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Invariant measures exist under a summability condition for unimodal maps

Tomasz Nowicki [1,*] **and Sebastian van Strien** [2]

[1] Mathematics Department, Warsaw University, Poland
[2] Mathematics Department, TU Delft, The Netherlands

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Summary. For unimodal maps with negative Schwarzian derivative a sufficient condition for the existence of an invariant measure, absolutely continuous with respect to Lebesgue measure, is given. Namely the derivatives of the iterations of the map in the (unique) critical value must be so large that the sum of (some root of) the inverses is finite.

1 Introduction and statement of results

The aim of this paper is to introduce a new, very weak, condition which guarantees the existence of an invariant probability measure, which is absolutely continuous with respect to the Lebesgue measure (acim in short), for unimodal maps of the interval. We believe this condition is so weak that it is even equivalent to the existence of acim's.

It is well known that there are three possibilities for such a map;

- f has a periodic attractor;
- there exist arbitrarily small intervals which are mapped into themselves by some iterate of f;
- the non-wandering set of f contains intervals.

In the last case such a map may have an acim. The purpose of such a measure is to describe the statistical properties of orbits: the frequency with which a trajectory falls into a set is given by the measure of this set. For several years it was believed that such maps have automatically an acim, but as was shown by Johnson [Jo] this is not true.

In general, the existence of an absolutely continuous measure of some interval map is related to the amount of expansion this map has. Indeed, if a map $f\colon [0,1] \to [0,1]$ is everywhere expanding (and therefore not smooth) then it has an acim [L-Y]. However, if the map has a critical point there is no universal

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expansion. Nevertheless by some analytical means one can estimate the counterplay between the contraction ruled by the derivative near the critical point and the expansion ruled by the derivative near the critical value [Ja, M, Sz, C-E, N1, N2, K1, K2, N-S, Str2]. In these results the expansion near the critical value was assumed to be exponential.

In what follows we prove that some weaker expansion is sufficient, and we conjecture that a condition similar in nature to ours is also necessary.

We shall deal with C^3 maps f of the interval $[0, 1]$ into itself with negative Schwarzian derivative, i.e., $Sf = f'''/f' - 3/2(f''/f')^2 < 0$. The maps f we deal with are unimodal. By this we mean that there is a unique $c \in [0, 1]$ such that $Df > 0$ on $[0, c)$ and $Df < 0$ on $(c, 1]$, where Df denotes the derivative of f. This point c is called the critical point of f. We say that the critical point c has order l if there are constants O_1, O_2 so that

$$O_1|x-c|^{l-1} \leqq |Df(x)| \leqq O_2|x-c|^{l-1} \tag{NF}$$

As usual let f^n be the n-th iterate of f and let $c_1 = f(c)$. Furthermore denote the Lebesgue measure of a measurable set I by $|I|$.

Main Theorem *Suppose that f is unimodal, C^3, has negative Schwarzian derivative and that the critical point of f is of order $l \geqq 1$. Moreover assume that the growthrate of $|Df^n(c_1)|$ is so fast that*

$$\sum_{n=0}^{\infty} |Df^n(c_1)|^{-1/l} < \infty$$

holds. Then f has a unique absolutely continuous invariant probability measure μ which is ergodic and of positive entropy. Furthermore there exists a positive constant K such that

$$\mu(A) \leqq K|A|^{1/l},$$

for any measurable set $A \subset (0, 1)$. Finally, the density ρ of the measure μ with respect to the Lebesgue measure is a $L^{\tau-}$ function where $\tau = l/(l-1)$, $L^{\tau-} = \bigcup_{1 \leqq t < \tau} L^t$ and $L^t = \{\rho \in L^1 : \int |\rho|^t dx < \infty\}$.

It is not hard to show that there exist many parameters a for which the quadratic map $f(x) = ax(1-x)$ satisfies the assumption of this theorem and not the condition that $|Df^n(c_1)|$ grows exponentially (which was introduced in [C-E]). So the condition from this paper is much weaker than the well-known Collet-Eckmann condition. Benedicks and Young [B-Y] proved the existence of acim's for maps with a non-flat critical point for which $|Df^n(c_1)|$ is at least $e^{\alpha\sqrt{n}}$ and for which moreover the distance of $f^n(c_1)$ to c is at least of the form $e^{-\alpha n}$. Clearly our result implies theirs. Moreover, we think that our proof is simpler than theirs.

Since the measure μ from this theorem is absolutely continuous, $\lambda_\mu := \int \log|Df| d\mu$ is strictly positive and since μ is ergodic this implies

$$\lim_{n\to\infty} \frac{1}{n} \log|Df^n(x)| = \lambda_\mu > 0 \quad \text{for } \mu - \text{almost all } x .$$

Of course the estimate $\mu(A) \leq K|A|^{1/l}$ shows that the poles of the invariant measure μ are at most of the form $|x - x_0|^{1/l-1}$. It is not hard to show that any absolutely continuous invariant probability measure has a pole of this order at the critical values $f^n(c)$, $n \geqq 1$, and therefore this estimate is optimal. Even for maps for which $|Df^n(c_1)|$ grows exponentially this result is new (the results in [C-E] and [N-S] only give some bounds for the order of the poles). Notice that the density of the invariant measure is always a $L^{\tau -}$ function where $\tau = l/(l-1)$, independently of the size of $\sum_{n=0}^{\infty}|Df^n(c_1)|^{-1/l} < \infty$. We conjecture that these maps either have an absolutely continuous probability invariant measure with a $L^{\tau -}$ density or do not have a finite absolutely continuous invariant measure at all. Finally it is a pleasure to thank Gerhard Keller, A.O. Lopes and P. Thieullen for some very useful comments.

2 A reformulation of the Main Theorem and an outline of its proof

In [BL1] and [BL2] it is shown that any unimodal map with negative Schwarzian derivative is ergodic (w.r.t. to the Lebesgue measure) and that any absolutely continuous invariant probability measure μ has positive metric entropy. More precisely, as was shown in [BL1] and [BL2] any forward invariant set of positive Lebesgue measure has the critical point as a density point. From this the ergodicity of any absolutely continuous invariant measure μ follows immediately. Furthermore if the entropy of the measure μ were zero, then f would be μ-almost everywhere invertible. But this would imply that the support of μ could have at most density 1/2 at the critical point of f, a contradiction with the previous statement. Therefore, in order to prove the Main Theorem it is enough to establish the existence of an absolutely continuous invariant probability measure μ.

In order to prove the existence of this invariant measure we will use the strategy of [N-S]. Usually invariant measures are constructed by considering iterations of the Perron-Frobenius operator. This operator associates to the density of a measure ν the density of $f_* \nu$. Of course $f_* \nu$ will have poles at the critical values of f even if ν does not. Therefore in order to show that iterations of the Perron-Frobenius operator (i.e. the densities $f_*^n \nu$) have a nice limit density, one has to choose a good 'topology' on a space of densities with infinitely many poles. In some cases one chooses L^p spaces, in other cases spaces with weighted norms.

Rather than to look at the densities of $f_*^n \nu$, in [N-S] it was proposed to compare the measures $f_*^n \nu$ with the Lebesgue measure. More precisely, using general arguments one can show that f has an absolutely continuous invariant probability measure provided that for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any measurable set A with $|A| < \delta$ one has that $|f^{-n}(A)| < \varepsilon$ for all $n > 0$. In fact in this paper we will prove the following more precise statement: there exists a constant K such that for every n and every measurable set A,

$$|f^{-n}(A)| < K|A|^{1/l}. \tag{1}$$

Let us first explain why (1) implies that f has an absolutely continuous invariant probability measure with a $L^{\tau -}$ density where $\tau = l/(l-1)$. For simplicity assume that $|I| = 1$ and let λ be the Lebesgue measure on I. Define $\lambda_n(A) = |f^{-n}(A)|$ (which is nothing but the probability measure $f_*^n \lambda$ from above when λ is the Lebesgue measure) and let $\mu_n = \frac{1}{n}\sum_{i=0}^{n-1} \lambda_i$, i.e., $\mu_n(A) = \frac{1}{n}\sum_{i=0}^{n-1} |f^{-i}(A)|$. Since the space of probability measures on I is compact (with respect to the weak topology), there exists a sequence $n_i \to \infty$ and a probability measure μ such that μ_{n_i} converges weakly to μ. From the definition of μ_n it follows easily that μ is invariant, $\mu(f^{-1}(A)) = \mu(A)$, and from (1) one has that $\mu(A) \leqq K|A|^{1/l}$ for each measurable set A. Hence μ is absolutely continuous. Let ρ be the density of μ with respect to the

Lebesgue measure, i.e., $\mu(A) = \int_A \rho(x)dx$. Informally speaking, the inequality $\mu(A) \leqq K|A|^{1/l}$ implies that the poles of the density can be at worse of the form $x^{1/l-1}$: if $d\mu \approx x^{1/l-1}dx$ then integrating gives $\int_0^\varepsilon x^{1/l-1}dx = \varepsilon^{1/l}$. So one expects the density to be in the space $L^{\tau-}$ where $\tau = l/(l-1)$. Let us make this argument precise. Take $t \geqq 1$ and $C_k = \{x; k \leqq \rho^t(x) \leqq k+1\}$ and $D_k = \bigcup_{l=k}^{\infty} C_l = \{x; \rho^t(x) \geqq k\}$. Since $\mu(A) \leqq K|A|^{1/l}$, $k^{1/t}|D_k| \leqq \int_{D_k} \rho dx = \mu(D_k) \leqq K \cdot |D_k|^{1/l}$ and therefore

$$|D_k| \leqq K' \cdot k^{-\frac{l}{(l-1)t}} .$$

Hence

$$\int \rho^\tau dx \leqq \sum_{k=0}^{\infty} (k+1)|C_k| = 1 + \sum_{k=0}^{\infty} |D_k| ,$$

i.e.,

$$\int \rho^\tau dx \leqq 1 + \sum_{k=0}^{\infty} K' \cdot k^{-\frac{l}{(l-1)t}} < \infty$$

whenever $t < \tau = l/(l-1)$. This shows that $\rho \in L^{\tau-}$.

One of the main results in [N-S] was to show that (1) can be deduced from the following: there exists a constant K' such that for any n and every $\varepsilon > 0$,

$$|f^{-n}(c_1 - \varepsilon, c_1)| < K'\varepsilon^{1/l} \tag{2}$$

where l is the order of the critical point of f. Because of the non-flatness condition at the critical point this is equivalent to: there exists a constant K'' such that for every $n > 0$ and every $\varepsilon > 0$

$$|f^{-n}(c - \varepsilon, c + \varepsilon)| < K''\varepsilon .$$

From all this it follows that the Main Theorem can be deduced from

Theorem A *Suppose that f is unimodal, C^3, has negative Schwarzian derivative and that the critical point of f is of order $l \geqq 1$. Moreover assume that*

$$\sum_{n=0}^{\infty} |Df^n(c_1)|^{-1/l} < \infty$$

holds. Then there exists a constant $K < \infty$ such that for each $\varepsilon > 0$,

$$|f^{-n}(c - \varepsilon, c + \varepsilon)| < K\varepsilon . \tag{3}$$

Let us say a few words about the proof of inequality (3). The main idea in [N-S] was to show that $f^{-n}(c - \varepsilon, c + \varepsilon)$ is contained in the union of sets of the form

$$f^{-(n-k)}\left(c - \frac{\varepsilon}{|Df^k(c_1)|^{1/l}}, c + \frac{\varepsilon}{|Df^k(c_1)|^{1/l}}\right),$$

where k runs over all integers such that $n - k > \log(\varepsilon)$. In this way it was possible to prove inductively that there exists $\delta > 0$ such that $|f^{-n}(c - \varepsilon, c + \varepsilon)| \leqq \varepsilon^\delta$ using the exponential growth of $|Df^k(c_1)|$.

In this paper a more refined version of this strategy is chosen. It is proved that each component of $f^{-n}(c - \varepsilon, c + \varepsilon)$ is either contained in or at least can be compared in size (this process we will call 'sliding') with a set of the form

$$f^{-(n-k)}\left(c - \frac{\varepsilon}{|Df^k(c_1)|^{1/l}}, c + \frac{\varepsilon}{|Df^k(c_1)|^{1/l}}\right).$$

Using this and the summability condition, inequality (3) will then be proved by induction.

3 Some notation and some prerequisites

Throughout this paper we will mean by (α, β) the interval with endpoints α and β irrespective of their ordering.

In this paper we assume that f has negative Schwarzian derivative, i.e., that $Sf = f'''/f' - 3/2\,(f''/f')^2 < 0$. From this it easily follows $Sf^n < 0$ and that $|Df^n|$ has no positive local minima. More generally such maps f have the following properties. A proof of the first property can be found in for example [M.S.1]. The second property was proved first used in [Str1], but see also [Str2], [M.S.2], [M.M.S.].

i (The cross-ratio is expanded) *For every k and every* $\alpha < \beta < \gamma < \delta$ *if* $Df^k_{|(\alpha,\delta)} \neq 0$ *then*

$$\frac{|f^k(I)|}{|I|}\frac{|f^k(T)|}{|T|} > \frac{|f^k(L)|}{|L|}\frac{|f^k(R)|}{|R|}, \qquad (EC)$$

where $L = (\alpha, \beta)$, $I = (\beta, \gamma)$, $R = (\gamma, \delta)$, $T = (\alpha, \delta)$.

ii (Koebe Lemma) *For every* $\tau > 0$ *there exists a constant* $\mathcal{O} = \mathcal{O}(\tau, KL) > 0$ *such that for every k and every* $\alpha < x < \beta$ *with* $Df^k_{|(\alpha,\beta)} \neq 0$ *if* $|f^k(\alpha, x)| > \tau\,|f^k(\alpha, \beta)|$ *then*

$$|Df^k(x)| > \mathcal{O}(\tau, KL)|Df^k(\beta)|\,. \qquad (KL)$$

In the proof we shall use the following convention. The symbol $\mathcal{O}$ will describe various finite and positive universal constants (i.e., they are independent on the iterate of f). All the estimates in this paper are based on the non-flatness condition, the fact that the cross-ratio is expanded or on the Koebe Lemma. $\mathcal{O}(NF)$ denotes a constant which is based on the non-flatness condition and $\mathcal{O}(\tau, KL)$ is the constant from the Koebe Lemma.

4 Branches which will be 'slided' later on

Let f^n be monotone on an interval I and assume that $f^n(I) = (c - \varepsilon, c + \varepsilon)$. Let T be the largest interval containing I on which f^n is monotone and lable the endpoints α and β of T so that $|f^n(\alpha) - c| \leqq |f^n(\beta) - c|$. Denote the endpoints of I by γ and δ so that either $\alpha < \gamma < \delta < \beta$ or $\alpha > \gamma > \delta > \beta$. In this section we will assume that

$$|f^n(\alpha) - f^n(\gamma)| \geqq 2\varepsilon\,. \qquad (*)$$

Let

$$A_0 = f^n(\alpha, \gamma),\ I_0 = f^n(I),\ R_0 = f^n(\delta, \beta)\,.$$

By $(*)$ one has

$$|R_0| \geqq |A_0| \geqq |I_0|\,. \qquad (**)$$

Later on we shall show that if $|R_0|$ is not too large compared to $|A_0|$ then the set $I_0 = f^n(I)$ can be 'slided'. In this section we will show that if I_0 cannot be 'slided' at least some smaller iterate $f^{k_s}(I)$ of I can be 'slided'. If $|A_0 \cup I_0| \geqq |R_0|$ then set

$s = 0$ and we are finished. Otherwise we shall define inductively a finite sequence of intervals $T^i = [\alpha_i, \alpha_{i-1}]$ and integers n_i as follows. Let $n_0 = n$, $\alpha_0 = \alpha$, $\alpha_{-1} = \beta$ and $T^0 = [\alpha_0, \alpha_{-1}] = [\alpha, \beta]$ (i.e. $T^0 = T$). By maximality of T one can choose n_1 such that $0 < n_1 < n$ and $f^{n_1}(\alpha_0) = c$. Now choose α_1 such that T^1 is the maximal interval of the form $T^1 = [\alpha_1, \alpha_0]$ which contains T^0 and on which f^{n_1} is monotone (of course one may have $T^1 = T$). Now assume that n_{i-1} and $T^{i-1} = [\alpha_{i-2}, \alpha_{i-1}]$ are defined. Then simply define $n_i < n_{i-1}$ such that $f^{n_i}(\alpha_{i-1}) = c$, and let T^i be the maximal interval of the form $[\alpha_i, \alpha_{i-1}]$ which contains $T^{i-1} = [\alpha_{i-2}, \alpha_{i-1}]$ and on which f^{n_i} is monotone. It follows that for $i \geqq 2$, T^i and T^{i-1} have precisely one common boundary point and that

$$I \subset T^0 \subset \ldots \subset T^i .$$

Let us now define the integers k_i and intervals I_i, R_i, A_i, L_i as follows:

$$k_i = n_i - n_{i+1}, I_i = f^{n_i}(I) ,$$

$$R_i = f^{n_i}(\alpha_{i-1}, \gamma)\backslash I_i, A_i = f^{n_i}(\alpha_i, \gamma)\backslash I_i, L_i = f^{n_i}(\alpha_{i-2}, \gamma)\backslash I_i .$$

In other words R_i is the component of $f^{n_i}(T^i\backslash I)$ which contains c and A_i is the other component. Furthermore L_i is contained in A_i and

$$f^{k_i}(I_{i+1}) = I_i, f^{k_i}(R_{i+1}) = A_i, f^{k_i}(L_{i+1}) = R_i ,$$

for all $i = 0, \ldots, s-1$. We stop the construction at $i = s$, when

$$|A_s \cup I_s| \geqq |R_s| . \qquad (***a)$$

In particular

$$|A_i \cup I_i| \leqq |R_i|, \quad \text{for } i = 0, 1, \ldots, s-1 . \qquad (***b)$$

The main result of this section is the following

Proposition 4.1 *There exist constants K', K'' such that*

$$|I_s| \leqq K' \frac{|f^n(I)|}{\prod_{i=0}^{s-1} K''|Df^{k_i}(c_1)|^{1/l}} .$$

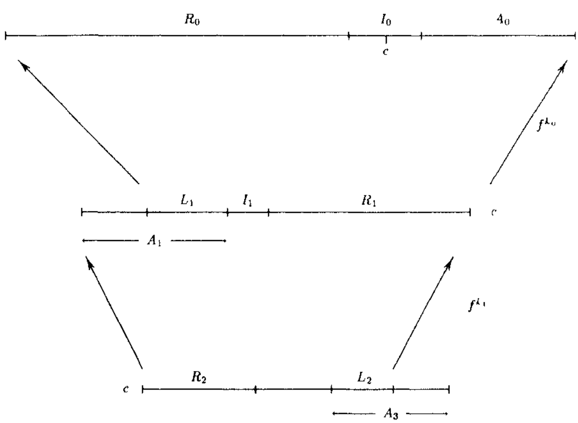

Fig. 1. The intervals R_i, I_i, L_i and A_i

In order to prove this proposition we need some lemmas:

Lemma 4.2

$$|I_s| \leqq |I_1| \frac{|A_1 \cup I_1 \cup R_1|}{|A_1|} \prod_{i=1}^{s-1} \frac{|R_{i+1}|}{|R_i|}.$$

Proof. By (EC) we have

$$\frac{|f^{k_i}(L_{i+1})|}{|L_{i+1}|} \frac{|f^{k_i}(R_{i+1})|}{|R_{i+1}|} < \frac{|f^{k_i}(I_{i+1})|}{|I_{i+1}|} \frac{|f^{k_i}(L_{i+1} \cup I_{i+1} \cup R_{i+1})|}{|L_{i+1} \cup I_{i+1} \cup R_{i+1}|}.$$

By definition of the sequences of intervals A, L, I, R this is equivalent to

$$|I_{i+1}| \leqq |I_i| \frac{|L_{i+1}|}{|A_i|} \frac{|R_i \cup I_i \cup A_i|}{|R_{i+1} \cup I_{i+1} \cup L_{i+1}|} \frac{|R_{i+1}|}{|R_i|}.$$

By induction we get

$$|I_s| \leqq |I_1| \frac{|A_1 \cup I_1 \cup R_1|}{|A_1|} \times \prod_{i=1}^{s-1} \frac{|R_{i+1}|}{|R_i|} \times \prod_{i=2}^{s-1} \left(\frac{|L_i|}{|A_i|} \frac{|R_i \cup I_i \cup A_i|}{R_i \cup I_i \cup L_i|} \right) \times \frac{|L_s|}{|L_s \cup I_s \cup R_s|}.$$

The last factor is clearly less than 1, and we can say the same about the last but one factor because these terms are of the form $\dfrac{l(a+w)}{a(l+w)}$ and because $l(a+w) < a(l+w)$ for positive l, a, w and $l < a$. □

Lemma 4.3 *Assume that f^k is a diffeomorphism on (c, w) and that for some $z \in (c, w)$ one has $f^k(z) = c$ and $|f^k(c, z)| < \tau |f^k(z, w)|$ for some $\tau \in (0, 1)$. Then*

$$\frac{|f^k(c, z)|}{|(c, z)|} \geqq \mathcal{O}(\tau) |Df^k(c_1)|^{1/l}.$$

Proof. Using the chain-rule, the non-flatness condition, (KL) and $f^k(z) = c$ one has

$$\begin{aligned} |Df^k(c_1)| &= |Df(f^{k-1}(c_1))| |Df^{k-1}(c_1)| \\ &\leqq \mathcal{O}(NF) |f^{k-1}(c_1) - c|^{l-1} |Df^{k-1}(c_1)| \\ &\leqq \mathcal{O}(NF) \mathcal{O}(KL, \tau) |f^{k-1}(c_1) - c|^{l-1} \frac{|f^{k-1}(f(c, z))|}{|f(c, z)|} \\ &= \mathcal{O}(NF) \mathcal{O}(KL, \tau) \frac{|f^k(c, z)|^l}{|f(c, z)|}. \end{aligned}$$

Using again the non-flatness condition gives the required estimate. □

Lemma 4.4 *Assume that f^k is a diffeomorphism on (c, z), that $f^k(z) = c$ and that for some $y \in (c, z)$ one has $|f^k(c, y)| < \tau |f^k(y, z)|$ for some $\tau \in (0, 1)$. Then*

$$\frac{|f^k(y, z)|}{|(c, y)|} \geqq \mathcal{O}(\tau) |Df^k(c_1)|^{1/l}.$$

Proof. From the non-flatness condition and since $|f^k(y, z)| \geqq \frac{1}{\tau} |f^k(c, y)|$ one has

$$\left(\frac{|f^k(y, z)|}{|c, y|} \right)^l \geqq \mathcal{O}(NF) \frac{|f^k(y, z)|^{l-1} |f^k(y, z)|}{|f(c, y)|} \geqq \frac{\mathcal{O}(NF)}{\tau} |f^k(y, z)|^{l-1} \frac{|f^k(c, y)|}{|f(c, y)|}.$$

Since $|f^k(c, y)| < \tau |f^k(y, z)|$ one gets from (KL) that the last factor is at least $\mathcal{O}(\tau, KL)|Df^{k-1}(c_1)|$. Moreover one has $|f^k(y, z)| \geqq \frac{1}{1+\tau}|f^k(c, z)| = \frac{1}{1+\tau}|f^k(c) - c|$. From all this

$$\left(\frac{|f^k(y, z)|}{|(c, y)|}\right)^l \geqq \mathcal{O}|f^k(c) - c|^{l-1}|Df^{k-1}(c_1)| .$$

By the non-flatness condition $|f^k(c) - c|^{l-1} \geqq \mathcal{O}(NF)|Df(f^k(c))|$, and therefore the lemma follows. □

Lemma 4.5 *If $s > 0$ then there exists a constant $K < \infty$ such that*

$$|I_1| \frac{|A_1 \cup I_1 \cup R_1|}{|A_1|} \leqq K \frac{|f^n(I)|}{|Df^{k_0}(c_1)|^{1/l}},$$

$$\frac{|f^{k_0}(R_1)|}{|R_1|} \geqq \frac{1}{K} |Df^{k_0}(c_1)|^{1/l},$$

$$\frac{|R_i|}{|R_{i+1}|} \geqq \frac{1}{K} |Df^{k_i}(c_1)|^{1/l} \textit{ for } i = 1, \ldots, s-1$$

Proof. By (EC) we have

$$|I_1| \frac{|A_1 \cup I_1 \cup R_1|}{|A_1|} \leqq |f^{k_0}(I_1)| \frac{|R_1|}{|f^{k_0}(R_1)|} \frac{|f^{k_0}(A_1 \cup I_1 \cup R_1)|}{|f^{k_0}(A_1)|} \leqq$$

$$\leqq |f^n(I)| \frac{|R_1|}{|f^{k_0}(R_1)|}\left(1 + \frac{|I_0 \cup A_0|}{|f^{k_0}(L_1)|}\right)$$

$$\leqq |f^n(I)| \frac{|R_1|}{|f^{k_0}(R_1)|}\left(1 + \frac{|I_0 \cup A_0|}{|R_0|}\right).$$

By $(**)$ this gives

$$|I_1| \frac{|A_1 \cup I_1 \cup R_1|}{|A_1|} \leqq |f^n(I)| \frac{|R_1|}{|f^{k_0}(R_1)|} \cdot 3 .$$

It is therefore enough to prove that $|f^{k_0}(R_1)|/|R_1| > \mathcal{O}|Df^{k_0}(c_1)|^{1/l}$. So let us prove this. One has $f^{k_0}(R_1) = A_0$. Let $R_1' \subset R_1 \cup I_1$ be the smallest interval containing R_1 such that $f^{k_0}(R_1')$ contains c. We want to apply Lemma 4.3 by taking (z, w) to be the interval T_1 and (c, z) the interval R_1'. Since $f^{k_0}(T_1 \backslash R_1') \supset f^{k_0}(L_1) = R_0$ and since $s > 0$ we get from (***b),

$$|f^{k_0}(T_1 \backslash R_1')| \geqq |R_0| \geqq |A_0 \cup I_0| \geqq |f^{k_0}(R_1')| .$$

From all this it follows that we can apply Lemma 4.3 and get that

$$|f^{k_0}(R_1')|/|R_1'| > \mathcal{O}|Df^{k_0}(c_1)|^{1/l} .$$

But since $|f^{k_0}(R_1')| = |A_0| + \varepsilon < 2|A_0| = 2|f^{k_0}(R_1)|$, this implies

$$\frac{|f^{k_0}(R_1)}{|R_1|} \geqq \frac{1}{2} \frac{|f^{k_0}(R_1')|}{|R_1'|} > \frac{\mathcal{O}}{2} |Df^{k_0}(c_1)|^{1/l} .$$

So the first two statements of the lemma follow.

To prove the third statement let as before $R'_{i+1} \subset R_{i+1} \cup I_{i+1}$ be the smallest interval containing R_{i+1} such that $f^{k_i}(R'_{i+1})$ contains c. Because the construction did not stop at i, $|f^{k_i}(L_{i+1})| = |R_i| \geqq |A_i \cup I_i| \geqq |f^{k_i}(R'_{i+1})|$. Therefore one can apply Lemma 4.4 and we get

$$\frac{|R_i|}{|R_{i+1}|} \geqq \frac{|R_i|}{|R'_{i+1}|} = \frac{|f^{k_i}(L_{i+1})|}{|R'_{i+1}|} \geqq \mathcal{O}|Df^{k_i}(c_1)|^{1/l} .$$

This proves the third statement of the lemma. □

Proof of Proposition 4.1 This follows immediately from Lemmas 4.2, 4.5. □

5 Estimates

In this section we shall prepare the estimates for the preimages of the intervals around the critical point c. So consider the set $E(\delta) = (c - \delta, c + \delta)$ and its preimages $E_n(\delta) = f^{-n}(E(\delta))$. For a given $\varepsilon > 0$ we shall subdivide the collection of components of $E_n(\varepsilon)$ into three subcollections.

Let σ be some positive number and let $\varepsilon \in (0, \frac{1}{2}\sigma)$. We define $\nu(\sigma)$ as $\inf\{k > 0 \colon |f^k(c) - c| < \sigma\}$. Clearly $\nu(\sigma)$ is monotone and $\nu(\sigma)$ tends to infinity as $\sigma \to 0$. Later on we shall choose σ appropriately.

Let I be a component of $E_n(\varepsilon)$. Suppose that $I \subset I' \subset I''$, where I' is a component of $E_n(2\varepsilon)$ and I'' is a component of $E_n(\sigma)$. If $Df^n_{|I''} \neq 0$ then I belongs to the collection $\mathcal{R}_n$. If $I \notin \mathcal{R}_n$ but $Df^n_{|I'} \neq 0$ then I belongs to the collection $\mathcal{S}_n$. All the other components form the collection $\mathcal{T}_n$.

5.1 The collection $\mathcal{R}_n$, the regular case

If $I \in \mathcal{R}_n$, then f^n is a diffeomorphism on I and there exists $\gamma \in I$ such that $f^n(\gamma) = c$. Let (α, β) be the maximal interval containing I on which f^n is a diffeomorphism. By definition of $\mathcal{R}_n$ we have $|f^n(\alpha, \gamma)|$, $|f^n(\beta, \gamma)| \geqq \sigma$. Therefore we can use (KL) on I and obtain:

Proposition 5.1 *There exists a constant K_R such that for $\varepsilon < \sigma/2$ and any regular component I as above, one has*

$$\frac{|I|}{|(\alpha, \beta)|} \leqq K_R \frac{\varepsilon}{\sigma}.$$

Proof. By (KL) one has $|Df^n(x)| \leqq \mathcal{O}(KL)|Df^n(y)|$ for any $x \in (\alpha, \beta)$ and $y \in I$. Therefore

$$\frac{\sigma}{|(\alpha, \beta)|} \leqq \frac{|f^n(\alpha, \beta)|}{|(\alpha, \beta)|} \leqq \mathcal{O}\frac{|f^n(I)|}{|I|} \leqq \mathcal{O}\frac{\varepsilon}{|I|} . \qquad \square$$

Corollary. *For $I \in \mathcal{R}_n$ let $\Delta_n(I)$ be the maximal interval on which f^n is a diffeomorphism. By the previous proposition we obtain*

$$\sum_{I \in \mathcal{R}_n} |I| \leqq \sum_{I \in \mathcal{R}_n} \frac{|I|}{|\Delta_n(I)|} |\Delta_n(I)| \leqq K_R \frac{\varepsilon}{\sigma} \sum_{I \in \mathcal{R}_n} |\Delta_n(I)| \leqq K_R \frac{\varepsilon}{\sigma} .$$

5.2 The collection $\mathscr{S}_n$, the case to slide

Let $I \in \mathscr{S}_n$. Then there exists $s \geqq 0$ and a sequence $n_s < n_{s-1} < \ldots < n_0 = n$ as in section 4 such that (in the terminology of that section) $|A_s \cup I_s| \geqq |R_s|$. Since for each $i = 0, \ldots, s$, f^{n_i} is a diffeomorphism on $T^i \supset I$, since $f^{n_i}(T^i) \supset R_i$ and since R_i contains c, there exists an interval $J^i \subset T^i$ such that $G_i = f^{n_i}(J^i)$ contains c and such that

$$|I| = |J^i|\,.$$

In other words by choosing an appropriate $x_i \in [x_i', c] = R_i \cup I_i$ one can assure that the preimage J^i of $G_i = [x_i, c]$ has the same size as I. Let $G = G_s$ and $J = J^s$. This process we call sliding. Note that $f^{n_i}(J^i)$ contains c. Because $|A_s \cup I_s| \geqq |R_s|$ we can use (KL) and obtain

$$\frac{|f^{n_s}(J)|}{|J|} \leqq \mathcal{O}(KL)\frac{|f^{n_s}(I)|}{|I|}$$

and therefore $|G| \leqq \mathcal{O}|I_s|$. Therefore by Proposition 4.1

$$|G| \leqq \frac{\mathcal{O}K'|f^n(I)|}{\prod_{j=0}^{s-1} K''|Df^{k_j}(c_1)|^{1/l}} \leqq \frac{\mathcal{O}K'2\varepsilon}{\prod_{j=0}^{s-1} K''|Df^{k_j}(c_1)|^{1/l}}\,.$$

So for each such component I of $E_n(\varepsilon)$, there exists an interval J as above such that $|I| = |J|$ and therefore such that $|I|$ is at most the size of the part of the f^{n_s}-preimage of

$$\left(c - \frac{\mathcal{O}K'2\varepsilon}{\prod_{j=0}^{s-1} K''|Df^{k_j}(c_1)|^{1/l}},\, c + \frac{\mathcal{O}K'2\varepsilon}{\prod_{j=0}^{s-1} K''|Df^{k_j}(c_1)|^{1/l}}\right)$$

that is contained in T^s. Now even for a given sequence of $n_s < n_{s-1} < \ldots < n_0 = n$, there may be several such components I in T^s. Even worse, some of these may give the same interval J (or at least overlapping intervals). But for every given sequence of $n_s < n_{s-1} < \ldots < n_0 = n$, there exist at most 2^s different components I of $E_n(\varepsilon)$ of type $\mathscr{S}_n$, such that the corresponding intervals J overlap. Indeed, at the first step of the construction two intervals I and $\tilde{I}$ can only slide onto overlapping intervals J^1 and $\tilde{J}^1$ if there is precisely one turning point of f^{n_1} between these two intervals. Similarly at the i-th step two intervals J^{i-1} and $\tilde{J}^{i-1}$ can only slide onto overlapping intervals J^i and $\tilde{J}^i$ if there is precisely one turning point of f^{n_i} between these two intervals. So at each step the number of intervals $I \in \mathscr{S}_n$ which correspond to overlapping intervals J can at most double. Thus we get

$$\sum_{I \in \mathscr{S}_n} |I| \leqq \sum_{\sum_{j=0}^{s-1} k_j \leqq n} 2^s |f^{-n_s}(c - |G|, c + |G|)|. \qquad (*)$$

Lemma 5.2 *There exists $\sigma_0 > 0$ such that $k_0, \ldots, k_{s-1} \geqq v(\sigma)$ for each $\sigma \in (0, \sigma_0)$.*

Proof. Choose $\sigma_0 > 0$ so small that for each $\sigma \in (0, \sigma_0)$ and each $k > v(\sigma)$ one has $|Df^k(c_1)|^{1/l} > 2K$ where K is the constant from Lemma 4.5. Because $v(\sigma) \to \infty$ as $\sigma \to 0$ and $|Df^k(c_1)| \to \infty$ as $k \to \infty$ this is possible.

By definition of $k_0, f^{k_0}(c)$ is contained in the closure of A_0. Since $I \in \mathscr{S}_n$, at least one critical value of $f^n|T_0$ is cotained in $(c - \sigma, c + \sigma)$. As $T_0 = [\alpha, \beta]$ and as $|f^n(\alpha) - c| \leqq |f^n(\beta) - c|$, this implies $f^n(\alpha) = f^{k_0}(c) \in (c - \sigma, c + \sigma)$. Hence $k_0 \geqq v(\sigma)$ and, observing the definition of A_0, $A_0 \subset (c - \sigma, c + \sigma)$.

Next notice that if for some $i = 1, \ldots, s-1$ one has

$$|R_i \cup I_i \cup A_i| \leqq \sigma \text{ implies } k_i \geqq v(\sigma) . \qquad (**)$$

Indeed, c is contained in the closure of $R_i \cup I_i \cup A_i$, this interval has at most length σ and $f^{k_i}(c)$ is contained in A_i. Therefore $k_i \geqq v(\sigma)$ follows from the definition of $v(\sigma)$.

So let us show by induction that for $\sigma \in (0, \sigma_0)$, $|R_i \cup I_i \cup A_i| \leqq \sigma$ for $i = 1, \ldots, s-1$. So assume $s \geqq 2$ and let us first prove that this inequality holds for $i = 1$. From Lemma 4.5, $|f^{k_0}(R_1)| \geqq K|Df^{k_0}(c_1)|^{1/l}|R_1|$. Therefore

$$|R_1 \cup I_1 \cup A_1| \leqq 2|R_1| \leqq 2K \frac{|f^{k_0}(R_1)|}{|Df^{k_0}(c_1)|^{1/l}}$$

$$= 2K \frac{|A_0|}{|Df^{k_0}(c_1)|^{1/l}} \leqq |A_0| ,$$

where the last inequality holds provided $\sigma \in (0, \sigma_0)$ because $k_0 \geqq v(\sigma)$, and $|Df^k(c_1)| > 2K$ for $k \geqq v(\sigma)$. Since $|A_0| \leqq \sigma$, the induction assertion is proved for $i = 1$. Similarly, we get for $i < s$, using the third inequality of Lemma 4.5, that

$$|R_i \cup I_i \cup A_i| \leqq 2|R_i| \leqq 2K \frac{|R_{i-1}|}{|Df^{k_{i-1}}(c_1)|^{1/l}} .$$

From the inductive assumption we know that $|R_{i-1} \cup I_{i-1} \cup A_{i-1}| \leqq \sigma$ and from (**) this implies $k_{i-1} > v(\sigma)$ and so we get again that for $\sigma \in (0, \sigma_0)$ sufficiently small that

$$|R_i \cup I_i \cup A_i| \leqq |R_{i-1}| \leqq |R_{i-1} \cup I_{i-1} \cup A_{i-1}| \leqq \sigma . \qquad \square$$

Therefore one has $k_j \geqq v(\sigma)$. Using this, (*) and the estimate for $|G|$ one gets the following

Proposition 5.3 *There exists a constant K_s such that for $\sigma \in (0, \sigma_0)$,*

$$\sum_{I \in \mathscr{S}_n} |I| \leqq \sum_{\substack{\{k_j \geqq v(\sigma)\} \\ \sum_{j=0}^{s-1} k_j \leqq n}} 2^s \left| f^{-n_s}\left(E\left(K_S \frac{\varepsilon}{\prod_{j=1}^{s-1} K''|Df^{k_j}(c_1)|^{1/l}} \right)\right)\right| . \qquad \square$$

5.3 The collection $\mathscr{T}_n$, the case to transport

We shall use the idea from [N-S], which is to reduce the estimation of the n-th preimage $I \in \mathscr{T}_n$ to the estimation of some k-th preimage, with $k < n$.

Let $I \in \mathscr{T}_n$ and let I' (resp. I'') be the component of $E_n(2\varepsilon)$ (resp. $E_n(\sigma)$) containing I. By definition f^n has at least one critical point in $I' \supset I$.

Since f^n has a critical point in I' there exists an integer $k < n$ such that $c \in f^k(I')$. Let k be the largest such integer. For simplicity we say that I belongs to the subcollection $\mathscr{T}_n^k$ of $\mathscr{T}_n$. From the properties just stated one has

$$f^{n-k}(c) \in (c - 2\varepsilon, c + 2\varepsilon) .$$

Since $c \notin f^i(I')$ for $i = k+1, \ldots, n-1$, f^{n-k-1} is clearly a diffeomorphism on $f^{k+1}(I')$.

Proposition 5.4 *There exists a constant K_T such that for every n*

$$\sum_{I\in\mathcal{T}_n}|I| = \sum_k \sum_{I\in\mathcal{T}_n^k}|I| \leqq \sum_k \left| f^{-k}\left(E\left(\frac{K_T\varepsilon}{|Df^{n-k}(c_1)|^{1/l}}\right)\right)\right| .$$

Proof. Let $f^{k+1}(I') = (x, c_1] \supset f^{k+1}(I)$. As we saw f^{n-k-1} is a diffeomorphism in (x, c_1). Moreover $f^n(I) \subset (c-\varepsilon, c+\varepsilon)$, $f^{n-k-1}(x) = c \pm 2\varepsilon$ and $f^n(I')\subset (c-2\varepsilon, c+2\varepsilon)$. Therefore one gets from the Koebe Lemma immediately that

$$\frac{|f^n(I)|}{|f^{k+1}(I)|} \geqq \mathcal{O}(KL)|Df^{n-k-1}(c_1)| .$$

Hence

$$|f^{k+1}(I)| \leqq \mathcal{O}\frac{\varepsilon}{|Df^{n-k-1}(c_1)|} .$$

From the non-flatness condition this gives

$$|f^k(I)| \leqq \mathcal{O}(NF)|f^{k+1}(I)|^{1/l} \leqq \mathcal{O}\left(\frac{\varepsilon}{|Df^{n-k-1}(c_1)|}\right)^{1/l} .$$

Since $f^{n-k-1}(c_1)\in(c-2\varepsilon, c+2\varepsilon)$, the non-flatness condition implies that

$$\left(\frac{\varepsilon}{|Df^{n-k-1}(c_1)|}\right)^{1/l} = \left(\frac{\varepsilon^l}{|\varepsilon^{l-1}Df^{n-k-1}(c_1)|}\right)^{1/l} \leqq \mathcal{O}\frac{\varepsilon}{|Df^{n-k}(c_1)|^{1/l}} ,$$

i.e.,

$$|f^k(I) \leqq \mathcal{O}\frac{\varepsilon}{|Df^{n-k}(c_1)|^{1/l}} .$$

Since $f^n(I) \subset (c-\varepsilon, c+\varepsilon)$ it follows that there exists a constant K_T such that

$$I \subset f^{-k}(f^k(I)) \subset f^{-k}E\left(\frac{K_T\varepsilon}{|Df^{n-k}(c_1)|^{1/l}}\right) .$$

The proposition follows. □

6 The proof of the Main Theorem

Let σ be fixed so small that for every n,

$$\sum_{\substack{\{k_j\geqq\nu(\sigma)\},\,1\leqq j\leqq s\\ \sum k_j\leqq n}} 3K_S\prod_{j=0}^{s-1}|K''Df^{k_j}(c_1)/2|^{-1/l} \leqq 1 ,$$

and

$$\sum_{k>\nu(\sigma)} 3K_T|Df^k(c_1)|^{-1/l} \leqq 1 .$$

This is possible by the summability condition, since $\nu(\sigma)$ tends to infinity as $\sigma\to 0$ and because of the following

Lemma 6.1 *Suppose that $d_k \geqq 0$ and $\sum_{k=0}^{\infty} d_k < \infty$. Then for any η, $\psi > 0$ there exists a v_0 such that*

$$P = \sum_n \prod_{\substack{\{k_j\geqq v_0\}\\ \sum_{j=0}^{s-1}k_j\leqq n}} (\eta d_{k_j}) < \psi .$$

Proof. Consider $S_{k_0} = \sum_{k>k_0} \eta d_k$. Then both S_{k_0} and $S = \sum_{s>0} S_{k_0}^s$ tend to zero when k_0 tends to infinity. Clearly $P \leqq S$ which proves the assertion. □

We shall now prove Theorem A in the following formulation:

Theorem B *For any n and $\varepsilon \leqq \sigma/2$*

$$|f^{-n}(c-\varepsilon, c+\varepsilon)| \leqq 3K_R \frac{\varepsilon}{\sigma}.$$

Proof. With the notations from the previous section we have

$$f^{-n}(c-\varepsilon, c+\varepsilon) = \bigcup_{I \in \mathscr{R}_n} I \cup \bigcup_{I \in \mathscr{S}_n} I \cup \bigcup_{I \in \mathscr{T}_n} I,$$

and

$$|f^{-n}(c-\varepsilon, c+\varepsilon)| \leqq \sum_{I \in \mathscr{R}_n} |I| + \sum_{I \in \mathscr{S}_n} |I| + \sum_{I \in \mathscr{T}_n} |I|.$$

Therefore

$$\begin{aligned} |f^{-n}(c-\varepsilon, c+\varepsilon)| &\leqq K_R \frac{\varepsilon}{\sigma} + \\ &+ \sum_{\substack{\{k_j \geqq \nu(\sigma)\}, 0 \leqq j \leqq s-1 \\ \sum k_j \leqq n}} 2^s \left| f^{-n_s}\left(E\left(K_S \frac{\varepsilon}{\prod_{j=0}^{s-1} K'' |Df^{k_j}(c_1)|^{1/l}} \right)\right)\right| + \\ &+ \sum_{k \geqq \nu(\sigma)} \left| f^{-k}\left(E\left(K_T \frac{\varepsilon}{|Df^{n-k}(c_1)|^{1/l}} \right)\right)\right|. \end{aligned}$$

We shall apply the induction. For n small only the first term is non-zero and the assertion of the theorem is true. Suppose that it is true for any $\varepsilon < \sigma/2$ and any $n < N$. Then by the choice of σ, the above formula and the induction assumption we have

$$\begin{aligned} |f^{-N}(c-\varepsilon, c+\varepsilon)| &\leqq K_R \frac{\varepsilon}{\sigma} + \\ &+ \sum_{\substack{\{k_j \geqq \nu(\sigma)\}, 0 \leqq j \leqq s-1 \\ \sum k_j \leqq N}} 2^s K_S \frac{3K_R \varepsilon/\sigma}{\prod_{j=0}^{s-1} K'' |Df^{k_j}(c_1)|^{1/l}} \\ &+ \sum_{\nu(\sigma) \leqq k \leqq N} K_T \frac{3K_R \varepsilon/\sigma}{|Df^{N-k}(c_1)|^{1/l}} \leqq 3K_R \frac{\varepsilon}{\sigma}, \end{aligned}$$

which completes the proof. □

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Note added in proof

It is not hard to show that if a map f as above has an absolutely continuous invariant probability measure μ with $\mu(A) \leqslant k.\, |A|^{1/l}$ for all measurable sets A then the summability condition is also satisfied. We conjecture that a weighted summability condition (where the weights are related to the position of $f'(c_1)$) is necessary for the existence of absolutely continuous invariant probability measures.