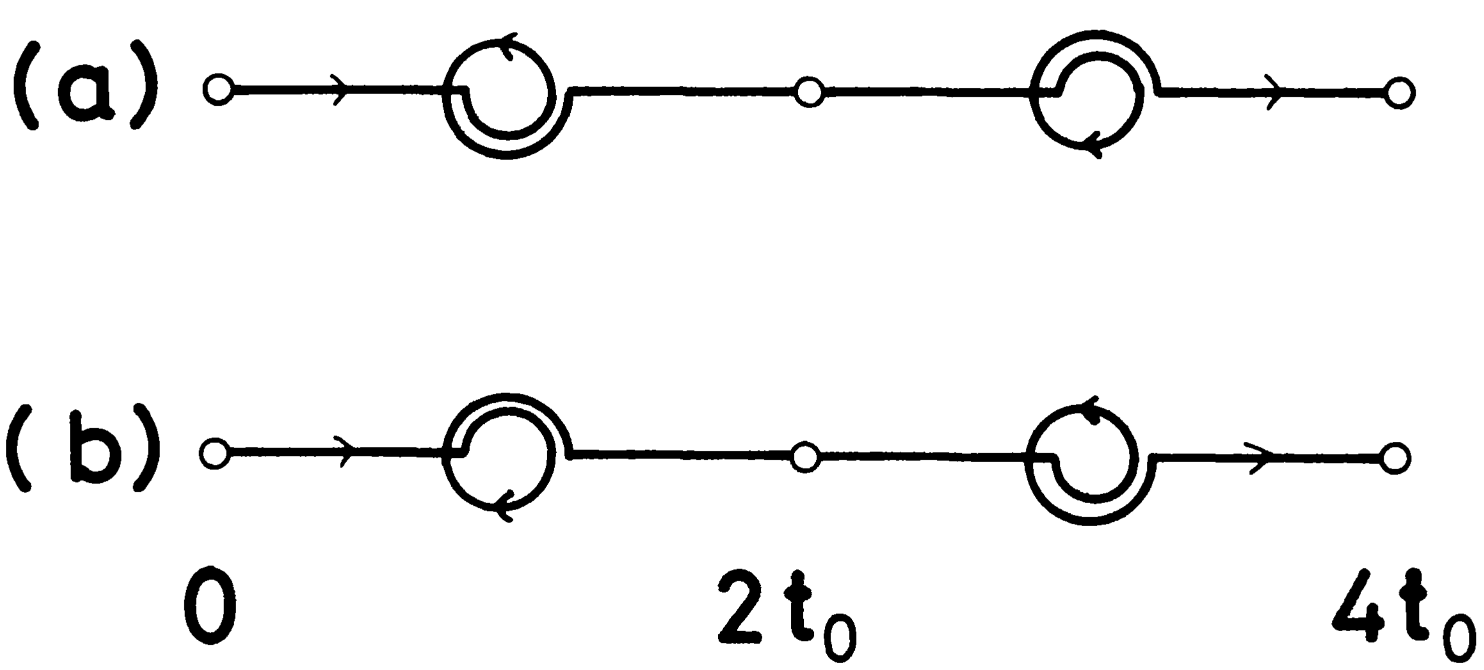

Fig. 4. Paths (Periods) (a): $T^{\langle 1\rangle}$ and (b): $T^{\langle 2\rangle}$ in the complex t-plane.

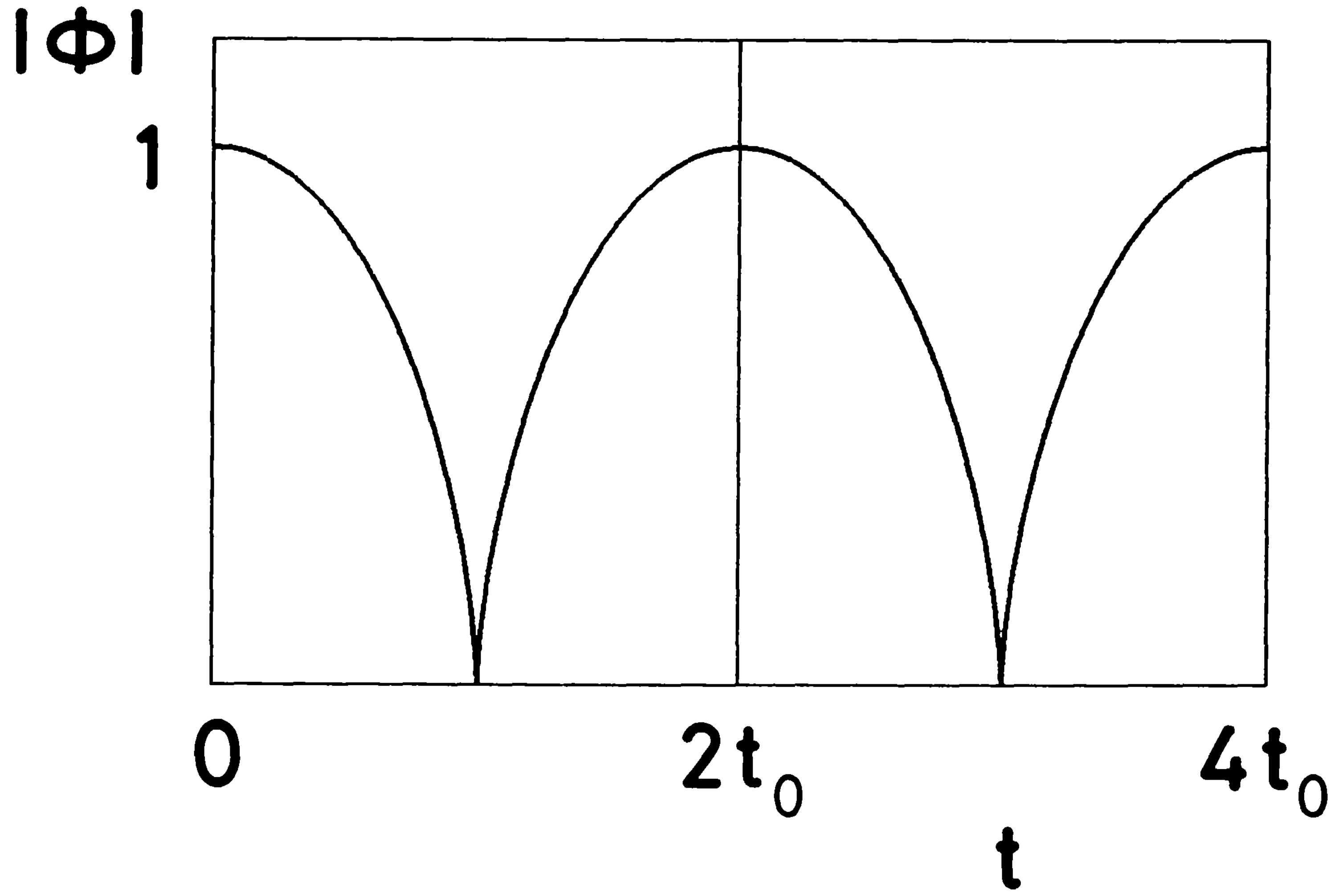

Fig. 5. Graph of $|\phi(t)|$ for one period.

3. Monodromy Matrices of the Variational Equations

The (linear) variational equations of (2.1) along the particular periodic solution (2.2) have the form

$$d^2\xi_i/dt^2 + \lambda_i\phi(t)^{-3}\xi_i = 0, \quad (i = 1, 2) \tag{3.1}$$

with $\xi_i = \delta q_i$, where $(\lambda_1, \lambda_2) = (\mu_1/\mu_2, -2)$ for the periodic solution with (2.5), and

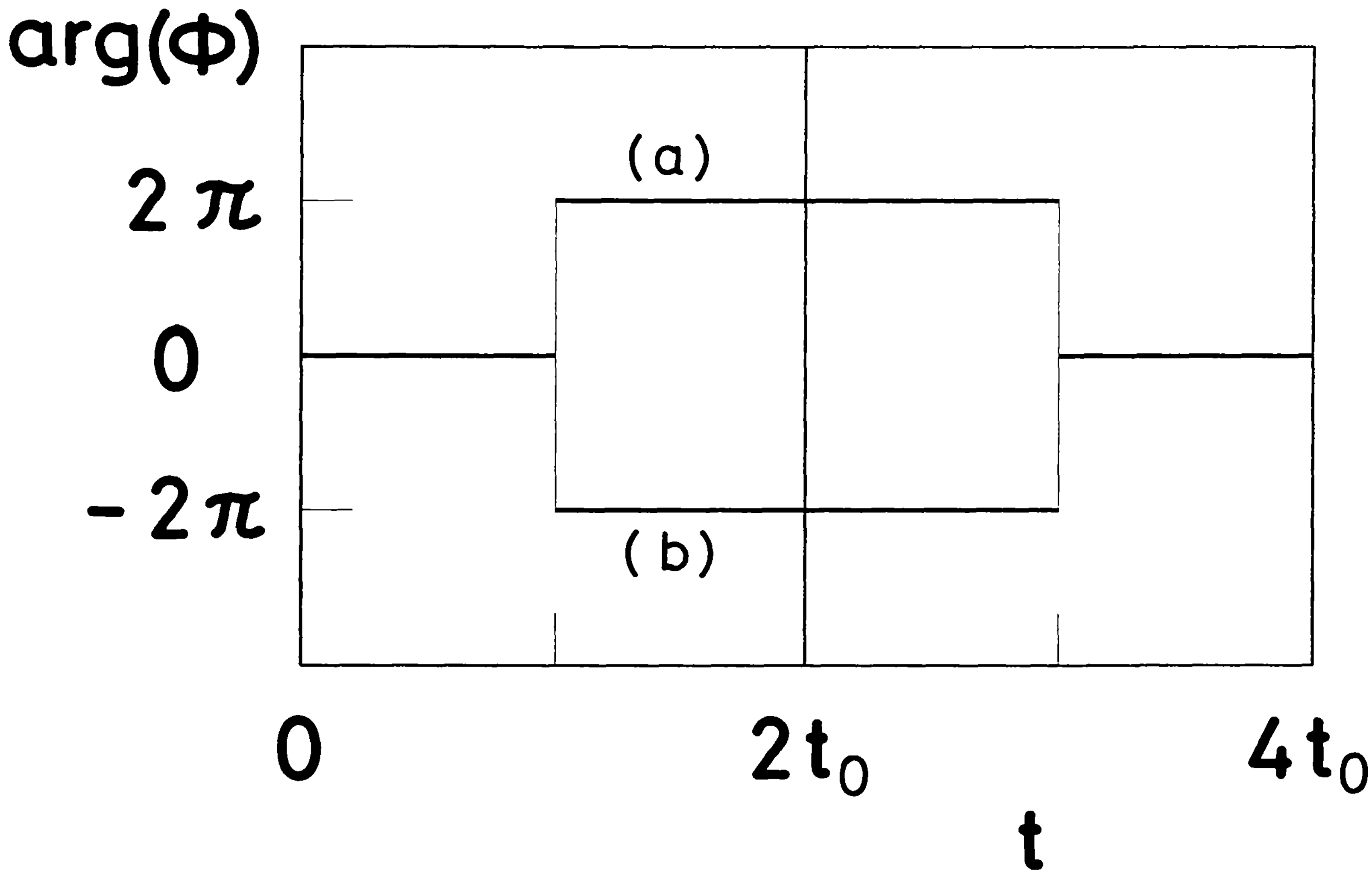

Fig. 6. Graph of (a): $\arg\phi^{\langle 1\rangle}(t)$ and (b): $\arg\phi^{\langle 2\rangle}(t)$.

$(\lambda_1, \lambda_2) = (-2, \mu_2/\mu_1)$ for the second one with (2.6). In (3.1), function $\phi(t)$ represents the periodic function $\phi^{\langle 1 \rangle}(t)$ or $\phi^{\langle 2 \rangle}(t)$ defined in section 2. Dropping the subscript i, we generally consider the equation, called the Hill's equation ([10], [12]),

$$d^2\xi/dt^2 + A(t)\xi = 0, \tag{3.2}$$

with periodic coefficient $A(t)$ of period T. Let $\xi^{(1)}(t)$ and $\xi^{(2)}(t)$ be two independent solutions of Equation (3.2), and make the fundamental system of solution $\Xi(t)$, by $\Xi(t) = [\xi^{(1)}(t), \xi^{(2)}(t)]$. Since $A(t)$ is periodic with period T, both $\Xi(t)$ and $\Xi(t+T)$ can be fundamental systems of solution. Therefore, there must be a linear relation of the type,

$$\Xi(t+T) = \Xi(t)M[T]. \tag{3.3}$$

The 2 by 2 constant matrix $M[T]$ is called the monodromy matrix of Equation (3.2), associated with the fundamental system $\Xi(t)$. Since (3.2) is derived by a Hamiltonian, it follows that $\det M[T] = 1$. Therefore, eigenvalues of the matrix $M[T]$, called characteristic multipliers, always appear as a pair, ρ and ρ^{-1}. Solution of Equation (3.2) is stable if $|\text{trace } M[T]| < 2$ (i.e. ρ is a complex number of unit modulus) and is exponentially unstable if and only if $|\text{trace } M[T]| > 2$ (i.e. ρ is real). ([1], [12])

There exists no universal procedure to give the explicit expression of the monodromy matrix $M[T]$, though direct numerical integration of Equation (3.2) could give the value of $M[T]$ up to any desired degree of precision ([1], p. 116). However, in our special case of Hill's equation

$$d^2\xi/dt^2 + \lambda\phi(t)^{-3}\xi = 0, \tag{3.4}$$

we can write down the monodromy matrix, explicitly. This relies on the fact that Equation (3.4) is transformed, by a change of independent variable t to z, defined by

$$z = 1/\phi(t), \tag{3.5}$$

to the Gauss hypergeometric equation [6], [13], [16]

$$z(1-z)d^2\xi/dz^2 + [c - (a+b+1)z]d\xi/dz - ab\xi = 0, \tag{3.6}$$

with special values of parameters

$$a + b = 3/2, \quad ab = \lambda/2, \quad c = 2. \tag{3.7}$$

Transformation of equation from (3.4) to (3.7) is given in Appendix A, in a more generalized form. It is noted that Equation (3.4) is discussed also by Nahon [18] with its another transformation to Gauss Equation (3.6).

Let $u^{(1)}(z)$ and $u^{(2)}(z)$ be a set of independent solutions of hypergeometric Equation (3.6). Then, by defining $U(z) = [u^{(1)}(z), u^{(2)}(z)]$, we see that $\Xi(t) = U(z) = U(1/\phi(t))$ gives a fundamental system of Equation (3.4). Thus, to evaluate the matrix $M[T]$, we have only to express $\Xi(t+T)$ in terms of $\Xi(t)$. A path, $t \to t + T$, in the complex t-plane, with T a period of $\phi(t)$, is mapped by (3.5) to a closed path γ in the complex z-plane. Let the change of the fundamental system $U(z)$ of hypergeometric

Equation (3.6) along the closed path γ be

$$U(z\gamma) = U(z)M[\gamma], \tag{3.8}$$

with a 2 by 2 constant matrix $M[\tau]$, also called the monodromy matrix or circuit matrix of hypergeometric Equation. ([6], [13]) In (3.8), $U(z\gamma)$ means the result of analytic continuation of $U(z)$ along the closed path γ which begins and ends at z. Since $U(z) = \Xi(t)$ and $U(z\gamma) = \Xi(t + T)$ by the definition of closed path γ, (3.3) and (3.8) give $M[T] = M[\gamma]$. The closed path γ is, in generally, not 0-contractable, because of the presence of singularities at $z = 0$, 1, *and* ∞. *We shall denote by* γ_0 *and* γ_1 two closed paths in the complex z-plane with a common fixed base point on the real z-axis ($0 < z < 1$), which make circuit the singularities $z = 0$ and $z = 1$, once in the positive direction (anti-clockwise), respectively. See Figure 7. Then any closed path in the complex z-plane with the same base point is expressed as a non-commutative product of γ_0, γ_1 and their inverse (inverse circuit), γ_0^{-1} and γ_1^{-1}.

A possible choice of two independent solutions of hypergeometric Equation (3.6) is to take

$$u^{(1)}(z) = \frac{\Gamma(c-b)\Gamma(b)}{\Gamma(c)} F(a, b; c; z), \tag{3.9}$$

$$u^{(2)}(z) = e^{-\pi i b} \frac{\Gamma(a+1-c)\Gamma(b)}{\Gamma(a+b+1-c)} F(a, b; a+b+1-c; 1-z), \tag{3.10}$$

where $F(a, b; c; z)$ represents the Gauss hypergeometric function defined by the Gauss hypergeometric series

$$F(a, b; c; z) = 1 + \frac{a \cdot b}{c \cdot 1} z + \frac{a(a+1)b(b+1)}{c(c+1)1 \cdot 2} z^2 + \cdots \tag{3.11}$$

and its analytic continuation beyond the circle of convergence $|z| = 1$ ([6], [16]). The scalar factors involving gamma function $\Gamma(x)$ in (3.9) and (3.10) are added to make later manipulations simple. In what follows, we assume that none of parameters a, b and c is zero or negative integer. With this assumption, (3.9) and (3.10) become, in fact, independent. To see this, let there be a linear relation

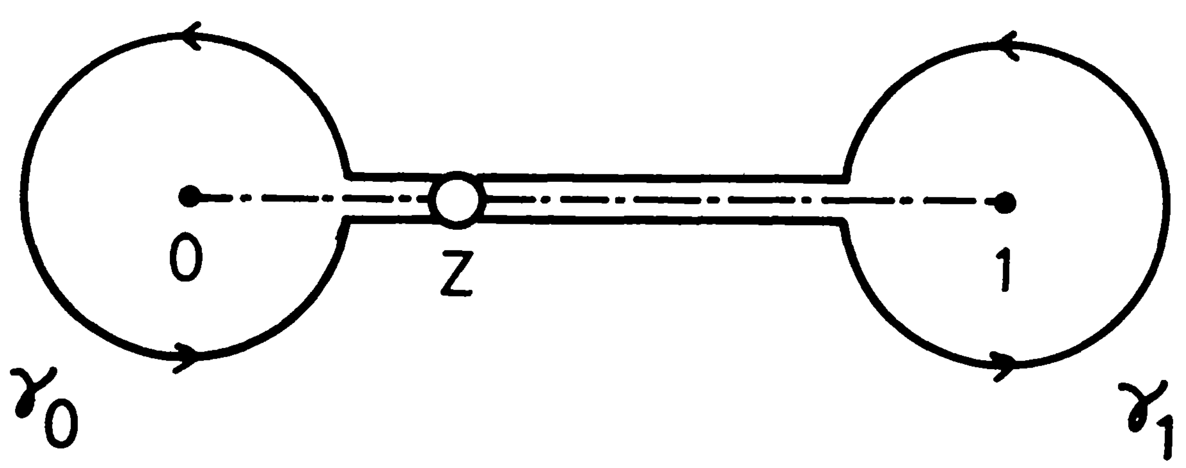

Fig. 7. Closed paths γ_0 and γ_1 in the complex z-plane.

$u^{(1)}(z) = \alpha u^{(2)}(z)$ with some constant α. Then, the function $u^{(1)}(z)$, which is holomorphic at $z = 0$, also becomes holomorphic at $z = 1$. Thus, function $u^{(1)}(z)$ becomes holomorphic for entire finite z, $|z| < \infty$. This implies that the hypergeometric series (3.11) terminates at some finite term, and becomes a polynomial. This occurs only when one of a and b is a negative integer or zero.

With these independent solutions, the changes of fundamental system $U(z)$ along the closed paths γ_0 and γ_1 are expressed as

$$U(z\gamma_0) = U(z)M[\gamma_0], \tag{3.12}$$

and

$$U(z\gamma_1) = U(z)M[\gamma_1], \tag{3.13}$$

where,

$$M[\gamma_0] = \begin{bmatrix} 1, & e^{-2\pi ib} - e^{-2\pi ic} \\ 0, & e^{-2\pi ic} \end{bmatrix}, \tag{3.14}$$

$$M[\gamma_1] = \begin{bmatrix} e^{2\pi i(c-a-b)}, & 0 \\ 1 - e^{2\pi i(c-a)}, & 1 \end{bmatrix}. \tag{3.15}$$

Derivation of (3.13) and (3.14) from the 'connection formula' of hypergeometric function is given in Appendix B, for generic values of parameters a, b and c. In our case of special value of parameter ($c = 2$), we need a direct proof which makes use of the Euler integral representation of (3.9) and (3.10) as seen, for example, in Plemelj [13], though both of which give the same expressions (3.13) and (3.14).

4. Proof of Theorem

The images of paths $T^{\langle 1\rangle}$ and $T^{\langle 2\rangle}$ of (2.11) in the complex t-plane by the conformal mapping (3.5), are closed paths in the complex z-plane, which are expressed as

$$\gamma^{\langle 1\rangle} = \gamma_0^{-1}\gamma_1\gamma_0\gamma_1, \tag{4.1}$$

for $T^{\langle 1\rangle}$, and

$$\gamma^{\langle 2\rangle} = \gamma_1\gamma_0\gamma_1\gamma_0^{-1}, \tag{4.2}$$

for $T^{\langle 2\rangle}$. Proof is given in Appendix C. Thus we have

$$\begin{aligned} M[T^{\langle 1\rangle}] &= M[\gamma^{\langle 1\rangle}] \\ &= M[\gamma_0]^{-1}M[\gamma_1]M[\gamma_0]M[\gamma_1], \end{aligned} \tag{4.3}$$

and

$$\begin{aligned} M[T^{\langle 2\rangle}] &= M[\gamma^{\langle 2\rangle}] \\ &= M[\gamma_1]M[\gamma_0]M[\gamma_1]M[\gamma_0]^{-1}. \end{aligned} \tag{4.4}$$

In our case of hypergeometric equation with special values of parameters (3.7), expressions (3.13) and (3.14) reduce to

$$M[\gamma_0]=\begin{bmatrix}1, & -B\\ 0, & 1\end{bmatrix}, \qquad M[\gamma_1]=\begin{bmatrix}-1, & 0\\ A, & 1\end{bmatrix}, \tag{4.5}$$

with

$$A=1-e^{-2\pi ia}, \quad B=1-e^{-2\pi ib}. \tag{4.6}$$

Note that (4.5) with (4.6) is valid at least when none of a and b is integer. Thus, from (4.3), (4.4) and (4.5), we find that, common to $T^{\langle 1\rangle}$ and $T^{\langle 2\rangle}$, the value of trace of monodromy matrix has the explicit expression

$$\begin{aligned}\text{trace } M[T] &= 2-A^2B^2\\ &= 2+4\cos^2\{\sqrt{9-8\lambda}\,\pi/2\}.\end{aligned} \tag{4.7}$$

Figure 8 shows the graph of trace $M[T]$ as a function of λ. From (4.7), we see that trace $M[T]>2$, except for the set of distinct values of λ, i.e.

$$\lambda=1,\ 0,\ -2,\ -5,\ -9,\ldots, \tag{4.8}$$

Therefore, except for the value of parameter λ in (4.8), solution of Equation (3.4) is known to be exponentially unstable. One comment is necessary here. We have derived the expression (4.7) on the assumption that none of a and b is an integer. In the case when a or b is an integer, or from (4.6) and (4.7), the case trace $M[T]=2$, our assumption becomes false. Nevertheless, expression (4.7) becomes valid for all values of λ, since trace $M[T]$ must be a continuous function of a parameter λ, which enters Equation (3.4) as a coefficient.

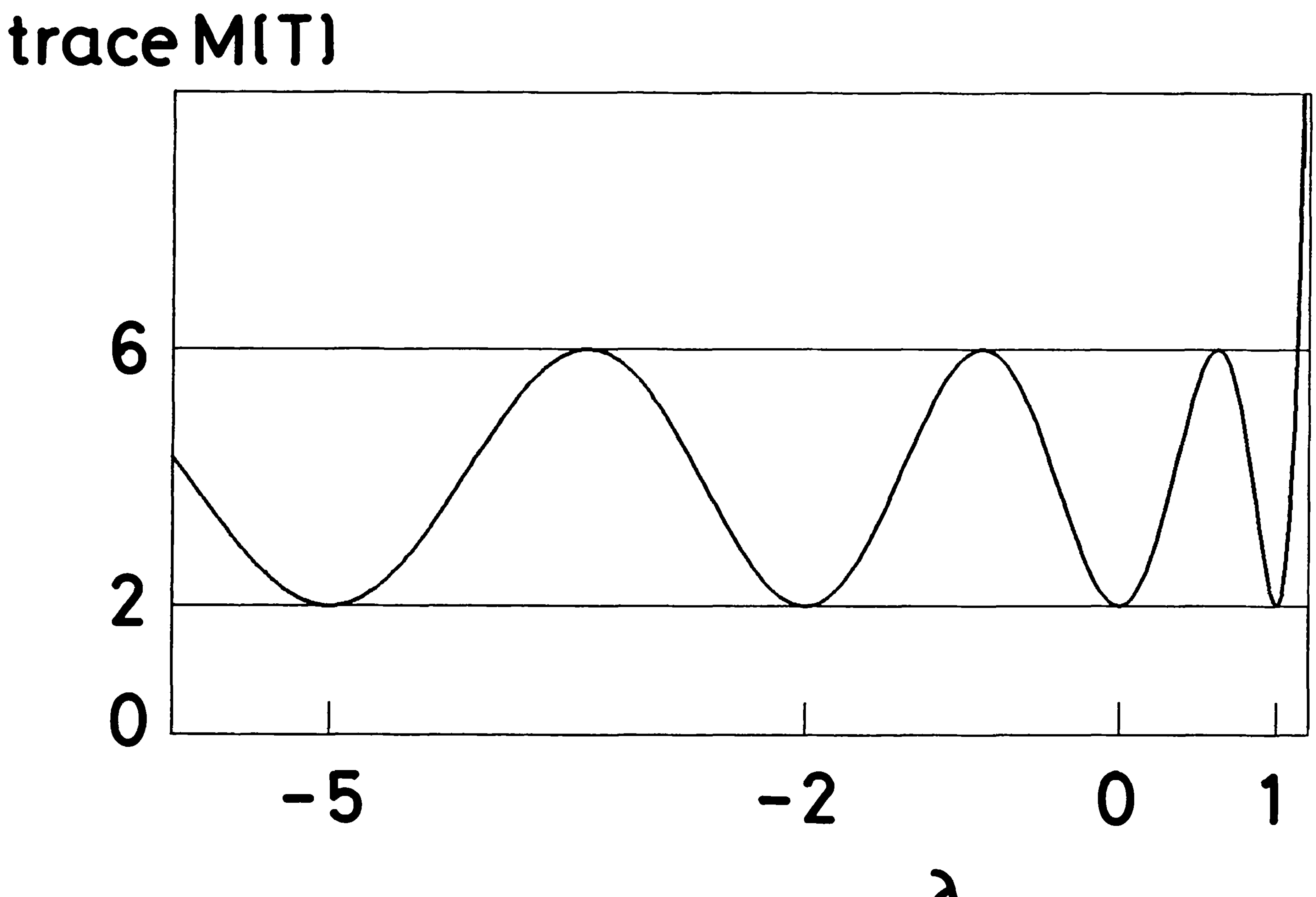

Fig. 8. Graph of trace $M[T]$ as a function of λ.

In our original variational Equations (3.1), we find that when $\mu_1 \neq \mu_2$ (i.e. μ_1/μ_2 is a positive integer not equal to one), one of λ_i in (3.1) is always out of values (4.8) for both collision periodic solutions on the q_1-axis and on the q_2-axis. This completes the proof of THEOREM.

Appendix A. Derivation of Hypergeometric Equation (3.6). [17]

Let $\phi(t)$ be the solution of the differential equation

$$d^2\phi/\mathrm{d}t^2 + \phi^{k-1} = 0. \tag{A.1}$$

with an integer k ($k = 0, \pm 1, \pm 2, \ldots$), and the fixed initial condition, $\phi = 1$ and $\mathrm{d}\phi/\mathrm{d}t = 0$ at $t = 0$. Then consider the linear equation

$$d^2\xi/\mathrm{d}t^2 + \lambda\phi(t)^{k-2}\xi = 0. \tag{A.2}$$

Equation (A.2) arises, as in (3.1), as a component of variational equations along homothetic straight-line solution of a Hamiltonian system

$$H = \tfrac{1}{2}p^2 + V(q), \tag{A.3}$$

with potential $V(q)$, a homogeneous function of degree k. Integrating (A.1) once, we have

$$\tfrac{1}{2}(\mathrm{d}\phi/\mathrm{d}t)^2 + \frac{1}{k}\phi^k = \frac{1}{k}, \tag{A.4}$$

which determines $\phi(t)$ as the inverse function of

$$t = \sqrt{\frac{k}{2}} \int_{\phi}^{1} \frac{\mathrm{d}u}{\sqrt{1-u^k}}. \tag{A.5}$$

We make the change of independent variable, from t to z by

$$z = [\phi(t)]^k. \tag{A.6}$$

Then, in (A.2),

$$d^2\xi/\mathrm{d}t^2 = (\mathrm{d}z/\mathrm{d}t)^2 \cdot d^2\xi/\mathrm{d}z^2 + d^2z/\mathrm{d}t^2 \cdot \mathrm{d}\xi/\mathrm{d}z. \tag{A.7}$$

Differentiating (A.6) with use of (A.1), (A.4) and (A.6) itself, we find that

$$(\mathrm{d}z/\mathrm{d}t)^2 = 2k\phi(t)^{k-2}z(1-z), \tag{A.8}$$

$$d^2z/\mathrm{d}t^2 = 2\phi(t)^{k-2}\{(1-3k/2)z + k - 1\}, \tag{A.9}$$

and finally that Equation (A.2) is transformed to Gauss hypergeometric Equation (3.6) with values of parameters

$$a + b = (k-2)/2k, \; ab = -\lambda/2k, \quad c = (k-1)/k. \tag{A.10}$$

When $k = -1$, that is our case, (A.1), (A.2), (A.6) and (A.10) reduce to (2.4), (3.2), (3.5) and (3.7), respectively.

Appendix B. Derivation of (3.14) and (3.15)

Among the Kummer's 24 solutions ([6], [16]) of hypergeometric Equation (3.6), we take the following 4 ones,

$$F_0^{(1)}(z) = F(a, b; c; z), \tag{B.1}$$

$$F_0^{(2)}(z) = z^{1-c}F(a+1-c, b+1-c; 2-c; z), \tag{B.2}$$

$$F_1^{(1)}(z) = F(a, b; a+b+1-c; 1-z), \tag{B.3}$$

$$F_1^{(2)}(z) = (1-z)^{c-a-b}F(c-a, c-b; c+1-a-b; 1-z). \tag{B.4}$$

Since the order of differential equation is two, among any three solution above, there exists a linear relation, called the connection formula of hypergeometric function. One of them is ([6], p. 107, formula (33))

$$F_0^{(1)}(z) = C_{11}F_1^{(1)}(z) + C_{12}F_1^{(2)}(z), \tag{B.5}$$

with $-\pi < \arg(1-z) < \pi$, and

$$C_{11} = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \quad c_{12} = \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}. \tag{B.6}$$

Another one is ([6], p. 107, formula (35))

$$F_1^{(1)}(z) = D_{11}F_0^{(1)}(z) + D_{12}F_0^{(2)}(z), \tag{B.7}$$

with $-\pi < \arg(z) < \pi$, and

$$D_{11} = \frac{\Gamma(a+b+1-c)\Gamma(1-c)}{\Gamma(a+1-c)\Gamma(b+1-c)}, \quad D_{12} = \frac{\Gamma(a+b+1-c)\Gamma(c-1)}{\Gamma(a)\Gamma(b)}. \tag{B.8}$$

which is also obtained from (B.5) by the substitutions, $z \to 1-z$ and $c \to a+b+1-c$. A comment is that formula (B.5) and (B.7) indeed hold for all values of parameters a, b and c, for which the gamma factors in (B.6) and (B.8) are finite. We shall assume this below.

$F_0^{(1)}(z)$ is holomorphic and single-value in the domain $|z| < 1$. This implies $F_0^{(1)}(z\gamma_0) = F_0^{(1)}(z)$, and consequently

$$u^{(1)}(z\gamma_0) = u^{(1)}(z), \tag{B.9}$$

since, $u^{(1)}(z)$ is only a scalar multiple of $F_0^{(1)}(z)$. Similarly, by the single-valuedness of $F_1^{(1)}(z)$ in the domain $|z-1| < 1$, we have the identity

$$u^{(2)}(z\gamma_1) = u^{(2)}(z). \tag{B.10}$$

We have to take care, when we evaluate $F_1^{(1)}(z\gamma_0)$, for example. In the common regions of $|z| < 1$ and $|z-1| < 1$, where the base point of γ_0 exists, we can express $F_0^{(1)}(z)$ uniquely as the right hand side of (B.7). Then, make a circuit $z \to z\gamma_0$. Because $F_0^{(2)}(z)$ is multiplied $e^{-2\pi i c}$ by this circuit, we have

$$F_1^{(1)}(z\gamma_0) = D_{11}F_0^{(1)}(z) + D_{12}F_0^{(2)}(z)e^{-2\pi i c}. \tag{B.11}$$

Elimination of $F_0^{(2)}(z)$ in (B.11) with re-use of (B.7), expresses the right hand side of (B.11) in terms of $F_0^{(1)}(z)$ and $F_1^{(1)}(z)$. Then, multiply the scalar factor in (3.10), and use the formula of gamma function

$$\Gamma(x)\Gamma(1-x) = \pi/\sin(\pi x). \tag{B.12}$$

Thus, we have finally

$$u^{(2)}(z\gamma_0) = u^{(1)}(z)[e^{-2\pi ib} - e^{-2\pi ic}] + u^{(2)}(z)e^{-2\pi ic}. \tag{B.13}$$

Next, expressing $F_0^{(1)}(z)$ in the right hand side of (B.5) and making the circuit $z \to z\gamma_1$, we have

$$F_0^{(1)}(z\gamma_1) = C_{11}F_1^{(1)}(z) + C_{12}F_1^{(2)}(z)e^{2\pi i(c-a-b)}. \tag{B.14}$$

Similar manipulation as above gives

$$u^{(1)}(z\gamma_1) = u^{(1)}(z)e^{2\pi i(c-a-b)} + u^{(2)}(z)[1 - e^{2\pi i(c-a)}]. \tag{B.15}$$

Combination of (B.9) and (B.13) proves the expression of $M[\gamma_0]$ in (3.14), and combination of (B.10) and (B.15), proves that of $M[\gamma_1]$ in (3.15).

Appendix C. Proof of (4.1) and (4.2)

Points $t = 0, 2t_0, 4t_0, \ldots$, in the complex t-plane are mapped by (3.5) to $z = 1$, which is a singularity of hypergeometric Equation (3.6), and this makes some difficulty in evaluating the images of mapping. To avoid this difficulty, we re-define the path $T_{3\pi}$ as a composition of the following five parts, with introduction of a sufficiently small real positive number ε. (The limit $\varepsilon \to 0$ will be taken finally.) [Figure 9]

(a) : Path on the circle $t = \varepsilon e^{i\theta}$, where θ increases from $-\pi/2$ to 0.
(b) : Path on the real t-axis, from $t = \varepsilon$ to $t = t_0 - \delta$.
(c) : Path on the circle $t = t_0 - \delta e^{i\psi}$, where ψ increases from 0 to 3π.
(d) : Path on the real t-axis, from $t = t_0 + \delta$ to $t = 2t_0 - \varepsilon$.
(e) : Path on the circle $t = 2t_0 - \varepsilon e^{i\theta}$, where θ increases from 0 to $\pi/2$.

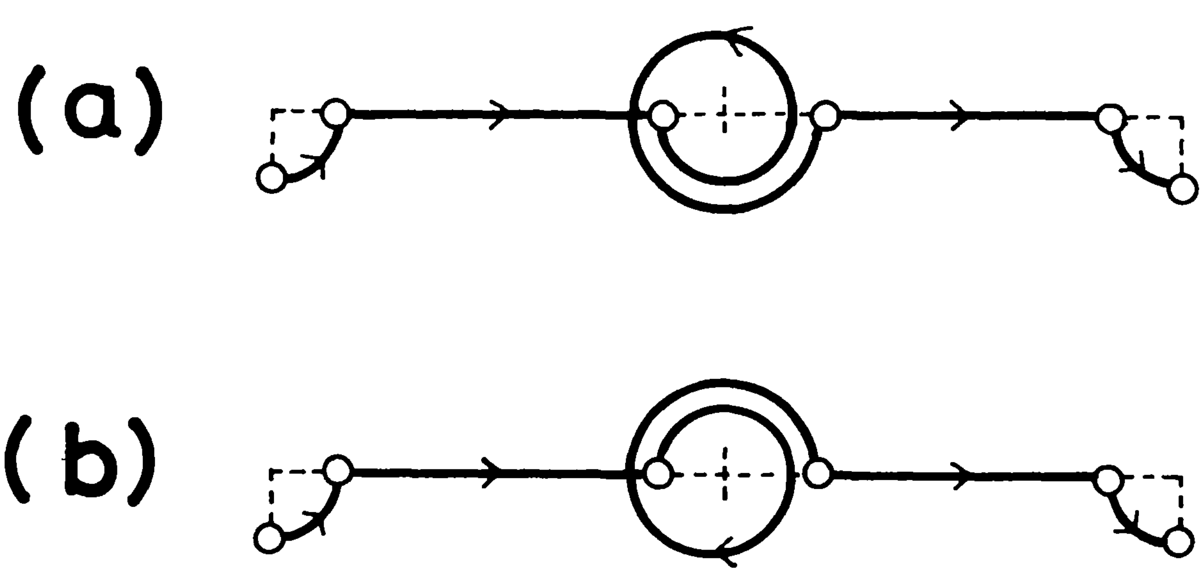

Fig. 9. Modification of paths $T_{3\pi}$ and $T_{-3\pi}$ in the complex t-plane.

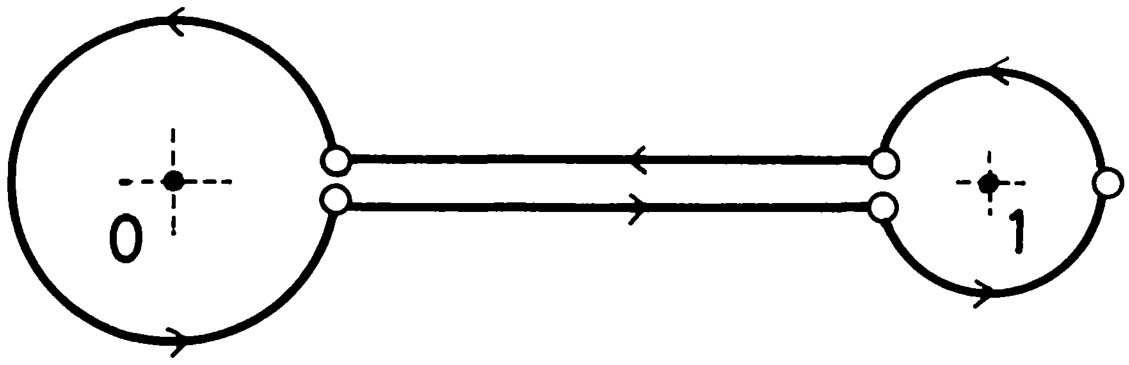

Fig. 10. Image of $T_{3\pi}$ by the mapping $w = \phi(t)$ in the complex w-plane.

Definition of $T_{-3\pi}$ is made only by replacing -3π instead of 3π in (c). First we examine the image of above path $T_{3\pi}$, by the mapping $w = \phi(t)$ in the complex w-plane. For the images of (***a***) and (***e***), we need the expansion of the function $\phi(t)$,

$$\phi(t) = 1 - (1/2)\tau^2 + O(\tau^4), \quad \tau = t - t_*, \tag{C.1}$$

around the points $t_* = 0,\ 2t_0,\ 4t_0, \ldots$ ○ For the image of (c), we make use of the expansion (2.10). These tell us that the image of path $T_{3\pi}$ in the complex w-plane becomes as follows. [Figure 10]

$\phi(\boldsymbol{a})$: Path on the circle $w = 1 + \varepsilon_1 e^{i\theta}$, where θ increases from 0 to π, and $\varepsilon_1 = (1/2)\varepsilon^2 + 0(\varepsilon^4)$.

$\phi(\boldsymbol{b})$: Path on the real w-axis from $w = 1 - \varepsilon_1$ to $w = \delta_1$, where $\delta_1 = \sqrt{2/9}\,\delta^{2/3}\{1 + 0(\delta^{2/3})\}$.

$\phi(\boldsymbol{c})$: Path on the circle $w = \delta_1 e^{i\psi}$, where ψ increases from 0 to 2π.

$\phi(\boldsymbol{d})$: Path on the real w-axis from $w = \delta_1$ to $w = 1 - \varepsilon_1$.

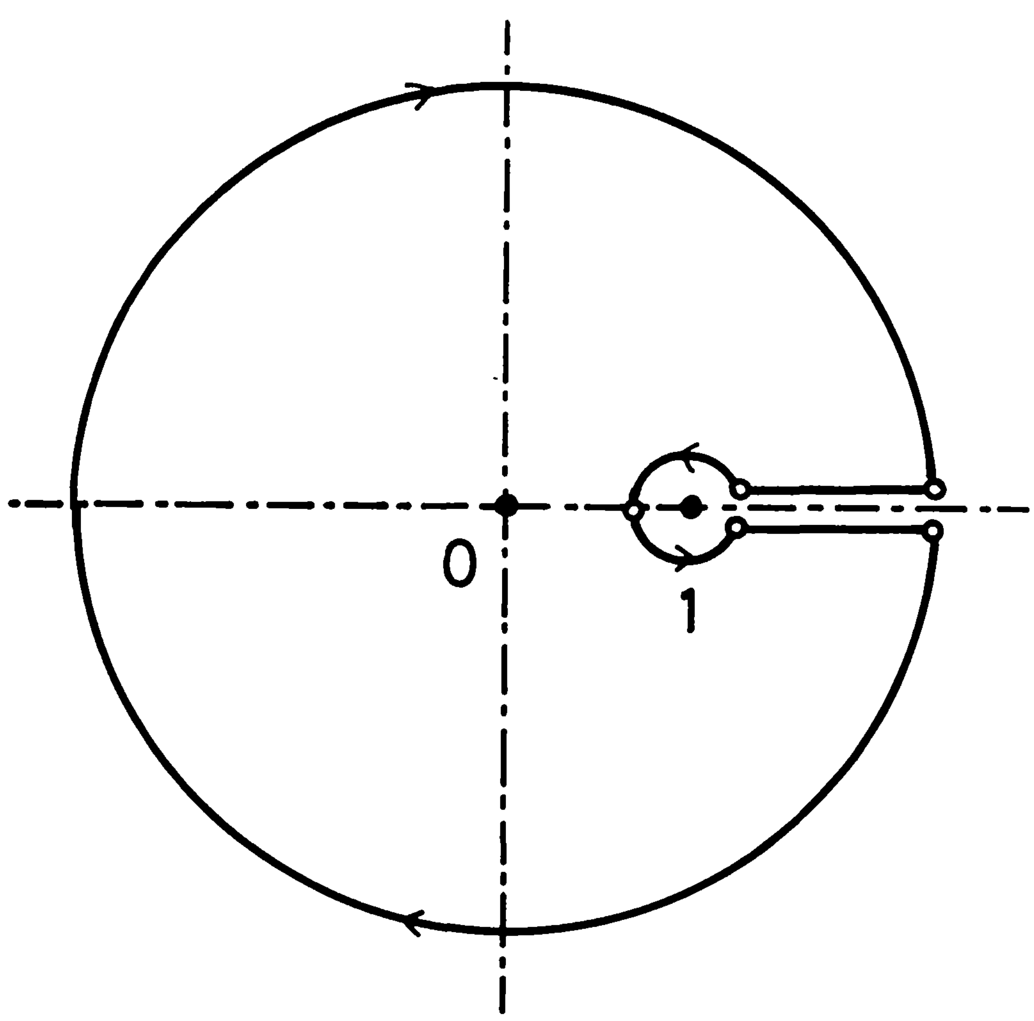

Fig. 11. Image of $T_{3\pi}$, by the mapping $z = 1/\phi(t)$, in the complex z-plane.

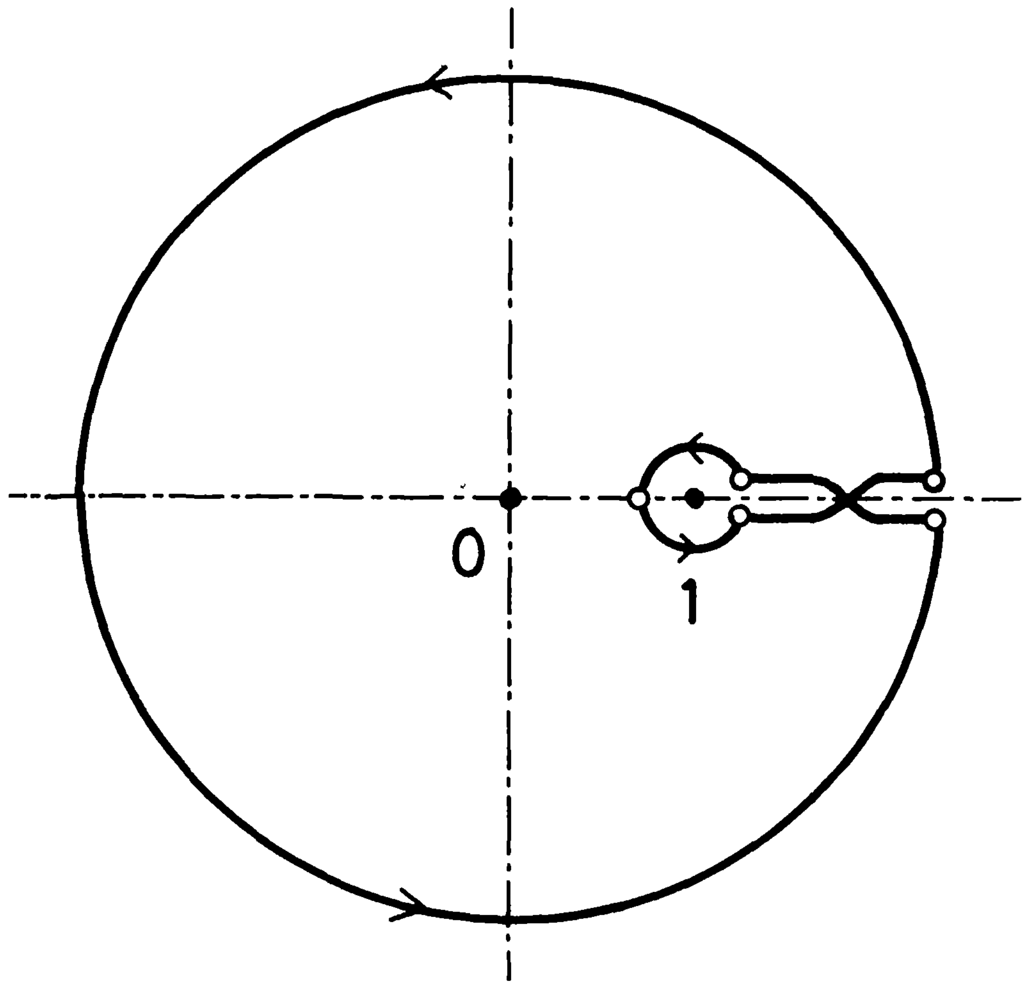

Fig. 12. Image of $T_{-3\pi}$, by the mapping $z = 1/\phi(t)$, in the complex z-plane.

$\phi(e)$: Path on the circle $w = 1 + \varepsilon_1 e^{i\theta}$, where θ increases from π to 2π.

Finally, by the mapping $z = 1/w$, we have the following image in the complex z-plane. [Figure 11]

$\phi(a)^{-1}$: Path on the 'small' circle $z = 1 + \varepsilon_2 e^{i\theta}$, where θ increases from $-\pi$ to 0, and $\varepsilon_2 = \varepsilon_1 + 0(\varepsilon_1^2)$.
$\phi(b)^{-1}$: Path on the real z-axis from $z = 1 + \varepsilon_2$ to $z = \delta_1^{-1}$.
$\phi(c)^{-1}$: Path on the 'large' circle $z = (1/\delta_1)e^{i\psi}$, where ψ decreases from 0 to -2π.
$\phi(d)^{-1}$: Path on the real z-axis from $z = \delta_1^{-1}$ to $z = 1 + \varepsilon_2$.
$\phi(e)^{-1}$: Path on the 'small' circle $z = 1 + \varepsilon_2 e^{i\theta}$, where θ increases from 0 to π.

The closed path, thus obtained, has the base point at $z = 1 - \varepsilon_2$, and makes a circuit $z = 0$ only once, in the negative direction. Therefore, the image can be written as γ_0^{-1}. Similar consideration, which only needs the change of $\phi(c)$ and $\phi(c)^{-1}$, shows that the image of $T_{-3\pi}$ is expressed as $\gamma_1\gamma_0\gamma_1$. [Figure 12] Thus, the definition of $T^{\langle 1\rangle}$ and $T^{\langle 2\rangle}$ in (2.11) proves the expressions (4.1) and (4.2).

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