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On the length of an extremal rational curve*

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The purpose of this paper is to supplement the cone theorem ([Ka1]) and prove the following (we refer to [KMM] for the notation):

Theorem 1 *Let $g: X \to Y$ be a projective morphism of algebraic varieties over a field of characteristic zero, and Δ a $\mathbb{Q}$-divisor on X such that the pair (X, Δ) has only log-terminal singularities. Let E be an irreducible component of the degenerate locus* $\mathrm{Exc}(g) := \{x \in X;$ *g is not an isomorphism at $x\}$, and let $n = \dim E - \dim g(E)$. Assume that $-(K_X + \Delta)$ is g-ample. Then E is covered by a family of rational curves $\{L_\lambda\}_{\lambda \in \Lambda}$ such that the $g(L_\lambda)$ are points and that $(-(K_X + \Delta) \cdot L_\lambda) < 2n$ (resp. $\leqq 2n$) if $E \neq X$ (resp. $E = X$).*

Let $f: X \to S$ be a projective morphism of algebraic varieties such that the pair (X, Δ) is log-terminal for a $\mathbb{Q}$-divisor Δ, and $g: X \to Y$ a contraction morphism associated to an extremal ray R for $f: X \to S$ with respect to the log-canonical divisor $K_X + \Delta$. Then g is an S-morphism which satisfies the conditions of Theorem 1. If X is smooth, S is a point and if $\Delta = 0$, then by Mori [M] and Ionescu [I], E is covered by a family of rational curves $\{L_\lambda\}_{\lambda \in \Lambda}$ such that $(-K_X \cdot L_\lambda) \leqq \dim X + 1 - 2\,\mathrm{codim}_X E$. The number $(-K_X \cdot L)$ is called the *length* of L. We note that our theorem is new even in the case where X has only terminal singularities and $\Delta = 0$. As a corollary we obtain an alternative proof of the discreteness of extremal rays:

Corollary ([Ko]). *For $f: X \to S$ as above, let H be an f-ample Cartier divisor and ε a positive rational number. Then the number of extremal rays R such that $((K_X + \Delta + \varepsilon H) \cdot R) < 0$ is finite.*

Proof. These extremal rays are generated by rational curves L such that $(H \cdot L) < (-(K_X + \Delta) \cdot L)/\varepsilon \leqq 2/\varepsilon \cdot \dim X$. Q.E.D.

The key to the proof of Theorem 1 is the following lemma on the adjunction which is proved similarly as [Ka2, Theorem A.1].

Lemma. *Let $f: X \to Y$ be a projective birational morphism of normal algebraic varieties, H an f-ample Cartier divisor on X, Δ an effective $\mathbb{Q}$-divisor on X, E*

* Dedicated to Professor Heisuke Hironaka on his sixtieth birthday

an irreducible component of the degenerate locus $\mathrm{Exc}(f)$ *of* f, $n=\dim E$, *and* $\nu\colon \bar{E}\to E$ *the normalization. Suppose that the pair* (X, Δ) *has only log-terminal singularities and* $f(E)$ *is a point. Then*

$$(H^{n-1}\cdot(K_X+\Delta)\cdot E)>((\nu^*H)^{n-1}\cdot K_{\bar{E}}).$$

Proof. We may assume that Y is affine and H is very ample. We may also assume that $n=1$. In fact, if $n>1$, then we replace X, E and $\bar{E}$ by a general member X_1 of $|H|$, $E_1=E\cap X_1$ and the normalization $\bar{E}_1$ of E_1, respectively. Since we have $K_{X_1}=(K_X+H)|_{X_1}$ and $K_{\bar{E}_1}=(K_{\bar{E}}+\nu^*H)|_{\bar{E}_1}$, the assertion for X follows from that for X_1.

Suppose that $n=1$ and $((K_X+\Delta)\cdot E)\leqq \deg K_{\bar{E}}$. Then there exists a Cartier divisor A_0 on E such that $\deg A_0\geqq((K_X+\Delta)\cdot E)\in\mathbb{Q}$ and $H^0(\bar{E}, K_{\bar{E}}-\nu^*A_0)\neq 0$. By the trace map, we have $H^0(E, \omega_E(-A_0))\neq 0$. We can extend A_0 to a Cartier divisor A on X if we replace X by a small analytic neighborhood of E. Since $A-(K_X+\Delta)$ is f-nef, we have $R^1f_*\mathcal{O}_X(A)=0$ by [KMM, 1.2.5] and [N, §3]. Hence $H^1(E, A_0)=0$, and $H^0(E, \omega_E(-A_0))=0$ by the Serre duality, a contradiction. Q.E.D.

Proof of Theorem 1. Let Y_1 be an affine open subset of Y which intersects $g(E)$ and Y_0 a generic linear space section of Y_1 whose codimension is equal to $\dim g(E)$. We set $X_0=g^{-1}(Y_0)$, $g_0=g|_{X_0}$, $\Delta_0=\Delta\cap X_0$, and $E_0=E\cap X_0$. Let H be a g_0-ample Cartier divisor on X_0, $\nu\colon \bar{E}_0\to E_0$ the normalization, and C the intersection curve of $n-1$ general elements of $|\nu^*H|$. We also let $M=-\nu^*(K_X+\Delta)$, which is an ample $\mathbb{Q}$-Cartier divisor on $\bar{E}_0$.

First, we assume that $E\neq X$. Then by Lemma, we have $\deg_C(K_{\bar{E}_0}|_C)<(-M\cdot C)<0$. In the case where $n=1$, since $R^1g_{0*}\mathcal{O}_{X_0}=0$ by [KMM, 1.2.5], we have $C=\bar{E}_0=E_0\simeq\mathbb{P}^1$, and $(-(K_X+\Delta)\cdot C)<2$. If $n>1$, then by replacing H by its multiple if necessary, we may assume that C is not a rational curve. By [MM], for an arbitrary point x on C, there exists a rational curve L on $\bar{E}_0$ passing through x such that $(M\cdot L)\leqq 2n(M\cdot C)/(-K_{\bar{E}_0}\cdot C)<2n$.

In the case where $E=X$, if $n=1$, then $X_0\simeq\mathbb{P}^1$, while if $n>1$, then there exists a rational curve L through a general point of X_0 such that $(M\cdot L)\leqq 2n(M\cdot C)/(-K_{X_0}\cdot C)\leqq 2n$, since $(\Delta_0\cdot C)\geqq 0$. Q.E.D.

We have another application of Lemma.

Theorem 2 *Let* $f\colon X\to Y$ *be a projective surjective morphism,* Δ *an effective* $\mathbb{Q}$-*divisor on* X, *and* E *an irreducible component of* $\{x\in X;\ \dim_x f^{-1}f(x)>\dim X-\dim Y\}$. *Suppose that the pair* (X, Δ) *has only log-terminal singularities and that* $-(K_X+\Delta)$ *is* f-*nef. Then* E *is covered by a family of rational curves.*

Proof. We may assume that Y is affine and $f(E)$ is a point. Let $\nu\colon \bar{E}\to E$ be the normalization, H a very ample divisor on X, X_1 the intersection of d general members of $|H|$ for $d=\dim X-\dim Y$, $\Delta_1=\Delta\cap X_1$, $E_1=E\cap X_1$, and $\bar{E}_1=\nu^{-1}(E_1)$. If $d=0$ and $\dim E=1$, then $E\simeq\mathbb{P}^1$ as before. Otherwise, by Lemma $\bar{E}_1$ is covered by a family of curves C such that $(K_{\bar{E}_1}\cdot C)<((K_{X_1}+\Delta_1)\cdot\nu_*C)\leqq d(\nu^*H\cdot C)$. Hence $(K_{\bar{E}}\cdot C)<0$ and E is covered by rational curves by [MM]. Q.E.D.

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