EQUATIONS OF MOTION OF THE RESTRICTED PROBLEM OF THREE BODIES WITH VARIABLE MASS

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Abstract. In this paper we have deduced the differential equations of motion of the restricted problem of three bodies with decreasing mass, under the assumption that the mass of the satellite varies with respect to time. We have applied Jeans law and the space time transformation contrast to the transformation of Meshcherskii. The space time transformation is applicable only in the special case $n = 1$, $k = 0$, $q = \frac{1}{2}$. The equations of motion of our problem differ from the equations of motion of the restricted three body problem with constant mass only by small perturbing forces.

1. Introduction

The two body problem with variable mass has been considered by Jeans (1928) in the study of the evolution of a binary system, since then many important results have been obtained from the physical and from the mathematical point of view.

The restricted problem of perturbed motion of two bodies with variable mass has been considered by T. B. Omarov (1963). He has shown that with the permissible rates of mass variation of binary system, the force is extremely small practically during all of the time of motion as compared to the modulus of the resultant of other two forces taken into consideration.

There are numerous practical problems where the mass does not remain constant. Recently it has been shown that the mass of Jupiter is increasing. The phenomenon of isotropic radiation or absorption of mass in stars, led us to think of the restricted three body problem with decreasing mass.

Meshcherskii assumed that mass is ejected isotropically from the two body system at very high velocities and is lost to the system. The change in orbits and the variation in quantities such as angular momentum and the energy of the system was examined by him. A number of authors (1975) do not realise that each layer of ejected mass takes with it a quantity of momentum determined by the instantaneous velocity of the ejecting point mass. This assumption constitutes the essential difference from the model for a two body system with increasing mass. In this way we have discussed the mechanics of the problem, as described above by Meshcherskii and others (1975).

In this paper we have found the differential equations of motion of our problem. We have supposed that the mass of the finite bodies remain constant whereas the mass of the satellite decreases with respect to time. The problem is restricted in the sense that the mass of satellite is taken to be infinitesimally small. Hence, the force of attraction on the two finite masses by the satellite is neglected.

With the help of Jeans law and the space time transformation in contrast to the

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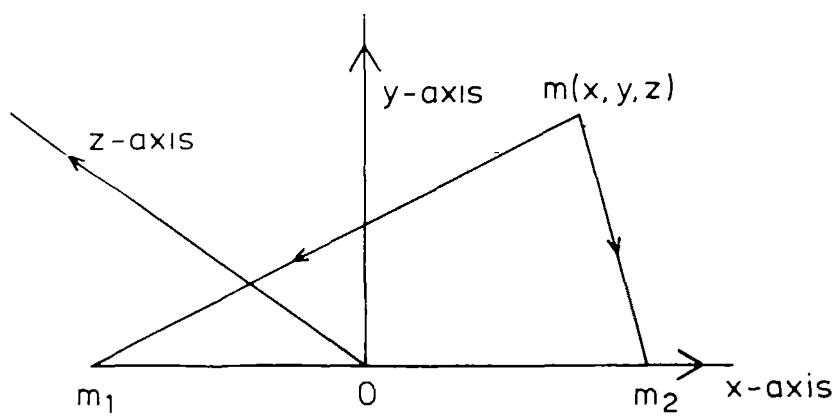
transformation of Meshcheriskii (1949) we have established the differential equations of motion of our problem. Consequently, we find that the equations of motion of our problem differ from the equations of motion of the restricted three body problem with constant mass only by a small perturbing force.

2. Equations of Motion

Let us suppose that mass of the finite bodies, i.e., m_1 and m_2 is constant and the mass of the satellite, i.e., m is variable with respect to time t.



Now let us consider a co-ordinate system 0XYZ, with origin at 0 and axes rotating relative to the initial space with angular velocity ω about Z axis. Without loss of generality we can choose the co-ordinate system such that the X-axis lies along the line joining the finite mass m_1 and m_2 with 0 at the barycentre. Let the distance between m_1 and m_2 be ρ, and the position of the satellite of mass m is given by the co-ordinate (x, y, z). Let the radius vector from m to m_1 and from m to m_2 be $\vec{\rho_1}$ and $\vec{\rho_2}$ respectively. As the motion of m_1 and m_2 are known we are only to find the motion of m.

The kinetic energy in the rotating frame of reference 0XYZ is given by:

$$\begin{aligned} T &= \tfrac{1}{2}m(\dot{x} - \omega y)^2 + (\dot{y} + \omega x)^2 + \dot{z}^2 \\ &= \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + m(x\dot{y} - \dot{x}y) + \tfrac{1}{2}m\omega^2(x^2 + y^2) \\ &= T_2 + T_1 + T_0. \end{aligned} \tag{1}$$

The potential energy is given by

$$V = -Km\left(\frac{m_1}{\rho_1} + \frac{m_2}{\rho_2}\right) \tag{2}$$

where K is gravitational constant.

$$\begin{aligned} \rho_1^2 &= (x + a)^2 + y^2 + z^2 \\ \rho_2^2 &= (x - b)^2 + y^2 + z^2 \end{aligned} \tag{3}$$

where $m_1(-a, 0)$ and $m_2(b, 0)$ are co-ordinates of the finite bodies. We know that Lagrangian

$$L = T - V.$$

Now let us introduce the modified potential energy

$$U = V - T_0.$$

Then Lagrangian can be written in the form

$$L = \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \tfrac{1}{2}m2\omega(x\dot{y} - \dot{x}y) - U. \tag{4}$$

The equation of motion can be written in the form:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} &= 0 \\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} &= 0 \\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{z}}\right) - \frac{\partial L}{\partial z} &= 0 \end{aligned} \tag{5}$$

i.e.,

$$\begin{aligned} \frac{\dot{m}}{m}(\dot{x} - \omega y) + \ddot{x} - 2\omega\dot{y} &= -\frac{1}{m}\frac{\partial U}{\partial x} \\ \frac{\dot{m}}{m}(\dot{y} + \omega x) + \ddot{y} + 2\omega\dot{x} &= -\frac{1}{m}\frac{\partial U}{\partial y} \\ \frac{\dot{m}}{m}\dot{z} + \ddot{z} &= -\frac{1}{m}\frac{\partial U}{\partial z}. \end{aligned} \tag{6}$$

These are the equations of motion of the restricted problem of three bodies when the mass of the satellite decreases with respect to time t.

Now from Jeans law

$$\frac{\mathrm{d}m}{\mathrm{d}t} = -\alpha m^n, \tag{7}$$

where α is constant coefficient and value of exponent n is within the limit $0.4 \leqslant n \leqslant 4.4$ for the star of the main sequence. The first and second law of Meshcherskii are obtained by integrating for $n = 2$ and $n = 3$ respectively.

Let us introduce the following space time transformation which in contrast to the transformation of Meshcherskii preserve the dimensions of the space and time.

$$\begin{aligned} &\xi = \gamma^q x, \qquad \eta = \gamma^q y, \qquad \zeta = \gamma^q z, \qquad \mathrm{d}\Gamma = \gamma^k\,\mathrm{d}t, \\ &r_1 = \gamma^q \rho_1, \qquad r_2 = \gamma^q \rho_2, \end{aligned} \tag{8}$$

where

$$\gamma = m/m_0 \tag{9}$$

and m_0 is mass of satellite when $t = 0$.

The space time transformation (8) is useful only in the special case $n = 1$, $k = 0$, $q = \frac{1}{2}$.

$$\frac{d\gamma}{dt} = \frac{dm}{dt}\frac{1}{m_0} = -\alpha m_0{}^{n-1}\gamma^n = -\beta\gamma^n \tag{10}$$

where

$$\beta = \alpha m_0{}^{n-1}.$$

Since

$$\xi = \gamma^q x, \qquad x = \xi\gamma^{-q}, \qquad \frac{dt}{d\Gamma} = \gamma^{-k}$$

$$\xi' = \gamma^{q-k}\dot{x} - q\beta\gamma^{n-k-1}\xi$$

then

$$\dot{x} = \xi'\gamma^{k-q} + q\beta\gamma^{n-q-1}\xi$$

$$\xi'' = \gamma^{q-k}\ddot{x}\frac{dt}{d\Gamma} + \dot{x}\frac{d}{dt}(\gamma^{q-k})\frac{dt}{d\Gamma} - \tag{11}$$

$$- q\beta\left[\frac{d}{dt}(\gamma^{n-k-1})\frac{dt}{d\Gamma}\xi + \gamma^{n-k-1}\xi^1\right]$$

$$= \gamma^{q-2k}\ddot{x} - \beta(q-k)\gamma^{q+n-2k-1}\dot{x} - q\beta\gamma^{n-k-1}\xi' +$$

$$+ q\beta^2(n-k-1)\xi\gamma^{2(n-k-1)}$$

then

$$\ddot{x} = \xi''\gamma^{2k-q} + (2q-k)\beta\xi'\gamma^{n+k-q-1} - q\beta^2(n-q-1)\gamma^{2(n-k-1)}\xi \tag{12}$$

Similarly $\dot{y}$, $\ddot{y}$, $\dot{z}$, and $\ddot{z}$ can be written in this form:

$$\dot{y} = \eta'\gamma^{k-q} + q\beta\gamma^{n-q-1}\eta \tag{13}$$

$$\ddot{y} = \eta''\gamma^{2k-q} + (2q-1)\beta\eta'\gamma^{n+k-q-1} - q\beta^2(n-q-1)\eta\gamma^{2n-q-2} \tag{14}$$

$$\dot{z} = \gamma^{k-q}\zeta' + q\beta\gamma^{n-q-1}\zeta \tag{15}$$

$$\ddot{z} = \zeta''\gamma^{2k-q} + (2q-k)\beta\zeta'\gamma^{n+k-q-1} - q\beta^2(n-q-1)\zeta\gamma^{2n-q-2}, \tag{16}$$

Substituting these new values of $x, \dot{x}, \ddot{x}$; $y, \dot{y}, \ddot{y}$ and $z, \dot{z}, \ddot{z}$ in the Equations (6),

$$\xi''\gamma^{2k-q} + (2q-k)\beta\,\xi'\gamma^{n+k-q-1} - q\beta^2(n-q-1)\xi\gamma^{2n-q-2} -$$

$$- 2\omega(\eta'\gamma^{k-q} + q\beta\eta\gamma^{n-q-1}) =$$

$$= -\frac{1}{m}\frac{\partial U}{\partial x} - [-\alpha m_0^{n-1}\gamma^{n-1}(\xi'\gamma^{k-q} + q\beta\gamma^{n-q-1}\xi - \omega\eta\gamma^{-q}]$$

$$= -\frac{1}{m}\frac{\partial U}{\partial x} + \beta\xi'\gamma^{n+k-q-1} + q\beta^2\gamma^{2n-q-2}\xi - \beta\omega\eta\gamma^{n-q-1}$$

i.e.,

$$\xi''\gamma^{2k-q} + (2q-k-1)\beta\xi'\gamma^{n+k-q-1} - q\beta^2(n-q)\xi\gamma^{2n-q-2} - \\ - 2\omega\eta'\gamma^{k-q} - \beta\omega\eta(2q-1)\gamma^{n-2k-1} = -\frac{1}{m_0}\gamma^{q-1}\frac{\partial U}{\partial \xi}$$

as

$$\frac{1}{m}\frac{\partial U}{\partial x} = \frac{1}{m_0}\gamma^{-1}\frac{\partial v}{\partial \xi}\frac{\partial \xi}{\partial x} = \frac{1}{m_0}\gamma^{q-1}\frac{\partial U}{\partial \xi}$$

i.e.,

$$\xi'' - 2\omega\eta'\gamma^{-k} + (2q-k-1)\beta\xi'\gamma^{n-k-1} - q\beta^2(n-q)\xi\gamma^{2(n-k-1)} - \\ - \beta\omega\eta(2q-1)\gamma^{n-2k-1} = -\frac{1}{m_0}\gamma^{2q-2k-1}\frac{\partial U}{\partial \xi}$$

$$\eta'' + 2\omega\xi'\gamma^{-k} + (2q-k-1)\beta\eta'\gamma^{n-k-1} - q\beta^2(n-q)\eta\gamma^{2(n-k-1)} + \\ + (2q-1)\omega\beta\xi\gamma^{n-2k-1} = -\frac{1}{m_0}\gamma^{2q-2k-1}\frac{\partial U}{\partial \eta}$$

$$\zeta'' + (2q-k-1)\beta\zeta'\gamma^{n-k-1} - q\beta^2(n-q)\gamma^{2(n-k-1)}\zeta = \\ = -\frac{1}{m_0}\gamma^{2q-2k-1}\frac{\partial U}{\partial \zeta} \tag{17}$$

where the prime denotes differentiation with respect to Γ. In order to free the Equations (17) from the factor which depends on the variation of mass, it is sufficient to put

$$n - k - 1 = 0$$

$$2q - 2k - 1 = 0$$

and

$$n = 1 \tag{18}$$

i.e.,

$$k = 0 \quad \text{and} \quad q = \tfrac{1}{2}. \tag{19}$$

Hence the Equations (17) can be written in the form:

$$\xi'' - 2\omega\eta' = -\frac{1}{m_0}\frac{\partial U}{\partial \xi} + \tfrac{1}{4}\beta^2/\xi$$

$$\eta'' + 2\omega\xi' = -\frac{1}{m_0}\frac{\partial U}{\partial \eta} + \tfrac{1}{4}\beta^2/\eta \tag{20}$$

$$\zeta'' = -\frac{1}{m_0}\frac{\partial U}{\partial \eta} + \tfrac{1}{4}\beta^2/\zeta.$$

These are the equations of motion of the restricted problem of three bodies with decreasing mass.

The Equations (20) differ from equations of motion of restricted problem of three bodies with constant mass only by a factor $\frac{1}{4}\beta^2$.

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