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Relations among the squares of the generators of the braid group

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The classical braid group $\mathscr{B}_n$ has presentation

$$\mathscr{B}_n = \langle \sigma_1, \sigma_2, \ldots, \sigma_{n-1} \colon \sigma_i\sigma_j = \sigma_j\sigma_i \quad (|i-j| \geqq 2),$$
$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \quad (1 \leqq i \leqq n-2)\rangle$$

see ([B, H] for general background and references). This presentation is also a particular case of the type of presentation used to define what have come to be called Artin groups although they seem first to have been explicitly considered in [T]. Such presentations are described in the following way.

Let $\{x_1, x_2, \ldots, x_n\}$ be an alphabet and let M be a symmetric $n \times n$ matrix of non-negative integers whose diagonal entries are all 0. For each pair (i, j) let u_{ij} be the word of length m_{ij} which begins with x_i and whose successive letters are alternatively x_i and x_j. The associated **Artin group** is the group G given by the presentation

$$G = \langle x_1, x_2, \ldots, x_n \colon u_{ij} = u_{ji}, \quad 1 \leqq i, j \leqq n\rangle.$$

According to Pride (see [P]) Tits has conjectured that in an Artin group G the subgroup $\langle x_i^2, 1 \leqq i \leqq n\rangle$ generated by the squares of the given generators of G has no relations other than those which are consequences of the relations $x_ix_j = x_jx_i$ corresponding to those $m_{ij} = m_{ji} = 2$. This conjecture has been verified under a variety of general hypotheses by Pride in [P], and for the braid groups $\mathscr{B}_n$, for $n \geqq 5$ by Droms et al. in [D-L-S]. The purpose of this note is to establish the Tits conjecture for arbitrary braid groups.

Let $\mathscr{H}_n$ be the *pure* braid group, that is the normal closure of the squares σ_i^2 of the generators of $\mathscr{B}_n$. The group $\mathscr{H}_n$ has a well-understood presentation – details and verification can be found in Appendix 1 by L. Gaede in [H] – in terms of the generators

$$a_{ij} = \sigma_{j-1} \ldots \sigma_{i+1}\sigma_i^2\sigma_{i+1}^{-1} \ldots \sigma_{j-1}^{-1}, \quad 1 \leqq i < j \leqq n.$$

Let $\mathscr{L}_n=\langle \sigma_1^2,\sigma_2^2,\ldots,\sigma_{n-1}^2\rangle=\langle a_{12},a_{23},\ldots,a_{(n-1)n}\rangle$ – our aim is to obtain defining relations for $\mathscr{L}_n$. Further let $\mathscr{A}_n$ be the subgroup generated by $x_1,\ldots,x_{n-1}$ where $x_i=a_{in}$. It well-known that $\mathscr{A}_n$ is free on the given generators.

There is a standard action of $\mathscr{B}_{n-1}$, and hence $\mathscr{H}_{n-1}$ on the free group of rank $n-1$ and so one may form the semidirect product $\mathscr{A}_n\rtimes\mathscr{H}_{n-1}$. This is, in fact, isomorphic to $\mathscr{H}_n$ – see Appendix 1 by L. Gaede in [H]. Hence $\mathscr{K}_n=\langle\mathscr{A}_n,\mathscr{L}_{n-1}\rangle=\mathscr{A}_n\rtimes\mathscr{L}_{n-1}$. The relations giving the action of $\mathscr{L}_{n-1}$ on $\mathscr{A}_n$ are obtained from those giving the action of $\mathscr{H}_{n-1}$ and, writing $y_r=a_{r(r+1)}$, $1\leqq r\leqq n-2$, are as follows:

(1) $$y_r^{-1}x_iy_r=x_i \quad i<r \text{ or } 1<r<i-1<n-2$$

(2) $$y_i^{-1}x_{i+1}y_i=x_ix_{i+1}x_i^{-1} \quad 1\leqq i\leqq n-2$$

(3) $$y_i^{-1}x_iy_i=x_ix_{i+1}x_ix_{i+1}^{-1}x_i^{-1} \quad 1\leqq i\leqq n-2.$$

In the presence of (2), (3) is equivalent to

(3′) $$y_ix_{i+1}y_i^{-1}=x_i^{-1}x_{i+1}x_i \quad 1\leqq i\leqq n-2.$$

It should be noted that the final type of relation on [H, p. 170] does not contribute since the inequality conditions on the subscripts are not consistent with the generators we consider.

The above shows that we obtain a presentation for the group $\mathscr{K}_n$ in terms of the generators $x_1,\ldots,x_{n-1},y_1,\ldots,y_{n-2}$ by writing down relations among $y_1,\ldots,y_{n-2}$, which are assumed to be known inductively, together with the relations (1), (2) and (3′). Sometimes we express (1), (2) and (3′) as

(1) $$[y_r,x_i]=1 \quad \text{where } 1\leqq i<r\leqq n-2 \text{ or } 1<r<i-1\leqq n-2$$

(2) $$[x_{i+1},y_ix_i]=1 \quad 1\leqq i\leqq n-2$$

(3′) $$[x_i,x_{i+1}y_i]=1 \quad 1\leqq i\leqq n-2.$$

We shall show that the full subpresentation determined by the generators $x_{n-1},y_1,\ldots,y_{n-2}$ defines $\mathscr{L}_n$. The idea of our argument is that instead of thinking of the generators of $\mathscr{A}_n$ as lying "underneath" $\mathscr{L}_{n-1}$ waiting to be acted on so as to construct $\mathscr{K}_n$, one can "turn the process upside down" and start with $\mathscr{L}_{n-1}$ as a base group to which the free generators of $\mathscr{A}_n$ can be added one at a time as stable letters in a chain of HNN extensions.

We shall also use the following notation and terminology. If $\mathscr{M}_1$, $\mathscr{M}_2$ are group presentations such that

(i) The generators of $\mathscr{M}_2$ consist of the generators of $\mathscr{M}_1$ and an additional generator, say, x;
(ii) the relations of $\mathscr{M}_2$ are just those of $\mathscr{M}_1$ together with a set of relations (equivalent to) $x^{-1}u_\lambda x=v_\lambda$, $\lambda\in\Lambda$ where u_λ, v_λ are words of $\mathscr{M}_1$;
(iii) the subgroups $\mathscr{N}^\alpha=\langle u_\lambda,\lambda\in\Lambda\rangle$ and $\mathscr{N}^\omega=\langle v_\lambda,\lambda\in\Lambda\rangle$ of $\mathscr{M}_1$ are isomorphic via a map sending $u_\lambda\mapsto v_\lambda$;

then we say $\mathscr{M}_1 \underset{x}{\leqq} \mathscr{M}_2$ is an **HNN-extension** (with stable letter x). We refer to $\mathscr{N}^\alpha$ and $\mathscr{N}^\omega$ as the edge groups. As is well-known the group (defined by) $\mathscr{M}_1$ is embedded in the group (defined by) $\mathscr{M}_2$. Finally let $\mathscr{M}_{i,j}$ denote the full subpresentation of (the presentation obtained for) $\mathscr{K}_n$ on the generators $y_1, \ldots, y_{n-2}, x_i, \ldots, x_j$, where $i \leqq j$.

Proposition. *The following are chains of* HNN-*extensions with the indicated stable letters*:

$$\mathscr{L}_{n-1} \underset{x_i}{\leqq} \mathscr{M}_{i,i} \underset{x_{i+1}}{\leqq} \mathscr{M}_{i,i+1} \underset{x_{i+2}}{\leqq} \cdots \underset{x_j}{\leqq} \mathscr{M}_{i,j}$$

$$\mathscr{L}_{n-1} \underset{x_j}{\leqq} \mathscr{M}_{j,j} \underset{x_{j-1}}{\leqq} \mathscr{M}_{j,j-1} \underset{x_{j-2}}{\leqq} \cdots \underset{x_i}{\leqq} \mathscr{M}_{i,j}.$$

Having established the proposition, the result is essentially immediate. Since the group defined by $\mathscr{M}_{n-1,n-1}$ is embedded in $\mathscr{K}_n = \mathscr{M}_{1,n-1}$ it follows that the required presentation of the subgroup $\mathscr{L}_n$ is just $\mathscr{M}_{n-1,n-1}$; thus (inductively)

$$\mathscr{L}_n = \langle y_1, y_2, \ldots, y_{(n-2)}, x_{n-1} \mid$$
$$[y_r, y_s] = 1 \ (|r-s| \geqq 2), \quad [x_{n-1}, y_s] = 1 \ (n-1-s \geqq 2)\rangle.$$

In terms of the original notation this gives:

Corollary. *The subgroup* $\langle \sigma_1^2, \sigma_2^2, \ldots, \sigma_{n-1}^2 \rangle$ *of the braid group*

$$\mathscr{B}_n = \langle \sigma_1, \sigma_2, \ldots, \sigma_{n-1} \colon \sigma_i\sigma_j = \sigma_j\sigma_i \ (|i-j| \geqq 2,$$
$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \quad (1 \leqq i \leqq n-2)\rangle$$

has presentation

$$\langle \sigma_1^2, \sigma_2^2, \ldots, \sigma_{n-1}^2 \mid [\sigma_r^2, \sigma_s^2] = 1 \quad (|r-s| \geqq 2).$$

It remains only to give the proof of the proposition.

Proof. We proceed by induction on $j-i$, proving both statements at once. If $j=i$ then we have $\mathscr{M}_{i,i}$ with base group $\mathscr{L}_{n-1}$ and the identity map between the coinciding edge groups so the result is immediate.

Now consider $\mathscr{M}_{i,j-1} \underset{x_j}{\leqq} \mathscr{M}_{i,j}$ where $i<j$. The argument requires a slight variation in the case when $j=n-1$ and so for the moment we shall assume that $j \neq n-1$. The map $\varphi_{i,j}$ between the edge groups is given by

$$y_r \mapsto y_r, \quad (r = 1, \ldots, j-2, j+1, \ldots, n-2),$$
$$y_{j-1}x_{j-1} \mapsto y_{j-1}x_{j-1}, \ x_{j-1} \mapsto y_{j-1}x_{j-1}y_{j-1}^{-1}.$$

In fact the edge groups coincide and we have to check that the map induces a well-defined automorphism of this common edge group

$$\mathscr{N}_{i,j} = \langle y_1, \ldots, y_{j-2}, y_{j+1}, \ldots, y_{n-2}, y_{j-1}, x_{j-1} \rangle.$$

In terms of the generators just displayed $\varphi_{i,j}$ is given by

$$y_r \mapsto y_r, \quad (r=1,\dots,j-2,j+1,\dots,n-2),$$
$$y_{j-1} \mapsto y_{j-1}x_{j-1}y_{j-1}x_{j-1}^{-1}y_{j-1}^{-1}, \; x_{j-1} \mapsto y_{j-1}x_{j-1}y_{j-1}^{-1}$$

and we shall show that $\varphi_{i,j}$ preserves all relations among these displayed generators. To do so we shall obtain a presentation of $\mathcal{N}_{i,j}$. By induction,

$$\mathcal{L}_{n-1} \underset{x_{j-1}}{\leqq} \mathcal{M}_{j-1,j-1} \underset{x_{j-2}}{\leqq} \mathcal{M}_{j-2,j-1} \underset{x_{j-3}}{\leqq} \cdots \underset{x_i}{\leqq} \mathcal{M}_{i,j-1}$$

is a chain of HNN-extensions. Thus

$$\begin{aligned}\mathcal{N}_{i,j} \leqq \mathcal{M}_{j-1,j-1} &= \langle \mathcal{L}_{n-1}, x_{j-1} \mid [x_{j-1}, y_r]=1, \quad (r=1,\dots,j-3,j,\dots,n-2)\rangle \\ &= \langle y_r, 1 \leqq r \leqq n-2, x_{j-1} \mid [y_r, y_s]=1 \quad (|r-s| \geqq 2) \\ &\qquad [x_{j-1}, y_r]=1, \quad (r=1,\dots,j-3,j,\dots,n-2)\rangle.\end{aligned}$$

From the relations which involve y_j it is immediate that $\mathcal{N}_{i,j-1} \underset{y_j}{\leqq} \mathcal{M}_{j-1,j-1}$ is an HNN-extension with the identity map between the edge groups. It follows therefore that $\mathcal{N}_{i,j}$ has defining relations

$$[y_r, y_s]=1 \quad (|r-s| \geqq 2,\ r, s \neq j),\ [x_{j-1}, y_r]=1 \quad (r \neq j-2, j-1, j).$$

To check that $\varphi_{i,j}$ preserves these relations, the only non-trivial cases to be considered are those which involve y_{j-1} and x_{j-1}. However inspection shows that in each of the two cases, the defining relations concerned specify that the generator in question commutes with

$$y_1, \dots, y_{j-3}, y_{j+1}, \dots, y_{n-2}$$

Since the images of y_{j-1} and x_{j-1} under $\varphi_{i,j}$ only involve y_{j-1} and x_{j-1} the desired conclusion is immediate.

The above shows that $\varphi_{i,j}$ induces an endomorphism of $\mathcal{N}_{i,j}$. Similarly the map $\psi_{i,j}$ given by

$$y_r \mapsto y_r, \quad (r=1,\dots,j-2,j+1,\dots,n-2),$$
$$y_{j-1} \mapsto x_{j-1}^{-1}y_{j-1}x_{j-1}, \quad x_{j-1} \mapsto x_{j-1}^{-1}y_{j-1}^{-1}x_{j-1}y_{j-1}x_{j-1}$$

induces a well-defined endomorphism of $\mathcal{N}_{i,j}$ which is easily seen to be inverse to $\varphi_{i,j}$.

We now turn to the extremal case, i.e. $j=n-1$. The difference between this and the previous case is that there is no generator y_i and so the edge group $\mathcal{N}_{i,j}$ coincides with the group $\mathcal{M}_{j-1,j-1} = \mathcal{M}_{n-2,n-2}$. But again the defining relations of $\mathcal{N}_{i,j}$ which involve y_{j-1} and x_{j-1} say that these commute with exactly the same set of generators and the argument proceeds as before. (It should be noted that when $j=n-2$, the general argument applies, even although there are no generators y_r, $r \geqq j+1$; the same goes for the case $j=2$ when there are no generators y_r, $r \leqq j-2$.

We also have to establish that $\mathcal{M}_{i+1,j} \underset{x_i}{\leqq} \mathcal{M}_{i,j}$ is an HNN-extension. The argument required is the dual of that given above. □

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