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Stable pairs, linear systems and the Verlinde formula

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0 Introduction

Let X be a smooth projective complex curve of genus $g \geqq 2$, let $\Lambda \to X$ be a line bundle of degree $d > 0$, and let (E, ϕ) be a pair consisting of a vector bundle $E \to X$ such that $\Lambda^2 E = \Lambda$ and a section $\phi \in H^0(E) - 0$. This paper will study the moduli theory of such pairs. However, it is by no means a routine generalization of the well-known theory of stable bundles. Rather, it will discuss at least three remarkable features of the moduli spaces of pairs:

1. Unlike bundles on curves, pairs admit many possible stability conditions. In fact, stability of a pair depends on an auxiliary parameter σ analogous to the weights of a parabolic bundle. This parameter was first detected by Bradlow [5] in the study of vortices on Riemann surfaces, and indeed the spaces we shall construct can also be interpreted as moduli spaces of rank 2 vortices. As σ varies, we will see that the moduli space undergoes a sequence of flips in the sense of Mori theory, whose locations can be specified quite precisely.
2. For some values of σ the moduli space $M(\sigma, \Lambda)$ is the blow-up of $\mathbb{P}H^1(\Lambda^{-1})$ along X, embedded as a complete linear system. Thus we can use $M(\sigma, \Lambda)$ to study the projective embeddings of X. In particular, we obtain a very general formula (7.8) for the dimension of the space of hypersurfaces of degree $m + n$ in $\mathbb{P}H^1(\Lambda^{-1})$ with a singularity at X of order n. This formula does not depend on the precise choice of X and Λ, only on g and d, which is rather surprising.
3. For other values of σ, stability of the pair implies semistability of the bundle, so $M(\sigma, \Lambda)$ plays the role in rank 2 Brill-Noether theory of the symmetric product in the usual case, and there is an Abel-Jacobi map from $M(\sigma, \Lambda)$ to the moduli space of semistable bundles. For large d this is generically a fibration, so we can use moduli spaces of pairs to study moduli spaces of bundles. In particular, we recover the known formulas for Poincaré polynomials [2, 14] and Picard groups [9]; more strikingly, we prove, and generalize, the rank 2 Verlinde formula (7.10) for both odd and even degrees.

We will not fully discuss the many other fascinating aspects of the subject, but we will briefly touch on one of them – the relation with Cremona transformations and Bertram's work on secant varieties – in an appendix, §8. We hope to treat the relation with vortices and Yang-Mills-Higgs theory in a later paper.

An outline of the other sections is as follows. In §1 we prove some basic facts about pairs, in analogy with bundles. Following Gieseker [11], we then use geometric invariant theory to construct the moduli space $M(\sigma, \Lambda)$ of σ-semistable pairs, and a universal family over the stable points of $M(\sigma, \Lambda)$. The choice of σ corresponds to a choice of linearization for our group action. In §2 we discuss the deformation theory of the moduli problem. In §3 we show that the $M(\sigma, \Lambda)$ are reduced, rational, and smooth at the stable points. We then show that as σ varies, $M(\sigma, \Lambda)$ undergoes a sequence of flips whose centres are symmetric products of X. We also define the rank 2 Abel-Jacobi map mentioned above. In §4 we calculate the Poincaré polynomial of (σ, Λ), and extract from it the Harder-Narasimhan formula for the Poincaré polynomial of the moduli space of rank 2 bundles of odd degree.

Thereafter we concentrate on studying the line bundles over $M(\sigma, \Lambda)$, and their spaces of sections. In §5 we compute the Picard group of $M(\sigma, \Lambda)$, and its ample cone. We explain how any section of a line bundle on $M(\sigma, \Lambda)$ can be interpreted as a hypersurface in projective space, singular to some order on an embedded X. We also make the connection with the Verlinde vector spaces. Finally in §§6 and 7 we use the Riemann-Roch theorem to calculate Euler characteristics of the line bundles on $M(\sigma, \Lambda)$. Combined with the information from §5, Kodaira vanishing, and some residue calculations which were carried out by Don Zagier, this gives a formula for the dimensions of the spaces of sections of line bundles on $M(\sigma, \Lambda)$, under some mild hypotheses. We conclude by extracting the Verlinde formula from this.

For convenience we work over the complex numbers, but much of the paper should be valid over any algebraically closed field: certainly §§1–3 and 5. Kodaira vanishing is of course crucial in §6, but the computation of the Euler characteristics ought to make sense in general, if integral cohomology is replaced with intersection theory.

A few notational habits should be mentioned: X_i refers to the ith symmetric product of X; π denotes any obvious projection, such as projection on one factor, or down from a blow-up; tensor products of vector bundles are frequently indicated simply by juxtaposition; and likewise a pullback such as f^*L is often called just L. Also, in §3 and thereafter, $M(\sigma, \Lambda)$ is referred to simply as M_i, where i depends on σ in a manner explained in §3. These conventions are not meant to be elliptical, but to clean up what would otherwise be some very messy formulas.

We also make the following assumptions, which are explained in the text but are repeated here for emphasis. We always assume $g \geqq 2$. In the geometric invariant theory construction of §1, we assume d is large, an assumption which is justified by (1.9) and the discussion following it. From §3 to the end we assume $d \geqq 3$. However, this assumption is implicit in other inequalities—so for example our main formula (7.8) is valid as it stands.

1 Constructing moduli spaces of σ-semistable pairs

Our main objects of study, which we refer to simply as *pairs*, will be pairs (E, ϕ) consisting of a rank 2 algebraic vector bundle E over our curve X, and a nonzero section $\phi \in H^0(E)$. A careful study of such pairs was made by Bradlow [5]. He defined a stability condition for pairs and proved a Narasimhan-Seshadri-type theorem relating stable pairs to vortices on a Riemann surface. The vortex equations depend on a positive real parameter τ, and so the stability condition also depends on τ. Bradlow and Daskalopoulos went on [6] to give a gauge-theoretic construction of the moduli space of τ-stable pairs, under certain conditions on τ and $\deg E$. Garcia-Prada later showed [10] that there always exists a projective moduli space, by realizing it as a subvariety of a moduli space of stable bundles on $X \times \mathbb{P}^1$. In this section we will give a geometric invariant theory construction of the moduli space of τ-stable pairs for arbitrary τ and $\deg E$ (though for convenience we assume rank $E = 2$). Aaron Bertram has informed me that he has done something similar [4], and I apologize to him for any overlap.

The Bradlow-Daskalopoulos stability condition is in general rather complicated, but in the rank 2 case it simplifies to the following. Let σ be a positive rational number. It is related to τ by $\sigma = \tau \operatorname{vol} X/4\pi - \deg E/2$, where vol X is the volume of X with respect to the metric chosen in [6].

(1.1) Definition. *The pair (E, ϕ) is* **σ-semistable** *if for all line bundles $L \subset E$,*

$$\deg L \leqq \tfrac{1}{2} \deg E - \sigma \text{ if } \phi \in H^0(L) \quad \text{and}$$

$$\deg L \leqq \tfrac{1}{2} \deg E + \sigma \text{ if } \phi \notin H^0(L).$$

It is **σ-stable** *if both inequalities are strict.*

The main result of this section is then the following.

(1.2) *Let $\Lambda \to X$ be a line bundle of degree d. There is a projective moduli space $M(\sigma, \Lambda)$ of σ-semistable pairs (E, ϕ) such that $\Lambda^2 E = \Lambda$, nonempty if and only if $\sigma \leqq d/2$.*

Our construction will be modelled on that of Gieseker [11]. We begin with a few basic facts about σ-stable and semistable pairs, parallel to those for bundles. We write Λ for $\Lambda^2 E$, and d for $\deg E = \deg \Lambda$.

(1.3) *For $\sigma > 0$, there exists a σ-semistable pair of determinant Λ if and only if $\sigma \leqq d/2$.*

Proof. If $\sigma > d/2$, then σ-semistability implies $\deg L < 0$ if $\phi \in H^0(L)$, which is absurd. If $\sigma \leqq d/2$, let $L \to X$ be a line bundle of degree $[d/2 - \sigma]$ having a nonzero section ϕ. Let E be a nonsplit extension

$$0 \to L \to E \to \Lambda L^{-1} \to 0.$$

Then the first inequality in Definition (1.1) is obvious. As for the second, if $M \subset E$ and $\deg M > d/2 + \sigma$, then there is a nonzero map $M \to \Lambda L^{-1}$. Since $\deg \Lambda L^{-1} < d/2 + \sigma + 1$, this is an isomorphism, so the extension is split, which is a contradiction. □

(1.4) *Let (E, ϕ) be a pair. There is at most one σ-destabilizing bundle $L \subset E$ such that $\phi \in H^0(L)$, and at most one σ-destabilizing $M \subset E$ such that $\phi \notin H^0(M)$. If both L and M exist, then $E = L \oplus M$.*

Proof. The first statement is obvious, and the second follows from the uniqueness of ordinary destabilizing bundles, since $\deg M \geqq d/2 + \sigma > d/2$. If both L and

M exist, then the map $M \to E \to \Lambda L^{-1}$ is nonzero since $\phi \in H^0(L)$ but $\notin H^0(M)$. But $\deg M \geq d/2 + \sigma \geqq \deg \Lambda L^{-1}$, so $M = \Lambda L^{-1}$ and E is split. □

(1.5) *Let* (E_1, ϕ_1) *and* (E_2, ϕ_2) *be σ-stable pairs of degree d, and let* $\psi: E_1 \to E_2$ *be a map such that* $\psi\phi_1 = \phi_2$. *Then* ψ *is an isomorphism.*

Proof. The kernel of ψ is a subsheaf of a locally free sheaf on a smooth curve, so it is locally free. If rank $\ker\psi = 2$, then ψ is generically zero, so $\psi = 0$ and $\psi\phi_1 \neq \phi_2$. If rank $\ker\psi = 1$, then $\ker\psi$ is a line subbundle L of E_1, since $E_1/\ker\psi$ is contained in the torsion-free sheaf E_2. Hence ψ descends to a map $\Lambda L^{-1} \to E_2$ (possibly with zeroes) such that $\phi_2 \in H^0(\Lambda L^{-1})$. Since (E_2, ϕ_2) is σ-stable, $\deg \Lambda L^{-1} < d/2 - \sigma$, so $\deg L > d/2 + \sigma$, contradicting the σ-stability of (E_1, ϕ_1). Finally, if rank $\ker\psi = 0$, then $\ker\psi = 0$ and ψ is injective. Moreover, coker ψ is a coherent sheaf on a curve with rank and degree 0, so coker $\psi = 0$ and ψ is an isomorphism. □

(1.6) *Let* (E, ϕ) *be a σ-stable pair. Then there are no endomorphisms of E annihilating* ϕ *except* 0, *and no endomorphisms preserving* ϕ *except the identity.*

Proof. Subtracting from the identity interchanges the two statements, so they are equivalent. We prove the first. Any endomorphism annihilating ϕ annihilates the subbundle L generated by ϕ, so descends to a map $E/L \to E$. But by σ-stability E/L is a line bundle of degree $\geqq d/2 + \sigma$, so the image of this map, if it were nonzero, would generate a line bundle of degree $\geqq d/2 + \sigma$, which would be destabilizing. □

(1.7) *Let* $(\mathbf{E}, \mathbf{\Phi})$, $(\mathbf{E}', \mathbf{\Phi}') \to T \times X$ *be two families over T parametrizing the same pairs. Then* $(\mathbf{E}, \mathbf{\Phi}) = (\mathbf{E}', \mathbf{\Phi}')$.

Proof. For any $t \in T$, the subspace of $H^0(X; \mathrm{Hom}(\mathbf{E}_t, \mathbf{E}'_t))$ consisting of homomorphisms ψ such that $\psi\mathbf{\Phi}_t = \lambda\mathbf{\Phi}'_t$ for some $\lambda \in \mathbb{C}$ is one-dimensional by (1.6). This determines an invertible subsheaf of the direct image $(R^0\pi)\,\mathrm{Hom}(\mathbf{E}_t, \mathbf{E}'_t)$. But this subsheaf is trivialized by the section $\lambda = 1$, which produces the required isomorphism. □

The notion of a Harder-Narasimhan filtration for rank 2 pairs is quite a simple one. For (E, ϕ) stable, define $\mathrm{Gr}(E, \phi) = (E, \phi)$. Otherwise, define $\mathrm{Gr}(E, \phi)$ to be a direct sum of line bundles, one of them containing the section ϕ, as follows. If L is the destabilizing bundle and $\phi \in H^0(L)$, define $\mathrm{Gr}(E, \phi) = (L \oplus \Lambda L^{-1}, \phi)$. If M is the destabilizing bundle and $\phi \notin H^0(M)$, project ϕ to a nonzero section $\phi' \in H^0(\Lambda M^{-1})$ and define $\mathrm{Gr}(E, \phi) = (M \oplus \Lambda M^{-1}, \phi')$. Note that if there are destabilizing bundles of both sorts, then by (1.4) $E = L \oplus \Lambda L^{-1}$ and the two definitions agree.

(1.8) *There exists a degeneration of* (E, ϕ) *to* $\mathrm{Gr}(E, \phi)$, *but* $\mathrm{Gr}(E, \phi)$ *degenerates to no semistable pair.*

Proof. The first statement is vacuous when (E, ϕ) is stable. If it is unstable, say with destabilizing bundle M, we can construct a pair $(\mathbf{E}, \mathbf{\Phi}) \to X \times \mathbb{C}$ such that $(\mathbf{E}_z, \mathbf{\Phi}_z) \cong (E, \phi)$ for $z \neq 0$, but $(\mathbf{E}_0, \mathbf{\Phi}_0) \cong \mathrm{Gr}\,(E, \phi)$, as follows. Pull back (E, ϕ) to

$X \times \mathbb{C}$, and tensor by $\mathcal{O}(0)$ when $\phi \notin H^0(M)$. This gives a pair $(\mathbf{E}', \mathbf{\Phi}') \to X \times \mathbb{C}$ such that $\mathbf{\Phi}'$ is annihilated by the natural map $\mathbf{E}' \to \Lambda M^{-1}|_{X \times \{0\}}$. Let $\mathbf{E}$ be the kernel of this map; then $\mathbf{\Phi}'$ descends to $\mathbf{\Phi} \in H^0(\mathbf{E})$, and it is straightforward to check that $(\mathbf{E}, \mathbf{\Phi})$ has the desired properties.

As for the second statement, suppose first that (E, ϕ) is stable. If C is a curve, $p \in C$, and $(\mathbf{E}, \mathbf{\Phi}) \to X \times C$ is a flat family of pairs such that $(\mathbf{E}_z, \mathbf{\Phi}_z) \cong (E, \phi)$ for $z \neq p$, then $\mathbf{\Phi}_p$ has the same zero-set D as ϕ, so E and $\mathbf{E}_p$ are both extensions of $L = \mathcal{O}(D)$ by $\Lambda(-D)$; indeed, $\mathbf{E}$ is a family of such extensions. The extension class varies continuously, so the extension class of $\mathbf{E}_p$ is in the same ray as that of E. If it is nonzero, $(E, \phi) \cong (\mathbf{E}_p, \mathbf{\Phi}_p)$, and if it is zero, $(\mathbf{E}_p, \mathbf{\Phi}_p)$ is destabilized by ΛL^{-1}.

Now suppose that (E, ϕ) is not stable, so that for some L, $\mathrm{Gr}(E, \phi) = L \oplus \Lambda L^{-1}$ and $\phi \in H^0(L)$. Then as above $\mathbf{E}_p$ is an extension of L by ΛL^{-1}, but now by continuity the extension class must be zero, so $\mathrm{Gr}(E, \phi) = (\mathbf{E}_p, \mathbf{\Phi}_p)$. ☐

(1.9) *If (E, ϕ) is σ-(semi)stable, then so is $(E(D), \phi(D))$ for any effective divisor D. Likewise, if ϕ vanishes on an effective divisor D and (E, ϕ) is σ-(semi)stable, then so is $(E(-D), \phi(-D))$.*

Proof. If $L \subset E$ is any line bundle, $\phi(D) \in H^0(L(D))$ if and only if $\phi \in H^0(L)$, and $\deg L(D) = \deg L + \deg D$. But $\frac{1}{2} \deg E(D) = \frac{1}{2} \deg E + \deg D$ also, so both inequalities are preserved by tensoring with D. The second statement is proved similarly. ☐

Hence if the moduli spaces $M(\sigma, \Lambda)$ exist for large enough d, then the moduli spaces for smaller d will be contained inside them as the locus of pairs (E, ϕ) such that ϕ vanishes on some effective D. So to prove our existence theorem (1.2) it suffices to construct $M(\sigma, \Lambda)$ for d large relative to g and σ, and *we will assume for the remainder of* §1 *that d is large in this sense.* For such a large d, we then have the following useful fact.

(1.10) *For fixed g and σ and large d, (E, ϕ) σ-semistable implies that $H^1(E) = 0$ and E is globally generated.*

Proof. Suppose that $H^1(E) \neq 0$. Then $H^0(KE^*) \neq 0$, so there is an injection $0 \to K^{-1}(D) \to E^*$ for some effective D. Hence there is an injection $0 \to K^{-1}\Lambda(D) \to E$. Since $\deg K^{-1}\Lambda(D) \geqq 2 - 2g + d$, the σ-semistability condition implies that $2 - 2g + d \leqq d/2 + \sigma$, so that $d \leqq 4g - 4 + 2\sigma$. So for d larger than this, $H^1(E) = 0$.

Similarly, if $d > 4g - 2 + 2\sigma$, then $H^1(E(-x)) = 0$ for all $x \in X$, so E is globally generated. ☐

Since we are assuming that d is large, the above lemma implies that for (E, ϕ) σ-stable, $\dim H^0(E) = \chi(E) = d + 2 - 2g$. Call this number χ. If we fix an isomorphism $s: \mathbb{C}^\chi \to H^0(E)$, we obtain a map $\Lambda^2 \mathbb{C}^\chi \xrightarrow{s} \Lambda^2 H^0(E) \xrightarrow{\wedge} H^0(\Lambda)$, which is nonzero because E is globally generated. Thus to any bundle E appearing in a σ-semistable pair, and any isomorphism s, we associate a point $T(E, s) \in \mathbb{P}\mathrm{Hom}(\Lambda^2 \mathbb{C}^\chi, H^0(\Lambda))$. We will consider the pair $(T(E, s), s^{-1}\phi) \in \mathbb{P}\mathrm{Hom} \times \mathbb{P}\mathbb{C}^\chi$, where $\mathbb{P}\mathrm{Hom}$ is short for $\mathbb{P}\mathrm{Hom}(\Lambda^2 \mathbb{C}^\chi, H^0(\Lambda))$. Roughly speaking, $M(\sigma, \Lambda)$ will be a geometric invariant theory quotient of the set of such pairs. The quotient is necessary to remove the dependence on the choice of s. Since two such isomorphisms are related by an element of $\mathrm{SL}(\chi)$, the group action will be the obvious diagonal action of $\mathrm{SL}(\chi)$ on $\mathbb{P}\,\mathrm{Hom} \times \mathbb{P}\mathbb{C}^\chi$. As usual in geometric invariant theory, we must *linearize* the action by choosing an ample line bundle and lifting the action of $\mathrm{SL}(\chi)$ to its dual. So let the ample bundle be any power of $\mathcal{O}(\chi + 2\sigma, 4\sigma)$, with the obvious lifting. (Of course $\chi + 2\sigma$ and 4σ may not be integers, but by abuse of notation we will refrain from clearing denominators, since the choice of power does not matter.) We can then define stable and semistable points in the sense of geometric invariant theory with respect to this linearization.

(1.11) *If (E, ϕ) is σ-(semi)stable, then $(T(E, s), s^{-1}\phi)$ is a (semi)stable point with respect to the linearization above.*

Proof. Suppose $T = (T(E, s), s^{-1}\phi)$ is not semistable. Then by Mumford's numerical criterion [19, 21] there exists a nontrivial 1-parameter subgroup $\lambda : \mathbb{C}^\times \to \mathrm{SL}(\chi)$ such that for any $\tilde{T}$ in the fibre of the dual of our ample bundle over T, $\lim_{t\to 0} \lambda(t) \cdot \tilde{T} = 0$. We interpret this limit concretely as follows. Any 1-parameter subgroup of $\mathrm{SL}(\chi)$ can be diagonalized, so there exists a basis e_i of $\mathbb{C}^\chi$ such that $\lambda(t) \cdot e_i = t^{r_i} e_i$, where $r_i \in \mathbb{Z}$ are not all zero and satisfy $\sum_i r_i = 0$ and $r_i \leqq r_j$ for $i \leqq j$. Then $\lim_{t\to 0} \lambda(t) \cdot \tilde{T} = 0$ means that any basis element $(e_i^* \wedge e_j^* \otimes v, e_k) \in \mathrm{Hom}(\Lambda^2 \mathbb{C}^\chi, H^0(\Lambda)) \oplus \mathbb{C}^\chi$ which is acted on with weight $\leqq 0$ has coefficient zero in the basis expansion of $\tilde{T}$. Because of our choice of linearization, this means that $T(E, s)(e_i, e_j) = 0$ whenever

(1.12)
$$r_i + r_j \leqq \frac{2\sigma}{\chi/2 + \sigma} r_\ell,$$

where $\ell = \max\{i : \text{coefficient of } e_i \text{ in } s^{-1}\phi \text{ is} \neq 0\}$. Let $L \subset E$ be the line bundle generated by $s(e_1)$. We distinguish between two cases, according to whether $\phi \in H^0(L)$.

First case. $\phi \in H^0(L)$. For $i \leqq \chi/2 - \sigma + 1$, note that

$$(\chi/2 - \sigma) r_1 + (\chi/2 + \sigma) r_i \leqq \sum_i r_i = 0,$$

since the left-hand side can be regarded as the integral over $[0, \chi)$ of a (two-step) step function whose value on $[j - 1, j)$ is $\leqq r_j$. Hence for $i \leqq \chi/2 - \sigma + 1$,

$$r_1 + r_i \leqq \frac{2\sigma}{\chi/2 + \sigma} r_1 \leqq \frac{2\sigma}{\chi/2 + \sigma} r_\ell,$$

so $T(E, s)(e_1, e_i) = s(e_1) \wedge s(e_i) = 0$. Hence $s(e_i)$ is a section of the same line bundle as $s(e_1)$, namely L. So $\dim H^0(L) > \chi/2 - \sigma$; since d is large relative to g and σ, this implies that $\deg L > d/2 - \sigma$, so (E, ϕ) is not σ-semistable.

Second case. $\phi \notin H^0(L)$. For $i \leqq \chi/2 + \sigma + 1$,

$$(\chi/2 + \sigma) r_1 + (\chi/2 - \sigma) r_i \leqq 0,$$

for the same reason as above. Hence

$$r_1 + r_i \leqq \frac{2\sigma}{\chi/2 + \sigma} r_i.$$

We claim that $\ell > \chi/2 + \sigma + 1$. If not, then for all $i \leqq \ell$,

$$r_1 + r_i \leqq \frac{2\sigma}{\chi/2 + \sigma} r_\ell,$$

so that $s(e_i)$ would be in the same line bundle as $s(e_1)$. Since ϕ is a linear combination of e_i for $i \leqq \ell$, we would conclude $\phi \in H^0(L)$, a contradiction. This proves the claim.

So for $i \leqq \chi/2 + \sigma + 1$, actually

$$r_1 + r_i \leqq \frac{2\sigma}{\chi/2 + \sigma} r_\ell;$$

hence $s(e_i) \in H^0(L)$ as in the first case. So $\dim H^0(L) > \chi/2 + \sigma$, and again (E, ϕ) is not σ-semistable.

The proof for stability is similar: the numerical criterion now just says $\lim_{t \to 0} \lambda(t) \cdot \tilde{T} \neq \infty$, so we replace the $\leqq$ in (1.12) by $<$. We just need to note that if $i < \chi/2 - \sigma + 1$, then

$$(\chi/2 - \sigma) r_1 + (\chi/2 + \sigma) r_i < 0$$

strictly, because either the two step functions are different just to the left of $\chi/2 - \sigma$, or the smaller one is identically $r_1 < 0$. □

(1.13) *Let (E, ϕ) be a pair, let $s : \mathbb{C}^\chi \to H^0(E)$ be a linear map, and let $v \in \mathbb{C}^\chi$ satisfy $s(v) = \phi$. Write T_s for the composition $\Lambda^2 \mathbb{C}^\chi \xrightarrow{s} \Lambda^2 H^0(E) \xrightarrow{\wedge} H^0(\Lambda)$. If (T_s, v) is semistable, then s is an isomorphism and (E, ϕ) is σ-semistable.*

Proof. First of all, note that if s is not injective, then (T_s, v) is certainly not semistable. Indeed, if $s(w) = 0$ for some w, put $e_1 = w$, $e_2 = v$, extend to a basis $\{e_i\}$ of $\mathbb{C}^\chi$, and then take the 1-parameter subgroup defined by $r_1 = -\chi + 2$, $r_2 = 0$, $r_3 = \cdots = r_\chi = 1$. Then $\ell = 2$, so

$$r_i + r_j \leqq \frac{2\sigma}{\chi/2 + \sigma} r_\ell$$

means just $r_i + r_j \leqq 0$. Hence either $i = 1$, or $j = 1$, or $i = j = 2$; in any case, clearly $T_s(e_i, e_j) = 0$.

Suppose then that s is injective and (E, ϕ) is σ-unstable. We will prove (T_s, v) is unstable. Let $L \subset E$ be the destabilizing bundle. We distinguish two cases, depending on the sign of $d - \deg L - 2g + 2$.

First case. $d - \deg L > 2g - 2$. Then $H^1(\Lambda L^{-1}) = 0$, but $H^1(L) = 0$ also since $\deg L > d/2 - \sigma$ which is large relative to g. Hence from the long exact sequence of

$$0 \to L \to E \to \Lambda L^{-1} \to 0 \tag{1.14}$$

we find that $H^1(E) = 0$, so $\dim H^0(E) = \chi$ and s is an isomorphism. Choose a basis $e_1, \ldots, e_p$ for $s^{-1}(H^0(L))$ and extend to a basis $e_1, \ldots, e_\chi$ for $\mathbb{C}^\chi$. Take the 1-parameter subgroup defined by $r_i = p - \chi$ for $i \leqq p$, p for $i > p$. Then $r_\ell = p - \chi$ if $\phi \in H^0(L)$, p if $\phi \notin H^0(L)$. Since L is destabilizing, $p > \chi/2 - \sigma$ if $\phi \in H^0(L)$,

$p > \chi/2 + \sigma$ if $\phi \notin H^0(L)$. Either way,

$$r_i + r_j \leqq \frac{2\sigma}{\chi/2 + \sigma} r_\ell$$

implies $i, j \leqq p$; if $\phi \in H^0(L)$, and say $i > p$, then

$$r_i + r_j - \frac{2\sigma}{\chi/2 + \sigma} r_\ell \geqq p + (p - \chi)\left(1 - \frac{2\sigma}{\chi/2 + \sigma}\right) = p \frac{\chi}{\chi/2 + \sigma} - \chi \frac{\chi/2 - \sigma}{\chi/2 + \sigma}$$

$$> (\chi/2 - \sigma)\frac{\chi}{\chi/2 + \sigma} - \chi \frac{\chi/2 - \sigma}{\chi/2 + \sigma} = 0,$$

whereas if $\phi \notin H^0(L)$, and say $j > p$, then

$$r_i + r_j - \frac{2\sigma}{\chi/2 + \sigma} r_\ell \geqq p - \chi + p\left(1 - \frac{2\sigma}{\chi/2 + \sigma}\right) = p \frac{\chi}{\chi/2 + \sigma} - \chi > \chi - \chi = 0.$$

But if $i, j \leqq p$, then $s(e_i), s(e_j) \in H^0(L)$, so $T_s(e_i, e_j) = 0$. Hence (T_s, v) is unstable.

Second case. $d - \deg L \leqq 2g - 2$. Then $\dim H^0(\Lambda L^{-1}) \leqq g$, so from the long exact sequence of (1.14) we deduce that the codimension of $H^0(L)$ in $H^0(E)$ is $\leqq g$. Hence the codimension of $s^{-1}(H^0(L))$ in $\mathbb{C}^\chi$ is $\leqq g$. Choose a basis $e_1, \ldots, e_p$ for $s^{-1}(H^0(L))$ and extend to a basis $e_1, \ldots, e_\chi$ for $\mathbb{C}^\chi$. Take the 1-parameter subgroup defined by $r_i = p - \chi$ for $i \leqq p$, p for $i > p$. Since $p \geqq \chi - g$ and $\chi = d + 2 - 2g$ is large relative to σ and g, certainly $p > \chi/2 + \sigma$. The remainder of the proof proceeds as in the first case.

So far we have proved that if (T_s, v) is semistable, then s is injective and (E, ϕ) is σ-semistable. But then by (1.10), $\dim H^0(E) = \chi$, so s is an isomorphism. □

(1.15) *Suppose (E_1, ϕ_1) and (E_2, ϕ_2) are σ-semistable, and there exist s_1, s_2 such that $(T(E_1, s_1), s_1^{-1}\phi_1) = (T(E_2, s_2), s_2^{-1}\phi_2)$. Then there is an isomorphism $(E_1, \phi_1) \cong (E_2, \phi_2)$ under which $s_1 \cong s_2$.*

Proof. By (1.10) each E_i is globally generated, so the components $s_i(e_j) \wedge s_i(e_k)$ of $T(E_i, s_i)$ give a map from X to the Grassmannian of $(\chi - 2)$-planes in $\mathbb{C}^\chi$ such that E_i is the pullback of the tautological rank 2 bundle, ϕ_i is the pullback of the section defined by $s_i^{-1}(\phi_i)$, and s_i is the natural map from $\mathbb{C}^\chi$ to the space of sections of the tautological bundle. So we can recover (E_i, ϕ_i) and s_i, up to isomorphism, from $(T(E_i, s_i), s_i^{-1}\phi_i)$. □

(1.16) *Let C be a smooth affine curve and $p \in C$. Let $(\mathbf{E}, \mathbf{\Phi})$ be a locally free family of pairs on $X \times C - \{p\}$, and suppose $\mathbf{E}$ is generated by finitely many sections s_i. Then after possibly rescaling $\mathbf{\Phi}$ by a function on $C - \{p\}$, $(\mathbf{E}, \mathbf{\Phi})$ and the s_i extend over p so that $\mathbf{E}$ is still locally free, $\mathbf{\Phi}_p \neq 0$, and the s_i generate $\mathbf{E}_p$ at the generic point.*

The reason for proving the last fact is to ensure that $T(E, s)$ is nonzero at p, so defines an element of $\mathbb{P}\,\mathrm{Hom}$.

Proof. Choose an ample line bundle L on $X \times C - \{p\}$ such that $\mathbf{E}^* \otimes L$ is globally generated. Then $\mathbf{E}$ embeds in a direct sum of copies of L, and $\oplus_j L$ can be extended over p as a sum of line bundles in such a way that the s_i extend too.

Consider the subsheaf of the extended $\oplus_j L$ generated by the s_i. This is a subsheaf of a locally free sheaf, so it is torsion-free, and hence [22] has singular set S of codimension $\geqq 2$. Furthermore, it injects into its double dual, whose singular set has codimension $\geqq 3$ [22], hence is empty. Hence the double dual is a locally free extension of $\mathbf{E}$ over p, and is generated by s_i away from S. As for $\mathbf{\Phi}$, it certainly extends with a possible pole at p, so it is just necessary to multiply it by a function on C vanishing to some order at p. □

We can finally proceed to construct the geometric invariant theory quotient. Consider the Grothendieck Quot scheme [13] parametrizing flat quotients of $\mathcal{O}_X^\chi$ with degree d, let $\mathrm{Quot}(\Lambda) \subset \mathrm{Quot}$ be the locally closed subset consisting of locally free quotients E with $\Lambda^2 E = \Lambda$, and let $U \subset \mathrm{Quot}(\Lambda)$ be the open set where the quotient induces an isomorphism $s: \mathbb{C}^\chi \to H^0(E)$. Then the pair E, s specifies a point in U. By (1.10), if (E_p, ϕ) is σ-semistable for any section ϕ, then $p \in U$.

Now U is acted upon by $\mathrm{SL}(\chi)$ in the obvious way, and the map

$$T \times 1: U \times \mathbb{P}\mathbb{C}^\chi \to \mathbb{P}\,\mathrm{Hom} \times \mathbb{P}\mathbb{C}^\chi$$

intertwines the group actions on the two sets. By (1.11) and (1.13), the σ-semistable set $V(\sigma) \subset U \times \mathbb{P}\mathbb{C}^\chi$ is the inverse image of the semistable set $V'(\sigma) \subset \mathbb{P}\,\mathrm{Hom} \times \mathbb{P}\mathbb{C}^\chi$ with respect to the linearization $\mathcal{O}(\chi + 2\sigma, 4\sigma)$. In future, we restrict $T \times 1$ to a map $V(\sigma) \to V'(\sigma)$.

Now Gieseker proves the following.

(1.17) *Let G be a reductive group and M_1 and M_2 be two G-spaces. Suppose that $f: M_1 \to M_2$ is a finite G-morphism and that a good quotient $M_2//G$ exists. Then a good quotient $M_1//G$ exists, and the induced morphism $M_1//G \to M_2//G$ is finite.*

So to show that $V(\sigma)$ has a good quotient it suffices to prove:

(1.18) *On $V(\sigma)$, $T \times 1$ is finite.*

Proof. By (1.15), $T \times 1$ is injective. We use the valuative criterion to check that $T \times 1$ is proper. Let C be a smooth curve, $p \in C$, and let $\Psi: C - \{p\} \to V(\sigma)$ be a map such that $(T \times 1)\Psi$ extends to a map $C \to V'(\sigma)$. On $C - \{p\}$, we then have a family $(\mathbf{E}, \mathbf{\Phi})$ of pairs such that $\mathbf{E}$ is generated by the sections $s(e_1), \ldots, s(e_\chi)$. By (1.16), on an open affine of C containing p, $(\mathbf{E}, \mathbf{\Phi})$ extends over p in such a way that $\mathbf{\Phi}_p \neq 0$ and the $s(e_i)$ generically generate $\mathbf{E}_p$. Thus $T(\mathbf{E}_p, s)$ is defined, and so by continuity $(T(\mathbf{E}_p, s), s^{-1}\mathbf{\Phi}_p) = ((T \times 1)\Psi)(p)$. Hence by (1.13) $s: \mathbb{C}^\chi \to H^0(\mathbf{E}_p)$ is an isomorphism and $(\mathbf{E}_p, \mathbf{\Phi}_p)$ is σ-semistable. So $(\mathbf{E}_p, s^{-1}\mathbf{\Phi}_p) \in V(\sigma)$ and Ψ extends to a map $C \to V(\sigma)$. □

Hence $V(\sigma)$ has a good projective quotient. By (1.8), the closure of the orbit of (E, ϕ) contains the orbit of $\mathrm{Gr}(E, \phi)$, which is closed in the σ-semistable set. But the closure of any orbit in the χ-semistable set contains only one closed orbit [21,3.14 (iii)]. Hence if two pairs are σ-semistable, then the closures of their orbits intersect if and only if they have the same Gr. This completes the proof of our main theorem (1.2). □

The stable subsets of these moduli spaces are actually fine:

(1.19) *There exists a universal pair over the σ-stable set $M_s(\sigma, \Lambda)$.*

Proof. There is a universal bundle $\mathbf{E} \to \mathrm{Quot}(\Lambda) \times X$ and a surjective map $\mathcal{O}^{\chi} \to \mathbf{E}$. Hence there is a natural $\mathrm{SL}(\chi)$-invariant section $\mathbf{\Phi} \in H^0(\mathrm{Quot}(\Lambda) \times \mathbb{P}\mathbb{C}^{\chi} \times X; \mathbf{E}(1))$, and $(\mathbf{E}(1), \mathbf{\Phi})$ is a universal pair. By (1.6) the only stabilizers of elements of the σ-stable subset of $V(\sigma)$ are the χth roots of unity. These act oppositely on $\mathbf{E}$ and on $\mathcal{O}(1)$, hence trivially on $\mathbf{E}(1)$, so on the σ-stable set $\mathbf{E}(1)$ is invariant under stabilizers. Hence by Kempf's descent lemma [9] $\mathbf{E}(1)$ descends to a bundle on $M_s(\sigma, \Lambda) \times X$, and the section $\mathbf{\Phi}$, being invariant, also descends. This pair over $M_s(\sigma, \Lambda) \times X$ then has the desired universal property. □

(1.20) Remark. If D is any effective divisor, by (1.9) there is an inclusion $\imath_D: M(\sigma, \Lambda) \subsetneq M(\sigma, \Lambda(2D))$ given by $(E, \phi) \mapsto (E(D), \phi(D))$. Indeed, if $(\mathbf{E}^{\Lambda}, \mathbf{\Phi}^{\Lambda})$ and $(\mathbf{E}^{\Lambda(2D)}, \mathbf{\Phi}^{\Lambda(2D)})$ are the corresponding universal pairs, there is a sequence

$$0 \to \mathbf{E}^{\Lambda} \xrightarrow{\imath_D} \imath_D^* \mathbf{E}^{\Lambda(2D)} \to \mathcal{O}_D(\imath_D^* \mathbf{E}^{\Lambda(2D)}) \to 0$$

such that $\imath_D(\mathbf{\Phi}^{\Lambda}) = \mathbf{\Phi}^{\Lambda(2D)}$.

2 Their tangent spaces

We now turn to the deformation theory of our pairs. By semicontinuity σ-stability is an open condition, so the Zariski tangent spaces to our moduli spaces at the σ-stable points will just be deformation spaces. Hence we may refer to $T_{(E,\phi)} M(\sigma, \Lambda)$ simply as $T_{(E,\phi)}$.

(2.1) *If $(E, \phi) \in M(\sigma, \Lambda)$ is σ-stable, then*

(i) (cf. [6]) *$T_{(E,\phi)}$ is canonically isomorphic to H^1 of the complex*

$$C^0(\mathrm{End}_0 E) \oplus \mathbb{C} \xrightarrow{p} C^1(\mathrm{End}_0 E) \oplus C^0(E) \xrightarrow{q} C^1(E),$$

where $p(g, c) = (dg, (g + c)\phi)$ and $q(f, \psi) = f\phi - d\psi$;

(ii) *H^0 and H^2 of this complex vanish;*

(iii) *there is a natural exact sequence*

$$0 \to H^0(\mathrm{End}\, E) \xrightarrow{\phi} H^0(E) \to T_{(E,\phi)} \to H^1(\mathrm{End}_0 E) \xrightarrow{\phi} H^1(E) \to 0.$$

Proof. Let $R = \mathbb{C}[\varepsilon]/(\varepsilon^2)$. By a well-known result [15, II Ex. 2.8] $T_{(E,\phi)}$ is the set of isomorphism classes of maps $\mathrm{Spec}\, R \to M(\sigma, \Lambda)$ such that $(\varepsilon) \mapsto (E, \phi)$. Since σ-stability is an open condition, $T_{(E,\phi)}$ is just the set of isomorphism classes of families $(\mathbf{E}, \mathbf{\Phi})$ of pairs on X with base $\mathrm{Spec}\, R$, such that $(\mathbf{E}, \mathbf{\Phi})_{(\varepsilon)} = (E, \phi)$ and $\Lambda^2 \mathbf{E}$ is the pullback of Λ. We will explain how to construct any such family.

The only open set in $\mathrm{Spec}\, R$ containing (ε) is $\mathrm{Spec}\, R$ itself, so any bundle $\mathbf{E}$ over $\mathrm{Spec}\, R \times X$ can be trivialized on $\mathrm{Spec}\, R \times U_\alpha$ for some open cover $\{U_\alpha\}$ of X. Thus if $\mathbf{E}_{(\varepsilon)} = E$, the transition functions give a Čech cochain of the form $1 + \varepsilon f_{\alpha\beta}$ where $f \in C^1$ $(\mathrm{End}\, E)$. In order for $\Lambda^2 \mathbf{E}$ to be isomorphic to the pullback of Λ, the transition functions of $\Lambda^2 \mathbf{E}$ must be conjugate to $1 \in C^0(\mathcal{O})$. But the transition functions are $\det(1 + \varepsilon f_{\alpha\beta}) = 1 + \varepsilon \operatorname{tr} f_{\alpha\beta}$, so we are asking that

$$(1 + \varepsilon g_\alpha)(1 + \varepsilon \operatorname{tr} f_{\alpha\beta})(1 - \varepsilon g_\beta) = 1$$

for some $g \in C^0(\mathcal{O})$, that is, $\operatorname{tr} f = -dg$. But if such a g exists, then $\tilde{f} = f + dg/2$ is trace-free, and $1 + \varepsilon\tilde{f}$ is obviously conjugate to $1 + \varepsilon f$, so determines the same bundle $\mathbf{E}$. Hence up to isomorphism we can obtain any $\mathbf{E}$ even if we consider only trace-free $f \in C^1(\operatorname{End}_0 E)$.

Now if there is a section $\mathbf{\Phi} \in H^0(\mathbf{E})$ such that $\mathbf{\Phi}_{(\varepsilon)} = \phi$, then with respect to the local trivializations of $\mathbf{E}$ described above, $\mathbf{\Phi} = \phi + \varepsilon\psi_\alpha$ for some Čech cochain $\psi \in C^0(E)$. Of course, ψ must be compatible with the transition functions; this means that

$$(1 + \varepsilon f_{\alpha\beta})(\phi + \varepsilon\psi_\beta) = (\phi + \varepsilon\psi_\alpha),$$

that is, $f\phi = d\psi$. Hence any pair $(\mathbf{E}, \mathbf{\Phi})$ having the desired properties can be obtained from some $(f, \psi) \in C^1(\operatorname{End}_0 E) \oplus C^0(E)$ satisfying $f\phi - d\psi = 0 \in C^1(E)$.

We now need only check which (f, ψ) give us isomorphic $(\mathbf{E}, \mathbf{\Phi})$. Of course the two choices will be related by a change of trivialization on $\operatorname{Spec} R \times U_\alpha$, but we may assume that the change of trivialization is of the form $1 + \varepsilon g_\alpha$ on U_α, since (E, ϕ) itself has no automorphisms (1.6). Furthermore, g must belong to $C^0(\operatorname{End}_0 E) \oplus \mathbb{C}$ in order to keep f trace-free, since the action of g is given by

$$1 + \varepsilon f_{\alpha\beta} \mapsto (1 + \varepsilon g_\alpha)(1 + \varepsilon f_{\alpha\beta})(1 - \varepsilon g_\beta),$$

that is, $f \mapsto f + dg$, and dg is trace-free if and only if $g \in C^0(\operatorname{End} E)$ is the sum of a trace-free cocycle and a constant. Similarly the action of g on ψ is

$$\phi + \varepsilon\psi_\alpha \mapsto (1 + \varepsilon g_\alpha)(\phi + \varepsilon\psi_\alpha),$$

that is, $\psi \mapsto \psi + g\phi$. Hence two pairs (f, ψ) and $(\tilde{f}, \tilde{\psi})$ determine isomorphic pairs $(\mathbf{E}, \mathbf{\Phi})$ if and only if they are in the same coset of the image of the map $C^0(\operatorname{End}_0 E) \oplus \mathbb{C} \to C^1(\operatorname{End}_0 E) \oplus C^0(E)$ given by $g + c \mapsto (dg, (g + c)\phi)$. This completes the proof of (i).

As for (ii) and (iii), substituting $H^0(\operatorname{End}_0 E) \oplus \mathbb{C} = H^0(\operatorname{End} E)$ into the long exact sequence of the double complex with exact rows

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] \arrow[d] & C^0(\operatorname{End}_0 E) \oplus \mathbb{C} \arrow[r] \arrow[d] & C^0(\operatorname{End}_0 E) \oplus \mathbb{C} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C^0(E) \arrow[r] \arrow[d] & C^1(\operatorname{End}_0 E) \oplus C^0(E) \arrow[r] \arrow[d] & C^1(\operatorname{End}_0 E) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C^1(E) \arrow[r] & C^1(E) \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}$$

gives

$$0 \to H^0 \to H^0(\operatorname{End} E) \to H^0(E) \to H^1 \to H^1(\operatorname{End}_0 E) \to H^1(E) \to H^2 \to 0,$$

where H^i is the cohomology of the complex from (i). But the map $H^0(\operatorname{End} E) \xrightarrow{\phi} H^0(E)$ is injective for (E, ϕ) σ-stable by (1.6), and the map $H^1(\operatorname{End}_0 E) \xrightarrow{\phi} H^1(\mathrm{E})$ is always surjective: indeed this is equivalent to the Serre dual map $H^0(KE^*) \xrightarrow{\phi} H^0(K \operatorname{End}_0 E^*)$ being injective, which is obvious since the map $KE^* \xrightarrow{\phi} K \operatorname{End}_0 E^*$ is an injection of sheaves. Hence H^0 and H^2 vanish, and we get the exact sequence in (iii). □

As a corollary, we obtain the following.

(2.2) *If $(E, \phi) \in M(\sigma, \Lambda)$ is σ-stable, then* $\dim T_{(E,\phi)} = d + g - 2$.

Proof. By (2.1 (iii))

$\dim T_{(E,\phi)} = \chi(E) - \chi(\mathrm{End}_0 E) - 1 = (d + 2 - 2g) - (3 - 3g) - 1 = d + g - 2.$ □

We will see in the next section that $\dim M(\sigma, \Lambda) = d + g - 2$; hence $M(\sigma, \Lambda)$ will be smooth at the stable points.

3 How they vary with σ

For obvious numerical reasons the σ-semistability condition remains the same, and implies σ-stability, for any $\sigma \in (\max(0, d/2 - i - 1), d/2 - i)$, where i is an integer between 0 and $(d - 1)/2$. Hence for σ in that interval we get a fixed projective moduli space $M(\sigma, \Lambda)$, which we will henceforth denote $M_i(\Lambda)$ or just M_i. The remainder of this paper will concentrate on these moduli spaces M_i, ignoring the special values of σ for which there exist σ-semistable pairs which are not σ-stable.

In the extreme case $i = 0$, it is then easy to construct the moduli space:

(3.1) $$M_0(\Lambda) = \mathbb{P}H^1(\Lambda^{-1}).$$

Proof. The first inequality in the σ-stability condition (1.1) says that $\phi \in H^0(L)$ implies $\deg L \leqq 0$. Hence $L = \mathcal{O}$, E is an extension of $\mathcal{O}$ by Λ, and $\phi \in H^0(\mathcal{O})$ is a constant section. The second inequality says that E has no subbundles of degree $\geqq d$: this is equivalent to not being split, since $M \to E \to \Lambda$ nonzero and $\deg M \geqq d = \deg \Lambda$ implies $M = \Lambda$. Hence $M_0(\Lambda)$ is simply the moduli space of nonsplit extensions of $\mathcal{O}$ by Λ, which is of course just $\mathbb{P}H^1(\Lambda^{-1})$. □

We will not attempt such a direct construction of $M_i(\Lambda)$ for $i > 0$. Rather, we will carefully study the relationship between M_{i-1} and M_i. Of course, this will only be of interest if there exists an M_i for $i > 0$, so *we will assume for the remainder of the paper that* $[(d - 1)/2] > 0$, *that is,* $d \geqq 3$. Anyhow, the first step is to construct families parametrizing those pairs which appear in M_i but not M_{i-1}, or M_{i-1} but not M_i. To do this, we first define two vector bundles over the ith symmetric product X_i.

Let $\pi: X_i \times X \to X_i$ be the projection and let $\Delta \subset X_i \times X$ be the universal divisor. Then define $W_i^- = (R^0\pi)\, \mathcal{O}_\Delta \Lambda(-\Delta)$ and $W_i^+ = (R^1\pi) \Lambda^{-1}(2\Delta)$. These are locally free sheaves of rank i and $d + g - 1 - 2i$, respectively.

(3.2) *For $i \leqq (d - 1)/2$, there is a family over $\mathbb{P}W_i^+$ parametrizing exactly those pairs which are represented in M_i but not M_{i-1}.*

Proof. As we pass from i to $i - 1$, the first inequality in the stability condition (1.1) gets stronger and the second gets weaker. So we look for pairs which almost violate the first inequality. That is, E must be an extension

$$0 \to \mathcal{O}(D) \to E \to \Lambda(-D) \to 0,$$

where $\deg D = i$, and ϕ is the section of $\mathcal{O}(D)$ vanishing on D. Conversely, any such pair is stable unless it splits $E = \mathcal{O}(D) \oplus \Lambda(-D)$. Indeed, if $L \subset E$ and $\phi \notin H^0(L)$, then the map $L \to \Lambda(-D)$ is nonzero, so $\deg L \leqq \deg \Lambda(-D) = d - i$, with equality only if $L = \Lambda(-D)$.

But $\mathbb{P}W_i^+$ is the base of a family parametrizing all such nonsplit pairs: indeed $\mathbf{E}$ is the tautological extension

$$0 \to \mathcal{O}(\Delta) \to \mathbf{E} \to \Lambda(-\Delta)(-1) \to 0,$$

and $\mathbf{\Phi}$ is the section of $\mathcal{O}(\Delta)$ vanishing on Δ. □

(3.3) *For $i \leqq (d-1)/2$, there is a family over $\mathbb{P}W_i^-$ parametrizing exactly those pairs which are represented in M_{i-1} but not M_i.*

Proof. This time the first inequality in (1.1) gets weaker and the second gets stronger. So we look for pairs which almost violate the second inequality. That is, E is an extension

$$0 \to M \to E \to \Lambda M^{-1} \to 0$$

where $\deg M = d - i$, and $\phi \notin H^0(M)$. Hence projecting ϕ in the exact sequence, we get a nonzero $\gamma \in H^0(\Lambda M^{-1})$ vanishing on a divisor D of degree i such that $\Lambda M^{-1} = \mathcal{O}(D)$. Then at D, ϕ lifts to $M = \Lambda(-D)$, so we get an element $p(E, \phi) \in H^0(\mathcal{O}_D \Lambda(-D))$, defined up to a scalar as usual.

On the other hand, we can recover (E, ϕ) from D and p. Indeed, choose a Čech cochain $\psi \in C^0(\Lambda(-D))$ such that $\psi|_D = p$. Then $d\psi|_D = dp = 0$, so $d\psi$ vanishes on D and descends to a closed cochain $f = d\psi/\gamma \in C^1(\Lambda(-2D))$. This determines an extension

$$0 \to \Lambda(-D) \to E' \to \mathcal{O}(D) \to 0.$$

The compatibility condition for $\gamma + \psi$ to define a section $\phi' \in H^0(E')$ is $\gamma f = d\psi$, which is automatic. Thus we get a new pair (E', ϕ') satisfying $p(E', \phi') = p$.

Up to isomorphism, (E', ϕ') is independent of the choice of ψ, since adding $\xi \in C^0(\Lambda(-2D))$ to ψ is simply equivalent to acting by $\left(\begin{smallmatrix} 1 & \xi \\ 0 & 1 \end{smallmatrix}\right)$ on the local splittings of E' with which the extension is defined. In particular, we can choose local splittings of the old E and let ψ be the projection of the old ϕ on $M = \Lambda(-D)$ with respect to these splittings. Then the construction of the previous paragraph recovers (E, ϕ), so $(E', \phi') = (E, \phi)$.

The construction above can be generalized to produce a family $(\mathbf{E}, \mathbf{\Phi}) \to \mathbb{P}W_i^- \times X$, as follows. Let $p: \mathbb{P}W_i^- \to X_i$ be the projection, and choose a cochain $\Psi \in C^0(\Lambda(-\Delta)(1))$ such that $\Psi|_{p^{-1}\Delta}$ is the tautological section. Then $d\Psi$ vanishes on $p^{-1}\Delta$, so descends to $C^1(\Lambda(-2\Delta)(1))$. This determines an extension

$$0 \to \Lambda(-\Delta)(1) \to \mathbf{E} \to \mathcal{O}(\Delta) \to 0,$$

and if $\gamma \in H^0(\mathcal{O}(\Delta))$ is the section vanishing on Δ, then $\gamma + \Psi$ defines the desired section $\mathbf{\Phi} \in H^0(\mathbf{E})$. □

By the universal properties of M_{i-1} and M_i, we thus get injections $\mathbb{P}W_i^+ \hookrightarrow M_i$ and $\mathbb{P}W_i^- \hookrightarrow M_{i-1}$. As an example, consider the case $i = 1$. By (3.1), $M_0 = \mathbb{P}H^1(\Lambda^{-1})$. Moreover, W_1^- is a line bundle and hence $\mathbb{P}W_1^- = X_1 = X$. Hence the inclusion of (3.3) is a map $X \hookrightarrow \mathbb{P}H^1(\Lambda^{-1})$; it can be identified explicitly as follows.

(3.4) *The inclusion $X \hookrightarrow \mathbb{P}H^1(\Lambda^{-1})$ is given by the complete linear system $|K_X\Lambda|$.*

Proof. There is an alternative way to see what pairs are represented in M_0 but not M_1. Any pair $(E, \phi) \in M_0$ is an extension

$$0 \to \mathcal{O} \to E \to \Lambda \to 0, \tag{3.5}$$

say with extension class $s \in H^1(\Lambda^{-1})$, and with $\phi \in H^0(\mathcal{O})$. Such a pair is the image of $x \in X$ under the injection of (3.3) if there is an inclusion $0 \to \Lambda(-x) \to E$ such that the composition $\gamma_x: \Lambda(-x) \to E \to \Lambda$ vanishes at x. Hence we ask for what extension classes $s \in H^1(\Lambda^{-1})$ the map $\gamma_x: \Lambda(-x) \to \Lambda$ lifts to E.

Twisting (3.5) by $\Lambda^{-1}(x)$ and taking the long exact sequence yields

$$H^0(E \otimes \Lambda^{-1}(x)) \to H^0(\mathcal{O}(x)) \xrightarrow{s} H^1(\Lambda^{-1}(x)),$$

where the second map is the cup product with s. Hence $\gamma_x \in H^0(\mathcal{O}(x))$ lifts to $H^0(E \otimes \Lambda^{-1}(x))$ as desired if and only if $\gamma_x s = 0$. That is, s must be in the kernel of the map $\gamma_x: H^1(\Lambda^{-1}) \to H^1(\Lambda^{-1}(x))$, or Serre dually, $\gamma_x: H^0(K_X\Lambda)^* \to H^0(K_X\Lambda(-x))^*$. Since γ_x is dual to the injection $H^0(K_X\Lambda(-x)) \to H^0(K_X\Lambda)$, it is surjective, so

$$\dim \ker \gamma_x = \dim H^0(K_X\Lambda(-x)) - \dim H^0(K_X\Lambda).$$

But since $\deg K_X\Lambda(-x) > 2g - 2$, this is 1. Hence for each $x \in X$, there is a unique $s \in \mathbb{P}H^1(\Lambda^{-1})$ such that $\gamma_x s = 0$.

What is this s? Regarded as a linear functional on $H^0(K_X\Lambda)$, $s \in \ker \gamma_x$ if it annihilates all sections vanishing at x. Certainly evaluation at x does this, so this is the s generating $\ker \gamma_x$. But it is also the image of x in the map $X \hookrightarrow \mathbb{P}H^1(\Lambda^{-1})$ given by $|K_X\Lambda|$. Hence the two maps are identical. □

(3.6) *The M_i are all smooth rational integral projective varieties of dimension $d + g - 2$, and for $i > 0$, there is a birational map $M_i \leftrightarrow M_1$, which is an isomorphism except on sets of codimension $\geqq 2$.*

Proof. By (3.1) and Riemann-Roch, the first statement is certainly true of M_0. For $i > 0$, suppose by induction on i that it is true of M_{i-1}. By (3.2) and (3.3) there is an isomorphism $M_{i-1} - \mathbb{P}W_i^- \leftrightarrow M_i - \mathbb{P}W_i^+$. But $\dim \mathbb{P}W_i^- = 2i - 1 < d - 1 < d + g - 2$, and $\dim \mathbb{P}W_i^+ = d + g - 2 - i < d + g - 2$, so $\dim M_i = \dim M_{i-1} = d + g - 2$ and M_i is birational to M_{i-1}, hence to M_0. Moreover by (2.2), the Zariski tangent space to M_i has constant dimension $d + g - 2$, so M_i is a smooth reduced variety. The second statement is also proved by induction: just note that for $i > 1$, $\operatorname{codim} \mathbb{P}W_i^-/M_{i-1} = d + g - 2i - 1 \geqq 2$ and $\operatorname{codim} \mathbb{P}W_i^+/M_i = i \geqq 2$. □

(3.7) *Let $(E, \phi) \in \mathbb{P}W_i^+$, let D be the zero-set of ϕ, and let γ be the map*

$$E \otimes \Lambda^{-1}(D) \to \Lambda(-D) \otimes \Lambda^{-1}(D) = \mathcal{O}.$$

Then $T_{(E,\phi)}\mathbb{P}W_i^+$ is canonically isomorphic to H^1 of the complex

$$C^0(E \otimes \Lambda^{-1}(D)) \oplus \mathbb{C} \xrightarrow{p} C^1(E \otimes \Lambda^{-1}(D)) \oplus C^0(\mathcal{O}(D)) \xrightarrow{q} C^1(\mathcal{O}(D)),$$

where $p(g, c) = (dg, (\gamma g + c)\phi)$ and $q(f, \psi) = \gamma f \phi - d\psi$. Moreover, H^0 and H^2 of this complex vanish.

Proof. The proof is modelled on that of (2.1). We regard $\mathbb{P}W_i^+$ as a moduli space of triples (L, E, ϕ), where L is a line bundle of degree i, E is an extension of L by ΛL^{-1}, and $\phi \in H^0(L)$, and consider the deformation theory of this moduli problem.

Let $R = \mathbb{C}[\varepsilon]/(\varepsilon^2)$ as before. Then $T_{(L,E,\phi)}\mathbb{P}W_i^+$ is the set of isomorphism classes of families $(\mathbf{L}, \mathbf{E}, \mathbf{\Phi})$ of triples on X with base $\operatorname{Spec} R$, such that $(\mathbf{L}, \mathbf{E}, \mathbf{\Phi})_{(\varepsilon)} = (L, E, \phi)$. We will explain how to construct any such family.

Any bundle over $\operatorname{Spec} R \times X$ can be trivialized on $\operatorname{Spec} R \times U_\alpha$ for some open cover $\{U_\alpha\}$ of X. Thus if $\mathbf{L}_{(\varepsilon)} = \mathcal{O}(D)$ and $\mathbf{E}_{(\varepsilon)} = E$, then the transition functions for $\mathbf{E}$ give a Čech cochain of the form $1 + \varepsilon f_{\alpha\beta}$ where $f \in C^1(\operatorname{End} E)$. Since $\mathbf{E}$ is to be a family of extensions of L by ΛL^{-1}, it must have $\Lambda^2 \mathbf{E} = \Lambda$, so as explained in the proof of (2.1) we may take $f \in C^1(\operatorname{End}_0 E)$. Furthermore, the transition functions must preserve $\mathbf{L}$, so if f' is the projection of f to $C^1(\Lambda(-2D))$ in the natural exact sequence

$$0 \to E \otimes \Lambda^{-1}(D) \to \operatorname{End}_0 E \to \Lambda(-2D) \to 0,$$

then $1 + \varepsilon f'_{\alpha\beta}$ must be conjugate to 1. Hence

$$(1 - \varepsilon g_\alpha)(1 + \varepsilon f'_{\alpha\beta})(1 - \varepsilon g_\beta) = 1$$

for some $g \in C^0(\Lambda(-2D))$, that is, $f' = dg$. But if such a g exists, then for any lifting $\tilde{g}$ of g to $C^0(\operatorname{End}_0 E)$, $\tilde{f} = f - d\tilde{g}$ projects to $0 \in C^1(\Lambda(-2D))$, and $1 + \varepsilon\tilde{f}$ is obviously conjugate to $1 + \varepsilon f$, so determines the same bundle $\mathbf{E}$. Hence up to isomorphism we can obtain any $\mathbf{E}$ that is an extension of some $\mathbf{L}$ by $\Lambda\mathbf{L}^{-1}$ even if we consider only those f in the kernel of $C^1(\operatorname{End}_0 E) \to C^1(\Lambda(-2D))$, that is, in $C^1(E \otimes \Lambda^{-1}(D))$. The transition functions for $\mathbf{L}$ are then just $1 + \varepsilon\gamma f_{\alpha\beta}$.

Now if there is a section $\mathbf{\Phi} \in H^0(\mathbf{L})$ such that $\mathbf{\Phi}_{(\varepsilon)} = \phi$, then with respect to the local trivializations of $\mathbf{E}$, $\mathbf{\Phi} = \phi + \varepsilon\psi_\alpha$ for some Čech cochain $\psi \in C^0(\mathcal{O}(D))$. Of course, ψ must be compatible with the transition functions; this means that

$$(1 + \varepsilon\gamma f_{\alpha\beta})(\phi + \varepsilon\psi_\beta) = (\phi + \varepsilon\psi_\alpha),$$

that is, $\gamma f \phi = d\psi$. Hence any triple $(\mathbf{L}, \mathbf{E}, \mathbf{\Phi})$ having the desired properties can be obtained from some $(f, \psi) \in C^1(E \otimes \Lambda^{-1}(D)) \oplus C^0(\mathcal{O}(D))$ satisfying $\gamma f \phi - d\psi = 0 \in C^1(\mathcal{O}(D))$.

We now need only check which (f, ψ) give us isomorphic $(\mathbf{L}, \mathbf{E}, \mathbf{\Phi})$. This part of the argument follows that of (2.1) exactly, except that g ends up being in $C^1(E \otimes \Lambda^{-1}(D)) \oplus \mathbb{C}$, and acts on ψ by $\psi \mapsto \psi + \gamma g \phi$. This completes the proof of the first statement.

As for the second, taking the long exact sequence of the double complex

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] \arrow[d] & C^0(E \otimes \Lambda^{-1}(D)) \oplus \mathbb{C} \arrow[r] \arrow[d] & C^0(E \otimes \Lambda^{-1}(D)) \oplus \mathbb{C} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C^0(\mathcal{O}(D)) \arrow[r] \arrow[d] & C^1(E \otimes \Lambda^{-1}(D)) \oplus C^0(\mathcal{O}(D)) \arrow[r] \arrow[d] & C^1(E \otimes \Lambda^{-1}(D)) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C^1(\mathcal{O}(D)) \arrow[r] & C^1(\mathcal{O}(D)) \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}$$

gives

$$\begin{aligned}
0 \to H^0 \to H^0(E \otimes \Lambda^{-1}(D)) \oplus \mathbb{C} \to H^0(\mathcal{O}(D)) \to H^1 \\
\to H^1(E \otimes \Lambda^{-1}(D)) \to H^1(\mathcal{O}(D)) \to H^2 \to 0,
\end{aligned}$$

where H^i is the cohomology of the complex in the statement. Now $H^0(\Lambda^{-1}(2D)) = 0$ since $\deg \Lambda^{-1}(2D) < 0$, and E is a nonsplit extension of $\mathcal{O}(D)$ by $\Lambda(-D)$, so

$$H^0(E \otimes \Lambda^{-1}(D)) = H^0(\mathrm{Hom}(\Lambda(-D), E)) = 0.$$

But the map $\mathbb{C} \to H^0(\mathcal{O}(D))$ is injective: indeed, it is multiplication by ϕ. Hence $H^0 = 0$. Likewise, the map $H^1(E \otimes \Lambda^{-1}(D)) \to H^1(\mathcal{O}(D))$ is surjective: indeed this is equivalent to the Serre dual map $H^0(K(-D)) \to H^0(E^* \otimes K\Lambda(-D))$ being injective, which is obvious since the map $K(-D) \to K \to E^* \otimes K\Lambda(-D)$ is an injection of sheaves. Hence $H^2 = 0$. ☐

The following proposition is proved similarly.

(3.8) *Let $(E, \phi) \in \mathbb{P}W_i^-$, and let $D = p(E, \phi)$. Then $T_{(E,\phi)}\mathbb{P}W_i^-$ is canonically isomorphic to H^1 of the complex*

$$C^0(E(-D)) \oplus \mathbb{C} \to C^1(E(-D)) \oplus C^0(E) \to C^1(E).$$

Moreover, H^0 and H^2 of this complex vanish.

(3.9) *The injection $\mathbb{P}W_i^+ \hookrightarrow M_i$ induces an exact sequence on $\mathbb{P}W_i^+$*

$$0 \to T\mathbb{P}W_i^+ \to TM_i|_{\mathbb{P}W_i^+} \to W_i^-(-1) \to 0.$$

Proof. The complex

$$C^0(\Lambda(-2\Delta)) \to C^1(\Lambda(-2\Delta)) \oplus C^0(\Lambda(-\Delta)) \to C^1(\Lambda(-\Delta))$$

with the obvious maps has $R^0\pi = 0$, $R^1\pi = W_i^-$ from the long exact sequence of the double complex

$$\begin{tikzcd}
0 \arrow[r] & C^0(\Lambda(-2\Delta)) \arrow[r] \arrow[d, "{(1,0)}"] & C^0(\Lambda(-2\Delta)) \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \begin{matrix} C^0(\Lambda(-2\Delta)) \\ \oplus C^1(\Lambda(-2\Delta)) \end{matrix} \arrow[r] \arrow[d, "{(0,1)}"] & \begin{matrix} C^0(\Lambda(-\Delta)) \\ \oplus C^1(\Lambda(-2\Delta)) \end{matrix} \arrow[r] \arrow[d] & C^0(\mathcal{O}_\Delta \Lambda(-\Delta)) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C^1(\Lambda(-2\Delta)) \arrow[r] & C^1(\Lambda(-\Delta)) \arrow[r] & C^1(\mathcal{O}_\Delta \Lambda(-\Delta)) \arrow[r] & 0.
\end{tikzcd}$$

Hence the result follows from the long exact sequence of the double complex

$$\begin{tikzcd}
0 \arrow[r] & C^0(\mathbf{E}\Lambda^{-1}(\Delta)) \oplus \mathbb{C} \arrow[r] \arrow[d, "{(1,0)}"] & C^0(\mathrm{End}_0\,\mathbf{E}) \oplus \mathbb{C} \arrow[r] \arrow[d, "p"] & C^0(\Lambda(-2\Delta))(-1) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \begin{matrix} C^1(\mathbf{E}\Lambda^{-1}(\Delta)) \\ \oplus C^0(\mathcal{O}(\Delta)) \end{matrix} \arrow[r] \arrow[d, "{(0,1)}"] & \begin{matrix} C^1(\mathrm{End}_0\,\mathbf{E}) \\ \oplus C^0(\mathbf{E}) \end{matrix} \arrow[r] \arrow[d, "q"] & \begin{matrix} C^1(\Lambda(-2\Delta))(-1) \\ \oplus C^0(\Lambda(-\Delta))(-1) \end{matrix} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C^1(\mathcal{O}(\Delta)) \arrow[r] & C^1(\mathbf{E}) \arrow[r] & C^1(\Lambda(-\Delta))(-1) \arrow[r] & 0,
\end{tikzcd}$$

together with (2.1) and (3.7). ☐

(3.10) *The map* $\mathbb{P}W_i^+ \hookrightarrow M_i$ *is an embedding.*

Proof. By (3.7), it is an injection, and by (3.9), so is its derivative. ☐

The following proposition and corollary are proved similarly, using (2.1) and (3.8).

(3.11) *The injection* $\mathbb{P}W_i^- \hookrightarrow M_{i-1}$ *induces an exact sequence on* $\mathbb{P}W_i^-$

$$0 \to T\mathbb{P}W_i^- \to TM_{i-1}|_{\mathbb{P}W_i^-} \to W_i^+(-1) \to 0.$$

(3.12) *The map* $\mathbb{P}W_i^- \hookrightarrow M_{i-1}$ *is an embedding.*

By (3.2) and (3.3) every pair in $M_i - \mathbb{P}W_i^+$ is also in $M_{i-1} - \mathbb{P}W_i^-$, and vice-versa. Hence there is a natural isomorphism $M_i - \mathbb{P}W_i^+ \to M_{i-1} - \mathbb{P}W_i^-$. Our next task is to extend this to a proper map. Let $\tilde{M}_i^+$ be the blow-up of M_i at $\mathbb{P}W_i^+$. Then by (3.9) the exceptional divisor is $E_i^+ = \mathbb{P}W_i^- \oplus \mathbb{P}W_i^+$, and $\mathcal{O}_{E_i^+}(E_i^+) = \mathcal{O}(-1, -1)$.

(3.13) *There is a map* $\tilde{M}_i^+ \to M_{i-1}$ *such that the following diagram commutes:*

$$\begin{tikzcd}
M_i - \mathbb{P}W_i^+ \arrow[r] \arrow[d, leftrightarrow] & \tilde{M}_i^+ \arrow[d] & E_i^+ \arrow[l] \arrow[d] \\
M_{i-1} - PW_i^- \arrow[r] & M_{i-1} & \mathbb{P}W_i^- . \arrow[l]
\end{tikzcd}$$

Proof. Let $(\mathbf{E}, \mathbf{\Phi}) \to \tilde{M}_i^+ \times X$ be the pullback of the universal family. We will construct a new family $(\mathbf{E}', \mathbf{\Phi}')$ of pairs all of which are in M_{i-1}.

By uniqueness of families (1.7), $(\mathbf{E}, \mathbf{\Phi})|_{E_i^+ \times X}$ is the pullback of the family over $\mathbb{P}W_i^+$ constructed in (3.2). Thus there is a surjective sheaf map $\mathbf{E} \to \mathcal{O}_{E_i^+ \times X}\Lambda(-\Delta)(0, -1)$ annihilating $\mathbf{\Phi}$. Define $\mathbf{E}'$ to be the kernel of this map, so that

$$0 \to \mathbf{E}' \to \mathbf{E} \to \mathcal{O}_{E_i^+ \times X}\Lambda(-\Delta)(0, -1) \to 0. \tag{3.14}$$

Then $\mathbf{E}'$ is locally free, and $\mathbf{\Phi}$ descends to $\mathbf{\Phi}' \in H^0(\mathbf{E}')$. For $z \in M_i - \mathbb{P}W_i^+$, clearly $(\mathbf{E}', \mathbf{\Phi}')_z = (\mathbf{E}, \mathbf{\Phi})_z$. So to prove the proposition it suffices to show that $(\mathbf{E}', \mathbf{\Phi}')_{E_i^+}$ is the pullback of the family over $\mathbb{P}W_i^-$ constructed in (3.3). The first promising thing to note is that there certainly is a surjection $\mathbf{E}' \to \mathcal{O}_{E_i^+ \times X}(\Delta) \to 0$, and $\Lambda^2\mathbf{E}' = \Lambda^2\mathbf{E}(-E_i^+ \times X)$, so we get an extension

$$0 \to \Lambda(-\Delta)(1, 0) \to \mathbf{E}'_{E_i^+ \times X} \to \mathcal{O}(\Delta) \to 0,$$

just as in the family of (3.3).

Now fix $s \in E_i^+$ over $(E, \phi) \in M_i$, and let D be the zero-set of ϕ. Let $R = \mathbb{C}[\varepsilon]/(\varepsilon^2)$ as before, and choose a map $\operatorname{Spec} R \to \tilde{M}_i^+$ representing an element of $T_s\tilde{M}_i^+ - T_sE_i^+$. Then (3.14) restricts to an exact sequence

$$0 \to \mathcal{O}_{\operatorname{Spec} R \times X}(\mathbf{E}') \to \mathcal{O}_{\operatorname{Spec} R \times X}(\mathbf{E}) \to \mathcal{O}_{(\varepsilon) \times X}\Lambda(-D) \to 0.$$

On some open cover $\{U_\alpha\}$ of X, E splits as

$$E|_{U_\alpha} = \mathcal{O}(D)|_{U_\alpha} \oplus \Lambda(-D)|_{U_\alpha}, \tag{3.15}$$

and this splitting can be extended to a splitting of $\mathbf{E}|_{\operatorname{Spec} R \times U_\alpha}$. Then

$$\mathbf{E}'|_{U_\alpha} = \mathcal{O}(D)|_{U_\alpha} \oplus \Lambda(D)|_{U_\alpha} \otimes \mathcal{I}_{(\varepsilon)}. \tag{3.16}$$

The section $\mathbf{\Phi}$ is then of the form $\phi + \varepsilon\psi_\alpha$ for some $\psi \in C^0(E)$, and the transition functions are $1 + \varepsilon f_{\alpha\beta}$ for some $f \in C^1(\mathrm{End}_0 E)$. The latter hence act as 1 on the second factor of (3.16).

Now decompose $\psi_\beta = \psi_\beta^{\mathcal{O}(D)} + \psi_\beta^{\Lambda(-D)}$ and $f_{\alpha\beta} = f_{\alpha\beta}^{\mathcal{O}(D)} + f_{\alpha\beta}^{\Lambda(-D)}$ corresponding to the splitting on U_β. If $\mathbf{E}'$ is restricted to $(\varepsilon) \times U_\alpha$, then $\varepsilon\psi_\beta^{\mathcal{O}(D)} = 0$ and $\varepsilon f_{\alpha\beta}^{\mathcal{O}(D)} = 0$, since everything divisible by ε is now set to zero. However, $\varepsilon\psi_\beta^{\Lambda(-D)}$ and $\varepsilon f_{\alpha\beta}^{\Lambda(-D)}$ are not necessarily zero, since not everything in their images is divisible by ε *in the module* $\Lambda(-D) \otimes \mathscr{I}_{(\varepsilon)}$. Hence $\mathbf{\Phi}_{(\varepsilon)} = \phi + \varepsilon\psi_\beta^{\Lambda(-D)}$ on U_β, and $\mathbf{E}_{(\varepsilon)}$ has transition functions $\begin{pmatrix} 1 & \varepsilon f_{\alpha\beta}^{\Lambda(-D)} \\ 0 & 1 \end{pmatrix}$ with respect to the splitting (3.16). In other words, the extension class of $E' = \mathbf{E}_{(\varepsilon)}$ is the projection of $f \in C^1(\mathrm{End}_0 E)$ to $C^1(\Lambda(-2D))$, and the lifting of ϕ' is the projection of $\psi \in C^1(E)$ to $C^1(\Lambda(-D))$. Hence (E', ϕ') is the bundle over the image of (E, ϕ) in $\mathbb{P}W_i^-$ in the family of (3.3). By uniqueness of families (1.7) this means that $(\mathbf{E}', \mathbf{\Phi}')|_{E_i^+ \times X}$ is the pullback of the family of (3.3). □

There is a result similar to (3.13) for the inverse map $M_{i-1} - \mathbb{P}W_i^- \to M_i - \mathbb{P}W_i^+$. Let $\tilde{M}_{i-1}^-$ be the blow-up of M_{i-1} at $\mathbb{P}W_i^-$. Hence by (3.11) the exceptional divisor is $E_i^- = \mathbb{P}W_i^- \oplus \mathbb{P}W_i^+$, and $\mathcal{O}_{E_i^-}(E_i^-) = \mathcal{O}(-1, -1)$. Note that there is an isomorphism $E_i^- \leftrightarrow E_i^+$.

(3.17) *There is a map* $\tilde{M}_{i-1}^- \to M_i$ *such that the following diagram commutes*:

$$\begin{tikzcd}
M_{i-1} - \mathbb{P}W_i^- \arrow[r] \arrow[d, leftrightarrow] & \tilde{M}_{i-1}^- \arrow[d] & E_i^- \arrow[l] \arrow[d] \\
M_i - \mathbb{P}W_i^+ \arrow[r] & M_i & \mathbb{P}W_i^+ . \arrow[l]
\end{tikzcd}$$

Proof. Let $(\mathbf{E}, \mathbf{\Phi}) \to \tilde{M}_{i-1}^- \times X$ be the pullback of the universal family. We will construct a new family $(\mathbf{E}', \mathbf{\Phi}')$ of pairs all of which are in M_i.

By uniqueness of families (1.7), $(\mathbf{E}, \mathbf{\Phi})|_{E_i^- \times X}$ is the pullback of the family over $\mathbb{P}W_i^-$ constructed in (3.3). Thus there is a surjective sheaf map $\mathbf{E} \to \mathcal{O}_{E_i^- \times X}(-\Delta)$. This time the map does not necessarily annihilate $\mathbf{\Phi}$. However, if we tensor by $\mathcal{O}(E_i^-)$, then the twisted map $\mathbf{E}(E_i^-) \to \mathcal{O}_{E_i^- \times X}(\Delta)(-1, -1)$ of course annihilates $\mathbf{\Phi}(E_i^-)$. If we define $\mathbf{E}'$ to be the kernel of this twisted map, so that

$$0 \to \mathbf{E}' \to \mathbf{E}(E_i^-) \to \mathcal{O}_{E_i^+ \times X}(\Delta)(-1, -1) \to 0,$$

then $\mathbf{E}'$ is locally free, and $\mathbf{\Phi}(E_i^-)$ descends to $\mathbf{\Phi}' \in H^0(\mathbf{E}')$. The remainder of the proof is analogous to that of (3.13). □

At last we come to the goal of all the above work.

(3.18) *There is a natural isomorphism* $\tilde{M}_i^+ \leftrightarrow \tilde{M}_{i-1}^-$ *such that the following diagram commutes*:

$$\begin{tikzcd}
M_i - \mathbb{P}W_i^+ \arrow[r] \arrow[d, leftrightarrow] & \tilde{M}_i^+ \arrow[d, leftrightarrow] & E_i^+ \arrow[l] \arrow[d, leftrightarrow] \\
M_{i-1} - \mathbb{P}W_i^- \arrow[r] & \tilde{M}_{i-1}^- & E_i^- . \arrow[l]
\end{tikzcd}$$

Proof. Both $\tilde{M}_i^+$ and $\tilde{M}_{i-1}^-$ are smooth, and by (3.13) and (3.17) they both inject into $M_{i-1} \times M_i$. Indeed, both injections are embeddings, since as is easily checked they annihilate no tangent vectors, and both have the same image. This image is precisely the closure of the graph of the isomorphism $M_i - \mathbb{P}W_i^+ \leftrightarrow M_{i-1} - \mathbb{P}W_i^-$, which proves the left-hand square; for both E_i^- and E_i^+ it is the map $\mathbb{P}W_i^- \oplus \mathbb{P}W_i^+ \to \mathbb{P}W_i^- \times \mathbb{P}W_i^+$, which proves the right-hand square. □

Note. In light of this result, we will henceforth refer to $\tilde{M}_i^+ = \tilde{M}_{i-1}^-$ simply as $\tilde{M}_i$, and $E_i^+ = E_i^-$ as E_i.

Thus M_i is obtained from M_{i-1} by blowing up $\mathbb{P}W_i^-$, and then blowing down the same exceptional divisor in another direction. Such a blow-up and blow-down is an example of what is called a flip (or more properly, a log flip) in Mori theory. This paper will not use any of the deep results of Mori theory, but we will see some of its basic principles in action.

In one case the flip degenerates to an ordinary blow-up.

(3.19) *The moduli space M_i is the blow-up of $M_0 = \mathbb{P}H^1(\Lambda^{-1})$ along X embedded via $|K_X\Lambda|$.*

Proof. Since W_i^- is a line bundle, there is nothing to blow down. □

The other extreme case is also of interest. Let $w = [(d-1)/2]$, so that M_w is the last moduli space in our sequence. Let N be the moduli space of ordinary rank 2 semistable bundles of determinant Λ.

(3.20) *There is a natural "Abel-Jacobi" map $M_w \to N$ with fibre $\mathbb{P}H^0(E)$ over a stable bundle E. It is surjective if $d > 2g-2$.*

Proof. If $i = w$, then $\sigma \in (0, [d/2] + 1 - d/2)$, so σ-stability of (E, ϕ) implies ordinary semistability of E. Thus there is a map $M_w \to N$. Moreover, ordinary stability of E implies σ-stability of (E, ϕ), so the fibre over a stable E is just $\mathbb{P}H^0(E)$. For $d > 2g-2$, any bundle E has a nonzero section ϕ by Riemann-Roch. Hence every stable bundle in N is certainly in the image of M_w. But M_w is complete, so its image is a complete variety containing the stable set, which must be N itself. □

We may sum up our findings in the following diagram.

$$\begin{tikzcd}
 & \tilde{M}_2 \arrow[dl] \arrow[dr] & & \tilde{M}_3 \arrow[dl] \arrow[dr] & & \tilde{M}_4 \arrow[dl] \arrow[dr] & & \tilde{M}_w \arrow[dl] \arrow[dr] & \\
M_1 \arrow[d] & & M_2 & & M_3 & & \cdots & & M_w \arrow[d] \\
M_0 & & & & & & & & N.
\end{tikzcd}$$

All the arrows are birational morphisms except sometimes the one to N.

4 Their Poincaré polynomials

Before going on to our main application in the next section, let us pause to see how the flips described above can be used to compute the Poincaré polynomials of our moduli spaces.

(4.1) $$P_t(M_i) = \frac{1}{1-t^2}\operatorname*{Coeff}_{x^i}\left(\frac{t^{2d+2g-2-4i}}{xt^4-1} - \frac{t^{2i+2}}{x-t^2}\right)\left(\frac{(1+xt)^{2g}}{(1-x)(1-xt^2)}\right).$$

Proof. Since $\tilde{M}_j$ is the blow-up of M_{j-1} at $\mathbb{P}W_j^-$, by the formula for the Poincaré polynomial of a blow-up [12, p. 605],

$$P_t(\tilde{M}_j) = P_t(M_{j-1}) + P_t(E_j) - P_t(\mathbb{P}W_j^-).$$

But $\tilde{M}_j$ is also the blow-up of M_j at $\mathbb{P}W_j^+$, so

$$P_t(\tilde{M}_j) = P_t(M_j) + P_t(E_j) - P_t(\mathbb{P}W_j^+)$$

as well. Hence

$$P_t(M_j) - P_t(M_{j-1}) = P_t(\mathbb{P}W_j^+) - P_t(\mathbb{P}W_j^-).$$

But the Poincaré polynomial of any projective bundle splits, so

$$\begin{aligned} P_t(\mathbb{P}W_j^+) - P_t(\mathbb{P}W_j^-) &= P_t(\mathbb{P}^{d+g-2-2j})P_t(X_j) - P_t(\mathbb{P}^{j-1})P_t(X_j) \\ &= \frac{t^{2j} - t^{2d+2g-2-4j}}{1-t^2}P_t(X_j). \end{aligned}$$

A formula for $P_t(X_j)$ was given by Macdonald [17]:

$$P_t(X_j) = \operatorname*{Coeff}_{x^j}\frac{(1+xt)^{2g}}{(1-x)(1-xt^2)}.$$

Hence

$$P_t(M_j) - P_t(M_{j-1}) = \operatorname*{Coeff}_{x^j}\frac{(t^{2j} - t^{2d+2g-2-4j})(1+xt)^{2g}}{(1-t^2)(1-x)(1-xt^2)}.$$

Notice that this formula also produces $P_t(M_0)$ when $j = 0$. So to sum up,

$$\begin{aligned} P_t(M_i) &= \frac{1}{1-t^2}\operatorname*{Coeff}_{x^i}\sum_{j=0}^{i}\frac{x^{i-j}(t^{2j} - t^{2d+2g-2-4j})(1+xt)^{2g}}{(1-x)(1-xt^2)} \\ &= \frac{1}{1-t^2}\operatorname*{Coeff}_{x^i}\left(\frac{x^{i+1} - t^{2i+2}}{x-t^2} + \frac{t^{2d+2g-2-4i}(1 - t^{4i-4}x^{i+1})}{xt^4-1}\right)\left(\frac{(1+xt)^{2g}}{(1-x)(1-xt^2)}\right), \end{aligned}$$

which agrees with the formula stated after the terms containing x^{i+1} are removed. □

We can use this formula to recover the formula of Harder and Narasimhan [14] for the Poincaré polynomial of the moduli space N of stable bundles of rank 2, determinant Λ, and odd degree d:

(4.2) $$P_t(N) = \frac{(1+t^3)^{2g} - t^{2g}(1+t)^{2g}}{(1-t^2)(1-t^4)}.$$

Proof. When $d > 2g - 2$ is odd and $i = w$, then by (3.20) there is a surjective map $M_w \to N$ with fibre $\mathbb{P}H^0(E)$ over a bundle E. If moreover $d > 4g - 4$, then $H^1(E) = 0$ for all stable E (see for example the proof of (1.10)), so M_w is then just the $\mathbb{P}^{d-2g+1}$-bundle $\mathbb{P}(R^0\pi)\mathbf{E}$, where $\mathbf{E}$ is a universal bundle over N, and

$$P_t(N) = \frac{1 - t^2}{1 - t^{2d-4g+4}} P_t(M_w).$$

For simplicity we may as well assume that $d = 4g - 3$. Then $w = 2g - 2$ and

$$P_t(N) = \frac{1}{1 - t^{4g-2}} \operatorname*{Coeff}_{x^{2g-2}} \left(\frac{t^{2g}}{xt^4 - 1} - \frac{t^{4g-2}}{x - t^2} \right) \left(\frac{(1 + xt)^{2g}}{(1 - x)(1 - xt^2)} \right).$$

The following argument, due to Don Zagier, then shows that this equals the Harder-Narasimhan formula. Let

$$F(a, b, c, t) = \operatorname*{Coeff}_{x^{2g-2}} \frac{(1 + xt)^{2g}}{(1 - ax)(1 - bx)(1 - cx)}.$$

Then

$$P_t(N) = \frac{t^{4g-4} F(1, t^2, t^{-2}, t) - t^{2g} F(1, t^2, t^4, t)}{1 - t^{4g-2}}.$$

On the other hand,

$$F(a, b, c, t) = \operatorname*{Res}_{x=0} \left\{ \frac{x^{1-2g}(1 + xt)^{2g}\,dx}{(1 - ax)(1 - bx)(1 - cx)} \right\};$$

since this has no pole at infinity, by the residue theorem

$$\begin{aligned} F(a, b, c, t) &= \left(-\operatorname*{Res}_{x=1/a} - \operatorname*{Res}_{x=1/b} - \operatorname*{Res}_{x=1/c} \right) \left\{ \frac{x^{1-2g}(1 + xt)^{2g}\,dx}{(1 - ax)(1 - bx)(1 - cx)} \right\} \\ &= \frac{(a + t)^{2g}}{(a - b)(a - c)} + \frac{(b + t)^{2g}}{(b - a)(b - c)} + \frac{(c + t)^{2g}}{(c - a)(c - b)}. \end{aligned}$$

After this substitution, it is a matter of high-school algebra to verify (4.2). □

5 Their ample cones

We now turn to a study of the line bundles over the M_i. Indeed, our goal is a formula for the dimension of the space of sections of any line bundle over any M_i. Since M_0 is just a projective space, the first interesting case is M_1; so we first of all ask what line bundles there are on M_1.

(5.1) Pic $M_1 = \mathbb{Z} \oplus \mathbb{Z}$, *generated by the hyperplane H and the exceptional divisor E_1.*

Proof. Obvious from (3.19). □

The case of M_1 will be crucial for us, so we introduce the notation

$$\mathcal{O}_1(m,n) = \mathcal{O}((m+n)H - nE_1),$$
$$V_{m,n} = H^0(M_1; \mathcal{O}_1(m,n)).$$

Pushing down to $M_0 = \mathbb{P}H^1(\Lambda^{-1})$ then yields $V_{m,n} = H^0(M_0; \mathcal{O}(m+n) \otimes \mathcal{I}_X^n)$. That is, an element of $\mathbb{P}V_{m,n}$ is a hypersurface of degree $m+n$ with a singularity of order n at X. The dimension of $V_{m,n}$, which we shall attempt to calculate, is thus a number canonically associated to X, Λ, m, and n.

Of course, in many cases this number is easy to compute. If $m < 0$, for example, then $V_{m,n} = 0$, since no hypersurface can have a singularity of order greater than its degree. If $n < 0$, then $V_{m,n} = H^0(M_0; \mathcal{O}(m+n) \otimes \mathcal{I}_X^n) = H^0(M_0; \mathcal{O}(m+n))$, because codim $X/M_0 = d + g - 3 > 1$ by our assumptions on d and g, and a section cannot have a pole on a set of codimension > 1. So in this case $\dim V_{m,n} = \binom{m+n+d+g-2}{m+n}$. However, for $m, n \geqq 0$, it is quite an interesting problem to calculate $\dim V_{m,n}$. When $n = 1$, these are of course precisely the spaces whose syzygies are studied by Green and Lazarsfeld [16], but for $n > 1$ very little appears to be known.

What about M_i for $i > 1$? These give exactly the same information as M_1, for the following simple reason.

(5.2) *For $i > 0$ there is a natural isomorphism* Pic M_1 = Pic M_i. *Moreover, if by abuse of notation we denote by $\mathcal{O}_i(m,n)$ the image of $\mathcal{O}_1(m,n)$ in* Pic M_i, *then for any m, n there is a natural isomorphism $V_{m,n} = H^0(M_i; \mathcal{O}_i(m,n))$.*

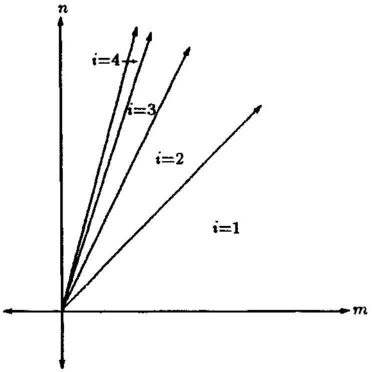
Proof. By (3.6), M_1 is isomorphic to M_i except on sets of codimension $\geqq 2$. Hence divisors, functions, line bundles, and sections can be pulled back from one to the other and extended over the bad sets in a unique way. ☐

However, we will certainly not ignore the higher M_i for the rest of the paper. Instead, they will be indispensable tools in the study of the cohomology of M_1, to be used as follows. A naive approach to calculating $\dim V_{m,n}$ would be to calculate $\chi(M_1; \mathcal{O}_1(m,n))$, which is easy using Riemann-Roch, and then to apply Kodaira vanishing to show that the higher cohomology all vanished. This will not work: the hypothesis of Kodaira vanishing, which is that $K_{M_1}^{-1}\, \mathcal{O}_1(m,n)$ must be ample, will not typically be satisfied, and the higher cohomology will not vanish. But this problem can be cured by shifting attention to some other M_i. Indeed, under some mild hypotheses on m and n, there will be some i such that $K_{M_i}^{-1}\, \mathcal{O}_i(m,n)$ will be ample on M_i. Hence $\dim V_{m,n} = \chi(M_i; \mathcal{O}_i(m,n))$, which will be calculated by an inductive procedure on i.

To carry out this programme, of course, we need to know the ample cone of each M_i. So our goal in this section will be to prove the following theorem.

(5.3) *For $0 < i < w$, the ample cone of M_i is bounded by $\mathcal{O}_i(1, i-1)$ and $\mathcal{O}_i(1, i)$. For $d > 2g - 2$, the ample cone of M_w is bounded by $\mathcal{O}_w(1, w-1)$ and $\mathcal{O}_w(2, d-2)$; for $d \leqq 2g - 2$, it is bounded on one side by $\mathcal{O}_w(1, w-1)$, and contains the cone bounded on the other side by $\mathcal{O}_w(2, d-2)$.*

So as we pass from $i - 1$ to i, the ample cone flips across the ray of slope $i - 1$, as illustrated for $d = 9$ in the figure. This is exactly the behaviour which is predicted by Mori theory; indeed, flips are so named for precisely this reason.



The first thing to notice is that, since all the M_i have unique universal pairs $(\mathbf{E}, \mathbf{\Phi}) \to M_i \times X$, an expression such as $\det \pi_! \mathbf{E}$, or $\Lambda^2 \mathbf{E}_x$ for some $x \in X$, defines line bundles on all the M_i, which agree with one another on the open sets where the maps between different M_i are defined, and which consequently correspond under the natural isomorphism of (5.2). Since $\Lambda^2 \mathbf{E}_x$ and $\det \pi_! \mathbf{E}$ are the canonical (indeed, essentially the only) examples, we work out what they are on M_1.

(5.4) *On* $M_1, \Lambda^2 \mathbf{E}_x = \mathcal{O}_1(0, -1)$ *and* $\det \pi_! \mathbf{E} = \mathcal{O}_1(-1, g-d)$; *that is,* $\mathcal{O}_1(m, n) = \det^{-m} \pi_! \mathbf{E} \otimes (\Lambda^2 \mathbf{E}_x)^{(d-g)m-n}$.

Proof. The universal pair on $M_0 \times X$ is easy to construct directly: it is the tautological extension

$$0 \to \mathcal{O} \to \mathbf{E}_0 \to \Lambda(-1) \to 0$$

determined by the class $\mathrm{id} \in \mathrm{End}\, H^1(X; \Lambda^{-1}) = H^0(\mathbb{P}H^1(\Lambda^{-1}); \mathcal{O}(1)) \otimes H^1(X; \Lambda^{-1}) = H^1((\mathbb{P}H^1(\Lambda^{-1}) \times X; \Lambda^{-1}(1))$, together with the constant section $\mathbf{\Phi}_0 \in H^0(\mathcal{O})$. Recall from (3.17) that the universal pair $(\mathbf{E}_1, \mathbf{\Phi}_1) \to M_1 \times X$ is constructed by pulling back $(\mathbf{E}_0, \mathbf{\Phi}_0)$, twisting by $\mathcal{O}(E_1^+)$, and modifying at E_1^+.

$$0 \to \mathbf{E}_1 \to \mathbf{E}_0(E_1^+) \to \mathcal{O}_{E_1^+ \times X}(\Delta)(-1) \to 0.$$

Hence $\Lambda^2(\mathbf{E}_1)_x = \Lambda^2(\mathbf{E}_0(E_1^+))_x \otimes \mathcal{O}(-E_1^+) = \Lambda^2 \mathbf{E}_0 \otimes \mathcal{O}(E_1^+) = \mathcal{O}_1(0, -1)$, and

$$\begin{aligned} \det \pi_! \mathbf{E}_1 &= \det \pi_! \mathbf{E}_0(E_1^+) \otimes \mathcal{O}((g-2)(E_1^+)) \\ &= \det \pi_! \mathcal{O}(E_1^+) \otimes \det \pi_! \Lambda(-1)(E_1^+) \otimes \mathcal{O}_1(g-2, 2-g) \\ &= \mathcal{O}_1(1-g, g-1) \otimes \mathcal{O}_1(0, -d-1+g) \otimes \mathcal{O}_1(g-2, 2-g) \\ &= \mathcal{O}_1(-1, g-d). \quad \square \end{aligned}$$

The next three results collect some basic information about pullbacks of $\mathcal{O}_i(m, n)$.

(5.5) *The restriction of* $\mathcal{O}_i(m, n)$ *to*

(i) *a fibre of* $\mathbb{P}W_i^+$ *is* $\mathcal{O}(n-(i-1)m)$;

(ii) *a fibre of* $\mathbb{P}W_i^-$ *is* $\mathcal{O}((i-1)m-n)$;

(iii) $f^{-1}(E) \subset M_w$, *where E is a stable bundle and f is the Abel-Jacobi map of* (3.20), *is* $\mathcal{O}(m(d-2)-2n)$.

Proof. By (3.2), the bundle $\mathbf{E}$ in the universal pair restricts to an extension

$$0 \to \mathcal{O}(D) \to \mathbf{E} \to \Lambda(-D)(-1) \to 0$$

on the fibre of $\mathbb{P}W_i^+$ over $D \in X_i$. Hence on this fibre $\Lambda^2 \mathbf{E}_x = \mathcal{O}(-1)$ and

$$\begin{aligned} \det \pi_! \mathbf{E} &= \det \pi_! \mathcal{O}(D) \otimes \det \pi_! \Lambda(-D)(-1) \\ &= \mathcal{O}(-\chi(\Lambda(-D))) = \mathcal{O}(-d+g-1+i). \end{aligned}$$

So by (5.4) $\mathcal{O}_i(m,n)$ restricts to $\mathcal{O}((d-g+1-i)m-(d-g)m+n) = \mathcal{O}((1-i)m+n)$, which proves (i). Similarly by (3.3), $\mathbf{E}$ restricts to an extension

$$0 \to \Lambda(-D)(1) \to \mathbf{E} \to \mathcal{O}(D) \to 0$$

on the fibre of $\mathbb{P}W_i^-$ over $D \in X_i$. Hence $\Lambda^2 \mathbf{E}_x = \mathcal{O}(1)$ and

$$\det \pi_! \mathbf{E} = \det \pi_! \Lambda(-D)(1) \otimes \det \pi_! \mathcal{O}(D) = \mathcal{O}(\chi(\Lambda(-D))) = \mathcal{O}(d-g+1-i).$$

So the previous situation is reversed, and $\mathcal{O}_i(m,n)$ restricts to $\mathcal{O}((i-1)m-n)$, which proves (ii). Finally, on a fibre $\mathbb{P}H^0(E)$ of the Abel-Jacobi map, the universal pair restricts to $E(1)$ with the tautological section. Hence on this fibre $\Lambda^2 \mathbf{E}_x = \mathcal{O}(2)$ and $\det \pi_! \mathbf{E} = \mathcal{O}(d+2-2g)$. So by (5.4) $\mathcal{O}_i(m,n)$ restricts to $\mathcal{O}((2g-2-d)m + 2((d-g)m-n)) = \mathcal{O}(m(d-2)-2n)$, which proves (iii). □

(5.6) *On* $\tilde{M}_i$, $\mathcal{O}_i(m,n) = \mathcal{O}_{i-1}(m,n)(((i-1)m-n)E_i)$.

Proof. Certainly $\mathcal{O}_i(m,n)$ and $\mathcal{O}_{i-1}(m,n)$ are isomorphic away from E_i, so $\mathcal{O}_i(m,n) = \mathcal{O}_{i-1}(m,n)(qE_i)$ for some q. But $\mathcal{O}_i(m,n)$ must be trivial on the fibres of $\mathbb{P}W_i^-$, and $\mathcal{O}_{E_i}(qE_i) = \mathcal{O}(-q,-q)$, so by (5.5 (ii) $q = (i-1)m-n$. □

(5.7) *For an effective divisor D, let* ι_D *be the inclusion of moduli spaces defined in* (1.20). *Then* $\iota_D^* \mathcal{O}_i(m,n) = \mathcal{O}_i(m, n-m|D|)$.

Proof. Choose $x \in X - D$. Then from (5.4) and the long exact sequence in (1.20), $\mathcal{O}_i(0,-1) = \Lambda^2 \mathbf{E}_x^\Lambda = \Lambda^2(\iota^* \mathbf{E}_x^{\Lambda(2D)}) = \iota^* \mathcal{O}_i(0,-1)$. Likewise,

$$\begin{aligned} \mathcal{O}_i(-1, g-d) &= \det \pi_! \mathbf{E}^\Lambda \\ &= \det \pi_! \iota^* \mathbf{E}^{\Lambda(2D)} \otimes \det{}^{-1} \pi_! \mathcal{O}_D(\mathbf{E}^{\Lambda(2D)}) \\ &= \det \pi_! \iota^* \mathbf{E}^{\Lambda(2D)} \otimes \bigotimes_{x \in D} (\Lambda^2 \mathbf{E}_x^{\Lambda(2D)})^{-1} \\ &= \iota^* \mathcal{O}_i(-1, g-d-2|D|) \otimes \iota^* \mathcal{O}_i(0,|D|) \\ &= \iota^* \mathcal{O}_i(-1, g-d-|D|). \quad \square \end{aligned}$$

We now pause to apply these ideas to compute the Picard group of the moduli space N of ordinary semistable bundles of determinant Λ:

(5.8) $$\operatorname{Pic} N = \mathbb{Z}.$$

Proof. If $g = 2$ and d is even, then $N = \mathbb{P}^3$ [20], so the result is obvious. Otherwise, the complement of the stable set $N_s \subset N$ has codimension $\geqq 2$; since N is normal [9], this implies $\operatorname{Pic} N_s = \operatorname{Pic} N$.

By (3.20) the Abel-Jacobi map $f: M_w \to N$ has fibre $\mathbb{P}H^0(E)$ over a stable bundle E. Tensoring by a line bundle, we may of course assume $d > 4g - 4$. But then $H^1(E) = 0$ (see for example the proof of (1.10)), so $\dim \mathbb{P}H^0(E) = d + 2g - 1$ always and f is locally trivial over N_s. Hence $\operatorname{Pic} N_s$ is the subgroup of $\operatorname{Pic} M_w$ whose restriction to each projective fibre of f is trivial. By (5.5 iii) this consists of the bundles $\mathcal{O}_w(k, k(d/2 - 1))$ for $k \in \mathbb{Z}$ (where k is even if d is odd). □

Denote by $\mathcal{O}(\Theta)$ the $\mathbb{Q}$-Cartier divisor class such that $f^* \mathcal{O}(\Theta) = \mathcal{O}_w(1, d/2 - 1)$. Note that this differs slightly from the normalization in [9]. The following is then true for any d, not just $d > 4g - 4$:

(5.9) $$f^* \mathcal{O}(\Theta) = \mathcal{O}_w(1, d/2 - 1).$$

Proof. True by definition if $d > 4g - 4$; follows otherwise from (5.7), since

$$\iota_D^* \mathcal{O}_w(1, d/2 + |D| - 1) = \mathcal{O}_w(1, d/2 - 1). \quad □$$

Now that we know Pic N, we can make the following definition.

(5.10) Definition. *The Verlinde vector spaces* are

$$Z_k(\Lambda) = H^0(N; \mathcal{O}(k\Theta)),$$

with the convention that $Z_k(\Lambda) = 0$ *if* d *and* k *are both odd.*

Verlinde's original papers [7, 24] conjectured a striking formula for the dimensions of these vector spaces, which has since been proved by several authors. We will give our own proof in §7; the first step, however, is the following result, originally due to Bertram [3].

(5.11) *For* $d > 2g - 2$, *there is a natural isomorphism* $Z_k(\Lambda) = V_{k, k(d/2 - 1)}$.

The proof requires the following lemma.

(5.12) *Let* M, N *be varieties with* N *normal, and let* $f: M \to N$ *be a morphism which is generically a projective bundle. Then* $f_* \mathcal{O}_M = \mathcal{O}_N$.

Proof. This is essentially Stein factorization. Let $U \subset N$ be the open set such that $f: f^{-1}(U) \to U$ is a projective bundle. Then certainly $f_* \mathcal{O}_{f^{-1}(U)} = \mathcal{O}_U$, so $N' = \mathbf{Spec} f_* \mathcal{O}_M$ is birational to N. By construction there is a map $f': M \to N'$ such that $f'_* \mathcal{O}_M = \mathcal{O}_{N'}$. On the other hand, since $f_* \mathcal{O}_M$ is a coherent sheaf of $\mathcal{O}_N$-algebras, the birational morphism $N' \to N$ is finite. But a birational finite morphism to a normal variety is an isomorpism—this is essentially Zariski's main theorem; the proof in [15, III 11.4] goes through, or see [18, III.9]. Hence $N' = N$ and $f_* \mathcal{O}_M = \mathcal{O}_N$. □

Proof of (5.11). Recall again from (3.20) that for $d > 2g - 2$, the Abel-Jacobi map $f: M_w \to N$ is surjective with fibre $\mathbb{P}H^0(E)$ over a stable bundle E. If $U \subset N$ is the set of bundles E such that E is stable and $\dim H^0(E)$ is minimal, then certainly $f: f^{-1}(U) \to U$ is a projective bundle; for example it is the descent of a trivial projective bundle over the Quot scheme. Moreover, N is always normal [9]. So by (5.12), $f_* \mathcal{O}_{M_w} = \mathcal{O}_N$. Hence $f_* f^* \mathcal{O}(k\Theta) = \mathcal{O}(k\Theta)$, so that

$$f^*: H^0(N; \mathcal{O}(k\Theta)) \to H^0(M_w; \mathcal{O}_w(k, k(d/2 - 1)))$$

has inverse f_*. □

It is worth mentioning, if not proving, a generalization of this result. Over the stable set $N_s \subset N$, let $\mathbf{E} \to N_s \times X$ be a universal bundle, normalized so that $\Lambda^2 \mathbf{E}|_{N_s \times \{x\}} = \mathcal{O}$. (Actually, such a normalization is impossible for d odd, and $\mathbf{E}$ will not even exist for d even! However, the obstructions are all in $\mathbb{Z}/2$, and will cancel in the cases we are considering; for details see [23].) Then let $U = (R^0\pi)\mathbf{E} \to N_s$.

(5.13) *For $d > 2g - 2$, there is a natural isomorphism $H^0(N_s; S^{m(d-2)-2n}U(m\Theta)) = V_{m,n}$ unless $g = 2$ and d is even.*

Sketch of proof. The complement of $f^{-1}(N_s) \subset M_w$ has codimension $\geqq 2$ unless $g = 2$ and d is even (in which case $N = \mathbb{P}^3$ [20]), so $V_{m,n} = H^0(f^{-1}(N_s); \mathcal{O}(m, n))$. Also $(R^0\pi)\mathcal{O}(m, n)|_{N_s} = S^{m(d-2)-2n}U(m\Theta)$, so

$$H^0(f^{-1}(N_s); \mathcal{O}(m, n)) = H^0(N_s; S^{m(d-2)-2n}U(m\Theta))$$

as in the proof of (5.11). □

Hence seeking a formula for dim $V_{m,n}$ can be regarded as seeking a generalization of the Verlinde formula.

At last we return to the determination of the ample cone of M_i. It can of course be quite difficult to decide whether a given line bundle on a projective variety is ample. However, a geometric invariant theory quotient is naturally endowed with an ample bundle, which is the descent of the ample bundle used in the linearization. So we shall work out how the line bundles used in the linearizations of §1 descend to M_i. Recall that the linearization was some power of $\mathcal{O}(\chi + 2\sigma, 4\sigma) \to \mathbb{P}\mathrm{Hom} \times \mathbb{P}\mathbb{C}^\chi$, or more precisely, its pullback to $\mathrm{Quot}(\Lambda) \times \mathbb{P}\mathbb{C}^\chi$, which by abuse of notation we still denote $\mathcal{O}(\chi + 2\sigma, 4\sigma)$. By further abuse of notation we refrain from worrying about whether $\chi + 2\sigma$ and 4σ are actually integers.

(5.14) *The bundle $\mathcal{O}(\chi + 2\sigma, 4\sigma) \to \mathrm{Quot}(\Lambda) \times \mathbb{P}\mathbb{C}^\chi$ descends to $\mathcal{O}_i(1, d - 1 - 2\sigma) \to M_i$.*

Proof. As in §1, let $U \subset \mathrm{Quot}(\Lambda)$ be the set of quotients $\mathcal{O}^\chi \to E \to 0$ of determinant Λ such that the induced map $\mathbb{C}^\chi \to H^0(E)$ is an isomorphism. If $\mathcal{O}^\chi \to \mathbf{E} \to 0$ is the universal quotient over $U \times X$, then as in (1.19) there is a universal pair $(\mathbf{E}(1), \boldsymbol{\Phi}) \to U \times \mathbb{P}\mathbb{C}^\chi \times X$ descending to the universal $(\mathbf{E}, \boldsymbol{\Phi})$ on each M_i. Hence $\det \pi_! \mathbf{E}(1) \to U \times \mathbb{P}\mathbb{C}^\chi$ descends to $\det \pi_! \mathbf{E} = \mathcal{O}_i(-1, g - d) \to M_i$, and for any $x \in X$, $\Lambda^2 \mathbf{E}(1)_x \to U \times \mathbb{P}\mathbb{C}^\chi$ descends to $\Lambda^2 \mathbf{E}_x = \mathcal{O}_i(0, -1) \to M_i$.

By [15, III Ex. 12.6(b)] $\mathrm{Pic}(U \times \mathbb{P}\mathbb{C}^\chi) = \mathrm{Pic}\, U \oplus \mathrm{Pic}\, \mathbb{P}\mathbb{C}^\chi$. So to determine a bundle on $U \times \mathbb{P}\mathbb{C}^\chi$, it suffices to determine it on $\{E\} \times \mathbb{P}\mathbb{C}^\chi$ and $U \times \{\phi\}$ for some $E \in U$, $\phi \in \mathbb{P}\mathbb{C}^\chi$.

On $\{E\} \times \mathbb{P}\mathbb{C}^\chi$, $\mathbf{E}(1) = E(1)$, so $\det \pi_! \mathbf{E}(1) = \mathcal{O}(\chi)$ and $\Lambda^2 \mathbf{E}_x = \mathcal{O}(2)$. On $U \times \{\phi\}$, $\mathbf{E}(1) = \mathbf{E}$, so $\det \pi_! \mathbf{E}(1) = \det \pi_! \mathbf{E}$. But for all $E \in U$, $H^0(E) = H^0(\mathcal{O}^\chi)$ and $H^1(E) = 0$. Consequently $\det \pi_! \mathbf{E} = \mathcal{O}$. Moreover, there is a canonical map

$$\Lambda^2 \mathbb{C}^\chi = \Lambda^2 H^0(\mathcal{O}^\chi) \to \Lambda^2 H^0(\mathbf{E}) \to H^0(\Lambda^2 \mathbf{E}),$$

so the pullback of $\mathcal{O}(1) \to \mathbb{P}\mathrm{Hom}(\Lambda^2 \mathbb{C}^\chi, H^0(\Lambda))$ to U, also denoted by $\mathcal{O}(1)$, is precisely $(R^0\pi)\, \mathrm{Hom}(\Lambda, \Lambda^2 \mathbf{E})$. This is clearly isomorphic to $\Lambda^2 \mathbf{E}_x = \mathrm{Hom}(\Lambda, \Lambda^2 \mathbf{E})_x$, since $\mathrm{Hom}(\Lambda, \Lambda^2 \mathbf{E})$ is trivial on every fibre of π.

Putting it all together, we find that $\mathcal{O}(0, \chi)$ descends to $\mathcal{O}_i(-1, g-d)$ and $\mathcal{O}(1, 2)$ descends to $\mathcal{O}_i(0, -1)$. The result follows after a little arithmetic. □

Proof of (5.3). For any $\sigma \in (\max(0, d/2 - i - 1), d/2 - i)$, the quotient of $U \times \mathbb{P}\mathbb{C}^\chi$ by the action of $\mathrm{SL}(\chi)$, linearized by $\mathcal{O}(\chi + 2\sigma, 4\sigma)$, gives the same quotient M_i. Hence the descent of $\mathcal{O}(\chi + 2\sigma, 4\sigma)$ to M_i is ample for any σ in that interval. By (5.14) and a little arithmetic these bundles span exactly the cones in the statement of (5.3). Hence those cones are contained in the ample cones of the M_i. It remains to show that no bundles over M_i outside those cones are ample, except possibly on one side for $i = w$ and $d \leqq 2g - 2$.

By (5.5)(i), the restriction of $\mathcal{O}_i(m, n)$ to a fibre of $\mathbb{P}W_i^+$ is $\mathcal{O}(n - (i-1)m)$. So $\mathcal{O}_i(m, n)$ can only be ample over M_i if this is positive, that is, if $(i-1)m < n$. Thus one side of the ample cone of M_i is where it should be.

Likewise by (5.5 ii) the restriction of $\mathcal{O}_{i-1}(m, n) \to M_{i-1}$ to a fibre of $\mathbb{P}W_i^-$ is $\mathcal{O}((i-1)m - n)$. So for $1 < i \leqq w$, that is, when the dimension of this fibre is positive, $\mathcal{O}_{i-1}(m, n)$ can only be ample over M_{i-1} if $(i-1)m > n$. Thus the other side of the ample cone of M_{i-1} is where it should be.

The only case we have not yet treated is the other side of the ample cone of M_w for $d > 2g - 2$. In that case there is by (3.20) a surjective map $M_w \to N$ onto the moduli space of semistable bundles of determinant Λ. It is not an isomorphism, since for example $\mathrm{Pic}\, M_w = \mathbb{Z} \oplus \mathbb{Z}$ while $\mathrm{Pic}\, N = \mathbb{Z}$. Hence the pullback of the ample bundle $\mathcal{O}(2\Theta) \to N$ is nef but not ample, that is, it is in the boundary of the ample cone. But by (5.9) this is precisely $\mathcal{O}(2, d-2)$. □

6 Their Euler chracteristics

Now that we know the ample cones of the M_i, we can calculate $\dim V_{m,n}$ following the programme outlined in the last section. We first need a formula for the canonical bundle of M_i:

(6.1) $$K_{M_i} = \mathcal{O}_i(-3, 4 - d - g).$$

Proof. Clearly the canonical bundle is preserved by the isomorphism of (5.2), so it suffices to work it out on M_1. But this is easy using (3.19) and the standard formulas for the canonical bundle of projective space and of a blow-up. □

(6.2) *Suppose that* $m, n \geqq 0$ *and that* $m(d-2) - 2n > -d + 2g - 2$. *Let* $b = [\frac{n+d+g-4}{m+3}] + 1$. *Then* $\dim V_{m,n} = \chi(M_b; \mathcal{O}_b(m, n))$.

The idea of the proof is that $\dim V_{m,n}$ will be an Euler characteristic by Kodaira vanishing provided that $\mathcal{O}(m, n)$ lies inside some cone in the translate of the ample fan by K. This is illustrated in the figure for the case $d = 9$.

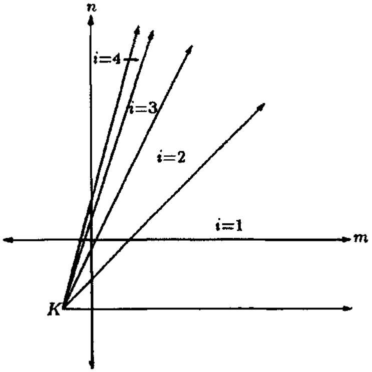
Proof of (6.2). Note first that the inequality can be rewritten

$$(d/2 - 1)(m + 3) > n + d + g - 4,$$

which guarantees that $b \leqq [(d-1)/2]$ and hence that M_b exists.

Suppose that $\frac{n+d+g-4}{m+3}$ is not an integer. Then $b(m+3) > n + d + g - 4 > (b-1)(m+3)$, so $\mathcal{O}_b(m + 3, n + d + g - 4)$, which by (6.1) equals $K_{M_b}^{-1}\mathcal{O}_b(m, n)$, is in the ample cone of M_b by (5.3). The result then follows from (5.2) and Kodaira vanishing.

If $\frac{n+d+g-4}{m+3}$ is an integer, then $\mathcal{O}_{b-1}(m+3, n+d+g-4)$ and $\mathcal{O}_b(m+3, n+d+g-4)$ are merely nef, so Kodaira vanishing does not apply. This case could be handled using the Kawamata-Viehweg vanishing theorem. However, we will take the more elementary approach of moving up to $\tilde{M}_b$. By (5.12) the 0th direct image of $\mathcal{O}_{\tilde{M}_b}$ in the projection $\tilde{M}_b \to M_b$ is $\mathcal{O}_{M_b}$, and by the theorem on cohomology and base change [15, III 12.11] the higher direct images vanish, so for all j, $H^j(\tilde{M}_b; \mathcal{O}_b(m,n)) = H^j(M_b; \mathcal{O}_b(m,n))$. By (6.1) and the standard formula for the canonical bundle of a blow-up, $K_{\tilde{M}_b} = \mathcal{O}_b(-3, 4-d-g)((b-1)E_b)$. Unfortunately $K^{-1}_{\tilde{M}_b}\mathcal{O}_b(m,n)$ may not be ample, so Kodaira vanishing still does not apply. Instead, we make the following two claims: first, that $H^j(\tilde{M}_b; \mathcal{O}_b(m,n)) = H^j(\tilde{M}_b; \mathcal{O}_b(m,n)((b-2)E_b))$ for all j, and second, that $\mathcal{O}_b(m+3, n+d+g-4)(-E_b)$ is ample on $\tilde{M}_b$. The desired result follows immediately from these claims, since at last Kodaira vanishing applies to $\mathcal{O}_b(m,n)((b-2)E_b)$.



To prove the first claim, note that for $0 < k < b$, $H^j(E_b; \mathcal{O}_b(m,n)(kE_b)) = 0$ for all j, since $\mathcal{O}_b(m,n)(kE_b)$ is $\mathcal{O}(-k)$ on each fibre of $\mathbb{P}^{b-1} \to E_b \to \mathbb{P}W_b^+$, so that every term in the Leray spectral sequence vanishes. Hence from the long exact sequence on $\tilde{M}_b$ of

$$0 \to \mathcal{O}_b(m,n)((k-1)E_b) \to \mathcal{O}_b(m,n)(kE_b) \to \mathcal{O}_b(m,n)\,\mathcal{O}_{E_b}(kE_b) \to 0,$$

we get isomorphisms $H^j(\tilde{M}_b; \mathcal{O}_b(m,n)((k-1)E_b)) = H^j(\tilde{M}_b; \mathcal{O}_b(m,n)(kE_b))$. The first claim follows by induction.

As for the second claim, note that on $\tilde{M}_b$, the line bundles $\mathcal{O}_{b-1}(1, b-2)$, $\mathcal{O}_b(1, b-1)$, and $\mathcal{O}_b(1,b)$ (or $\mathcal{O}_b(2, 2b-1)$ if $b = w$) are all nef, since they are pulled back from nef bundles on M_{b-1} or M_b. It is easy using (5.6), the constraints on m and n, and a little arithmetic to check that $\mathcal{O}_b(m+3, n+d+g-4)(-E_b)$ is in the interior of the cone generated by these three bundles. □

We will have to assume in future that

(6.3) $$m(d-2) - 2n > -d + 2g - 2,$$

since otherwise there is no analogue of the last result and $K_{M_i}^{-1}\ \mathcal{O}_i(m, n)$ may not be ample for any i. However, for $d \geqq 2g$, we still get a complete answer to our problem, for the following reason.

(6.4) *For $d \geqq 2g$ and $m(d-2) - 2n < 0$, $V_{m,n} = 0$.*

Proof. By Riemann-Roch $\deg \mathrm{E} \geqq 2g$ implies $\dim H^0(E) \geqq 2$, so for any stable bundle E, by (3.20) the fibre $f^{-1}(E)$ of the Abel-Jacobi map is a projective space of positive dimension. By (5.5 iii), the restriction of $\mathcal{O}_w(m, n)$ to this is $\mathcal{O}(m(d-2) - 2n)$, so any section of $\mathcal{O}_w(m, n)$ must vanish on $f^{-1}(E)$. Hence it must vanish on the inverse image $f^{-1}(N_s)$ of the stable subset of N. But this is open, so it must vanish everywhere. ☐

Let $L_i \to X_i$ be the line bundle defined by $L_i = \det^{-1} \pi_! \Lambda(-\Delta) \otimes \det^{-1} \pi_! \mathcal{O}(\Delta)$. Also put $q_i = n - (i-1)m$.

(6.5) *The restriction of $\mathcal{O}_{i-1}(m, n)$ to $\mathbb{P}W_i^-$ is $L_i^m(-q_i)$.*

Proof. Easy from (5.4) and the description of the universal pair over $\mathbb{P}W_i^-$ in (3.3). ☐

Now let $U_i \to X_i$ be the vector bundle $(W_i^-) \oplus (W_i^+)^*$, and define numbers

$$N_i = \chi(X_i; L_i^m \otimes \Lambda^i W_i^- \otimes S^{q_i - i} U_i),$$

with of course the convention that this is zero when $q_i - i < 0$. On M_0, which is just projective space, make the additional convention that $\mathcal{O}_0(m, n) = \mathcal{O}(m+n)$.

$$\textbf{(6.6)} \qquad N_0 = \chi(M_0; \mathcal{O}_0(m, n)) = \binom{m + n + d + g - 2}{m + n}.$$

Proof. Since X_0 is just a point and $W_0^- = 0$, $U_0 = (W_0^+)^*$ is just the vector space $H^1(\Lambda^{-1})^*$. Hence $S^{m+n} U_0 = H^0(M_0; \mathcal{O}_0(m, n))$ with our conventions and the result follows. ☐

(6.7) *Let $0 < i \leqq b$, and suppose that $m, n \geqq 0$ satisfy* (6.3). *Then*

$$\chi(M_i; \mathcal{O}_i(m, n)) - \chi(M_{i-1}; \mathcal{O}_{i-1}(m, n)) = (-1)^i N_i.$$

Proof. By (5.12) the 0th direct image of $\mathcal{O}_{\tilde{M}_i}$ in the projection $\tilde{M}_i \to M_i$ is $\mathcal{O}_{M_i}$, and by the theorem on cohomology and base change [15, III 12.11] the higher direct images vanish, so $\chi(\tilde{M}_i; \mathcal{O}_i(m, n)) = \chi(M_i; \mathcal{O}_i(m, n))$. Likewise $\chi(\tilde{M}_i; \mathcal{O}_{i-1}(m, n)) = \chi(M_{i-1}; \mathcal{O}_{i-1}(m, n))$, so it suffices to work on $\tilde{M}_i$.

Suppose first that $q_i \leqq 0$, so that $N_i = 0$. For $0 < j \leqq -q_i$, consider the exact sequence

$$0 \to \mathcal{O}_{i-1}(m, n)((j-1)E_i) \to \mathcal{O}_{i-1}(m, n)(jE_i) \to \mathcal{O}_{i-1}(m, n) \otimes \mathcal{O}_{E_i}(jE_i) \to 0.$$

By (6.5) the restriction of $\mathcal{O}_{i-1}(m, n)$ to $E_i = \mathbb{P}W_i^- \oplus \mathbb{P}W_i^+$ is $L_i^m(-q_i, 0)$, and $\mathcal{O}_{E_i}(E_i) = \mathcal{O}(-1, -1)$, so the third term of the exact sequence becomes $\mathcal{O}(-q_i - j, -j)$ and we get

$$\chi(\tilde{M}_i; \mathcal{O}_{i-1}(m, n)(jE_i)) - \chi(\tilde{M}_i; \mathcal{O}_{i-1}(m,n)((j-1)E_i)) = \chi(E_i; L_i^m(-q_i - j, -j)).$$

Summing over j and using (5.6) yields

$$\chi(\tilde{M}_i; \mathcal{O}_i(m,n)) - \chi(\tilde{M}_i; \mathcal{O}_{i-1}(m,n)) = \sum_{j=1}^{q_i} \chi(E_i; L_i^m(-q_i - j, \ -j)).$$

However, for $0 < i \leqq b$ and $m, n, d, g \geqq 0$, a little high-school algebra shows $-q_i < d + g - 1 - 2i$. Hence for all j in the sum above, $0 < j < d + g - 1 - 2i$, so every term in the Leray sequence of the fibration $\mathbb{P}^{d+g-2-2i} \to E_i \to \mathbb{P}W_i^-$ vanishes. Hence all terms are zero, as desired.

Now suppose $q_i > 0$. By an argument similar to the one above,

$$\chi(\tilde{M}_i; \mathcal{O}_i(m,n)) - \chi(\tilde{M}_i; \mathcal{O}_{i-1}(m,n)) = \sum_{j=0}^{q_i - 1} \chi(E_i; L_i^m(-q_i + j, \ +j)).$$

Each term of the right-hand side can be evaluated using the Leray sequence of the fibration $\mathbb{P}^{i-1} \times \mathbb{P}^{d+g-2-2i} \to E_i \to X_i$. Because $-q_i + j < 0 \leqq j$, the only nonzero direct image of $L_i^m(-q_i + j, j)$ is the ith, which is just $L_i^m \otimes \Lambda^i W_i^- \otimes S^{q_i - j - i}(W_i^-) \otimes S^j(W_i^+)^*$. Here the factor of $\Lambda^i W_i^-$ comes from Serre duality, since the isomorphism $\mathcal{O}(-i) = K_{\mathbb{P}^{i-1}}$ is not canonical unless the right-hand side is tensored by such a factor. Hence

$$\chi(E_i; L_i^m(-q_i + j, j)) = (-1)^i \chi(X_i; L_i^m \otimes \Lambda^i W_i^- \otimes S^{q_i - j - i}(W_i^-) \otimes S^j(W_i^+)^*).$$

Of course the right-hand side is zero if $q_i - j - i < 0$, so the sum need only run up to $q_i - i$. The result follows because certainly

$$S^{q_i - i} U_i = \bigoplus_{j=0}^{q_i - i} S^{q_i - j}(W_i^-) \otimes S^j(W_i^+)^*. \quad \square$$

(6.8) *For $i > b$, $N_i = 0$.*

Proof. It suffices to show that if $i > b$, then $q_i - i < 0$, that is, $(m+n)/(m+1) < i$. But using $m, n \geqq 0$, the definition of b, and the inequality (6.3), it is a matter of high-school algebra to check $(m+n)/(m+1) \leqq b$. ☐

(6.9)

$$\dim V_{m,n} = \sum_{i=0}^{\infty} (-1)^i N_i.$$

Proof. Put together (6.2), (6.6), (6.7), and (6.8). ☐

Since each N_i can be evaluated using Riemann-Roch on X_i, the right-hand side depends only on g, d, m, and n, not on the precise geometry of X and Λ. So even before doing the hard work of the next section, we have found that $\dim V_{m,n}$ depends only on g, d, m, and n, which is rather surprising.

7 Don Zagier to the rescue

All of the results in this section (except (7.4) and (7.5)) are due to Don Zagier, who communicated them to the author.

In this section we will compute the N_i, using the Riemann-Roch theorem and Macdonald's description [17] of the cohomology ring of X_i. So we begin with a review of Macdonald's results. Let $e_i, \ldots, e_g, e_1', \ldots, e_g' \in H^1(X; \mathbb{Z})$ be generators such that the intersection form is $\sum_j e_j \otimes e'_j$. Define classes $\xi, \xi' \in H^1(X_i; \mathbb{Z})$ and $\eta \in H^2(X_i; \mathbb{Z})$ as the Künneth components of the divisor $\Delta \subset X_i \times X$, regarded as belonging to $H^2(X_i \times X; \mathbb{Z})$:

$$\Delta = \eta + \sum_j (\xi'_j e_j - \xi_j e'_j) + iX.$$

These generate the ring $H^*(X_i; \mathbb{Z})$. Moreover, if we put $\sigma_j = \xi_j \xi'_j$, then for any multiindex I without repeats,

$$\langle \eta^{i-|I|} \sigma_I, X_i \rangle = 1. \tag{7.1}$$

This implies that for any two power series $A(x)$, $B(x)$,

$$\begin{aligned} \langle A(\eta) \exp(B(\eta)\sigma), X_i \rangle &= \sum_{k=0}^{\infty} \langle A(\eta) B(\eta)^k \sigma^k / k!, X_i \rangle \\ &= \sum_{k=0}^{g} \binom{g}{k} \operatorname*{Res}_{\eta=0} \left\{ \frac{A(\eta) B(\eta)^k}{\eta^{i-k+1}} \, d\eta \right\} \\ &= \operatorname*{Res}_{\eta=0} \left\{ \frac{A(\eta)(1 + \eta B(\eta))^g}{\eta^{i+1}} \, d\eta \right\}, \end{aligned} \tag{7.2}$$

where $\sigma = \sum_j \sigma_j$. Note that since $\sigma_j^2 = 0$, $\sigma^k / k!$ is just the kth symmetric polynomial in the σ_j.

Since we will be doing Riemann-Roch, we need to know the Todd class of X_i; luckily this can be worked out in a useful form.

$$\operatorname{td} X_i = \left(\frac{\eta}{1 - e^{-\eta}} \right)^{i-g+1} \exp\left(\frac{\sigma}{e^{\eta} - 1} - \frac{\sigma}{\eta} \right). \tag{7.3}$$

Proof. Macdonald [17] shows that the total Chern class of the tangent bundle of X_i is

$$c(X_i) = (1 + \eta)^{i-2g+1} \prod_{j=1}^{g} (1 + \eta - \sigma_j).$$

Let $h(x) = x/(1 - e^{-x})$, so that

$$\operatorname{td} X_i = h(\eta)^{i-2g+1} \prod_{j=1}^{g} h(\eta - \sigma_j).$$

Expanding $h(\eta - \sigma_j)$ in a power series around η and using $\sigma_j^2 = 0$,

$$\begin{aligned} \operatorname{td} X_i &= h(\eta)^{i-g+1} \prod_{j=1}^{g} \left(1 - \sigma_j \frac{h'(\eta)}{h(\eta)} \right) \\ &= h(\eta)^{i-g+1} \sum_{k=0}^{\infty} (-1)^k \frac{\sigma^k}{k!} \left(\frac{h'(\eta)}{h(\eta)} \right)^k \\ &= h(\eta)^{i-g+1} \exp\left(-\sigma \frac{h'(\eta)}{h(\eta)} \right), \end{aligned}$$

which yields the desired formula. □

(7.4) *For any line bundle $M \to X$ and any $k \in \mathbb{Z}$,*

$$\operatorname{ch} \pi_! M(k\Delta) = ((\deg M + ki + 1 - g) - k^2\sigma)e^{k\eta}.$$

Proof. By Grothendieck-Riemann-Roch

$$\begin{aligned} \operatorname{ch} \pi_! M(k\Delta) &= \pi_* \operatorname{ch} M(k\Delta) \operatorname{td} X \\ &= \pi_* \exp((\deg M + ki)X + k\Xi + k\eta)(1 + (1-g)X) \\ &= \pi_*(1 + (\deg M + ki)X)(1 + k\Xi - k^2\sigma X)e^{k\eta}(1 + (1-g)X) \\ &= ((\deg M + ki + 1 - g) - k^2\sigma)e^{k\eta}, \end{aligned}$$

where $\Xi = \sum_j (\xi_j' e_j - \xi_j e_j')$, so that $\Xi^2 = -2\sigma X$. □

(7.5) (i) $\operatorname{ch}(L_i) = \exp((d - 2i)\eta + 2\sigma)$;
(ii) $\operatorname{ch}(\Lambda^i W_i^-) = \exp((d - 3i + 1 - g)\eta + 3\sigma)$;
(iii) $\operatorname{ch}(U_i) = (d - i + 1 - 2g)e^{-\eta} + (2g - 2)e^{-2\eta} + \sum_{j=1}^{g} e^{-\eta-\sigma_j}$.

Proof. Since $L_i = \det^{-1} \pi_! \Lambda(-\Delta) \otimes \det^{-1} \pi_! \mathcal{O}(\Delta)$, by (7.4)

$$\begin{aligned} c_1(L_i) &= -c_1(\pi_! \Lambda(-\Delta)) - c_1(\pi_! \mathcal{O}(\Delta)) \\ &= (d - i + 1 - g)\eta + \sigma + (-i - 1 + g)\eta + \sigma = (d - 2i)\eta + 2\sigma, \end{aligned}$$

which implies (i). From the exact sequence

$$0 \to \Lambda(-2\Delta) \to \Lambda(-\Delta) \to \mathcal{O}_\Delta \Lambda(-\Delta) \to 0,$$

it follows that $W_i^- = \pi_! \mathcal{O}_\Delta \Lambda(-\Delta) = \pi_! \Lambda(-\Delta) - \pi_! \Lambda(-2\Delta)$ in K-theory. Hence by (7.4)

$$\operatorname{ch} W_i^- = ((d - i + 1 - g) - \sigma)e^{-\eta} - ((d - 2i + 1 - g) - 4\sigma)e^{-2\eta}.$$

In particular

$$\begin{aligned} c_1(\Lambda^i W_i^-) = c_1(W_i^-) &= -(d - i + 1 - g)\eta - \sigma + 2(d - 2i + 1 - g)\eta + 4\sigma \\ &= (d - 3i + 1 - g)\eta + 3\sigma, \end{aligned}$$

which implies (ii). Again by (7.4),

$$\operatorname{ch}(W_i^+)^* = \operatorname{ch} \pi_! \Lambda^{-1}(2\Delta) = ((d - 2i + g - 1) - 4\sigma)e^{-2\eta}.$$

Hence

$$\begin{aligned} \operatorname{ch} U_i &= \operatorname{ch}(W_i^-) \oplus (W_i^+)^* \\ &= ((d - i + 1 - g) - \sigma)e^{-\eta} + (2g - 2)\, e^{-2\eta} \\ &= (d - i + 1 - 2g)e^{-\eta} + (2g - 2)\, e^{-2\eta} + \sum_{j=1}^{g} e^{-\eta-\sigma_j}, \end{aligned}$$

which is (iii). □

(7.6) $\operatorname{ch}(L_i^m \otimes \Lambda^i W_i^- \otimes S^{q_1 - i} U_i)$

$$= \operatorname*{Coeff}_{t^{q_1 - i}} \left[e^{(m(d-2) - 2n)\eta} \exp\left((2m + 3)\sigma - \frac{t\sigma}{e^{-\eta} - t} \right) \frac{(e^{-\eta} - t)^{-d+i-1+g}}{(1 - t)^{2g-2}} \right].$$

Proof. The Chern roots of $S^k U_i$ are the sums of k (not necessarily distinct) Chern roots of U_i, so by (7.5 iii)

$$\begin{aligned}\sum_{k=0}^{\infty} \operatorname{ch}(S^k U_i) t^k &= \prod_{\substack{\text{Chern roots} \\ \alpha \text{ of } U_i}} \frac{1}{1 - te^{\alpha}} \\ &= \left(\frac{1}{1 - te^{-\eta}}\right)^{d-i+1-2g} \left(\frac{1}{1 - te^{-2\eta}}\right)^{2g-2} \prod_{j=1}^{g} \left(\frac{1}{1 - te^{-\eta-\sigma_j}}\right) \\ &= \frac{(1 - te^{-\eta})^{-d+i-1+g}}{(1 - te^{-2n})^{2g-2}} \exp\left(\frac{-t\sigma}{e^{\eta} - t}\right).\end{aligned}$$

Replacing t by te^{2n} and taking coefficients of t^{q_i-i} yields

$$\operatorname{ch}(S^{q_i-i} U_i) = \operatorname*{Coeff}_{t^{q_i-i}} \left[e^{-2(q_i-i)\eta} \frac{(1 - te^{\eta})^{-d+i-1+g}}{(1-t)^{2g-2}} \exp\left(\frac{-t\sigma}{e^{-\eta} - t}\right) \right].$$

The result then follows using (7.5 i) and (ii) and the pleasing identity

$$m(d - 2i) + (d - 3i + 1 - g) - 2(q_i - i) = m(d - 2) - 2n + (d - i + 1 - g). \quad \square$$

We are now ready to perform our Riemann-Roch calculation:

$$\begin{aligned} N_i &= \langle \operatorname{ch}(L_i^m \otimes \Lambda^i W_i^- \otimes S^{q_i-i} U_i) \operatorname{td}(X_i), X_i \rangle \\ &= \operatorname*{Coeff}_{t^{q_i-i}} \Big\langle e^{(m(d-2)-2n)\eta} \exp\left((2m+3)\sigma - \frac{t\sigma}{e^{-\eta} - t}\right) \frac{(e^{-\eta} - t)^{-d+i-1+g}}{(1-t)^{2g-2}} \\ &\quad \times \left(\frac{\eta}{1 - e^{-\eta}}\right)^{i-g+1} \exp\left(\frac{\sigma}{e^{\eta} - 1} - \frac{\sigma}{\eta}\right), X_i \Big\rangle \\ &= \operatorname*{Coeff}_{t^{q_i-i}} \operatorname*{Res}_{\eta=0} \Bigg\{ \frac{e^{((d-2)m-2n)\eta} (e^{-\eta} - t)^{-d+i-1+g}}{(1+t)^{2g-2} (1 - e^{-\eta})^{i+1}} \\ &\qquad \left(e^{-\eta} + \left(2m + 3 - \frac{t}{e^{-\eta} - t}\right)(1 - e^{-\eta})\right)^{g} d\eta \Bigg\}; \end{aligned} \tag{7.7}$$

the first equality by Riemann-Roch, the second by (7.3) and (7.6), and the third by taking

$$A(x) = \left(\frac{x}{1 - e^{-x}}\right)^{i-g+1} e^{((d-2)m-2n)x} \frac{(e^{-x} - t)^{-d+i-1+g}}{(1+t)^{2g-2}}$$

and

$$B(x) = 1/(e^x - 1) - 1/x + 2m + 3 - t/(e^{-x} - t)$$

in (7.2), then combining gth powers.

The term in braces is the product of $(\frac{e^{-\eta}-t}{1-e^{-\eta}})^i$ with something independent of i, so make the substitution

$$y = \frac{e^{-\eta} - t}{1 - e^{-\eta}}, \quad e^{-\eta} = \frac{1 + ty}{1 + y}, \quad 1 - e^{-\eta} = \frac{(1-t)y}{1+y},$$

$$e^{-\eta} - t = \frac{1-t}{1+y}, \quad d\eta = \frac{(1-t)dy}{(1+y)(1+ty)}.$$

Then the residue in (7.7) becomes

$$\operatorname*{Res}_{y=0}\left\{\frac{a(y)\,dy}{y^{i+1}}\right\} = \operatorname*{Coeff}_{y^i} a(y)$$

for

$$a(y) = \frac{(1+ty)^{2q_{d/2}-1}(1+y)^{-2q_{d/2}+d-2g+1}}{(1-t)^{d+g-1}}(1+(2m+3)(1-t)y-ty^2)^g.$$

Then since $q_i - i = (m+n) - (m+1)i$,

$$\begin{aligned}\dim V_{m,n} &= \sum_{i=0}^{\infty}(-1)^i N_i\\ &= \sum_{i=0}^{\infty}(-1)^i \operatorname*{Coeff}_{t^{q_i-i}}\operatorname*{Coeff}_{y^i} a(y)\\ &= \operatorname*{Coeff}_{t^{m+n}}\left(\sum_{i=0}^{\infty}(-t^{m+1})^i \operatorname*{Coeff}_{y^i} a(y)\right)\\ &= \operatorname*{Coeff}_{t^{m+n}} a(-t^{m+1}).\end{aligned}$$

Thus we obtain the following theorem. We repeat the definition of $V_{m,n}$ for convenience.

(7.8) *Let X be embedded in $\mathbb{P}H^1(\Lambda^{-1})$ via the linear system $|K_X\Lambda|$. For any $m, n \geqq 0$, let $V_{m,n} = H^0(\mathbb{P}H^1(\Lambda^{-1}); \mathcal{O}(m+n)\otimes\mathcal{I}_X^n)$. Define*

$$F(t) = \frac{(1-t^{m+2})^{-h-1}(1-t^{m+1})^{-h'-1}}{(1-t)^{d+g-1}t^{m+n}}(1-(2m+3)(1-t)t^{m+1}-t^{2m+3})^g,$$

where $h = (d-2)m - 2n$ and $h' = -h - d + 2g - 2$. Then if $m(d-2) - 2n > -d + 2g - 2$,

$$\dim V_{m,n} = \operatorname*{Res}_{t=0}\left\{\frac{F(t)\,dt}{t}\right\},$$

that is, the constant term in the Laurent expansion of $F(t)$ at $t = 0$. Moreover, if $d \geqq 2g$ and $m(d-2) - 2n < 0$, then $V_{m,n} = 0$.

This is the most explicit formula for dim $V_{m,n}$ we will obtain in general. However, in some cases we could obtain completely explicit formulas. If $m + n$ is small, for example, we could calculate directly, since we would then be looking at the residue of a function with a pole of low order; for fixed $m + n$, we would get an explicit polynomial in g, d, m, and n. Otherwise, we can still use the residue theorem, which says that the sum of the residues at all the poles of $F(t)dt/t$ is zero. These poles are of five possible kinds: $t = 0$, $t = \infty$, $t = 1$, $t^{m+1} = 1$ but $t \neq 1$, and $t^{m+2} = 1$ but $t \neq 1$ (note that the last two cases are disjoint). But in fact $t = 1$ is never a pole, since at that point $1 - (2m+3)(1-t)t^{m+1} - t^{2m+3}$ has a triple zero, and hence the order of $F(t)$ is

$$(-h-1) + (-h'-1) - (d+g-1) + 3g = 1 \geqq 0.$$

Also, it is straightforward to check that $F(1/t) = -F(t)$, which implies that

$$\operatorname*{Res}_{t=\infty}\left\{\frac{F(t)\,dt}{t}\right\} = \operatorname*{Res}_{t=0}\left\{\frac{F(t)\,dt}{t}\right\}.$$

Hence

(7.9) $$-2\dim V_{m,n} = \left(\sum_{\substack{\zeta^{m+1}=1\\ \zeta\neq 1}}\operatorname*{Res}_{t=\zeta} + \sum_{\substack{\zeta^{m+2}=1\\ \zeta\neq 1}}\operatorname*{Res}_{t=\zeta}\right)\left\{\frac{F(t)\,dt}{t}\right\}.$$

There are poles at the $(m+2)$th roots of unity if and only if $h \geqq 0$, and at the $(m+1)$th roots of unity if and only if $h' \geqq 0$. Thus $\dim V_{m,n}$ is a sum over the residues at the $(m+2)$th roots if $h' < 0 \leqq h$, a sum over the residues at the $(m+1)$th roots if $h < 0 \leqq h'$, and is 0 if $h, h' < 0$. (Note that this last case agrees with (6.4).) For $h \geqq 0$ it is necessary to calculate the residue of a function with a pole of order $1 + h$, which gets more and more difficult as h grows. However, when $h = 0$, the calculation is easy, and we can prove the celebrated Verlinde formula.

(7.10) $$\dim Z_k(\Lambda) = \left(\frac{k+2}{2}\right)^{g-1}\sum_{j=1}^{k+1}\frac{(-1)^{d(j+1)}}{\left(\sin\frac{j\pi}{k+2}\right)^{2g-2}}.$$

Proof. If d and k are both odd, then on symmetry grounds the right-hand side is zero as desired. So assume d and k are not both odd. By (5.11) $\dim Z_k(\Lambda) = \dim V_{k,k(d/2-1)}$ for any $d > 2g-2$. Then $h = 0$ and $h' < 0$, so by (7.9)

$$-2\dim V_{k,k(d/2-1)} = \sum_{\substack{J^{k+2}=1\\ \zeta\neq 1}}\operatorname*{Res}_{t=\zeta}\left(\frac{-dt/t}{t^{k+2}-1}\right)\frac{(1-\zeta^{-1})^{d-2g+1}}{(1-\zeta)^{d+g-1}\zeta^{kd/2}}$$
$$\times(1-(2k+3)(\zeta^{-1}-1)-\zeta^{-1})^{g}.$$

But $(1-(2k+3)(\zeta^{-1}-1)-\zeta^{-1}) = (2k+4)(1-\zeta^{-1})$, the residue is $-1/(k+2)$, and

$$\frac{(1-\zeta^{-1})^d}{(1-\zeta)^d\zeta^{kd/2}} = \frac{(1-\zeta^{-1})^d}{(1-\zeta)^d\zeta^{-d}\zeta^{(k+2)d/2}} = (-1)^d\zeta^{(k+2)d/2},$$

so

$$\dim V_{k,k(d/2-1)} = (2k+4)^{g-1}\sum_{\substack{\zeta^{k+2}=1\\ \zeta\neq 1}}(-1)^d\zeta^{(k+2)d/2}\left(\frac{-\zeta}{(1-\zeta)^2}\right)^{g-1}$$
$$= \tfrac{1}{2}(2k+4)^{g-1}\sum_{\substack{\xi^{2k+4}=1\\ \xi\neq\pm 1}}\frac{(-1)^{d+g-1}\xi^{(k+2)d}}{(\xi^{-1}-\xi)^{2g-2}},$$

which is equivalent to the Verlinde formula. □

8 Relation with Bertram's work

In this appendix we explain briefly, without proving anything, how this paper is related to Bertram's work on secant varieties.

In [3], Bertram considers how to resolve the rational map $\mathbb{P}H^1(\Lambda^{-1}) \to N$. He shows that blowing up first $X \subset \mathbb{P}H^1(\Lambda^{-1})$, then the proper transform of each of its secant varieties in turn, produces after $[(d-1)/2]$ steps a smooth variety $\tilde{\mathbb{P}}$ having a morphism to N that agrees with the rational map away from the blow-ups. The existence of the morphism is proved by constructing a sequence of families of bundles, each obtained by an elementary transformation of the last, starting with the pullback of the tautological family on $\mathbb{P}H^1(\Lambda^{-1}) \times X$, and ending with a family of bundles that are all semistable. Bertram's families of bundles can be interpreted, after some twisting, as families of pairs in our sense, and it follows that his $\tilde{\mathbb{P}}$ dominates all of the M_i. In other words, he performs all of our blow-ups but none of our blow-downs. In particular, our blow-up loci are birational to his, that is, our $\mathbb{P}W_i^-$ in M_{i-1} is the proper transform of the ith secant variety in $\mathbb{P}H^1(\Lambda^{-1}) = M_0$. This makes sense, since both are essentially $\mathbb{P}^{i-1}$-bundles over X_i.

However, this correspondence is a little more delicate than it seems, because the $\mathbb{P}^{i-1}$-bundles are different: ours is $\mathbb{P}W_i^- = \mathbb{P}(R^0\pi)\,\mathcal{O}_\Delta \Lambda(-\Delta)$, but as Bertram explains, the secant variety is the image in $\mathbb{P}H^1(\Lambda^{-1})$ of $\mathbb{P}(R^0\pi)\,\mathcal{O}_\Delta K\Lambda$. How is one projective bundle transformed into another? If we pull back the lower secant varieties to $\mathbb{P}(R^0\pi)\,\mathcal{O}_\Delta K\Lambda$ we find that blowing them up and down induces a *Cremona transformation* on each fibre of the projective bundle. For example, consider the $\mathbb{P}^2$ fibre over $x_1 + x_2 + x_3 \in X_3$ of the 3rd secant variety. This of course meets $X \subset \mathbb{P}H^1(\Lambda^{-1})$ in the 3 points x_1, x_2, x_3, so if X is blown up, then $\mathbb{P}^2$ gets blown up at those 3 points. The proper transform of the 2nd secant variety meets this blown-up $\mathbb{P}^2$ in the proper transforms of the 3 lines between the points, so blowing it up does nothing, and blowing it down blows down the 3 lines. All in all we have blown up the vertices of a triangle in the plane, then blown down the proper transforms of the edges. This is well-known to recover $\mathbb{P}^2$ [15, V 4.2.3]; indeed it is given in coordinates by $[z_0, z_1, z_2] \mapsto [z_1 z_2, z_0 z_2, z_0 z_1]$.

If we do the same thing to $\mathbb{P}^3$, we find ourselves blowing up the vertices of a tetrahedron, then blowing up and down – that is to say, flipping – the proper transforms of the edges, and finally blowing down the proper transforms of the faces. Notice that by the time we get to the faces, they have already undergone Cremona transformations themselves. More generally, starting with a simplex in $\mathbb{P}^n$, we may flip all of the subsimplices, starting with the vertices and working our way up. The varieties we obtain thus fit into a diagram shaped exactly like that at the end of §3. It is not so well-known that this recovers $\mathbb{P}^n$, or that it is given in coordinates by $[z_i] \mapsto [z_0 \cdots z_{i-1} z_{i+1} \cdots z_n]$, but these facts can be proved using the theory of toric varieties.

Even that is not quite the end of the story, since over divisors in X_i with multiple points the transformations are somewhat different. Over $2x_1 + x_2 \in X_3$, for example, we want to blow up one reduced point and one doubled point, then blow down one reduced line and one doubled line. In coordinates, this is $[z_0, z_1, z_2] \mapsto [z_0^2, z_0 z_1, z_1 z_2]$. It is an amusing exercise to work out coordinate expressions for the Cremona transformations over other divisors with multiple points.

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