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CR-structures on three dimensional circle bundles

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1 Introduction

In [BuEp2] we considered the problem of embedding three dimensional CR-manifolds in $\mathbb{C}^d$ for some d. The primary focus of that paper was to understand the structures near to the standard structure on the three sphere. This restriction was largely due to the fact that we made extensive use of the spherical harmonic expansions unique to that context. Recently several researchers have considered similar questions from other points of view. In particular Bland, Duchamp and Lempert, see [BlDu] and [Le]. In the work of these authors a central role is played by a $U(1)$-action rather than $SU(2)$.

After learning of this general approach and in particular, the paper of Bland and Duchamp, [BlDu], it became clear that there was a very close connection between these two approaches, and that in fact much of the analysis in [BuEp2] only required an action by $U(1)$. In this paper we extend some of the results of the earlier paper to the more general context of perturbations of $U(1)$-invariant CR-structures on circle bundles over surfaces. If the structure is $U(1)$-invariant then it is a result of Baouendi, Rothschild and Treves, and Lempert that it can always be embedded in $\mathbb{C}^d$ for some d. Lawson and Yau have shown that it can always be realized as a hypersurface in an affine algebraic variety with a linear action by $\mathbb{C}^*$, see [LaYa, Sect. 2]. We discuss this from a slightly different perspective in Appendix A.

In the next two sections we show how, in the presence of certain additional structure one can actually embed perturbations of a $U(1)$-invariant structure represented by deformations of the CR-structure with 'non-negative' Fourier coefficients, in an appropriate sense. This viewpoint was clearly delineated in [BlDu]. In Sect. 2 we study the solvability of the $\bar{\partial}_b$-equation with respect to a $U(1)$-invariant structure. Our method is to introduce a normalized contact form, θ, and local sections, $\bar{Z}$ of $T^{0,1}$ which allow us to control the $\ker\bar{\partial}_b^*$. We assume that they satisfy

$$\mathfrak{L}_{\bar{Z}}\theta \wedge d\theta = 0 \quad \text{and} \quad U_{\phi *}\bar{Z} = e^{im\phi}\bar{Z}$$

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for some integer m. A contact form defining such a pseudo-hermitian structure arises from an asymptotic solution of the Fefferman Monge-Ampére equation. Here and in the sequel we denote the Lie derivative in the direction X by $\mathfrak{L}_X$ and the action of $e^{i\phi} \in U(1)$ by U_ϕ.

In an open subset U of M perturbations of this structure are represented by $\bar{Z} + \psi Z$ where $\psi \in \mathscr{C}^\infty(U)$. These functions patch together to define a global section of a flat line bundle over M. If ψ satisfies

$$U_\phi^* \psi = e^{-2im\phi} \psi$$

then $\bar{Z} + \psi Z$ represents another $U(1)$-invariant structure. This allows us to define what is meant by 'positive' or 'non-negative' Fourier coefficients. For an open set U let $F_n(U)$ denote the functions defined on U which transform according to the rule

$$U_\phi^* f = e^{in\phi} f.$$

If the projection of ψ into F_n is zero for $n \leqq -2m$ then we say that the perturbed structure $\bar{Z} + \psi Z$ has 'positive' Fourier coefficients, if the projections are zero for $n < -2m$ we say that $\bar{Z} + \psi Z$ has 'non-negative' Fourier coefficients. This is consistent with the usage in [BlDu] and [Le1, Theorem 4.1]. We show that sufficiently small perturbations of $U(1)$-invariant structures with 'positive' Fourier coefficients are embeddable as small perturbations of certain embeddings. In Sect. 3 we use the geometry of line bundles to obtain the normalizations posited in Sect. 2 for circle bundles of degree -1. This entails the construction of a distinguished contact form. This contact form is shown to be biholomorphically invariant for circle bundles over surfaces of genus at least 2. In Sect. 4 the embedding problem is considered for deformations of $U(1)$-invariant structures with 'non-negative' Fourier coefficients on circle bundles of arbitrary degree by using covers and equivariant immersions to reduce to the case of bundles of degree -1.

In [Le1] it is shown that a pseudoconvex, three dimensional CR-manifold that admits an 'inner S^1-action' is the boundary of a compact surface. This implies that it is embeddable in $\mathbb{C}^N$ for some N. I have been informed by John Bland that having 'positive' Fourier coefficients is essentially equivalent to admitting an 'inner S^1-action', [Bl1]. Bland has also obtained a similar result, see [Bl2]. In Sect. 5 we examine the relationship between the representation of deformations and the concept of 'non-negative' Fourier coefficients and give a simple proof of the Bland-Lempert result using our normalizations. In Sect. 6 and Sect. 7 we generalize the variational analysis of the $\Box_b$-operator in [BuEp2]. This leads to generic non-embeddability of real analytic perturbations of real analytic $U(1)$-invariant structures. In Sect. 8 we use the explicit representation of CR-functions for the perturbed structures in terms of the unperturbed structures obtained in Sect. 2 and the analyticity results obtained in [BuEp2] to study the stability of given embeddings among perturbations with 'non-negative' Fourier coefficients. These results were partly inspired by the recent preprint of Catlin and Lempert, [CaLe] and a conversation with John Bland, [Bl1]. We have omitted a review of the basic facts of CR-geometry. This can be found in several papers including [BlDu], [Le1] or [We1].

2 Perturbing normalized $U(1)$-invariant structures

Suppose that M is a contact manifold diffeomorphic to a circle bundle over a compact surface with a free transverse contact action by $U(1)$. We further suppose that M is endowed with a $U(1)$-invariant strictly pseudoconvex CR-structure. In this section we assume that M has a contact form θ for which the pseudohermitian structure is especially simple, see [We1]. Let $\{U_\alpha; \alpha \in A\}$ be an open cover of M by a finite collection of $U(1)$-invariant sets. We assume that in each U_α there is a one form θ_α^1 of type $(1,0)$ which satisfies

$$\begin{aligned} d\theta &= i\theta_\alpha^1 \wedge \theta_\alpha^{\bar{1}}, \\ d\theta_\alpha^1 &= iB_\alpha \theta \wedge \theta_\alpha^1 \bmod \theta_\alpha^{\bar{1}}, \; B_\alpha \in C^\infty(U_\alpha), \\ U_\phi^* \theta_\alpha^1 &= e^{-im\phi}\theta_\alpha^1, \end{aligned} \tag{2.1}$$

for some integer m. On the overlaps, $U_\alpha \cap U_\beta$, the different sections are related by

$$\theta_\alpha^1 = e^{ig_{\alpha\beta}}\theta_\beta^1. \tag{2.2}$$

From the structure equations (2.1) it follows easily that the coefficients $\{e^{ig_{\alpha\beta}}\}$ are constants of modulus one. They define a 1-cocycle relative to the cover U_α. Once the contact form is fixed the conditions in (2.1) define a flat line bundle, λ over M. When we represent a CR-structure in terms of such normalized sections its deformations are represented by sections of λ^{-2}.

Let Z_α denote the $(1,0)$-vector field defined on U_α dual to $(\theta, \theta_\alpha^1, \theta_\alpha^{\bar{1}})$. From (2.1) we conclude that

$$\begin{aligned} U_{\phi *} Z_\alpha &= e^{-im\phi} Z_\alpha, \\ \mathfrak{L}_{\bar{Z}_\alpha} \theta \wedge d\theta &= 0. \end{aligned} \tag{2.3}$$

The choice of pseudohermitian structure fixes a differential operator representing $\bar{\partial}_b$. In U_α it is given by:

$$\bar{\partial}_b u = \bar{Z}_\alpha u \theta_\alpha^{\bar{1}}.$$

Since θ is a contact form $\mathrm{dV} = \theta \wedge d\theta$ defines a volume form on M. From (2.2) it follows that if ω is a $(0,1)$-form such that

$$\omega \restriction_{U_\alpha} = f_\alpha \theta_\alpha^{\bar{1}}, \quad \alpha \in A,$$

then

$$|f_\alpha| = |f_\beta| \quad \text{in} \quad U_\alpha \cap U_\beta. \tag{2.4}$$

Formula (2.4) implies that the norm of a $(0,1)$-form ω is well defined, we denote the common value by $|f|$. Define an L^2-inner product on sections of $\Lambda^{0,1}(M)$ by setting

$$\langle \omega, \omega \rangle = \int_M |f|^2 \mathrm{dV}. \tag{2.5}$$

Proposition 2.6 *With respect to the inner product defined in* (2.5) *the adjoint of* $\bar{\partial}_b$ *is given by*

$$\bar{\partial}_b^* f_\alpha \theta_\alpha^{\bar{1}} = -Z_\alpha f_\alpha,$$

in U_α.

Proof. Since the computation of the L^2-adjoint is local we can suppose that η is supported in a single set U_α where it equals $f_\alpha\theta_\alpha^{\bar{1}}$. Then

$$\langle\bar{\partial}_b u, \eta\rangle = \int_{U_\alpha} (\bar{Z}_\alpha u)\bar{f}_\alpha \mathrm{dV},$$
$$= -\int_{U_\alpha} u\overline{(Z_\alpha f_\alpha)}\mathrm{dV} + u\bar{g}_\alpha \mathfrak{L}_{\bar{Z}_\alpha}(\mathrm{dV}).$$

The conclusion follows from (2.3).

The action on M by $U(1)$ induces an orthogonal splitting of $L^2(M; \mathrm{dV})$ into invariant subspaces. We let F_n denote the functions in L^2 which satisfy

$$U_\phi^* f = e^{in\phi} f.$$

Define

$$\mathscr{F}_n = \bigoplus_{j=n}^{\infty} F_j. \tag{2.7}$$

To specify the set on which the functions are defined we sometimes replace F_n by $F_n(U)$ and $\mathscr{F}_n$ by $\mathscr{F}_n(U)$, etc.

It is a consequence of Proposition A8, in Appendix A, that if M is strictly pseudoconvex then all CR-functions have 'non-negative' Fourier coefficients. In other words

$$\ker\bar{\partial}_b \subset \mathscr{F}_0. \tag{2.8}$$

The Fourier decomposition extends to (0, 1)-forms. Such a form is specified by its restrictions to the sets in the open cover $\{U_\alpha\}$. If

$$\omega\restriction_{U_\alpha} = f_\alpha\theta_\alpha^{\bar{1}} \quad \text{then}$$
$$f_\alpha = e^{-ig_{\alpha\beta}} f_\beta \quad \text{on} \quad U_\alpha \cap U_\beta. \tag{2.9}$$

We say that $\omega \in F_n$ if $f_\alpha \in F_n(U_\alpha)$ for each $\alpha \in A$. This slightly ambiguous notation should not lead to any confusion. From (2.9) it is evident that this is well defined. Since the volume form is $U(1)$-invariant we can define orthogonal projections onto the different Fourier components. We obtain a decomposition

$$\omega = \sum_{n=-\infty}^{\infty} \omega_n \quad \text{where}$$
$$\omega_n \restriction_{U_\alpha} = f_{\alpha n}\theta_\alpha^{\bar{1}}, \quad f_{\alpha n} \in F_n(U_\alpha).$$

Note that

$$f_{\alpha n} = e^{-ig_{\alpha\beta}} f_{\beta n} \quad \text{on} \quad U_\alpha \cap U_\beta.$$

Lemma 2.10 *Under the normalizations in* (2.1)

$$\bar{\partial}_b : F_n \to F_{n-m}; \quad \bar{\partial}_b^* : F_n \to F_{n+m}. \tag{2.11}$$

Proof. Let $u \in F_n$, then

$$U_\phi^* \bar{\partial}_b u = \bar{\partial}_b U_\phi^* u$$
$$= e^{in\phi}\bar{\partial}_b u.$$

On U_α we have

$$U_\phi^* \bar\partial_b u = U_\phi^*(\bar Z_\alpha u) U_\phi^* \theta_\alpha^{\bar 1} = e^{im\phi}(U_\phi^* \bar Z_\alpha u)\theta_\alpha^{\bar 1} .$$

Taken together, these two computations show that

$$U_\phi^* \bar Z_\alpha u = e^{i(n-m)\phi} \bar Z_\alpha u .$$

A similar computation proves the other statement.

Since M is a circle bundle over a surface with a free transverse $U(1)$-action by CR-automorphisms the quotient, Σ is a smooth surface,

$$\pi: M \to M/U(1) = \Sigma .$$

A complex structure on Σ is defined by

$$T^{0,1}\Sigma = \pi_* T^{0,1} M .$$

The functions on M which belong to F_n can be identified with sections of a holomorphic line bundle over Σ. This is defined by using the unitary characters of $U(1)$:

$$\chi_n(e^{i\phi}) = e^{in\phi}, \quad n \in \mathbb{Z} ,$$

and forming the fiber product. We define L_n as

$$L_n = M \times_{\chi_n} \mathbb{C} ,$$

where the equivalence relation on $M \times \mathbb{C}$ is

$$(p, z) \sim (U_\phi p, e^{in\phi} z) .$$

We have a commutative diagram:

$$\begin{array}{ccccc} M \times \mathbb{C} & \xrightarrow{\cong} & \pi^*(L_n) & \xrightarrow{\pi_*} & L_n \\ & & \downarrow & & \downarrow \\ & & M & \xrightarrow[\pi]{} & \Sigma . \end{array}$$

Functions in F_n clearly define sections of L_n and conversely a section of L_n pulls back to define a function in F_n. A simple calculation in local coordinates verifies that a section s of L_n is holomorphic if and only if the pull back $\pi^*(s)$ is a CR-function on M. Appendix A covers this construction in greater detail. M itself is biholomorphic to a hypersurface in L_{-1}. M bounds a relatively compact strictly pseudoconvex domain in L_{-1} if and only if the $\deg L_{-1} < 0$.

Using the relationship between holomorphic sections of line bundles over Σ and CR-functions we can study the kernel of $\bar\partial_b^*$.

Lemma 2.12 *If the* $\deg L_{-1} < 0$ *then the kernel of* $\bar\partial_b^*$ *is orthogonal to* $\mathscr{F}_1$.

Proof. Suppose that η is a (0, 1)-form in the $\ker \bar\partial_b^*$. From Proposition 2.6 it follows that in each set U_α we have a representation

$$\eta \restriction_{U_\alpha} = f_\alpha \theta_\alpha^{\bar 1}, \quad Z_\alpha f_\alpha = 0 \quad \text{with} ,$$

$$f_\alpha = e^{-ig_{\alpha\beta}} f_\beta \quad \text{in} \quad U_\alpha \cap U_\beta . \tag{2.13}$$

As in the discussion leading up to Proposition A11, the cocycle $\{e^{ig_{\alpha\beta}}\}$ defines a flat holomorphic line bundle, λ over Σ. If

$$\eta = \sum_{n=-\infty}^{\infty} \eta_n$$

is the Fourier decomposition of η then $\bar{\eta}_n$ defines a section, s_n of $L_{-n} \otimes \lambda$. Lemma 2.10 implies that $\eta \in \ker \bar{\partial}_b^*$ if and only if $\eta_n \in \ker \bar{\partial}_b^*$ for every n. From Proposition A11 and (2.13) it follows that $\eta_n \in \ker \bar{\partial}_b^*$ if and only if s_n is holomorphic. Since the degree of L_{-1} is negative and

$$\deg(L_{-n} \otimes \lambda) = n \deg L_{-1}$$

it is an elementary fact from Riemann surface theory that this bundle does not have holomorphic sections if $n > 0$, [GrHa, p. 155]. Thus $\eta_n = 0$ if $n \geqq 1$; this completes the proof of the lemma.

The $\Box_b$-operator associated to $\bar{\partial}_b$ is given by $\Box_b u = \bar{\partial}_b^* \bar{\partial}_b u$. From (2.11) it follows that

$$\Box_b : F_n \to F_n \,. \tag{2.14}$$

From (2.11) and (2.14) it is immediate that

$$\begin{gathered} \bar{\partial}_b : \mathscr{F}_n \to \mathscr{F}_{n-m}, \quad \bar{\partial}_b^* : \mathscr{F}_n \to \mathscr{F}_{n+m}, \\ \Box_b : \mathscr{F}_n \to \mathscr{F}_n \,. \end{gathered} \tag{2.15}$$

Now we are ready to consider perturbations of the CR-structure defined by $\{\bar{Z}_\alpha\}$. On each open set U_α a perturbation is defined by setting

$$\bar{Z}_\alpha^\psi = \bar{Z}_\alpha + \psi_\alpha Z_\alpha \,. \tag{2.16}$$

From (2.2) it follows that

$$Z_\beta = e^{ig_{\alpha\beta}} Z_\alpha \quad \text{on } U_\alpha \cap U_\beta$$

and therefore in order for a collection of functions $\psi_\alpha \in \mathscr{C}^\infty(U_\alpha)$, $\alpha \in A$ to define a global CR-structure on M it is necessary that

$$\psi_\beta = e^{-2ig_{\alpha\beta}} \psi_\alpha \quad \text{on} \quad U_\alpha \cap U_\beta \,. \tag{2.17}$$

These are sections of a flat line bundle over M which we denote by $\mathfrak{D} = \lambda^{-2}$. The Fourier decomposition extends to sections of $\mathfrak{D}$ in an obvious way. We use $F_n(\mathfrak{D})$ to denote sections of $\mathfrak{D}$ with $\psi_\alpha \in F_n(U_\alpha)$ and $\mathscr{F}_n(\mathfrak{D})$ for sections with $\psi_\alpha \in \mathscr{F}_n(U_\alpha)$ for each α. From (2.17) it follows that if ψ is a section of $\mathfrak{D}$ then the (0, 1)-forms

$$\bar{\partial}_b^\psi u \restriction_{U_\alpha} = (\bar{Z}_\alpha + \psi_\alpha Z_\alpha) u \theta_\alpha^{\bar{1}}, \quad \alpha \in A \,,$$

patch together to define a global (0, 1)-form. Thus we can decompose the perturbed $\bar{\partial}_b$-operator as

$$\bar{\partial}_b^\psi = \bar{\partial}_b + P_\psi \,.$$

We want to solve the perturbation equation

$$\bar{\partial}_b^\psi (z + \xi) = 0 \tag{2.18}$$

where ψ is a section of $\mathfrak{D}$ and $z \in \ker \bar{\partial}_b \cap \mathscr{F}_{k+1}$. Arguing as in the proof of Theorem 5.3 of [BuEp2] we define an iteration:

$$\xi_0 = 0, \quad \bar{\partial}_b \xi_n = - P_\psi(z + \xi_{n-1}) . \tag{2.19}$$

Proposition 2.20 *If* $\deg L_{-1} < 0$, $z \in \ker \bar{\partial}_b \cap \mathscr{F}_{1+k}$, $k \geqq 0$ *and*

$$\psi \in \mathscr{F}_{-2m}(\mathfrak{D}) \quad \text{if } m \leqq 0, \quad \text{or} \quad \psi \in \mathscr{F}_{-(M+m)}(\mathfrak{D}), \quad M = \min\{m, k\} \quad \text{if } m > 0 ,$$

then (2.19) *has a solution* $\xi_n \in \mathscr{F}_{1+k}$ *for all* $n = 1, 2, \ldots$

Proof. Since $z \in \mathscr{F}_{1+k}$ it follows from (2.14) that

$$P_\psi z \in \mathscr{F}_{1-m+k} \quad \text{if } m \leqq 0 ,$$

$$\text{if } m > 0, \quad P_\psi z \in \mathscr{F}_1 \quad \text{for} \quad k < m, \quad P_\psi z \in \mathscr{F}_{1-m+k} \quad \text{for } k \geqq m .$$

Since $\ker \bar{\partial}_b^*$ is orthogonal to $\mathscr{F}_1$ it follows, by using the partial inverse, $\mathscr{Q}$, to the Kohn-Laplace operator, $\Box_b$, as in [BuEp2, p. 832], that we can solve for ξ_1:

$$\xi_1 = - \mathscr{Q}(\bar{\partial}_b^* P_\psi z) .$$

From (2.15) it follows that

$$\xi_1 \in \mathscr{F}_{1+k} .$$

Inductively we assume that we can solve for $\xi_1, \ldots, \xi_n$ and that

$$\xi_j \in \mathscr{F}_{1+k} . \tag{2.21}$$

From (2.21) it follows, as above for $n = 0$, that

$$\begin{aligned} &P_\psi(z + \xi_n) \in \mathscr{F}_{1-m+k} \quad \text{if } m \leqq 0 , \\ &\text{if } m > 0, \quad P_\psi(z + \xi_n) \in \mathscr{F}_1 \quad \text{for } k < m, \\ &P_\psi(z + \xi_n) \in \mathscr{F}_{1-m+k} \quad \text{for } k \geqq m . \end{aligned} \tag{2.22}$$

From (2.22) it follows that we can solve for ξ_{n+1} and from (2.15) it follows that

$$\xi_{n+1} \in \mathscr{F}_{1+k} .$$

This completes the proof of the proposition.

To complete the perturbation theory all that remains is to prove that the sequence $\{\xi_n\}$ converges in an appropriate topology for ψ sufficiently small. This argument is really done in the proof of Theorem (5.3) of [BuEp2]. The analytic part of the proof is entirely formal employing the apparatus of the Heisenberg calculus and applies *mutatis mutandis* to the present situation. For latter applications we state a slightly more precise result, which follows from the argument above and the analysis in [BuEp2, pp. 831–2].

Theorem 2.23 *Under the hypothesis* (2.1) *if* $\deg L_{-1} < 0$, $z \in \ker \bar{\partial}_b \cap \mathscr{F}_{k+1}$, $k \geqq 0$ *and*

$$\psi \in \mathscr{F}_{-2m}(\mathfrak{D}) \quad \text{if } m \leqq 0, \quad \psi \in \mathscr{F}_{-(M+m)}(\mathfrak{D}), \quad M = \min\{m, k\} \quad \text{if } m > 0 ,$$

is of sufficiently small C^4-norm then the equation

$$\bar{\partial}_b^\psi (z + \xi) = 0$$

has a unique solution orthogonal to $\ker \bar{\partial}_b$ *with*

$$\xi \in \mathscr{F}_{k+1} .$$

If $\psi \in C^{\infty}(\mathfrak{D})$ *of sufficiently small* C^4*-norm then the map*

$$z \to \xi$$

is bounded in the C^{∞}*-topology with constants depending linearly on* ψ.

As an immediate corollary we have:

Corollary 2.24 *Under the hypothesis* (2.1) *with* $\deg L_{-1} < 0$, *the* CR-*structures defined by* ψ *with*

$$\psi \in \mathscr{F}_{-2m}(\mathfrak{D}) \quad \text{if } m \leqq 0, \qquad \psi \in \mathscr{F}_{-m}(\mathfrak{D}) \quad \text{if } m > 0 ,$$

of sufficiently small C^4*-norm, embed as small perturbations of an embedding of* $(M, \bar{\partial}_b)$.

Note that in this corollary any sufficiently small perturbation of $\bar{\partial}_b$ which satisfies the hypothesis can be realized as a deformation of an *arbitrary* embedding of $(M, \bar{\partial}_b)$.

In the sequel it will be of interest to consider equivariant immersions as well as embeddings. If $x_1, \ldots, x_d$, define an immersion of M with

$$U_{\phi}^{*} x_j = e^{ik\phi} x_j, \quad k \in \mathbb{N}$$

then we can consider perturbations of this immersion by restricting to functions $\psi \in \mathscr{F}_{-2m}(\mathfrak{D})$ whose projection into F_j is non-zero only for $j = -2m + nk$, $n \in \mathbb{N}_0$. For integers l, m define

$$\mathscr{F}_l^m = \bigoplus_{j=0}^{\infty} F_{l+jm} , \tag{2.25}$$

with the obvious modifications for sections of flat line bundles. Using (2.14) instead of (2.15) and the fact that

$$U_{\phi}^{*} \Box_b = \Box_b U_{\phi}^{*} ,$$

in the above argument we easily obtain

Corollary 2.26 *If* $\deg L_{-1} < 0$, $z \in F_{nk}$, $n \in \mathbb{N}$ *and*

$$\psi \in \mathscr{F}^{k}_{-2m}(\mathfrak{D}) \quad \text{if } m \leqq 0 ,$$

$$\psi \in \mathscr{F}^{k}_{-2m+lk}(\mathfrak{D}),\ (l + n)k \geqq m + 1 \quad \text{if } m > 0$$

is of sufficiently small C^4*-norm then there is a unique solution to*

$$\bar{\partial}_b^{\psi}(z + \xi) = 0$$

orthogonal to $\ker \bar{\partial}_b$ *which belongs to* $\mathscr{F}^{k}_{nk}$.

The hypothesis in Corollary 2.24 that $\psi \in \mathscr{F}_{-m}(\mathfrak{D})$ if $m > 0$, is non-optimal in that it does not include all structures with 'positive' Fourier coefficients, i.e. $\psi \in \mathscr{F}_{1-2m}(\mathfrak{D})$. If we assume that $z \in \mathscr{F}_{m+1}$ instead of $\mathscr{F}_1$ then we can allow $\psi \in \mathscr{F}_{-2m}(\mathfrak{D})$. As follows from Theorem A16 this suffices for applications of this theorem to study embeddability of perturbations of $U(1)$-invariant structures.

However it suggests that in certain cases there may exist embeddable perturbations which cannot be realized as perturbations of a particular embedding. This is closely related to a construction of Catlin and Lempert. They have constructed smooth families of CR-structures on circle bundles with the property that the embeddings are unstable under deformations. More precisely, they have constructed a family of CR-structures J_t depending smoothly on the parameter t such that all structures are embeddable. However the structures for $t \neq 0$ cannot be realized as perturbations of a fixed size of a given embedding of J_0, see [CaLe]. The question of the stability of embeddings is taken up in Sect. 8.

3 Ricci flat volume forms

In the previous section we showed how to solve the perturbed $\bar{\partial}_b$-equation under certain additional hypotheses. In this section we examine the geometric implications of these hypotheses. To recap we have a CR-manifold with a free transverse $U(1)$-action by CR-automorphisms. Let Σ and L_n be as defined in Sect. 2. Since the action on the $\mathbb{C}$-factor is unitary the bundles L_n come equipped with a canonical metric. Another local coordinate calculation verifies that the unit circle in the bundle L_{-1} is biholomorphic to M itself. Thus we have M realized as the boundary of a compact complex manifold. As a hypersurface in L_{-1}, M is either pseudo-concave of pseudoconvex. If the degree of the line bundle L_{-1} is negative then M is pseudoconvex.

In this section we obtain a contact form, θ and local sections, θ^1_α as in (2.1), for the canonical bundle of Σ. Using the theory of line bundles over a Riemann surface we show how to extend these constructions to any bundle of degree $2(g-1)$. Similar techniques were employed in [BuEp1, Sect. 6] for other purposes. The holomorphic line bundles over Σ are parametrized by their degree, or Chern class and the Picard variety. Let κ denote the canonical bundle of Σ. If $g = 0$ then κ is negative, if $g \geqq 2$ then it is positive and if $g = 1$ it is trivial. First suppose that Σ is of genus $g \neq 1$.

These line bundles have a very special property, there is a non-vanishing, holomorphic (2, 0)-form, ω defined on the total space which satisfies

$$U^*_\phi \omega = e^{i\phi} \omega \,. \tag{3.1}$$

This is the canonical holomorphic symplectic form. To define this form we recall that a point in κ is a one-form of type $(1, 0)$ on Σ and therefore if p denotes the projection to Σ then at $q \in \kappa$ we define

$$\eta_q(v) = q(p_* v) \quad \text{and} \quad \omega = d\eta \,. \tag{3.2}$$

If z is a local holomorphic coordinate defined in an open subset $V \subset \Sigma$ and we use dz to trivialize κ, with w the fiber coordinate, then

$$\omega(z, \omega) = dw \wedge dz \,. \tag{3.3}$$

From (3.3) it follows that ω is nonvanishing, holomorphic and transforms according to (3.1). For the moment we suppose that M is the unit circle bundle in the canonical bundle of Σ relative to some Hermitian metric.

In [BuEp1, Sect. 6] it is shown that if one has a Ricci flat volume form v defined in the neighborhood of a strictly pseudoconvex hypersurface in complex manifold then one can define a third order asymptotic solution to the Monge-Ampére equation along this hypersurface. Setting

$$\nu = \omega \wedge \bar{\omega}$$

defines a Ricci flat volume form on κ which is invariant under the $U(1)$-action. Let u denote the third order solution to the Monge-Ampére equation along $M \subsetneq L_{-1}$. Since u transforms by a power of the absolute value of the Jacobian of a biholomorphic mapping it is invariant under the $U(1)$-action. Thus if we define a contact form by

$$\theta = \operatorname{Im} \partial u$$

then it too is invariant under the $U(1)$-action.

Using the contact form we define a vector field $\mathscr{T}$ of type $(1, 0)$ in a neighborhood of $M \subsetneq L_{-1}$ by the conditions

$$i_{\mathscr{T}} \partial\bar{\partial} u = a \bar{\partial} u \,,$$

$$i_{\mathscr{T}} \partial u = 1 \,. \tag{3.4}$$

Here a is a smooth function defined in a neighborhood of M. A holomorphic coordinate system z_1, z_2 is called unimodular if

$$\nu = dz_1 \wedge dz_2 \wedge d\bar{z}_1 \wedge d\bar{z}_2 \,.$$

The coordinates (w, z) introduced above are unimodular. On p. 66 of [BuEp1] it is shown that if the vector field $\mathscr{T}$ is given in terms of a unimodular coordinate system by

$$\mathscr{T} = t_1 \partial_w + t_2 \partial_z$$

then the $(1, 0)$-form

$$\theta^1 = t_2\, dw - t_1\, dz \,, \tag{3.5}$$

satisfies

$$d\theta = i\theta^1 \wedge \theta^{\bar{1}} \,,$$

$$d\theta^1 = iB\theta \wedge \theta^1 \,, \tag{3.6}$$

for some function B. Since the conditions (3.4) are $U(1)$-invariant it follows that

$$U_{\phi *} \mathscr{T} = \mathscr{T} \,.$$

This implies that

$$U_\phi^* \theta^1 = e^{i\phi} \theta^1 \,. \tag{3.7}$$

If we set $U = \pi^{-1}(V)$ then U is a $U(1)$-invariant open subset of M in which we can find a $(1, 0)$-form θ^1 so that the conditions in (2.1) are fulfilled.

Let $\{V_\alpha, \alpha \in A\}$ denote an open cover of Σ by holomorphic coordinate neighborhoods in which κ is trivial. Let (z_α, w_α), $\alpha \in A$ denote the local trivializations defined above. Set θ_α^1 equal to the $(1, 0)$-form defined in $U_\alpha = \pi^{-1}(V_\alpha)$ by the foregoing construction. Thus we have obtained an open cover of M by $U(1)$-invariant sets in which $(1, 0)$-forms satisfying (2.1) are defined. This shows that the hypothesis (2.1) can be verified for a unit circle bundle in the canonical bundle of a Riemann surface provided the genus is not 1. A simple calculation shows that the flat line bundle λ is trivial in this case. Thus we have a globally defined (1, 0)-form satisfying (2.1).

Any other line bundle, L over Σ of degree $2(g-1)$ can be represented as $\kappa \otimes \lambda$ where $\lambda = \kappa^{-1} \otimes L$ is a flat line bundle. The flat line bundle, λ determines and is determined by a 1-cocycle consisting of constants of modulus 1. To obtain this presentation of the flat line bundles one represents Σ as the quotient of the unit disk by a Fucshian group and uses the unitary characters of $\pi_1(\Sigma)$ to construct the line bundles of degree zero. After possibly refining the cover constructed above we denote the cocycle by $\{e^{il_{\alpha\beta}}\}$. If $\{m_{\alpha\beta}\}$ denote the transition functions defining κ relative to the refined cover,

$$w_\beta = m_{\alpha\beta} w_\alpha \quad \text{in } U_\alpha \cap U_\beta ,$$

then $\{e^{il_{\alpha\beta}} m_{\alpha\beta}\}$ is the 1-cocycle defining $\kappa \otimes \lambda$.

Since the cocycles differ by constants of modulus 1 it is clear that the metric defined on κ by M and the Ricci flat volume form can be transferred to $\kappa \otimes \lambda$. This in turn defines a $U(1)$-invariant hypersurface in $\kappa \otimes \lambda$. Denote the hypersurface by M_λ. Using the Ricci flat volume form we can repeat the constructions of the contact form and family of (1, 0)-forms satisfying (3.6) and (3.7) for $M_\lambda \subsetneq \kappa \otimes \lambda$. Note that M_λ is locally biholomorphic to M and one can actually use the contact form and (1, 0)-forms constructed for κ, only the 'gluing instructions' are changed.

Starting with a hypersurface M_λ in $\kappa \otimes \lambda$ the process outlined above can be reversed to obtain a hypersurface M in κ. From this observation we conclude

Theorem 3.8 *Suppose that M is a strictly pseudoconvex or pseudoconcave, three dimensional, $U(1)$-invariant,* CR-*manifold such that the quotient Riemann surface has genus $g \neq 1$. Suppose further that,*

$$L_{-1} = \kappa \otimes \lambda ,$$

where λ is the flat line bundle defined by the cocycle $\{e^{il_{\alpha\beta}}\}$, then there is a $U(1)$-invariant contact form, a cover of M by $U(1)$-invariant subset $\{U_\alpha\}$ and (1, 0)-*forms θ^1_α defined in U_α such that the conditions* (2.1) *are satisfied with $m = -1$. Moreover*

$$\theta^1_\beta = e^{il_{\alpha\beta}} \theta^1_\alpha \quad \textit{in } U_\alpha \cap U_\beta .$$

Suppose that $\tilde{M}$ is a CR-manifold with a free transverse $U(1)$-action which equivariantly covers a CR-manifold M satisfying the hypotheses of the theorem. The contact form, the $U(1)$-invariant open cover and the (1, 0)-forms can all be lifted to $\tilde{M}$. Denote their lifts by $\tilde{\theta}$, $\{\tilde{U}_\alpha\}$ and $\{\tilde{\theta}^1_\alpha\}$ respectively. The lifts satisfy (2.1). If the cover is m-sheeted then

$$U_{\phi *} \tilde{\theta}^1_\alpha = e^{im\phi} \tilde{\theta}^1_\alpha .$$

For applications it is most useful to work on bundles of degree ± 1 as every bundle of non-zero degree is a quotient of such a bundle. We restate Theorem 3.8 in this context. The case $g = 0$ is covered in the previous theorem.

Proposition 3.9 *If M is a three dimensional* CR-*manifold with a transverse, free $U(1)$* CR-*action such that $\Sigma = M/U(1)$ is a Riemann surface of genus $g \neq 0, 1$ and the line bundle $M \times_{\chi_{-1}} \mathbb{C}$ is of degree ± 1 then there is a contact form θ, an open cover by $U(1)$-invariant subsets and* (1, 0)-*forms $\{\theta^1_\alpha\}$ which satisfy the conditions* (2.1) *with*

$$m = \mp 2(g-1) .$$

If the genus of Σ is one then the construction is more direct. Suppose that Γ is the period lattice for Σ so that

$$\Sigma \simeq \mathbb{C}/\Gamma .$$

We can normalize so that Γ is generated by $1, \tau$ with $\operatorname{Im}\tau > 0$. From the classical theory of line bundles over abelian varieties it follows that every line bundle over a surface of genus 1 arises as a quotient of the trivial bundle over $\mathbb{C}$, see [GrHa, p. 316].

To every holomorphic line bundle, L, over Σ there corresponds a non-vanishing holomorphic function $e(z)$. The line bundle over Σ is defined as the quotient of $\mathbb{C} \times \mathbb{C}$ by the equivalence relation generated by:

$$(z, w) \sim (z + \tau, e(z)w); \quad (z, w) \sim (z + 1, w) . \tag{3.10}$$

From this construction it is easy to see that any holomorphic line bundle over an 2-torus has a non-vanishing holomorphic (2, 0)-form, defined in the complement of the zero section, which is invariant under the action of $U(1)$.

Using the (z, w)-coordinates appearing in (3.10) we define

$$\omega = \frac{dz \wedge dw}{w} . \tag{3.11}$$

This form is clearly invariant under the diffeomorphisms defining the equivalence relation in (3.10) and therefore descends to the quotient. The local formula also shows that it is holomorphic and invariant under the action of $U(1)$.

We suppose that the line bundle L defined by (3.10) has non-zero degree and that a metric has been chosen so that the unit circle bundle has non-vanishing curvature. Denote the unit circle bundle by M. The curvature condition implies that M is either pseudoconcave or pseudoconvex. We need to introduce $x = \log w$ in order to have a unimodular coordinate. The $U(1)$-action lifts to a translation by $i\mathbb{R}$. The construction above of a defining function u and vector field $\mathcal{T}$ can now be repeated. Both of these objects are invariant under the action by $i\mathbb{R}$ and therefore can be re-expressed in terms of the (z, w)-coordinates, for example

$$\theta^1 = t_2 \frac{dw}{w} - t_1\, dz .$$

This (1, 0)-form satisfies

$$U_\phi^* \theta^1 = \theta^1 . \tag{3.12}$$

A calculation shows that θ^1 is invariant under the action defined in (3.10) and therefore descends to the quotient, M. This extends the foregoing considerations to the genus one case.

Proposition 3.13 *If M is a three dimensional,* CR-*manifold with a transverse, free $U(1)$* CR-*action such that $\Sigma = M/U(1)$ is a Riemann surface of genus one and the line bundle $M \times_{\chi^{-1}} \mathbb{C}$ is of non-zero degree then there is a $U(1)$-invariant contact form and* (1, 0)-*form θ_α^1 such that the conditions* (2.1) *are satisfied with $m = 0$.*

We now prove that the contact form defined on a $U(1)$-invariant CR-manifold is a biholomorphic invariant if the genus of Σ is at least 2. Therefore the flat line bundle defined by (2.1) is as well. This constitutes a first step in the construction of a normal form for certain classes of three dimensional CR-manifolds analogous to that constructed in [BlDu, pp. 68–9] for linearly convex domains. Throughout this subsection we assume that the $U(1)$-action is transverse to the contact field and free.

Proposition 3.14 *Suppose that M_1 and M_2 are circle bundles over a surface of genus at least 2 with $U(1)$-invariant* CR*-structures. Then any orientation preserving,* CR-*diffeomorphism intertwines the $U(1)$-actions and therefore extends to a biholomorphic bundle map of the line bundles $M_i \times_{\chi^{-1}} \mathbb{C}$.*

Proof. This proposition follows by considering the possible automorphism groups which could arise and then applying results of Cartan, see [Ca]. Let ψ denote a biholomorphism from M_1 to M_2 and let T_1, T_2 denote the infinitesimal generators of the respective $U(1)$ actions. The statement of the proposition is equivalent to

$$\psi_* T_1 = \alpha T_2 , \tag{3.15}$$

where α is a constant. In any case $T_1' = \psi_* T_1$ defines a transverse, free, CR $U(1)$-action on M_2. Let $\mathrm{Aut}(M_2)$ denote the identity component in the group of CR-transformations of M_2. It follows from Theorem 5.1 in [ChMo] that $\mathrm{Aut}(M_2)$ is a finite dimensional Lie group. We will show that the identity component of $\mathrm{Aut}(M_2)$, $\mathrm{Aut}_0(M_2)$, must be one dimensional from which (3.15) follows.

Suppose that T_1' is not proportional to T_1 at some point on M_2, then

$$\dim \mathrm{Aut}_0(M_2) \geqq 2 . \tag{3.16}$$

We split our considerations into two cases.

Aut_0 is compact

Suppose the $\dim \mathrm{Aut}_0(M_2) = 2$; this implies that $\mathrm{Aut}_0(M_2)$ is abelian.

Lemma 3.17 *Suppose that M is a three dimensional, unit circle bundle with a $U(1)$-invariant* CR-*structure. If* $\mathrm{Aut}_0(M)$ *contains a two dimensional abelian subgroup containing the given $U(1)$-action then*

$$M/U(1) \simeq \mathbb{S}^2 \text{ or } \mathbb{T}^2 . \tag{3.18}$$

Remark. Since we are assuming that the $U(1)$-action is transverse to the contact field and free, M is automatically orientable and the quotient of M by $U(1)$ is a smooth orientable Riemann surface.

Proof of lemma. Let ϕ_t, ψ_s denote two one-parameter subgroups generating the abelian subgroup, with the given $U(1)$-action defined by ϕ_t. Let Σ denote the quotient Riemann surface defined by ϕ_t, with π the quotient map. Since the group is two dimensional there must be some open subset of M where the infinitesimal generators of the given subgroups are not proportional. Denote these by X, Y respectively. Because the group is abelian we have

$$\phi_t \circ \psi_s = \psi_s \circ \phi_t, \quad \text{and} \quad \phi_{t*} Y = Y, \quad \psi_{s*} X = X . \tag{3.19}$$

The formulae in (3.19) imply that for each s the map

$$\psi_s'(q) = \pi \circ \psi_s \circ \pi^{-1}(q) ,$$

of Σ to itself is well defined. Using the infinitesimal statement in (3.19) and the definition of the complex stucture on Σ it follows that these maps are holomorphic self maps. Since Y is not parallel to X it follows that these maps are non-constant for sufficiently small values of s and therefore depend in a non-trivial way on the parameter. The only oriented Riemann surfaces with continuous families of holomorphic self maps are $\mathbb{S}^2$ and $\mathbb{T}^2$. This completes the proof of the lemma.

This allows us to conclude that the group is not two dimensional as we have assumed that genus of Σ is at least 2. So if the group is compact and (3.15) fails then

$$\dim \mathrm{Aut}_0(M_2) \geqq 3 . \tag{3.20}$$

If the dimension is 3 then a theorem of Cartan implies that $\mathrm{Aut}_0(M_2)$ is transitive, see [Ca, Sects. 28–29]. Thus M_2 would be one of the CR-manifolds on Cartan's list. However an examination of the list shows that this cannot be the case since the genus of Σ is at least 2. This completes the analysis in case Aut_0 is compact.

Aut_0 is non-compact

It follows from a theorem of Webster, [We2], that M is spherical, that is locally biholomorphic to $\mathbb{S}^3$ with its standard structure. Again suppose that

$$\dim \mathrm{Aut}_0(M_2) = 2 . \tag{3.21}$$

There are only two real Lie algebras, up to isomorphism, of dimension 2, one abelian and one non-abelian. It follows from Lemma 3.17 that $\mathrm{Aut}_0(M_2)$ cannot be abelian. Any non-abelian two dimensional Lie algebra has generators X, Y such that

$$[X, Y] = 2Y . \tag{3.22}$$

The corresponding simply connected Lie group is $\mathbb{R}^2$ represented as 2×2 matrices

$$\mathfrak{G} = \left\{ \begin{pmatrix} e^{\lambda} & xe^{\lambda} \\ 0 & e^{-\lambda} \end{pmatrix}, \quad x, \lambda \in \mathbb{R} \right\} . \tag{3.23}$$

This group has neither compact subgroups nor discrete normal subgroups. Since $\mathrm{Aut}_0(M_2)$ contains a compact subgroup it cannot be isomorphic to a quotient of $\mathfrak{G}$. Hence if $\dim \mathrm{Aut}_0(M_2) \neq 1$ then

$$\dim \mathrm{Aut}_0(M_2) \geqq 3 .$$

In Sect. 63 of [Ca] it is shown that in this case $\mathrm{Aut}_0(M_2)$ is a subgroup of $SU(2, 1)$. These subgroups are then classified according to which three dimensional subgroup of $SU(2, 1)$ they contain. In particular the action of $\mathrm{Aut}_0(M_2)$ would be transitive and therefore M_2 would have to be on Cartan's list. This has already been ruled out and therefore

$$\dim \mathrm{Aut}_0(M_2) = 1$$

in this case as well.

This proposition implies that

$$\psi_* T_1 = \alpha T_2$$

for some function α on M_2. Since the dimension of $\mathrm{Aut}_0(M_2)$ is 1, α must be a constant. As the periods of T_1 and T_2 are normalized to equal 2π the constant must be ± 1. Since ψ preserves orientation we see the $\alpha = 1$. This completes the proof of the proposition.

For the moment we restrict to unit circle bundles in line bundles of degree $2(1-g)$ over surfaces, Σ of genus $g \geqq 2$. Later we consider the 'universal case' of bundles of degree -1. Such bundles are of the form $T^{1,0}\Sigma \otimes \lambda$ for a flat line bundle λ. Using the construction above we obtained a Ricci flat volume form on the complement of the zero section in any such bundle. It is obtained by transference from the bundle $T^{1,0}\Sigma$ itself. Let Λ_1 and Λ_2 be line bundles of the stated form. Denote the Ricci flat volume forms by v_1, v_2 respectively.

Lemma 3.24 *If Ψ is a biholomorphic bundle map from Λ_1 to Λ_2 then*

$$\Psi^*(v_2) = c^3 v_1$$

for some positive constant c.

Proof. To prove (3.24) we introduce local coordinates charts $\{z_\alpha, V_\alpha\}$ on Σ_2 over which Λ_2 is trivialized and such that Λ_1 is trivialized over their inverse images

$$\{\xi_\alpha = \Psi^*(z_\alpha),\ V'_\alpha = \Psi^{-1}(V_\alpha)\}\ .$$

We represent Λ_j, $j = 1, 2$ as twists of the bundles $T^{1,0}\Sigma_j, j = 1, 2$ by flat line bundles $\lambda_j, j = 1, 2$. Let ∂_{z_α} and ∂_{ξ_α} be used to trivialize $T^{1,0}\Sigma_j, j = 1, 2$ with fiber variables w_α, η_α respectively. We also use these parameters to trivialize Λ_1 and Λ_2. The Ricci flat volume forms are given in all cases by

$$v_1 = \frac{d\xi_\alpha \wedge d\eta_\alpha \wedge d\bar{\xi}_\alpha \wedge d\bar{\eta}_\alpha}{|\eta_\alpha|^4}, \quad v_2 = \frac{dz_\alpha \wedge dw_\alpha \wedge d\bar{z}_\alpha \wedge d\bar{w}_\alpha}{|w_\alpha|^4}. \tag{3.25}$$

Let ψ' denote the biholomorphic mapping of Σ_1 onto Σ_2 defined by Ψ. This induces a biholomorphic bundle map of $T^{1,0}\Sigma_1$ onto $T^{1,0}\Sigma_2$ which we denote by $\hat{\psi}$. In terms of these local parameters this map is given by

$$(\xi_\alpha, \eta_\alpha) \to (z_\alpha, m_\alpha(z_\alpha)w_\alpha)\ ,$$

for some non-vanishing holomorphic function m_α. As a consequence of Proposition 3.14 one can show that Ψ in these local coordinates is also given by

$$(\xi_\alpha, \eta_\alpha) \to (z_\alpha, n_\alpha(z_\alpha)m_\alpha(z_\alpha)w_\alpha)\ , \tag{3.26}$$

where n_α is another non-vanishing holomorphic function. By comparing the representations for different coordinate charts we deduce that $\{n_\alpha\}$ is a non-vanishing holomorphic section of the flat line bundle $\Psi^{-1*}(\lambda_1^{-1}) \otimes \lambda_2$. Since this bundle is represented in these coordinates by constant transition functions of modulus one it follows that each n_α is constant and

$$|n_\alpha| = |n_\beta| \quad \text{on} \quad V_\alpha \cap V_\beta\ . \tag{3.27}$$

The lemma follows from (3.25)–(3.27).

From the work of Fefferman it follows that the solution of the Monge-Ampére equation on a Kähler manifold is invariantly a section of the cube root of $\kappa \otimes \bar{\kappa}$ where κ is the canonical bundle, see [Fe, Sect. II]. This of course also holds for formal asymptotic solutions. If u_1, u_2 denote the asymptotic solutions constructed above in unimodular coordinate frames then

$$\Psi^*(u_2 v_2^{1/3}) = u_1 v_1^{1/3} + O(u_1^3)\ . \tag{3.28}$$

Combining Lemma 3.24 with (3.28) we obtain

(3.29) $$\Psi^*(u_2) = \frac{u_1}{c} + O(u_1^3).$$

Let

$$\theta_j = ia_j \partial u_j \restriction_{M_j}, \quad j = 1, 2,$$

where the constants a_j are chosen so that

(3.30) $$\int_{M_j} \theta_j \wedge d\theta_j = 1, \quad j = 1, 2.$$

From (3.29) and (3.30) it follows that

(3.31) $$\Psi^*(\theta_2) = \theta_1.$$

Now take M_1 and M_2 to be unit circle bundles in line bundles of degree -1 over surfaces of genus g. Assume that they are biholomorphically equivalent via a map Ψ. It follows from Proposition 3.14 that Ψ intertwines the $U(1)$-actions and therefore defines a biholomorphic mapping of the quotient of $M_1/\mathbb{Z}_{2(g-1)}$ onto $M_2/\mathbb{Z}_{2(g-1)}$. From this and the argument given above we conclude that if θ_1, θ_2 are the canonical contact forms on M_1, M_2 respectively then

$$\Psi^*(\theta_2) = \theta_1.$$

This completes the proof of

Theorem 3.32 *Suppose that M is a unit circle bundle diffeomorphic to the unit circle in a line bundle of degree -1 over a surface of genus $g \geqq 2$. The contact form θ defined by the third order asymptotic solution to the Monge-Ampère equation normalized by* (3.30) *is biholomorphically invariant.*

Remark. This is a normal form for the $U(1)$-invariant structures themselves. In the work of Bland, Duchamp and Lempert a solution to the homogeneous Monge-Ampère equation is used to construct normal forms for linearly convex domains.

To conclude this section we consider embeddings of $U(1)$-invariant structures on $\mathbb{S}^3$. The three sphere, with its canonical structure, is the unit circle bundle in the square root of the canonical bundle of the two sphere, $\kappa^{1/2}$. The uniformization theorem implies that there is a unique complex structure on the two sphere. Thus if M is $\mathbb{S}^3$ endowed with a $U(1)$-invariant CR-structure then $M/U(1) = \Sigma$ is canonically biholomophic to the unit sphere in $\mathbb{R}^3$. Let Ψ denote the biholomorphism.

The manifold M is CR-equivalent to the unit circle bundle in the square root of the canonical bundle of Σ relative to some hermitian metric. The mapping Ψ lifts to define a biholomorphic bundle map between the square root of the canonical bundle of Σ and that of $\mathbb{S}^2$. Evidently M can be realized as a $U(1)$-invariant hypersurface in $\kappa^{1/2}$ which does not intersect the zero section. Since the total space of $\kappa^{1/2}$ maps holomorphically into $\mathbb{C}^2$, with the zero section blown down to a point, it follows that M can be realized as a hypersurface in $\mathbb{C}^2$ invariant under the standard linear action of $U(1)$. This completes the proof of

Theorem 3.33 *Any* CR-*structure on the three sphere which is invariant under the standard action of $U(1)$ can be realized as a hypersurface in $\mathbb{C}^2$ invariant under the standard linear action of $U(1)$ on $\mathbb{C}^2$.*

In fact it is clear from the construction that the embedding is complex star shaped relative to the origin in $\mathbb{C}^2$. I would like to thank Dan Burns for simplifying the proof of this theorem. Implicit in the proof of this theorem is an application of the theorem of Ahlfors and Bers on the solvability of the Beltrami equation on the sphere, see [AhBe]. Since the CR-embedding theorem for small perturbations can be proved directly one obtains a new proof of the Ahlfors-Bers theorem in this case.

4 Embeddable perturbations of quotients

Along with the theory of line bundles over Riemann surfaces, Propositions 3.9, 3.3 and Corollary 2.21 allow a reasonably complete treatment of the perturbation theory of $U(1)$-invariant CR-structures on three dimensional circle bundles. As we saw in the previous section a $U(1)$-invariant structure can be realized as the unit circle bundle in a holomorphic line bundle with a hermitian metric. If the underlying Riemann surface has genus one then one can proceed directly as a unit circle in any such bundle has an invariant contact form and $(1, 0)$-form satisfying (2.1). For Riemann surfaces of other genera the holomorphic line bundles of degree ± 1 are 'universal' in that every other line bundle, of non-zero degree, can be realized as a quotient.

As remarked above the unit circle in the canonical bundle is pseudoconvex if $g = 0$ and pseudoconcave if the genus is two or more. There is a biholomorphic map from the complement of the zero section in a degree -1 bundle to the complement of the zero section in the dual degree $+1$ bundle. This map carries the unit circle bundle relative to some metric onto the unit circle bundle relative to the dual metric. If we have defined a contact form and normalized $(1, 0)$-forms by the procedure outlined in Sect. 3 on the degree $+1$ bundle then we can use the biholomorphic map to pull them back to the degree -1 bundle. Let θ denote the pulled back contact form and θ_α^1 the pulled back $(1, 0)$-forms. If Σ has genus g then

$$U_\phi^* \theta_\alpha^1 = e^{2i(1-g)\phi} \theta_\alpha^1 .$$

In this section we assume that $g \geqq 2$ unless otherwise stated.

Let $\mathscr{L}$ denote a holomorphic line bundle of degree -1 over Σ a Riemann surface of genus $g \geqq 2$ and M the unit circle bundle in $\mathscr{L}$ relative to some hermitian metric of positive curvature. To embed M into $\mathbb{C}^d$ for some d we need to construct sufficiently many CR-functions. Let h_n denote CR-functions which belong to F_n. In Appendix A we prove that the CR-functions in h_n are in one to one correspondence with holomorphic sections of $\mathscr{L}^{-n}$. According to Theorem A16 we can embed M into $\mathbb{C}^d$ in such a way that all the embedding functions belong to $\mathscr{F}_n$. This allows us to apply Theorem 2.23 with $k = 2(g - 1)$ so that we can study all structures with 'non-negative' Fourier coefficients, that is, those with $\psi \in \mathscr{F}_{4(1-g)}(\mathfrak{T})$. Thus Proposition 2.20 implies that the sections $\psi \in \mathscr{F}_{4(1-g)}(\mathfrak{T})$ of sufficiently small C^4-norm define embeddable perturbations of the $U(1)$-invariant structures. To obtain such results for hypersurfaces in bundles which are covered by $\mathscr{L}$ we simply use the equivariant perturbation result, Corollary 2.26 and embeddings of the quotients defined by holomorphic sections of large powers of the underlying line bundle.

Let M be a $U(1)$-invariant strictly pseudoconvex CR-manifold, set

$$\Sigma = M/U(1) \quad \text{and} \quad L = M \times_{\chi^{-1}} \mathbb{C} , \tag{4.1}$$

with the induced complex structures. As proved in Appendix A, L is a holomorphic line bundle over Σ. Lemma A4 implies that M has a CR-embedding into L; in the sequel we frequently identify M with its image in L. Since M is pseudoconvex it follows that L is a bundle of non-zero degree. After possibly reversing the direction of the infinitesimal generator of the $U(1)$-action, we can assume without loss of generality, that $\deg L < 0$. The classical theory of holomorphic line bundles over Riemann surfaces implies the existence of a holomorphic line bundle of degree -1 over Σ which covers L. We denote such a bundle by $\mathscr{L}$ with p denoting the holomorphic projection

$$p\colon \mathscr{L} \to L\,.$$

Set $k = \operatorname{card}\{p^{-1}(x)\}$, $x \in M$ and

$$\tilde{M} = p^{-1}(M)\,.$$

If $\tilde{M}$ is given the CR-structure induced from $\mathscr{L}$ then p restricted to $\tilde{M}$ is a CR-immersion. Functions on M pull back via p^* to functions on $\tilde{M}$ lying in the subspace

$$\mathscr{F}^k(\tilde{M}) = \bigoplus_{j=-\infty}^{\infty} F_{kj}(\tilde{M})\,.$$

For integers l, m define the subspaces of functions on $\tilde{M}$ by

$$\mathscr{F}_l^m(\tilde{M}) = \bigoplus_{j=0}^{\infty} F_{l+jm}(\tilde{M})\,.$$

The pullback identifies functions in $\mathscr{F}_1(M)$ with functions in $\mathscr{F}_k^k(\tilde{M})$. The CR-functions in $\mathscr{F}_n(M)$ pull back to functions in $\mathscr{F}_{nk}^k(\tilde{M})$.

Let g denote the genus of the Riemann surface, Σ defined in (4.1). We let $\tilde{\mathfrak{D}}$ be the flat line bundle parametrizing the deformations of $\tilde{M}$. Using this covering the deformations of the CR-structure on M can be parametrized by a space of sections of $\tilde{\mathfrak{D}}$.

Proposition 4.2 *Suppose that M is a $U(1)$-invariant CR-manifold and $\tilde{M}$ the k sheeted cover defined above. Let $\{\bar{Z}_\alpha\}$ denote the normalized sections of $T^{0,1}\tilde{M}$ which satisfy*

$$U_{\phi *}\bar{Z}_\alpha = e^{2i(g-1)\phi}\bar{Z}_\alpha\,. \tag{4.3}$$

The map

$$\psi_\alpha \to p_*(\bar{Z}_\alpha + \psi_\alpha Z_\alpha)$$

defines a surjection from $\mathscr{F}^k_{4(1-g)+k}(\tilde{\mathfrak{D}}) \oplus \mathscr{F}^{-k}_{4(1-g)}(\tilde{\mathfrak{D}})$ onto deformations of the given CR-structure on M.

Remark. If $g = 1$ then $\tilde{M}$ can be taken equal to M and $k = 1$.

Proof. It is clear that if

$$\psi \in \mathscr{F}^k_{4(1-g)+k}(\tilde{\mathfrak{D}}) \oplus \mathscr{F}^{-k}_{4(1-g)}(\tilde{\mathfrak{D}})$$

then $p_*(\bar{Z}_\alpha + \psi_\alpha Z_\alpha)$ defines a CR-structure on M. Moreover since p is a CR-immersion of $\tilde{M}$ onto M the section $\psi = 0$ corresponds to the initial $U(1)$-invariant structure on M. What remains is to show that every deformation of the given structure on M arises in this way.

Let H denote the real 2-plane field on M underlying the given CR-structure. As a consequence of the theorem of Gray on the stability of contact structures, [Gray] any deformation of this structure is specified by choosing a line subbundle $l \subset H \otimes \mathbb{C}$ which does not coincide with $T^{1,0}M$ at any point. The condition

$$p_*(\bar{Z}_{\alpha q} + \psi_\alpha(q) Z_{\alpha q}) \in l_{p(q)} ,$$

defines a section ψ of $\tilde{\mathfrak{D}}$. The transformation formula (4.3) implies that ψ_α is of the required form.

To study the embeddability of such structures we apply Corollary 2.26 to solve the perturbation equation

$$[p_* \bar{\partial}_b^\psi](x + \xi) = 0$$

on M by pulling back to $\tilde{M}$. It $\tilde{x}$ is the pullback of a CR-function from $\mathscr{F}_n(M)$, n sufficiently large, and $\psi \in \mathscr{F}^k_{4(1-g)}(\tilde{\mathfrak{D}})$ is of sufficiently small C^4-norm then there is a function $\xi \in \mathscr{F}^k_{nk}(\tilde{M})$ which satisfies

$$\bar{\partial}_b^\psi(\tilde{x} + \xi) = 0 .$$

Clearly $\tilde{x} + \xi$ is the pull back of a CR-function on M relative to the pushed forward structure. This argument shows that if the C^4-norm is small enough then the immersion of $\tilde{M}$ defined by an embedding of M can be perturbed to give a k-equivariant immersion of the perturbed structure on $\tilde{M}$ which in turn defines an embedding of the pushed forward structure on M.

To summarize we have shown:

Theorem 4.4 *Let M be a three dimensional pseudoconvex* CR*-manifold with a free transverse* CR*-action by $U(1)$. Suppose that the associated Riemann surface, Σ, has genus g and that $M \to \Sigma$ is the unit circle bundle in a line bundle of degree $k < 0$. With $\tilde{M}$ and $\tilde{\mathfrak{D}}$ as defined above the sections in $\mathscr{F}^k_{4(1-g)}(\tilde{\mathfrak{D}})$ of sufficiently small C^4-norm parametrize a space of embeddable* CR*-structures on M.*

Remarks. 1. In the genus one case the covering argument can be dispensed with as the line bundle $\mathfrak{D}$ is trivial and thus embeddable perturbations are parametrized by $\mathscr{F}_0(M)$.
2. If $k > 2(g-1)$ then the non-constant CR-functions on M lift to $\mathscr{F}_{2g-1}(\tilde{M})$. Thus we see that if $\iota: M \to \mathbb{C}^d$ is any embedding then all sufficiently small perturbations with 'non-negative' Fourier coefficients can be realized as *small perturbations* of ι. On the other hand if $1 \leqq k \leqq 2(g-1)$ it is conceivable that certain perturbations with 'non-negative' Fourier coefficients cannot be realized as deformations of a given embedding of M. At least our iteration scheme for solving the perturbation equations may fail. This is considered in greater detail in Sect. 8.

5 Special framings and 'positive' Fourier coefficients

So far we have insisted on pseudohermitian structures and framings that satisfy the very strict conditions in (2.1). These conditions are useful for constructing normal forms and studying the solvability of $\bar{\partial}_b$ but are more restrictive than is necessary to define what is meant by 'positive' or 'non-negative' Fourier coefficients. In this section we introduce a more general class of pseudohermitian structures and framings for which these notions agree.

In order for the Fourier decomposition of (0, 1)-forms to make sense all that is required is that the contact form be $U(1)$-invariant and the local framing of $\Lambda^{1,0}$ satisfy

$$U_\phi^* \theta^1 = e^{-im\phi}\theta^1 . \tag{5.1}$$

The dual (1, 0)-vector field Z satisfies

$$U_{\phi *} Z = e^{-im\phi} Z . \tag{5.2}$$

Suppose that $\theta, \tilde{\theta}$ are two $U(1)$-invariant contact forms and $\theta^1, \tilde{\theta}^1$ are (1, 0)-forms defined in an open set U, which satisfy (5.1) and

$$d\theta = i\theta^1 \wedge \theta^{\bar{1}}, \quad d\tilde{\theta} = i\tilde{\theta}^1 \wedge \tilde{\theta}^{\bar{1}} ,$$

respectively. If $Z, \tilde{Z}$ denote the dual (1, 0)-vector fields then

$$\tilde{Z} = uZ ;$$

from (5.1) it follows that u is $U(1)$-invariant. Suppose that ψ defines a local perturbation of Z by setting

$$\bar{Z}^\psi = \bar{Z} + \psi Z .$$

In the normalization defined by $\tilde{Z}$ this CR-structure is represented by

$$\bar{\tilde{Z}} + \tilde{\psi}\tilde{Z}, \quad \text{where } \tilde{\psi} = \frac{\bar{u}}{u}\psi . \tag{5.3}$$

Since u is $U(1)$-invariant it follows from (5.3) that

$$\psi \in \mathscr{F}_l^k(U) \quad \text{if and only if } \tilde{\psi} \in \mathscr{F}_l^k . \tag{5.4}$$

Thus we see that the notion of an equivariant perturbation with 'non-negative' Fourier coefficients is well defined relative to all local representations of the CR-structure satisfying (5.1).

The reason we introduced normalized sections was to prove Proposition 2.6. This result carries over in a simple way to the more general context presently under consideration. Suppose that θ, θ^1 are normalized according to (2.1) and that $\tilde{\theta}, \tilde{\theta}^1$ are as above. Let $\bar{\partial}_b^v$ denote the $\bar{\partial}_b$-operator defined by $\tilde{\theta}$, it is given locally by

$$\bar{\partial}_b^v f = \bar{\tilde{Z}} f \tilde{\theta}^1 .$$

The L^2-norm of a (0, 1)-form is given locally by

$$\langle f\tilde{\theta}^{\bar{1}}, f\tilde{\theta}^{\bar{1}} \rangle_v = \int_U |f|^2 \tilde{\theta} \wedge d\tilde{\theta} . \tag{5.5}$$

Using (5.5) we derive that

$$\bar{\partial}_b^{v*}(g\tilde{\theta}^{\bar{1}}) = -v^{-2} Zuv^2 g \quad \text{if } \tilde{Z} = uZ, \quad \tilde{\theta} = v\theta . \tag{5.6}$$

Since u and v are $U(1)$-invariant we see from (5.6) that $\ker \bar{\partial}_b^{v*}$ is also orthogonal to $\mathscr{F}_1$. The above remarks make clear that a (0, 1)-form belongs to $\mathscr{F}_1$ relative to θ^1 if and only if it belongs to $\mathscr{F}_1$ relative to $\tilde{\theta}^1$.

The cocycles defining the flat line bundles introduced in the previous sections can be trivialized over the sheaf of smooth $U(1)$-invariant functions. That is we can find non-vanishing functions $\phi_\alpha \in \mathscr{C}^\infty(U_\alpha)$ which are $U(1)$-invariant and satisfy

(5.7) $$\frac{\phi_\beta}{\phi_\alpha} = e^{il_{\alpha\beta}} \quad \text{in} \quad U_\alpha \cap U_\beta .$$

Define a (1, 0)-vector field by

$$W \restriction_{U_\alpha} = \phi_\alpha Z_\alpha .$$

This vector field is easily seen to be globally defined on M. It satisfies (5.2), therefore we can use this globally defined vector field to parametrize deformations of the CR-structure by functions in $\mathscr{C}^\infty(M)$ with sup-norm less than 1. As remarked above the deformations with 'non-negative' Fourier coefficients relative to this normalization coincide with that introduced previously. If the quotient surface Σ has genus g and the line bundle L has degree k then $\mathscr{F}^k_{4(1-g)}(\tilde{M})$ parametrizes these structures.

So far we have only considered deformations of the CR-structure with the same underlying contact structure. The theorem of Gray on the stability of contact structures implies that these deformations actually include representatives of all deformations, [Gray, Sect. 5]. If H_1, H_2 are two contact structures on M invariant under $U(1)$-actions we would like to find a contact transformation from H_1 to H_2 which intertwines the actions. The following argument indicates that this is not an unreasonable expectation:

Topological Lemma 5.8 *If M is a oriented circle bundle over an oriented surface, Σ of genus at least 2 then $\pi_1(M)$ is a central extension of $\pi_1(\Sigma)$. The center of $\pi_1(M)$ is generated by the fiber of $M \to \Sigma$.*

Proof. Let N denote the subgroup of $\pi_1(M)$ generated by the fiber. In the case at hand Proposition II.4.5 in [JS] implies that if $\alpha \in N$ then the centralizer of α, $C_\alpha = \pi_1(M)$. On the other hand if $\alpha \notin N$ then Proposition II.4.7 in [JS] implies that C_α is abelian. Since $\pi_1(M)$ is non-abelian this implies that the center of $\pi_1(M)$ is exactly N. Since

$$\pi_1(\Sigma) = \pi_1(M)/N$$

this proves the lemma.

From this lemma we deduce that the surfaces

(5.9) $$\Sigma_j = M/U^j, \quad \pi_j\colon M \to \Sigma_j \quad j = 1, 2 ,$$

are diffeomorphic and the line bundles

$$L_j = M \times_{\chi^j_{-1}} \mathbb{C}$$

have the same degree. Let θ_j be contact forms which are invariant under the actions U^j normalized so that

(5.10) $$\int_{\pi_j^{-1}(q)} \theta_j = 2\pi, \quad \text{for all } q \in \Sigma_j .$$

It follows from the $U(1)$-invariance and (5.10) that the forms $i\theta_j$ define connection forms for the principle bundles defined in (5.9). The Euler classes of the bundles L_j are given by $d\theta^j/2\pi$. Since these bundle are diffeomorphic it follows that

$$\int_{\Sigma_1} d\theta_1 = \int_{\Sigma_2} d\theta_2 \, . \tag{5.11}$$

Thus we have defined volume forms, $d\theta_j$, on Σ_j, $j = 1, 2$ with the same total volume. The theorem of Moser, see [Mos], implies that there is a diffeomorphism

$$\psi: \Sigma_1 \to \Sigma_2$$

such that

$$\psi^* d\theta_2 = d\theta_1 \, . \tag{5.12}$$

We would like to use the connections to lift ψ and obtain a map Ψ which makes the following diagram commute:

$$\begin{array}{ccc} M & \xrightarrow{\Psi} & M \\ \pi_1 \downarrow & & \downarrow \pi_2 \\ \Sigma_1 & \xrightarrow[\psi]{} & \Sigma_2 \, . \end{array} \tag{5.13}$$

This can of course be done over contractible subsets of Σ_1, the lifted map is a contact transformation. However ψ defines a monodromy representation of $\pi_1(\Sigma_1)$ in $U(1)$. The map can be lifted if and only if this representation is trivial. As this would take us too far afield we leave it as a conjecture that, at least for maps sufficiently close to the identity, by composing ψ with area preserving maps on the right and left a map ψ' can be found for which the monodromy representation is trivial.

Assuming this conjecture we observe that the notion of 'non-negativity' of Fourier coefficients has strong invariance properties. Let W, Z define two $U(1)$-invariant CR-structures relative to the actions U^1 and U^2 respectively. Let Φ denote a contact transformation conjugating U^1 into U^2. Let F_n and G_n denote the Fourier subspaces relative to U^1 and U^2 respectively, with the notations $\mathscr{F}_l^k$ and $\mathscr{G}_l^k$ defined by (2.25) with the obvious alphabetic modifications. Since Φ conjugates the two $U(1)$-actions it follows that

$$\Phi^*(G_n) = F_n \, . \tag{5.14}$$

Thus we see that if $\Phi^* W$ is represented by $\bar{Z} + \varphi Z$, for a function $\varphi \in F_{-2m}$ then a deformation $\bar{W} + \psi W$ is represented by

$$(\bar{Z} + \varphi Z) + \psi'(Z + \bar{\varphi}\bar{Z}), \quad \text{where } \psi' = \frac{\alpha}{\bar{\alpha}} \Phi^*(\psi), \quad \alpha \in F_0 \, . \tag{5.15}$$

Thus $\psi \in \mathscr{G}_l^k$ if and only if $\psi' \in \mathscr{F}_l^k$. Rewriting (5.15) in the normalization associated with Z we obtain that $\bar{W} + \psi W$ is represented by

$$\bar{Z} + \frac{\psi' + \varphi}{1 + \bar{\varphi}\psi'} Z \, . \tag{5.16}$$

Since $\|\psi'\|_{L^\infty}$, $\|\varphi\|_{L^\infty} < 1$ so is $\|(\psi' + \varphi)/(1 + \bar{\varphi}\psi')\|_{L^\infty}$. Suppose that $\psi \in \mathscr{G}^k_{-2m}$. Since $\varphi \in F_{-2m}$ it follows that

(5.17) $$(1 + \bar{\varphi}\psi')^{-1} \in \mathscr{F}^k_0 \,.$$

From (5.16) and (5.17) we conclude that

(5.18) $$\frac{\psi' + \varphi}{1 + \bar{\varphi}\psi'} \in \mathscr{F}^k_{-2m} \,.$$

To complete this part of the paper we show that deformations of a CR-structure defined by ψ with 'non-negative' Fourier coefficients can be extended to integrable almost complex structures on the unit disk in the total space of the line bundle containing M. This was proved in [BlDu] for CR-structures on $\mathbb{S}^{2n-1}$, John Bland suggested that this was also true in the case at hand, [Bl1]. For simplicity we only consider the case where the line bundle L has degree -1. The general case is treated by taking quotients. This result is the content of Theorem 4.1 in [Le1]; Lempert allows somewhat more general circle actions.

Suppose that M is a $U(1)$-invariant CR-manifold embedded as the unit circle bundle in a line bundle L of degree -1 over a Riemann surface Σ with genus $g \neq 0, 1$. We cover Σ by holomorphic coordinate neighborhoods $\{V_\alpha\}$ with lifts $\{U_\alpha\}$ to L. Let z_α denote a local holomorphic coordinate in V_α and x_α a fiber variable for $L \restriction_{U_\alpha}$ obtained, as in Sect. 3, by transference from a bundle which covers the canonical bundle of Σ. There are sections Z_α of $T^{1,0}(U_\alpha \cap M)$ which satisfy

(5.19) $$Z_\alpha = e^{il_{\alpha\beta}} Z_\beta \quad \text{in } U_\alpha \cap U_\beta \cap M \,.$$

As a submanifold of L, $M \cap U_\alpha$ is given by

(5.20) $$h_\alpha(z_\alpha)|x_\alpha|^2 = 1 \,,$$

where $h_\alpha(z_\alpha)$ is a positive function of the coordinate z_α. The (0, 1)-vector field $\bar{Z}_\alpha$, is given in these local coordinates, by

$$\bar{Z}_\alpha = \bar{x}_\alpha^{2(g-1)} \left[\partial_{\bar{z}_\alpha} - \frac{\partial_{\bar{z}_\alpha} h_\alpha}{h_\alpha} \bar{x}_\alpha \partial_{\bar{x}_\alpha} \right] .$$

These vector fields also satisfy

$$U_{\phi *} \bar{Z}_\alpha = e^{2i(g-1)} \bar{Z}_\alpha \,.$$

Suppose that $\psi \in \mathscr{F}_{4(1-g)}(\mathfrak{D})$ is defined along M and has sup-norm less than one, so that

$$\bar{Z}^\psi_\alpha = \bar{Z}_\alpha + \psi_\alpha Z_\alpha$$

defines a strictly pseudoconvex CR-structure on M. Observe that $\bar{Z}_\alpha$ and Z_α are defined in $L \cap U_\alpha$ and satisfy

(5.21) $$Z_\alpha = e^{il_{\alpha\beta}} Z_\beta \quad \text{in } U_\alpha \cap U_\beta \,.$$

Using the local coordinates, (z_α, x_α) we expand ψ_α to obtain

(5.22) $$\psi_\alpha(z_\alpha, x_\alpha) = \sum_{j=4(1-g)}^{\infty} a_{j\alpha}(z_\alpha) x_\alpha^j \,.$$

The expansion is uniformly convergent for $h_\alpha(z_\alpha)|x_\alpha|^2 = 1$ and defines an extension of $\psi_\alpha(z_\alpha, x_\alpha)$ as a function meromorphic along the fiber of $L \restriction_{U_\alpha}$ with a pole of order $4(g-1)$ along the zero section. Let $\hat{\psi}_\alpha$ denote this extension, it is a section of $\mathfrak{D}$ pulled back to $L\setminus\{0\}$.

To extend the perturbed CR-structure to an almost complex structure on L we set

$$\hat{\bar{Z}}_\alpha^\psi = (h_\alpha x_\alpha)^{2(1-g)} \bar{Z}'_\alpha$$

$$\text{(5.23)} \qquad \text{where} \quad \bar{Z}'_\alpha = \left[\left(\partial_{\bar{z}_\alpha} - \frac{\partial_{\bar{z}_\alpha} h_\alpha}{h_\alpha} \bar{x}_\alpha \partial_{\bar{x}_\alpha} \right) + (h_\alpha x_\alpha)^{2(g-1)} \hat{\psi}_\alpha Z_\alpha \right].$$

From (5.20) it follows that

$$\hat{\bar{Z}}_\alpha^\psi = (h_\alpha |x_\alpha|^2)^{2(1-g)} \bar{Z}_\alpha + \hat{\psi}_\alpha Z_\alpha ,$$

agrees with $\bar{Z}_\alpha^\psi$ along U_α. Since $h_\alpha(z_\alpha)|x_\alpha|^2$ is simply a scalar function on L in local coordinates, the extended vector fields satisfy (5.21). Moreover (5.23) implies that

$$\text{(5.24)} \qquad \bar{Z}'_\alpha = e^{il_{\alpha\beta}} \left(\frac{\bar{x}_\alpha}{\bar{x}_\beta} \right)^{2(1-g)} \bar{Z}'_\beta, \quad \text{in } U_\alpha \cap U_\beta ,$$

at least on the complement on the zero section. The ratio $\bar{x}_\alpha/\bar{x}_\beta$ actually depends only on z_α and therefore (5.24) extends to the zero section as well.

The vector field $\bar{Z}'_\alpha$ is smooth and non-vanishing on the closed unit disk bundle. It is defined on this subset up to a non-vanishing multiple. Observe that the perturbation term in (5.23) can be rewritten as

$$\text{(5.25)} \qquad h_\alpha^{2(g-1)} x_\alpha^{4(g-1)} \hat{\psi}_\alpha(z_\alpha, x_\alpha) \left(\partial_{z_\alpha} - \frac{\partial_{\bar{z}_\alpha} h_\alpha}{h_\alpha} \partial_{x_\alpha} \right).$$

From (5.25) we see that this vector field is smooth and holomorphic as a function of x_α up to the zero section. Since

$$|h_\alpha^{2(g-1)} x_\alpha^{4(g-1)} \hat{\psi}_\alpha(x_\alpha, z_\alpha)| < 1 \quad \text{where } h_\alpha |x_\alpha|^2 = 1$$

the maximum principle implies that this inequality holds for $h_\alpha|x_\alpha|^2 \leqq 1$. Therefore $\{\bar{Z}'_\alpha\}$ and $\{\partial_{\bar{x}_\alpha}\}$ define an almost complex structure on the closed unit disk bundle extending the perturbed CR-structure on the boundary. Since $x_\alpha^{4(g-1)} \hat{\psi}_\alpha$ is holomorphic in x_α it follows from (5.23) that

$$\text{(5.26)} \qquad [\bar{Z}'_\alpha, \partial_{\bar{x}_\alpha}] = \frac{\partial_{\bar{z}_\alpha} h_\alpha}{h_\alpha} \partial_{\bar{x}_\alpha} .$$

From (5.26) we conclude that the extended almost complex structure is integrable. Thus we have proved:

Theorem 5.27 (Bland-Lempert) *If M is the strictly pseudoconvex unit circle bundle in a holomorphic line bundle, L of degree -1 over a surface of genus g then the perturbations of the* CR-*structure represented by functions*

$$\psi \in \mathscr{F}_{4(1-g)}(\mathfrak{D}) \quad \text{with} \quad \|\psi\|_{L^\infty(M)} < 1$$

extend to integrable almost complex structures on the unit disk bundle in L.

Remark. The zero section of L remains a compact holomorphic curve relative to the perturbed CR-structure. As pointed out by Lempert this implies that there are nearby embeddable structures which are not parametrized by functions with 'non-negative' Fourier coefficients. For example let V be the null variety in $\mathbb{C}^3$ of a homogeneous polynomial $p(z)$ and M be the intersection of V with the unit sphere. The polynomial can be chosen so that M is covered by a manifold satisfying the hypotheses of Theorem 5.27. For small ε, the intersection of $p(z) = \varepsilon \neq 0$ with the unit sphere is a strictly pseudoconvex deformation of M. The variety it bounds contains no compact holomorphic curves, [Le2].

If $\psi \in \mathscr{F}^k_{4(1-g)}(\mathfrak{D})$ then the extended structure has the same equivariance properties as the CR-structure on M defined by $\bar{Z} + \psi Z$. This allows the extension of Theorem 5.27 to the general $U(1)$-invariant case treated above. We leave this to the interested reader.

As a corollary of this theorem we have:

Corollary 5.28 *Any structure satisfying the hypotheses of Theorem 5.27 is embeddable in $\mathbb{C}^d$ for some d.*

Proof. Applying the construction in Sect. 5 of [Le1] we obtain a defining function for $M \subsetneq L$ which is strictly plurisubharmonic on a neighborhood of the image of M in the unit disk bundle. This allows us to apply the "Newlander-Nirenberg with boundary" proved in [Cat] to conclude that the unit disk bundle in L has the structure of a complex manifold with boundary where the CR-structure on the boundary is that defined by ψ. Finally we apply Theorem 5.3 in [Ko] to conclude that the range of $\bar{\partial}_b$ is closed and therefore the structure is embeddable.

6 Generic non-embeddability in the holomorphically trivial case

In [BuEp2] it is demonstrated that the generic CR-structure on the three sphere with the standard underlying contact field in non-embeddable. This of course was well known for generic $\mathscr{C}^\infty$-structures, however in the cited paper this is shown for generic real analytic structures as well, see [BuEp2, p. 830]. Perhaps more important than the generic non-embeddability is the construction of an explicit obstruction to embedding. This obstruction is obtained by considering one parameter families of structures of the form $\bar{Z}^{t\psi} = \bar{Z} + t\psi Z$. It is shown that the associated $\Box_b$-operators form an analytic family of operators. The analyticity statement is actually proved for any three dimensional CR-manifold with a trivial holomorphic tangent bundle.

In this section M is a strictly pseudoconvex, three dimensional, CR-manifold with trivial holomorphic tangent bundle and volume form dV. Let $\bar{Z}$ denote a global real analytic section of $T^{0,1}M$ and $\bar{Z}^*$ the L^2-adjoint of $\bar{Z}$ relative to dV. The perturbations of the CR-structure defined by $\bar{Z}$ are parametrized by functions $\psi \in \mathscr{C}^\infty(M)$, with $\|\psi\|_{L^\infty} < 1$, by setting

$$\bar{Z}^\psi = \bar{Z} + \psi Z .$$

The associated $\Box_b$-operator is given by

$$\Box_b^\psi = (\bar{Z}^* + Z^*\bar{\psi})(\bar{Z} + \psi Z) . \tag{6.1}$$

It is a non-negative, self-adjoint operator on $L^2(M; \mathrm{dV})$. We use $\mathbb{E}^{\psi}_{[0,\lambda]}$ to denote the family of spectral projections defined by $\Box^{\psi}_b$. The normalization in (6.1) is slightly different from that used in [BuEp2] but this is immaterial for what follows.

Suppose that $0 < \lambda \notin \operatorname{spec}(\Box^{\psi}_b)$, then the operator

$$\mathscr{A}^{\psi} = \Box^{\psi}_b \mathbb{E}^{\psi}_{[0,\lambda]}, \tag{6.2}$$

defines an obstruction to embeddability. The structure is non-embeddable if and only if the rank of this operator is infinite. As this operator is difficult to study directly we introduce the analytic one parameter family of operators $\mathscr{A}^{t\psi}$. If any derivative of $\mathscr{A}^{t\psi}$ at $t = 0$ has infinite rank then so does $\mathscr{A}^{t\psi}$ for all but countably many values of t.

For $M = \mathbb{S}^3$, Theorem 14.1 in [Le1] implies that there exists a number $\lambda > 0$ such that if ψ is sufficiently small and

$$\operatorname{rk} \mathscr{A}^{\psi} > 0,$$

then actually the rank is infinite and the structure is non-embeddable. In light of this it follows that if any derivative of $\mathscr{A}^{t\psi}$ at zero has non-zero rank then the structures are non-embeddable for all sufficiently small, non-zero values of t. In [BuEp2] we conjectured that certain Fourier coefficients of ψ provided local moduli for perturbations of the standard structure on $\mathbb{S}^3$. This fact was recently proved by John Bland, [Bl2, Theorem 14.1]. In fact Bland discovered two normal forms; we call the one conjectured in [BuEp2] the *inner normal form* and the other the *outer normal form*. By combining Theorem 5.25 in [BuEp2] with Theorem 13.1 in [Le1] and the continuity of the map between the inner and outer normal forms established in Sect. 14 of [Bl2], one can show that a structure in inner normal form which is a small perturbation of the standard sphere is embeddable if and only if the second derivative of $\mathscr{A}^{t\psi}$ at $t = 0$ is zero. The detailed definitions and proof of this statement would carry us too far afield and will be left to a latter publication. In this section we extend the variational analysis presented in Sects. 2–4 [BuEp2] to arbitrary $U(1)$-invariant structures on three dimensional CR-manifolds with trivial holomorphic tangent bundle.

The first step is to compute the first and second variations of $\mathscr{A}^{t\psi}$ at $t = 0$. The calculation is essentially identical to the derivation given in Sect. 3 in [BuEp2]. We simply state the result.

Proposition 6.3 *Let M be an embeddable, three dimensional* CR*-manifold with $\bar{Z}$ a non-vanishing section of $T^{0,1}M$ and* dV *a volume form. Let $\psi \in \mathscr{C}^{\infty}(M)$ and $\mathscr{A}^{t\psi}$ be defined by* (6.2), *we have*

$$\partial_t \mathscr{A}^{t\psi} \restriction_{t=0} = 0, \quad \partial_t^2 \mathscr{A}^{t\psi} \restriction_{t=0} = 2\mathscr{C}^*(\psi)\mathscr{C}(\psi),$$

where

$$\mathscr{C}(\psi) = \bar{\mathscr{S}} \psi Z \mathscr{S}. \tag{6.4}$$

Here $\mathscr{S}$ is the orthogonal projection on $\ker \bar{Z}$ and *$\bar{\mathscr{S}}$ is the orthogonal projection onto* $\ker \bar{Z}^*$ *relative to the volume form* dV.

Remark. We have corrected a sign error in (3.3) of [BuEp2].

The second derivative of $\mathscr{A}^{t\psi}$ has infinite rank if and only if $\mathscr{C}(\psi)$ does. This operator depends linearly on ψ. As noted above, in the case of perturbations of the standard structure on $\mathbb{S}^3$, it represents the complete obstruction to embeddability. Hopefully it will prove useful in general. In this section we show that this is a smoothing operator and the map $\psi \to \mathscr{C}(\psi)$ is a continuous from $C^k(M)$, for some finite k, to bounded operators on $L^2(M)$ with the uniform topology, $B_{\text{unif}}(L^2(M))$.

To show that $\mathscr{C}(\psi)$ is generically of infinite rank we use the following general proposition.

Proposition 6.5 *Let $\mathscr{B}$ be a Banach space and $B(H)$ denote bounded operators on a Hilbert space with the strong topology. Suppose that $x \to K(x)$ defines a continuous linear map from $\mathscr{B}$ to $B(H)$. Either there is a constant $N < \infty$ such that*

$$\operatorname{rk} K(x) \leqq N \quad \text{for all } x \in \mathscr{B}$$

or there is subset $\mathscr{G} \subset \mathscr{B}$ which is a dense G_δ such that

$$\operatorname{rk} K(x) = \infty \quad \text{for } x \in \mathscr{G} .$$

Proof. To prove the proposition we require an elementary lemma.

Lemma 6.6 *Let A_n be a sequence of operators in $B(H)$ which converge to $A \in B(H)$. Suppose that for some $m \leqq \infty$*

$$\operatorname{rk} A_n \leqq m ,$$

then

$$\operatorname{rk} A \leqq m$$

as well.

Proof of lemma. Suppose that $\operatorname{rk} A > m$. Then we can find vectors u_i, v_i, $i = 1, \dots, m+1$ such that the matrix,

$$a_{ij} = \langle Au_i, v_j \rangle ,$$

has rank $m + 1$. In light of the strong convergence the sequence of matrices,

$$a_{ij}^n = \langle A_n u_i, v_j \rangle, \quad n = 1, 2, \dots ,$$

converges to a_{ij}. The hypothesis implies that

$$\operatorname{rk} a_{ij}^n \leqq m$$

and therefore the lower semicontinuity of the rank for finite dimensional matrices implies

$$\operatorname{rk} a_{ij} \leqq m .$$

The contradiction proves the lemma.

From the lemma we conclude that the sets

$$R_n = \{x \in \mathscr{B}; \operatorname{rk} K(x) \leqq n\}$$

are closed for each n. Suppose that for some N the set R_N has non-empty interior. We claim that

$$\mathscr{B} = R_N . \tag{6.7}$$

Suppose that this were not the case. Let x belong to the interior of R_N and let $y \in \mathscr{B} \setminus R_N$. In virtue of the linearity of K it is clear that $K(ty + (1-t)x)$ is an analytic family of operators. Since x is in the interior of R_N the rk $K(ty + (1-t)x)$ is at most N for t in a neighborhood of 0. Let $u_i, v_i \in H$, $i = 1, \ldots, m+1$ be chosen so that

$$a_{ij}^t = \langle K(ty + (1-t)x)u_i, v_j \rangle$$

has rank $m + 1$ when $t = 1$. As a_{ij}^t is an analytic matrix valued function, if its rank is at most N on an open subset then

$$\operatorname{rk} a_{ij}^t \leqq N \quad \text{for all } t .$$

This contradiction proves (6.7). If (6.7) does not hold for any N, then the interior of R_n is empty for every n. The Baire category theorem then implies that

$$\mathscr{G} = \bigcap_{n=0}^{\infty} R_n^c$$

is a dense G_δ. This completes the proof of the proposition.

Thus to show that $\operatorname{rk} \mathscr{C}(\psi)$ is generically infinite for $\psi \in \mathscr{B}$ it suffices to construct a sequence $\psi_n \in \mathscr{B}$ for which $\operatorname{rk} \mathscr{C}(\psi_n)$ is unbounded. As in [BuEp2] such a formulation gives us considerable latitude in the choice of function space $\mathscr{B}$. To show that $\mathscr{C}(\psi)$ is a smoothing operator we use the representation of the Szegö kernel as a Fourier integral operator with complex phase and the calculus of wave front sets for such operators. To show that $\mathscr{C}(\psi)$ defines a continuous map from $C^k(M)$ to $B_{\text{unif}}(L^2(M))$ we use the simpler representation afforded by the Heisenberg calculus. This is a symbol filtered algebra of operators contained in $S^*_{\frac{1}{2},\frac{1}{2}}$. Very briefly, one begins with an odd dimensional manifold, N and a codimension one subbundle, $\mathscr{V}$ of TN. The Heisenberg symbol classes are denoted by $S^m_{\mathscr{V}}(N)$, corresponding to operators in $\Psi^m_{\mathscr{V}}(N)$. The algebra is closed under compositions and adjoints; it has a symbolic composition formula and complete asymptotic expansions. The interested reader is referred to the excellent account presented in [BeGr].

Let M be a strictly pseudoconvex, embeddable three dimensional CR-manifold. The theorem of Harvey and Lawson implies that M smoothly bounds a variety, V. Let ρ be a plurisubharmonic defining function for $M \subsetneq V$. The contact form

$$\theta = -i\partial\rho \restriction_M$$

defines the real two plane bundle underlying the CR-structure on M.

Theorem 6.8 *Suppose that M is a strictly pseudoconvex, three dimensional, embeddable* CR-*manifold. For $\psi \in \mathscr{C}^\infty(M)$ the operator $\mathscr{C}(\psi)$, defined in* (6.4), *is a smoothing operator. There is a finite k such that the map, $\psi \to \mathscr{C}(\psi)$, is continuous from $C^k(M)$ to $B_{\text{unif}}(L^2(M))$.*

Proof. As is shown in [BeGr, Sect. 25] the operators $\mathscr{S}, \bar{\mathscr{S}} \in \Psi^0_{\ker\theta}(M)$ and therefore

$$\bar{\mathscr{S}}\psi Z \mathscr{S} \in \Psi^1_{\ker\theta}(M) . \tag{6.9}$$

We deduce the second statement in the theorem from the first statement and (6.9). These two assertions imply that the composition

$$\bar{\mathscr{S}}\psi Z \mathscr{S} \in \Psi^{-\infty}_{\ker\theta}(M) .$$

Since the Heisenberg calculus has a symbolic composition formula it follows from (6.9) that for any finite N we have an expansion of the symbol of $\mathscr{C}(\psi)$ into homogeneous terms with a smaller remainder term,

$$\sigma_{\ker\theta}(\mathscr{C}(\psi))(p,\xi)=\sum_{j=0}^{N} a_{1-j}(p,\xi)+r_N(p,\xi)\,. \tag{6.10}$$

Here $a_{1-j}\in S^{1-j}_{\ker\theta}(M)$ and the remainder term r_N is of order at most $-N$. On the other hand since $\mathscr{C}(\psi)$ is a smoothing operator the terms

$$a_{1-j}=0 \quad \text{for } j=0,\ldots,N\,. \tag{6.11}$$

For each N the computation of the remainder term requires only finitely many derivatives of ψ. Using the Fourier representation for the Schwartz kernel of $\mathscr{C}(\psi)$ it follows from (6.10) and (6.11) that we can estimate the sup-norm of the kernel in terms of finitely many derivatives of ψ. The continuity statement follows from this, the linearity of the map $\psi\to\mathscr{C}(\psi)$ and the closed graph theorem.

In [BoSj] it is shown that $\mathscr{S}$ is a Fourier integral operator with complex phase. From Proposition 2.16 in [BoSj] it follows that the canonical relation determined by $\mathscr{S}$ is

$$C^+=\bigcup_{p\in M}\{(p,t\theta_p;\,p,t\theta_p);\,t\in(0,\infty)\}\subset T^*M\times T^*M\,.$$

An essentially identical argument shows that $\bar{\mathscr{S}}$ is a Fourier integral operator with complex phase associated to the canonical relation

$$C^-=\bigcup_{p\in M}\{(p,t\theta_p;\,p,t\theta_p);\,t\in(-\infty,0)\}\subset T^*M\times T^*M\,.$$

Set

$$L^+=\{(p,t\theta_p);\,p\in M,\,t\in(0,\infty)\}\quad\text{and}\quad L^-=\{(p,t\theta_p);\,p\in M,\,t\in(-\infty,0)\}\,.$$

From (25.2.2) in [Hö] and Sect. 7 in [MeSj] it follows that if $u\in C^{-\infty}(M)$ then

$$\mathrm{WF}(\mathscr{S}u)\subset C^+(\mathrm{WF}(u))=\mathrm{WF}(u)\cap L^+$$

$$\mathrm{WF}(\bar{\mathscr{S}}u)\subset C^-(\mathrm{WF}(u))=\mathrm{WF}(u)\cap L^-\,.$$

Since ψZ is a differential operator it follows from this calculation that

$$\mathrm{WF}(\mathscr{C}(\psi)u)\subset\mathrm{WF}(u)\cap L^+\cap L^-=\varnothing\,.$$

Thus $\mathscr{C}(\psi)$ defines a map from $\mathscr{C}^{-\infty}(M)$ to $\mathscr{C}^{\infty}(M)$ which is the first assertion of the theorem.

Suppose that $\mathscr{B}$ is a Banach space of functions on M such that the norm $\|\cdot\|_{\mathscr{B}}$ satisfies

$$\|f\|_{C^k}\leqq c_k\|f\|_{\mathscr{B}}\,, \tag{6.12}$$

for some constant c_k. It follows from Theorem 6.8 that $\mathscr{C}(\psi)$ extends as a continuous map

$$\mathscr{C}:\mathscr{B}\to B_{\mathrm{unif}}(L^2(M))\,.$$

Proposition 6.5 implies that if there exists a sequence $\{f_n\}\subset\mathscr{B}$ for which $\operatorname{rk}\mathscr{C}(f_n)$ is unbounded then $\operatorname{rk}(\mathscr{C}(f))$ is generically infinite for $f\in\mathscr{B}$. We first construct such a sequence using smooth functions then, using a result of Morrey to obtain uniform analytic approximations, we extend to the real analytic case.

For the remainder of the section we restrict our considerations to M a *real analytic* 3-dimensional CR-manifold with a real analytic, fixed point free, transverse, $U(1)$ CR-action. Let

$$\Sigma = M/U(1)$$

endowed with the induced complex structure and L the holomorphic line bundle defined as the fiber product $M \times_{\chi^{-1}} \mathbb{C}$. Since M and the $U(1)$-action are real analytic, Σ and L have induced real analytic structures compatible with their holomorphic structures. The embedding of M into L as the unit circle bundle is real analytic. We use these observations to construct a real analytic trivialization of $T^{1,0}M$.

In Sect. 3 we defined open covers of Σ and M which we denoted by $\{V_\alpha\}$ and $\{U_\alpha\}$ respectively. We can assume that the sets $\{V_\alpha\}$ are a basis of real analytic coordinate neighborhoods for Σ. We also constructed local sections $\{Z_\alpha\}$ of $T^{1,0}M$ which satisfied (5.2) and

$$Z_\beta = e^{-ig_{\alpha\beta}} Z_\alpha \quad \text{in} \quad U_\alpha \cap U_\beta \, .$$

Since the embedding of M into L is real analytic and the holomorphic and real analytic structures are compatible it follows that these local sections are also real analytic. To obtain a global real analytic section we need to show that the flat line bundle, λ defined by the cocycle $\{e^{ig_{\alpha\beta}}\}$ has a non-vanishing real analytic section. This follows from the Oka-Grauert principle.

Suppose that N is a compact real analytic manifold. A complex neighborhood of N is a complex manifold, $\mathcal{N}$ such that

$$\dim_{\mathbb{C}} \mathcal{N} = \dim_{\mathbb{R}} N \quad \text{and} \quad N \hookrightarrow \mathcal{N}$$

as a totally real submanifold. Using the holomorphic extensions of real analytic transitions functions, defining the real analytic structure on N, one easily constructs a complex neighborhood $\mathcal{N}$. Choose a Riemannian metric on $\mathcal{N}$ and define $\rho(p)$, $p \in \mathcal{N}$ to be the square of the distance from p to N relative to this metric. Since N is smooth and totally real, there exists an open neighborhood U of $N \subset \mathcal{N}$ in which ρ is smooth, strictly plurisubharmonic and $d\rho \neq 0$ for $\rho > 0$. Setting

$$\mathcal{N}_\mu = \{p \in \mathcal{N}; \rho(p) \leqq \mu\} \, ,$$

it follows that, for small enough μ, $\mathcal{N}_\mu$ is a smooth Stein manifold, see [Gr, Sect. 3].

We complexify Σ by extending the real analytic transition functions relative to the cover $\{V_\alpha\}$. Denote the complexification by Σ'. The flat line bundle λ clearly extends to define a holomorphic line bundle on Σ'. This bundle is topologically trivial. Let Σ'_0 denote a relatively compact neighborhood of Σ in Σ' which is a Stein manifold. According to the Oka-Grauert principle the extended line bundle restricted to Σ'_0 must also be holomorphically trivial, see [JL, Sect. 2]. Let $\{\phi_\alpha\}$ denote a non-vanishing holomorphic section of λ relative to the cover $\{V_\alpha\}$. It satisfies

$$\frac{\phi_\beta}{\phi_\alpha} = e^{ig_{\alpha\beta}} \quad \text{in} \quad U_\alpha \cap U_\beta \, .$$

The restrictions of the ϕ_α to Σ are real analytic as are their lifts to M. Denote these lifts by $\{\hat{\phi}_\alpha\}$. Define a real analytic section of $T^{1,0}M$ by

$$Z\restriction_{U_\alpha} = \hat{\phi}_\alpha Z_\alpha \,. \tag{6.13}$$

These are easily seen to piece together to define a global section which satisfies (5.2).

The following classical lemma forms the basis of our construction.

Lemma 6.14 *Suppose that Σ is a Riemann surface and E is a positive, holomorphic line bundle over Σ. Given a positive integer n and non-negative integers i_j, $j=1,\ldots,n$ there exist a positive integer m, distinct points $p_1,\ldots,p_n \in \Sigma$ and holomorphic sections $s_1,\ldots,s_n$ of $E^{\otimes m}$ such that s_i vanishes to order at least i_j at p_j, $i \neq j$ and to order exactly i_i at p_i.*

Proof. This is a consequence of the Riemann-Roch theorem.

Remarks. If $k_1,\ldots,k_n$ is another set of non-negative integers with $k_j \leqq i_j$ then one can use the same points $p_1,\ldots,p_n$ and the same integer m to find sections s_i' which vanish to order at least k_j at $p_j, j \neq i$ and exactly order k_i at p_i. One should also note that the condition on the points $\{p_1,\ldots,p_n\}$ is generic relative to the Zariski topology on Σ^n.

The assumption that M is strictly pseudoconvex is equivalent to the negativity of L. To apply Lemma 6.14 we use the relationship between holomorphic sections of line bundles over Σ and CR-functions established in Proposition A8 as well as the connection between the $\ker \bar{\partial}_b^*$ and holomorphic sections of $L^{-n} \otimes \lambda^{-1}$. Suppose that $\{U_\alpha\}$ is an open cover of M by $U(1)$-invariant open sets with θ, θ_α^1 satisfying (2.1) and (2.2). Proposition 2.11 implies that $\{f_\alpha \theta_\alpha^{\bar{1}}\}$ defines a global section $\Lambda^{0,1}$ in $\ker \bar{\partial}_b^*$ provided

$$f_\alpha = e^{ig_{\alpha\beta}} f_\beta \quad \text{in } U_\alpha \cap U_\beta \quad \text{and} \quad Z_\alpha f_\alpha = 0\,. \tag{6.15}$$

Taking conjugates in (6.15) we see that $\{\bar{f}_\alpha\}$ satisfies

$$\bar{f}_\alpha = e^{-ig_{\alpha\beta}} \bar{f}_\beta \quad \text{in } U_\alpha \cap U_\beta \quad \text{and} \quad \bar{Z}_\alpha \bar{f}_\alpha = 0\,. \tag{6.16}$$

Thus we see that if

$$\{f_\alpha \theta_\alpha^{\bar{1}}\} \in F_n \cap \ker \bar{\partial}_b^*\,,$$

then $\{\bar{f}_\alpha\} \in F_{-n}$ are locally CR-functions.

The section, $s_{\bar{f}}$ of $L^{-n} \otimes \lambda^{-1}$ defined by these functions is holomorphic. Conversely if s is a holomorphic section of $L^{-n} \otimes \lambda^{-1}$ corresponding to $\{\bar{f}_\alpha\}$ then $\{f_\alpha \theta_\alpha^{\bar{1}}\} \in F_n \cap \ker \bar{\partial}_b^*$. This reduces the construction of (0, 1)-forms in the $F_n \cap \ker \bar{\partial}_b^*$ to the construction of holomorphic sections of $L^{-n} \otimes \lambda^{-1}$.

Let $\{\phi_\alpha\}$ be a C^ω, $U(1)$-invariant cochain which trivializes λ and let Z be defined by (6.13). For $\{f_\alpha \theta_\alpha^{\bar{1}}\} \in F_n \cap \ker \bar{\partial}_b^*$ we define a

$$g\restriction_{U_\alpha} = \bar{\hat{\phi}}_\alpha f_\alpha\,, \tag{6.17}$$

to obtain a globally defined function in $F_n \cap \ker \bar{Z}^*$. Thus the construction of functions in $F_n \cap \ker \bar{Z}^*$ is also reduced to the construction of holomorphic sections of $L^{-n} \otimes \lambda^{-1}$.

We use Lemma 6.14 in two special cases. First we construct holomorphic sections $\{s_i; i=1,\ldots,n\}$ of $L^{-m}\otimes\lambda^{-1}$ such that

$$s_i(p_j)=0 \quad \text{if } i\neq j, \quad s_i(p_i)\neq 0 .$$

Secondly we construct sections $\{\sigma_i; i=1,\ldots,n\}$ of L^{-m} such that

$$\sigma_i(p_j)=\partial\sigma_i(p_j)=0 \quad \text{if } i\neq j, \quad \sigma_i(p_i)=0, \quad \partial\sigma_i(p_i)\neq 0 .$$

We can pull these sections back to M to obtain functions. Denote the pull back of σ_i to a function on M by $\tilde{\sigma}_i$. From Proposition A8 it follows that $Z\tilde{\sigma}_i$ vanishes if and only if $\partial\sigma_i$ vanishes at the corresponding point on $M/U(1)$. Let $\tilde{s}_i$ denote the section s_i pulled back to M, conjugated and multiplied by the cochain $\{\hat{\phi}_\alpha\}$ as in (6.17). These are functions in $\ker\bar{Z}^*$. If we let $\{q_i\}\subset M$ denote a choice of preimages of $\{p_i\}$ then we have proved the following proposition:

Proposition 6.18 *For each $n>0$ there exist distinct points $q_1,\ldots,q_n\in M$ and functions $\tilde{s}_i$, $\tilde{\sigma}_i$, $i=1,\ldots,n$ such that*

$$\bar{Z}^*\tilde{s}_j=0, \quad \tilde{s}_i(q_j)=\delta_{ij}; \quad \bar{Z}\tilde{\sigma}_j=0, \quad Z\tilde{\sigma}_i(q_j)=\delta_{ij} .$$

Let $\{\tau_m\}$ be a sequence of functions in $\mathscr{C}^\infty(M)$ which converges, in the sense of distributions, to a sum of δ-masses concentrated at the points $q_1,\ldots,q_n$. Since $\{\tilde{s}_j\}\subset\ker\bar{Z}^*$ and $\{\tilde{\sigma}_j\}\subset\ker\bar{Z}$, we have

$$A_m=\langle\mathscr{C}(\tau_m)\tilde{\sigma}_i,\tilde{s}_j\rangle ,$$

$$=\langle\tau_m Z\tilde{\sigma}_i,\tilde{s}_j\rangle . \tag{6.19}$$

This sequence of matrices converges to the $n\times n$ identity matrix. Since n is arbitrary, Proposition 6.5 implies that $\operatorname{rk}\mathscr{C}(\psi)$ is unbounded for $\psi\in\mathscr{C}^\infty(M)$.

Morrey proved that a compact real analytic manifold has a real analytic embedding into $\mathbb{R}^l$ for some l, see [Mo].

Morrey's theorem. *If N is a compact, real analytic manifold then for some l there exists an analytic embedding*

$$\imath: N\to\mathbb{R}^l .$$

The following approximation result is a simple consequence of Morrey's theorem; for the sake of completeness we include a proof.

Proposition 6.20 *If N is a compact real analytic manifold with a family of complex neighborhoods, $\mathscr{N}_\mu$, then for μ sufficiently small the restriction of functions holomorphic on $\mathscr{N}_\mu$ to N is dense in $\mathscr{C}^\infty(N)$.*

Proof. Let $\imath$ denote an analytic embedding of N into $\mathbb{R}^l$ with coordinate functions $(x_1,\ldots,x_l)$. Since N is compact there is a $\mu_0>0$ such that the functions x_i, $i=1,\ldots,l$ have holomorphic extensions to $\mathscr{N}_{\mu_0}$. This in turn defines a holomorphic map,

$$\hat{\imath}: \mathscr{N}_{\mu_0}\to\mathbb{C}^l$$

which extends $\imath$.

If $\psi\in\mathscr{C}^\infty(N)$ a standard argument shows that there is a function $\Psi\in\mathscr{C}_c^\infty(\mathbb{R}^l)$ such that

$$\imath^*(\Psi)=\psi .$$

By convolving Ψ with the heat kernel for $\mathbb{R}^l$ we can obtain a sequence of functions $\{\Psi_m\}$ which extend as entire functions to $\mathbb{C}^l$ and such that

$$\lim_{m\to\infty} \|\Psi_m - \Psi\|_{C^k(\mathbb{R}^l)} = 0 ,$$

for any k. Setting

$$\psi_m = \hat{\imath}^*(\Psi_m) ,$$

we obtain a sequence of holomorphic functions defined on $\mathcal{N}_{\mu_0}$ for which

$$\lim_{m\to\infty} \|\psi_m - \psi\|_{C^k(M)} = 0 ,$$

for any k.

Let $\mathcal{M}_\mu$ denote a family of complex neighborhoods of M as in Proposition 6.20. Let $C^\omega_\mu(M)$ denote real analytic functions on M which have a bounded holomorphic extension to $\mathcal{M}_\mu$. We use

$$\|u\|_\mu = \sup_{p\in\mathcal{M}_\mu} |f(p)|$$

to define a norm on $C^\omega_\mu(M)$. Standard Cauchy estimates imply that these norms satisfy (6.12) for any value of k. Proposition 6.20 implies that if μ is sufficiently small then we can uniformly approximate functions in $\mathscr{C}^\infty(M)$ by functions in $C^\omega_\mu(M)$.

Proposition 6.21 *If μ is sufficiently small then for any $n\in\mathbb{N}$ there exists a sequence $\{\varphi_m\}\subset C^\omega_\mu(M)$ such that*

$$\lim_{m\to\infty} \operatorname{rk}\mathscr{C}(\varphi_m) = n . \tag{6.22}$$

Proof. Fix a positive integer n, let $\{\tau_m\}$ denote the sequence constructed in the proof of Proposition 6.18. It follows from the Proposition 6.20 that if we choose a positive sequence $\{\delta_m\}$ then, for sufficiently small μ, there is a sequence

$$\{\varphi_m\}\subset C^\omega_\mu(M)$$

such that

$$\|\varphi_m - \tau_m\|_{L^\infty(M)} \leqq \delta_m . \tag{6.23}$$

Note that μ is a fixed positive number. By choosing a sequence $\{\delta_m\}$ tending to zero sufficiently fast we can arrange that

$$\langle \mathscr{C}(\varphi_m)\tilde{\sigma}_i, \tilde{s}_j\rangle$$

has rank n for large m. This proves the proposition.

As a corollary of Theorem 6.8, Propositions 6.5 and 6.21 and the standard Cauchy estimates, we have

Corollary 6.24 *For μ sufficiently small the operator $\mathscr{C}(\psi)$ has infinite rank for a dense G_δ in $C^\omega_\mu(M)$.*

This corollary, Proposition 6.3 and Lemma 2.33 in [BuEp2] give the generic nonembeddability result.

Corollary 6.25 *If M is a $U(1)$-invariant, real analytic three dimensional* CR-*manifold which has a real analytic global section of $T^{1,0}M$ that satisfies* (5.2) *then, for*

μ sufficiently small, there is a dense G_δ, $\mathscr{G}_\mu \subset C^\omega_\mu(M)$ such that if $\psi \in \mathscr{G}_\mu$ then $\bar{Z} + \psi Z$ is not embeddable.

Remark. It would be interesting to obtain a precise description of those ψ for which $\mathscr{C}(\psi) \neq 0$. In case that $M = Sl(2, \mathbb{R})/\Gamma$, where Γ is a co-compact discrete subgroup this may be possible using representation theory. I would like to thank Bruce Kleiner for help clarifying the material in this section.

7 Generic non-embeddability for quotients

In this section we extend Corollaries 6.24 and 6.25 to $U(1)$-invariant CR-manifolds which are quotients of CR-manifolds of the type considered in the previous section. We adopt the approach used in Sect. 4 of replacing analysis on the quotient with 'equivariant' analysis on the cover. Let M denote a real analytic $U(1)$-invariant CR-manifold. Define Σ and L as in (4.1) and $\mathscr{L}$ and $\tilde{M}$ as in Sect. 4. The line bundle $\mathscr{L}$ is of degree -1 and $\tilde{M}$, the unit circle bundle in $\mathscr{L}$, defines a k-sheeted CR-cover of M. Let p denote the projection of $\tilde{M}$ to M.

We suppose that $\tilde{M}$ is endowed with a real analytic section $\bar{Z}$ of $T^{0,1}\tilde{M}$ and a contact form, so that (5.2) is satisfied with $m = 2(g-1)$. Then $\Box_b = -\bar{Z}^*\bar{Z}$ is $U(1)$-invariant, that is

$$U^*_\phi \Box_b = \Box_b U^*_\phi \,. \tag{7.1}$$

Since the contact form is $U(1)$-invariant it descends to the quotient. Using this contact form we obtain a representation of $\bar{\partial}_b$ on M as a differential operator. The contact form also defines a volume form which in turn defines an adjoint for $\bar{\partial}_b$ and an associated $\Box_b$-operator on M. Denote this operator by $\hat{\Box}_b$. From this construction and (7.1) it follows that

$$p^* \hat{\Box}_b = \Box_b p^* \,. \tag{7.2}$$

Suppose that Σ has genus g, then according to Proposition 4.2 the deformations of the given $U(1)$-invariant CR-structure on M are parametrized by $\mathscr{F}^k_{4(1-g)+k}(\tilde{M}) \oplus \mathscr{F}^{-k}_{4(1-g)}(\tilde{M})$. Let ψ belong to this subspace, set

$$\bar{Z}^\psi = \bar{Z} + \psi Z \,,$$

and let $\Box^\psi_b$ denote the associated $\Box_b$-operator. This operator no longer commutes with the action of $U(1)$, however if

$$\phi_j = \frac{2\pi j}{k}, \quad j \in \mathbb{Z} \,,$$

then

$$U^*_{\phi_j} \Box^\psi_b = \Box^\psi_b U^*_{\phi_j}, \quad j = 0, \ldots, k \,. \tag{7.3}$$

From this we conclude that $\Box^\psi_b$ satisfies

$$p^* \hat{\Box}^\psi_b = \Box^\psi_b p^* \,, \tag{7.4}$$

with $\hat{\Box}^\psi_b$ denoting the operator induced on $\mathscr{C}^\infty(M)$ by the pushed forward CR-structure and contact form.

The pullback defines an isomorphism between $\mathscr{C}^\infty(M)$ and $\mathscr{F}^k(\tilde{M}) \cap \mathscr{C}^\infty(\tilde{M})$. From (7.3) and (7.4) we conclude that

$$\square_b^\psi \mathscr{F}^k(\tilde{M}) \subset \mathscr{F}^k(\tilde{M}) . \tag{7.5}$$

Proposition 7.6 *Suppose that*

$$\psi \in \mathscr{F}^k_{4(1-g)+k}(\tilde{M}) \oplus \mathscr{F}^{-k}_{4(1-g)}(\tilde{M}) , \tag{7.7}$$

the self adjoint extension of $\hat{\square}_b^\psi$ on $L^2(M)$ is unitarily equivalent to the self adjoint extension of $\square_b^\psi \restriction_{\mathscr{F}^k(\tilde{M}) \cap \mathscr{C}^\infty(\tilde{M})}$.

Proof. This is immediate from (7.3), (7.4) and (7.5).

Let $\mathbb{E}^\psi_{[0,\lambda]}$ denote the spectral projections defined by $\square_b^\psi$. As a consequence of (7.5) and the functional calculus it follows that $\mathscr{F}^k(\tilde{M})$ is an invariant subspace of $\mathbb{E}^\psi_{[0,\lambda]}$ for all λ. If $\hat{\mathbb{E}}^\psi_{[0,\lambda]}$ denotes the spectral projections defined by $\hat{\square}_b^\psi$ then Proposition 7.6 implies that

$$\hat{\mathscr{A}}^\psi = \hat{\square}_b^\psi \hat{\mathbb{E}}^\psi_{[0,\lambda]}$$

is unitarily equivalent to

$$\mathscr{A}^\psi_k = \square_b \hat{\mathbb{E}}^\psi_{[0,\lambda]} \restriction_{\mathscr{F}^k(\tilde{M})} .$$

From these observations we conclude

Proposition 7.8 *If ψ satisfies (7.7) then $\partial_t^2 \hat{\mathscr{A}}^{t\psi} \restriction_{t=0}$ is unitarily equivalent to $\partial_t^2 \mathscr{A}^{t\psi}_k \restriction_{t=0}$.*

This proposition reduces the variational calculation on M to an equivariant calculation on $\tilde{M}$.

From Proposition 6.3 and elementary linear algebra it follows that

$$\operatorname{rk} \partial_t^2 \mathscr{A}^{t\psi}_k \restriction_{t=0} = \operatorname{rk} \mathscr{C}(\psi) \restriction_{\mathscr{F}^k(\tilde{M})} .$$

Let $\mathscr{C}_k(\psi)$ denote $\mathscr{C}(\psi) \restriction_{\mathscr{F}^k(\tilde{M})}$. As in the previous section we first produce a smooth sequence

$$\{\tau_m\} \subset \mathscr{F}^k_{4(1-g)+k}(\tilde{M}) \oplus \mathscr{F}^{-k}_{4(1-g)}(\tilde{M}) \tag{7.9}$$

so that $\operatorname{rk}(\mathscr{C}_k(\tau_m))$ tends to infinity and then approximate by real analytic functions.

Applying Lemma 6.14 we conclude that for each n we can find points $\{p_1, \ldots, p_n\}$ and a positive integer m such that there are holomorphic sections $\{\sigma_1, \ldots, \sigma_n\}$ of $\mathscr{L}^{-mk}$ and holomorphic sections, $\{s_1, \ldots, s_n\}$, of $\mathscr{L}^{2(1-g)-mk} \otimes \lambda^{-1}$ which satisfy

$$\partial \sigma_i(p_j) = \delta_{ij}, \quad s_i(p_j) = \delta_{ij} . \tag{7.10}$$

For $i = 1, \ldots, n$ let $\{q_{ij}, j = 1, \ldots, k\} \subset \tilde{M}$, denote a choice of $\mathbb{Z}_k$-orbit lying over p_i. The sections σ_i, $i = 1, \ldots, n$ lift to CR-functions on $\tilde{M}$ in F_{mk}. As before we lift the sections $\{s_i\}$ to sections in $F_{mk-2(1-g)}(\lambda^{-1})$. Denote by $\{\tilde{s}_i\}$ the functions obtained by conjugating the lifted sections and multiplying by the cochain $\{\bar{\tilde{\phi}}_\alpha\}$ defined in (6.17). As before these functions lie in $\ker \bar{Z}^* \cap F_{2(1-g)-mk}$. Formula (7.10) implies that, by multiplying by constants, the lifted sections can be normalized to satisfy

$$Z\tilde{\sigma}_m(q_{ij}) = e^{4\pi i \frac{(j-1)(g-1)}{k}} \delta_{im}, \quad \tilde{s}_m(q_{ij}) = e^{4\pi i \frac{(j-1)(1-g)}{k}} \delta_{im} . \tag{7.11}$$

Finally we can choose a smooth sequence,

$$\{\tau_m\} \subset \mathscr{F}^{k}_{4(1-g)+k}(\tilde{M}) \oplus \mathscr{F}^{-k}_{4(1-g)}(\tilde{M}),$$

converging to

$$\sum_{i,j} e^{8\pi i \frac{(j-1)(1-g)}{k}} \delta_{q_{ij}}(p). \tag{7.12}$$

Let $A_m = \langle \mathscr{C}_k(\tau_m)\tilde{\sigma}_i, \tilde{s}_j \rangle$; from (6.23), (7.11) and (7.12) it follows that

$$\lim_{m\to\infty} A_m = k\,\mathrm{Id}_n. \tag{7.13}$$

Let $\tilde{\mathscr{M}}_\mu$ denote a family of complex neighborhoods of $\tilde{M}$. The assumption that the $U(1)$-action is real analytic implies that U_ϕ extends to a biholomorphism of some complex neighborhood $\tilde{\mathscr{M}}_{\mu_0}$. By choosing the Riemannian metric on $\tilde{\mathscr{M}}$ to be $U(1)$-invariant we can suppose that $\tilde{\mathscr{M}}_{\mu_0}$ is carried to itself by the $U(1)$-action. By possibly decreasing μ_0 we can assume that the real analytic embedding of $\tilde{M}$ into $\mathbb{R}^l$ extends to a holomorphic mapping of $\tilde{\mathscr{M}}_{\mu_0}$ into $\mathbb{C}^l$.

Let $\{\delta_m\}$ be a sequence of positive numbers and $\mu < \mu_0$ be a fixed positive number. Applying Proposition 6.20 we obtain a sequence $\{\psi_m\}$ of holomorphic functions defined on $\tilde{\mathscr{M}}_\mu$ which satisfy

$$\|\psi_m - \tau_m\|_{L^\infty(\tilde{M})} < \delta_m. \tag{7.14}$$

Since the action of U_ϕ on $\tilde{\mathscr{M}}_\mu$ is biholomorphic, the Fourier components of the functions ψ_m are also holomorphic on $\tilde{\mathscr{M}}_\mu$. Let $\{\psi'_m\}$ denote the projections of $\{\psi_m\}$ into the subspace $\mathscr{F}^{k}_{4(1-g)+k}(\tilde{M}) \oplus \mathscr{F}^{-k}_{4(1-g)}(\tilde{M})$. These functions are also holomorphic on $\tilde{\mathscr{M}}_\mu$. Since the sequence $\{\tau_m\}$ satisfies (7.9) we can choose a sequence $\{\delta_m\}$ tending to zero so rapidly that

$$\lim_{m\to\infty} \operatorname{rk}\mathscr{C}_k(\psi'_m) = n. \tag{7.15}$$

The real analytic perturbations of M are parametrized by

$$C^{\omega}_{\mu,k}(\tilde{M}) = C^{\omega}_{\mu}(\tilde{M}) \cap \mathscr{F}^{k}_{4(1-g)+k}(\tilde{M}) \oplus \mathscr{F}^{-k}_{4(1-g)}(\tilde{M})$$

for different values of μ. The functions $\psi'_m \in C^{\omega}_{\mu,k}$. Since the integer n in (7.15) is arbitrary Theorem 6.8 implies

Theorem 7.16 *Suppose that M is a real analytic, three dimensional* CR-*manifold with a free, transverse, real analytic action by $U(1)$ then for each $\mu < \mu_0$ there is a dense G_δ, $\mathscr{G}_{\mu,k} \subset C^{\omega}_{\mu,k}(\tilde{M})$ such that if $\psi \in \mathscr{G}_{\mu,k}$ then the* CR-*structure $p_*(\bar{Z} + \psi Z)$ is non-embeddable.*

8 Small eigenvalues and the stability of embeddings

Let $(M, \bar{\partial}^t_b)$, $t \in [-\varepsilon, \varepsilon]$ be a continuous family of strictly pseudoconvex CR-structures on a manifold M. We say that a CR-function, v relative to $\bar{\partial}^0_b$ is stable if there is a $0 < \delta \leqq \varepsilon$ and a continuous family of functions $\{v_t;\, t \in (-\delta, \delta)\}$, such that

$$v_0 = v, \quad \bar{\partial}^t_b v_t = 0. \tag{8.1}$$

For the present discussion we leave vague the sense in which v_t depends continuously on t. A CR-function is unstable if a continuous family satisfying (8.1) does not exist for any $\delta > 0$. We say that the family $(M, \bar{\partial}_b^t)$ is *stably embeddable at* 0 if every CR-function relative to $\bar{\partial}_b^0$ is stable. If the families in (8.1) can be taken to be continuous relative to the C^2-topology on M then this condition implies that any CR-embedding

$$i: (M, \bar{\partial}_b^0) \hookrightarrow \mathbb{C}^N ,$$

can be included in a continuous family of CR-embeddings

$$i_t: (M, \bar{\partial}_b^t) \hookrightarrow \mathbb{C}^N, \quad t \in (-\delta, \delta), \quad \delta > 0 ,$$

with $i_0 = i$. In general δ may have to be taken less than ε.

If $(v_1, \ldots, v_N)$ defines a CR-embedding of $(M, \bar{\partial}_b^0)$ and one of the coordinate functions is unstable then we say that i is an unstable embedding. In Sects. 2 and 4 we observed that certain embeddings of $U(1)$-invariant CR-manifolds may be unstable. In fact Lempert and Catlin have produced an example of an unstable embedding for a family of $U(1)$-invariant structures. In this section we give conditions in terms of the spectrum of the associated $\square_b$-operators and the algebraic structure of the ring of CR-functions for a family to be stably embeddable. In some sense these conditions are necessary and sufficient.

Let M be a circle bundle of degree -1 over a surface Σ of genus $g \geqq 2$. We suppose that a $U(1)$-invariant CR-structure is fixed on M. From Proposition 3.9 we can suppose that there is a normalized contact form θ, a $U(1)$-invariant covering $\{U_\alpha : \alpha \in A\}$ and sections θ_α^1 of $\Lambda^{1,0} U_\alpha$ normalized by (2.1). Let Z_α denote the sections of $T^{1,0} U_\alpha$ dual to θ_α^1. As in the previous sections we treat this 'universal' case in detail, with the quotients following by using equivariant analysis on $\tilde{M}$. Unless otherwise stated, an object 'v_α', indexed by α, refers to a globally defined object whose restriction to U_α is v_α.

Recall that in terms of the normalized frame we have

$$\bar{\partial}_b u = \bar{Z}_\alpha u \theta_\alpha^{\bar{1}}; \quad \bar{\partial}_b^*(f_\alpha \theta_\alpha^{\bar{1}}) = -Z_\alpha f_\alpha . \tag{8.2}$$

The deformations are parametrized by sections $\{\psi_\alpha\}$ of the flat line bundle $\mathfrak{D}$ defined in (2.17). The normalized contact form defines a volume form, $dV = \theta \wedge d\theta$, and therefore an L^2-structure on M. As in Sect. 2, we define

$$\bar{\partial}_b^\psi u \restriction_{U_\alpha} = (\bar{Z}_\alpha + \psi_\alpha Z_\alpha) u \theta_\alpha^{\bar{1}} .$$

Since the frame $\theta_\alpha^{\bar{1}}$ satisfies (2.2) we can use dV to define an L^2-structure on (0, 1)-forms. This in turn defines the adjoint of $\bar{\partial}_b^\psi$:

$$\bar{\partial}_b^{\psi *}(u_\alpha \theta_\alpha^{\bar{1}}) = -(Z_\alpha + \bar{Z}_\alpha \bar{\psi}_\alpha) u_\alpha .$$

The associated Laplace operator is given by

$$\square_b^\psi = \bar{\partial}_b^{\psi *} \bar{\partial}_b^\psi .$$

We let $\mathscr{S}^\psi$ denote the orthogonal projection onto the $\ker \bar{\partial}_b^\psi$ and $\mathscr{Q}^\psi$ denote the partial inverse to $\square_b^\psi$, they satisfy:

$$\mathscr{Q}^\psi \square_b^\psi = \square_b^\psi \mathscr{Q}^\psi = \mathrm{Id} - \mathscr{S}^\psi . \tag{8.3}$$

For simplicity we use $\mathscr{S}$ and $\mathscr{Q}$ to denote these operators relative to the unperturbed structure. If $\eta = f_\alpha \theta_\alpha^{\bar{1}}$ is orthogonal to the kernel of $\bar{\partial}_b^{\psi *}$ then the unique solution of the equation

$$\bar{\partial}_b^\psi u = \eta ,$$

orthogonal to $\ker \bar{\partial}_b^\psi$, is given by

$$u = \mathscr{Q}^\psi \bar{\partial}_b^{\psi *} \eta . \tag{8.4}$$

Otherwise stated

$$\bar{\partial}_b^\psi \mathscr{Q}^\psi \bar{\partial}_b^{\psi *} = \mathrm{Id} - \bar{\mathscr{S}}^\psi , \tag{8.4'}$$

where $\bar{\mathscr{S}}^\psi$ is the orthogonal projection onto $\ker \bar{\partial}_b^{\psi *}$. We denote the $\ker \bar{\partial}_b^\psi$ by $\mathfrak{H}^\psi$; as above $\mathfrak{H}$ refers to $\ker \bar{\partial}_b$.

Suppose that v is in $\mathfrak{H}^\psi$, this implies that

$$\bar{\partial}_b v = -(\psi_\alpha Z_\alpha v)\theta_\alpha^{\bar{1}} . \tag{8.5}$$

Since the right hand side of (8.5) lies in the range of $\bar{\partial}_b$ it follows from (8.2) and (8.4) that

$$\tilde{v} = \mathscr{Q}\bar{\partial}_b^*(\psi_\alpha Z_\alpha v \theta_\alpha^{\bar{1}})$$

satisfies

$$\bar{\partial}_b \tilde{v} = -(\psi_\alpha Z_\alpha v)\theta_\alpha^{\bar{1}} . \tag{8.6}$$

From (8.5)–(8.6) we conclude that $h = v - \tilde{v}$ belongs to the $\mathfrak{H}$. To summarize we have shown that if $v \in \mathfrak{H}^\psi$ then there is a function $h \in \mathfrak{H}$ such that

$$(\mathrm{Id} - \mathscr{Q}\bar{\partial}_b^* \psi_\alpha Z_\alpha)v = h . \tag{8.7}$$

Proposition 8.8 *Suppose that $\{\psi_\alpha\}$ is a section of $\mathfrak{D}$ and the L^2-norm of the operator $\mathscr{Q}\bar{\partial}_b^* \psi_\alpha Z_\alpha$ is less than 1 then the operator $(\mathrm{Id} - \mathscr{Q}\bar{\partial}_b^* \psi_\alpha Z_\alpha)$ defines an injection of $\mathfrak{H}^\psi$ into $\mathfrak{H}$. Conversely, under these hypotheses, if $h \in \mathfrak{H}$ then*

$$v = (\mathrm{Id} - \mathscr{Q}\bar{\partial}_b^* \psi_\alpha Z_\alpha)^{-1} h$$

belongs to $\mathfrak{H}^\psi$ if and only if

$$\bar{\mathscr{S}}(\psi_\alpha Z_\alpha v \theta_\alpha^{\bar{1}}) = 0 .$$

Proof. If $\mathscr{Q}\bar{\partial}_b^* \psi_\alpha Z_\alpha$ has L^2-norm less than 1 then the operator in (8.7) is invertible and the first claim follows easily. Formula (8.4′) implies that if we apply $\bar{\partial}_b$ to both sides of (8.7) we obtain

$$\bar{\partial}_b^\psi v + \bar{\mathscr{S}}(\psi_\alpha Z_\alpha v \theta_\alpha^{\bar{1}}) = 0 .$$

From this the second statement is immediate.

Remark. Note that Proposition 8.8 does not require ψ to have 'non-negative' Fourier coefficients. John Bland has also obtained a result of this sort, [Bl1].

Under the hypotheses of the proposition we can use a Newmann series to represent $(\mathrm{Id} - \mathscr{Q}\bar{\partial}_b^* \psi_\alpha Z_\alpha)^{-1}$. Several important corollaries follow from this representation. For each $m \in \mathbb{Z}$, let

$$\mathfrak{H}_m^\psi = \mathfrak{H}^\psi \cap \mathscr{F}_m .$$

Since $\mathcal{Q}$ belongs to the functional calculus defined by $\square_b$, (7.1) implies that

$$\mathcal{Q}: F_m \to F_m, \quad \text{for } m \in \mathbb{Z} .$$

Thus for $\{\psi_\alpha\} \in \mathcal{F}^k_{kl+4(1-g)}(\mathfrak{D})$ it follows from (2.11) that

$$(8.9) \qquad (\mathcal{Q}\bar{\partial}_b^* \psi_\alpha Z_\alpha)^j : \mathcal{F}^k_m \to \mathcal{F}^k_{m+jkl}, \quad \text{for } m \in \mathbb{Z},\ j \in \mathbb{N}_0 .$$

Let $\psi_{\alpha j}$ denote the $[j + 4(1-g)]$-Fourier component of ψ_α.

Corollary 8.10 *If $\psi_\alpha \in \mathcal{F}_{4(1-g)}(\mathfrak{D})$ is such that the operators $\mathcal{Q}\bar{\partial}_b^* \psi_\alpha Z_\alpha$ and $\mathcal{Q}\bar{\partial}_b^* \psi_{\alpha 0} Z_\alpha$ have L^2-norm less than 1 then for $m \in \mathbb{N}_0$ the map in (8.7) defines an injection of $\mathfrak{H}^\psi_m$ into $\mathfrak{H}_m$. The subspace $\mathfrak{H}^\psi_0$ equals $\mathfrak{H}^\psi$ moreover*

$$(8.11) \qquad \dim(\mathfrak{H}^\psi_m / \mathfrak{H}^\psi_{m+1}) \leqq \dim(\mathfrak{h}_m / \mathfrak{H}_{m+1}) .$$

If $m > 2(g-1)$ then the map from $\mathfrak{H}^\psi_m$ to $\mathfrak{H}_m$ is an isomorphism.

Proof. The assumption that ψ has 'non-negative' Fourier coefficients, (8.9) and the assumption on the norm of the operator $\mathcal{Q}\bar{\partial}_b^* \psi_\alpha Z_\alpha$ imply the first statement in the corollary.

Let $\{v_1, \ldots, v_d\} \subset \mathfrak{H}^\psi_m$ be representatives for a basis of $\mathfrak{H}^\psi_m / \mathfrak{H}^\psi_{m+1}$ and set

$$h_i = (I - \mathcal{Q}\bar{\partial}_b^* \psi_\alpha Z_\alpha) v_i \in \mathfrak{H}_m .$$

We claim that the functions $\{h_1, \ldots, h_d\}$ are linearly independent mod $\mathfrak{H}_{m+1}$. Were this not the case then we could find constants, $\{a_1, \ldots, a_d\}$, not all zero, such that

$$(8.12) \qquad \sum_i a_i h_i \in \mathfrak{H}_{m+1} .$$

Set $v = a_1 v_1 + \cdots + a_d v_d$, as the v_i define a basis for $\mathfrak{H}^\psi_m / \mathfrak{H}^\psi_{m+1}$, we conclude that

$$v \not\equiv 0 \bmod \mathfrak{H}^\psi_{m+1} .$$

From formula (8.12) it follows that

$$(8.13) \qquad (\mathrm{Id} - \mathcal{Q}\bar{\partial}_b^* \psi_\alpha Z_\alpha) v \in \mathfrak{H}_{m+1} .$$

We decompose v:

$$v = v' + v'', \quad \text{where } 0 \neq v' \in F_m, \quad v'' \in \mathcal{F}_{m+1} .$$

Formula (8.9) implies

$$(\mathrm{Id} - \mathcal{Q}\bar{\partial}_b^* \psi_\alpha Z_\alpha) v'' \in \mathcal{F}_{m+1} ,$$

from (8.13) we deduce that

$$(\mathrm{Id} - \mathcal{Q}\bar{\partial}_b^* \psi_{\alpha 0} Z_\alpha) v' = 0 .$$

As $v' \neq 0$ this contradicts the assumption on the norm of the operator $\mathcal{Q}\bar{\partial}_b^* \psi_{\alpha 0} Z_\alpha$ and completes the proof of (8.11).

From the Neumann series representation for $(\mathrm{Id} - \mathcal{Q}\bar{\partial}_b^* \psi_\alpha Z_\alpha)^{-1}$ and (8.9) we conclude that if $h \in \mathfrak{H}_m$ then $v \in \mathcal{F}_m$. Thus if $m \geqq 2(g-1) + 1$ then

$$\psi_\alpha Z_\alpha (\mathrm{Id} - \mathcal{Q}\bar{\partial}_b^* \psi_\alpha Z_\alpha)^{-1} h \theta^{\bar{1}}_\alpha \in \mathcal{F}_{m-2(g-1)} \subset \mathcal{F}_1 .$$

The final statement follows from this and Lemma 2.12.

Remark. If ψ is in $F_{4(1-g)}$ so that $\bar{\partial}_b^\psi$ is also $U(1)$-invariant and one could freely interchange the roles of $\bar{\partial}_b$ and $\bar{\partial}_b^\psi$ in (8.11) then it would follow that the inequalities in (8.11) are equalities. The reason one cannot do this in general is that the norm of $\mathcal{Q}^\psi$ is determined by the smallest eigenvalue of $\square_b^\psi$. Thus $\mathcal{Q}\bar{\partial}_b^*\psi_\alpha Z_\alpha$ may have very small norm while

$$\mathcal{Q}^\psi(Z_\alpha + \bar{Z}_\alpha\bar{\psi}_\alpha)\psi_\alpha(Z_\alpha + \bar{\psi}_\alpha\bar{Z}_\alpha)$$

has very large norm. The latter operator can be used, as in (8.7), to represent CR-functions relative to $\bar{\partial}_b$ in terms of CR-functions relative to $\bar{\partial}_b^\psi$. This prevents one from establishing the reverse of the inequalities in (8.11). This is a reflection of the classical fact that the dimension of the space of holomorphic sections of a bundle with degree between 1 and $2(g-1)$ depends on the bundle.

These considerations lead to an algebraic condition for a continuous family of perturbations to be stably embeddable. In the remainder of this section we let $\psi(t)$ denote a one parameter family of sections of $\mathfrak{D}$ with $\psi(0) = 0$. For simplicity we replace notations like $\mathfrak{H}^{\psi(t)}$ by $\mathfrak{H}^t$.

Theorem 8.14 *If $\psi(t) \subset \mathcal{F}_{4(1-g)}(\mathfrak{D})$, $t \to \psi(t)$ is continuous with respect to the C^k-topology on sections, for some sufficiently large $k \geqq 1$, and the dimensions,*

$$\dim(\mathfrak{H}^t_m/\mathfrak{H}^t_{m+1}), \quad m = 0, 1, \ldots \tag{8.15}$$

are independent of t then the family $(M, \bar{\partial}_b^t)$ is stably embeddable at 0.

Proof. As shown in [BuEp2, 5.13] the L^2-norms of the operators $\mathcal{Q}\bar{\partial}_b^*\psi_\alpha Z_\alpha$ and $\mathcal{Q}\bar{\partial}_b^*\psi_{\alpha 0} Z_\alpha$ are bounded by a constant times the C^1-norms of ψ_α and $\psi_{\alpha 0}$ respectively. Thus we can apply the previous corollary to conclude that if t is sufficiently small then

$$A_t = (\mathrm{Id} - \mathcal{Q}\bar{\partial}_b^*\psi_\alpha(t)Z_\alpha) \tag{8.16}$$

defines an injection of $\mathfrak{H}^t_m$ into $\mathfrak{H}_m$. From Corollary 8.10 it follows that if m is large enough then this injection is actually an isomorphism. Let m_0 be the least such m for which this is true; in fact $m_0 = 0$. Suppose that this were not the case, then there would be a proper subspace $V \subset \mathfrak{H}_{m_0-1}$ such that

$$A_t^{-1}V = \mathfrak{H}^t_{m_0-1}\,.$$

However this would imply that

$$\begin{aligned}\dim(\mathfrak{H}^t_{m_0-1}/\mathfrak{H}^t_{m_0}) &= \dim(V/\mathfrak{H}_{m_0}) \\ &< \dim(\mathfrak{H}_{m_0-1}/\mathfrak{H}_{m_0})\,,\end{aligned}$$

contradicting (8.15).

On pp. 832–3 of [BuEp2] it is shown that, since A_t is a pseudodifferential operator of order 0 and type $(\frac{1}{2}, \frac{1}{2})$, for each positive number s there is a constant $C_s(t)$ such that

$$\|(A_t - \mathrm{Id})u\|_{H^s} \leqq C_s(t)\|u\|_{H^s}\,. \tag{8.17}$$

For each s the constant $C_s(t)$ depends on some finite number, $k(s)$, of derivatives of $\psi(t)$. *If* $\psi(t)$ is continuous with respect to the $C^{k(s)}$-topology on sections then the constant $C_s(t)$ tends to zero as t tends to zero. This implies that, for sufficiently small t, we can sum the Neumann series for A_t^{-1} and obtain a continuous family of operators from H^s to H^s tending to the identity as t tends to 0.

Let f be a CR-function relative to the unperturbed structure. Suppose that $\psi(t)$ is continuous relative to the $C^{k(s)}$-topology on sections. The functions

$$f^t = A_t^{-1} f$$

form a continuous one parameter family in H^s with

$$f^t \in \mathfrak{H}^t, \quad f^0 = f.$$

This proves the theorem.

Let $(f_1, \ldots, f_N)$ define a CR-embedding of $(M, \bar{\partial}_b^0)$ and define

$$f_i^t = A_t^{-1} f_i .$$

If $\psi(t)$ is continuous with respect to the $C^{k(4)}$-topology on sections then it follows, from the Sobolev embedding theorem, that the mappings

$$i_t(p) = (f_1^t(p), \ldots, f_N^t(p)) ,$$

have uniform estimates in $C^2(M)^N$. Since i_0 is an embedding it follows from the compactness of M that there is an interval $(-\delta, \delta)$ such that i_t is an embedding of M into $\mathbb{C}^N$ for $t \in (-\delta, \delta)$.

The next order of business is to relate the changes in the dimensions of the quotients in (8.11) to the appearance of small eigenvalues for $\Box_b^t$. In [BuEp2, Sect. 2] it is shown that if the real and imaginary parts of $\psi(t)$ are real analytic functions of t then $\Box_b^t$ is an analytic family of operators. The Friedrichs' extension of $\Box_b^t$ is self adjoint for real values of t. In the cited paper it is shown that the spec $\Box_b^t \setminus \{0\}$ consists of discrete eigenvalues of finite multiplicity, however accumulation can occur at $\{0\}$. This prevents the direct application of analytic perturbation to obtain the existence of analytic families of eigenvalues and eigenprojections for the 'small eigenvalues.' We make this notion more precise. Let $\psi(t)$ denote a real analytic family of smooth sections of $\mathfrak{D}$ such that $\psi(0)$ defines an embeddable structure and r denote a positive real number such that

$$(0, r] \cap \operatorname{spec}(\Box_b^0) = \emptyset .$$

In [BuEp2, p. 821] it is shown that there is a $\delta > 0$ such that if $|t| < \delta$ then $r \notin \operatorname{spec}(\Box_b^t)$. For values of t with $|t| < \delta$ an eigenvalue of $\Box_b^t$ lying in the interval $(0, r)$ is called a small eigenvalue. Let

$$\mathscr{A}^t = \Box_b^{\psi(t)} \mathbb{E}_{[0,r]}^{\psi(t)} ;$$

for $|t| < \delta$ this is a real analytic family of compact operators. The non-zero eigenvalues of $\mathscr{A}^t$ are precisely the small eigenvalues of $\Box_b^t$.

Proposition 8.18 *Let $\psi(t)$ denote a real analytic family of smooth sections of $\mathfrak{D}$ with $r, \mathscr{A}^t$ and δ as above. Suppose that $\operatorname{rk}(\mathscr{A}^t)$ is finite for all $t \in (-\delta, \delta)$ and not identically zero then there is an $m \in \mathbb{N}$ and analytic functions $\mu_i(t)$, $i = 1, \ldots, m$ such that for $t \in (-\delta, \delta) \setminus \{0\}$*

$$\operatorname{spec}(\mathscr{A}^t) \cap (0, r) = \{\mu_1(t), \ldots, \mu_m(t)\} . \tag{8.19}$$

Moreover there are projection valued analytic functions $\mathbb{P}_i^t$, $i = 1, \ldots, m$ *such that*

(8.20) $$\mathscr{A}^t \mathbb{P}_i^t = \mathbb{P}_i^t \mathscr{A}^t = \mu_i(t) \mathbb{P}_i^t .$$

Proof. From Proposition 6.5 it follows that there is a non-negative integer N such that

(8.21) $$\operatorname{rk}(\mathscr{A}^t) \leqq N ,$$

with equality on a dense set. For real values of t define the '$N \times N$ characteristic polynomial' of $\mathscr{A}^t$ by

$$D_N(\lambda; t) = \sum_{j=0}^{N} \lambda^j (-1)^{N-j} \sigma_{N-j}(\mathscr{A}^t) ,$$

here $\sigma_k(A)$ is the kth elementary symmetric function in the eigenvalues of the matrix A. If we set

$$\tau_k^t = \operatorname{tr}(\mathscr{A}^t)^k ,$$

then the formulae of Newton imply that there are polynomials, $p_j(x_1, \ldots, x_j)$ such that

$$\sigma_k(\mathscr{A}^t) = p_k(\tau_1^t, \ldots, \tau_k^t), \quad k = 1, \ldots, N$$

see [Chr, pp. 436–7]. From these equations it follows easily that the coefficients of $D_N(\lambda; t)$ depend analytically on t. Using elementary linear algebra one can show that, for real values of t, the non-zero eigenvalues of $\mathscr{A}^t$ are among the roots of $D_N(\lambda; t)$. A standard argument from finite dimensional self adjoint, analytic perturbation theory, shows that these roots are given by analytic functions $\{\lambda_1(t), \ldots, \lambda_N(t)\}$, see [Ka, Chap. 2, Sects. 1, 6].

From among these functions we can select a maximal set of distinct non-zero, functions which we denote by $\{\mu_1(t), \ldots, \mu_m(t)\}$. This completes the proof of (8.19). The set of values of t for which any of the equations

$$\mu_i(t) = \mu_j(t), \ i \neq j, \quad \mu_i(t) = 0, \ i = 1, \ldots, m ,$$

has a solution is finite. Let E denote this set, for $t \notin E$ let $\mathbb{P}_i^t$ denote the orthogonal projection onto the $\ker(\mathscr{A}^t - \mu_i(t))$. For each $i \in \{1, \ldots, m\}$ define an analytic family of operators by

(8.22) $$P_i^t = \int_{|\lambda| = r} \lambda \prod_{j \neq i} (\lambda - \mu_j(t))(\lambda - \square_b^t)^{-1} \frac{d\lambda}{2\pi i} .$$

For $t \notin E$ it follows from the spectral theorem for self adjoint operators and the functional calculus that

(8.23) $$P_i^t = \mu_i(t) \prod_{j \neq i} (\mu_i(t) - \mu_j(t)) \mathbb{P}_i^t .$$

Because the norm of $\mathbb{P}_i^t$ is at most 1 we conclude that, for any functions $u, v \in L^2(M)$, the divisor of the analytic function $\langle P_i^t u, v\rangle$, in a small disk about 0, satisfies the inequality

(8.24) $$(\langle P_i^t u, v\rangle) \geqq \left(\mu_i(t) \prod_{j \neq i} (\mu_i(t) - \mu_j(t)) \right) .$$

From (8.24) it follows that

$$\tilde{\mathbb{P}}_i^t = P_i^t\left[\mu_i(t)\prod_{j\neq i}(\mu_i(t)-\mu_j(t))\right]^{-1}, \tag{8.25}$$

defines a weakly analytic family of operators in a small disk about 0. From (8.23) it follows that for $t \notin E$

$$\tilde{\mathbb{P}}_i^t = \mathbb{P}_i^t . \tag{8.26}$$

Since the notions of weak and uniform analyticity coincide for bounded families of operators, $\mathbb{P}_i^t$ define analytic families of projection operators. For $t \notin E$ the operators $\mathbb{P}_i^t$ are in the functional calculus of $\Box_b^t$ and therefore

$$\mathscr{A}^t\mathbb{P}_i^t = \mathbb{P}_i^t\mathscr{A}^t = \mu_i(t)\mathbb{P}_i^t, \quad i = 1,\dots,m \quad \text{for } t \notin E . \tag{8.27}$$

Since E is a finite set this extends, by continuity, to $t \in E$ as well. This completes the proof of the proposition.

Above we observed that the CR-functions in $\mathfrak{H}_m$ are stable for sufficiently large m in that for each function $u \in \mathfrak{H}_m$ there is a continuous family of functions $u_t \in \mathfrak{H}_m^t$ with $u_0 = u$. Using the analytic family of projections obtained in the proposition we can show that the small eigenvalues are connected with unstable CR-functions. Since the eigenvalues $\mu_i(t)$ tend to 0 at $t = 0$ it follows that

$$\operatorname{range}\mathbb{P}_i^0 \subset \mathfrak{H}^0 .$$

Proposition 8.28 *Suppose that the quotient of M by $U(1)$ is a surface of genus g, then* range($\mathbb{P}_i^0$) *has a trivial projection into $\mathfrak{H}_{2g-1}^0$, for $i = 1,\dots,m$.*

Proof. The structure corresponding to $t = 0$ is $U(1)$-invariant thus the Fourier components of a CR-function are also CR-functions. As the $\mathbb{P}_i^0$, $i = 1,\dots,m$ are orthogonal projections onto subspaces of $\mathfrak{H}^0$ it follows that for each i there is an orthonormal set of CR-functions $\{u_{ij}, j = 1,\dots,n_i\}$, $i = 1,\dots,m$ such that

$$\mathbb{P}_i^0 f = \sum_{j=1}^{n_i} \langle f, u_{ij}\rangle u_{ij} . \tag{8.29}$$

Suppose that for some i, j u_{ij} has a non-trivial projection into $\mathfrak{H}_{2g-1}^0$, denote it by v. Using Corollary 8.10 we conclude that

$$v_t = (\mathrm{Id} - \mathscr{Q}\bar{\partial}_b^*\psi_\alpha(t)Z_\alpha)^{-1}v$$

is an analytic family of functions which satisfies

$$\Box_b^t v_t = 0; \quad v_0 = v . \tag{8.30}$$

From (8.27) it follows that

$$\Box_b^t\mathbb{P}_i^t v_i = \mathbb{P}_i^t\Box_b^t v_t = 0 . \tag{8.31}$$

On the other hand it follows from (8.29) that $\mathbb{P}_i^0 v_0 \neq 0$ and thus $\mathbb{P}_i^t v_t \neq 0$ for t in some open interval about 0. The function $\mu_i(t)$ does not vanish in a deleted neighborhood of 0, therefore in this neighborhood

$$\Box_b^t\mathbb{P}_i^t v_t = \mu_i(t)\mathbb{P}_i^t v_t \neq 0 .$$

This contradicts (8.31) and completes the proof of the proposition.

To connect this result to the question of stable embeddability we consider the following question: given a function $v \in \mathfrak{H}^0$ is there a continuous family of functions $v_t \in \mathfrak{H}^t$ with $v_0 = v$?

Proposition 8.32 *If v lies in the range of $\mathbb{P}_i^0$ for some $i \in \{1, \ldots, m\}$ then there does not exist an L^2-continuous family, v_t, with*

$$v_t \in \mathfrak{H}^t \quad \text{and} \quad v_0 = v .$$

Proof. Suppose that such a family exists, call it v_t. Since the family is continuous in the L^2 topology and

$$\mathbb{P}_i^0 v_0 = v_0$$

it follows that for some open interval

$$\mathbb{P}_i^t v_t \neq 0 . \tag{8.33}$$

From (8.29) we conclude that

$$\square_b^t \mathbb{P}_i^t v_t = \mathbb{P}_i^t \square_b^t v_t = 0 . \tag{8.34}$$

On the other hand (8.20) implies that

$$\square_b^t \mathbb{P}_i^t v_t = \mu_i(t) \mathbb{P}_i^t v_t .$$

Since $\mu_i(t) \neq 0$, for t in a deleted neighborhood of 0, this contradicts (8.34) thus proving the proposition.

Corollary 8.35 *A real analytic family, $\psi(t) \subset \mathscr{F}_{4(1-g)}(\mathfrak{D})$ of* CR-*structures is stably embeddable at* 0 *if and only if* $\operatorname{rk} \mathscr{A}(t) = 0$ *for t in a neighborhood of* 0.

Proof. With r as above we define an analytic family of operators by

$$\tilde{\mathscr{S}}^t = \int_{|\lambda| = r} (\lambda - \square_b^t)^{-1} \frac{d\lambda}{2\pi i} .$$

If the $\operatorname{rk} \mathscr{A}(t) = 0$ then $\tilde{\mathscr{S}}^t$ is the Szegö projector for the structure $\bar{\partial}_b^t$. If f is a CR-function relative to $(M, \bar{\partial}_b^0)$ then

$$f^t = \tilde{\mathscr{S}}^t f$$

is a continuous family of CR-functions relative to $\bar{\partial}_b^t$ with $f^0 = f$. Since $\tilde{\mathscr{S}}^t$ is an analytic family of operators it is easy to establish the continuity of the family $\{f^t\}$ relative to the H^s topology for any $s \geqq 0$.

If $(M, \bar{\partial}_b^t)$ is stably embeddable at zero then any CR-function relative to $\bar{\partial}_b^0$ can be included into a continuous family. If $\operatorname{rk} \mathscr{A}(t)$ were not 0 for any value of t then we could easily derive a contradiction by applying Proposition 8.32.

We can now relate the algebraic conditions obtained at the beginning of this section to the small eigenvalues of the associated $\square_b$-operators. As above we assume that $\psi(t) \subset \mathscr{F}_{4(1-g)}(\mathfrak{D})$ defines analytic family of CR-structures with $\psi(0) = 0$. The ranges of the projection operators, $\mathbb{P}_i^t$, are mutually orthogonal. Define

$$\mathbb{P}^t = \sum_{i=1}^{m} \mathbb{P}_i^t ,$$

and set

$$h_m = \mathfrak{H}^0 \cap F_m; \quad H = h_0 \oplus \cdots \oplus h_{2(g-1)} .$$

Proposition 8.28 implies that the range of $\mathbb{P}^0$ is contained in H. Denote the complementary orthogonal subspaces of H by

$$H_1 = \mathbb{P}^0 H \quad \text{and} \quad H_2 = (\mathrm{Id} - \mathbb{P}^0) H .$$

From the spectral theory of $\square_b^t$ it follows that if $\mu_i(t) \neq 0$, for all $i = 1, \ldots, m$, then

$$\mathscr{S}^t = \tilde{\mathscr{S}}^t - \mathbb{P}^t$$

is the Szegö projector for the structure $\bar{\partial}_b^t$. This is also an analytic family of operators for t in an interval about 0. Let Π_m denote the orthogonal projection onto $F_0 \oplus \cdots \oplus F_m$ and define the operator

$$R = (\mathrm{Id} - \mathbb{P}^0)\Pi_{2(g-1)} . \tag{8.36}$$

Lemma 8.37 *For t in a deleted neighborhood of 0 the dimension*

$$\dim(\mathfrak{H}_0^t / \mathfrak{H}_{2(g-1)+1}^t) ,$$

is constant.

Proof. Observe that for each t we can find a set of functions

$$\{h_1, \ldots, h_d\} \subset H \quad \text{such that } v_i = A_t^{-1} h_i ,$$

$$\text{is a basis for } \mathfrak{H}_0^t / \mathfrak{H}_{2(g-1)+1}^t . \tag{8.38}$$

Here and in the sequel A_t is as defined in (8.16). This follows easily from Corollary 8.10. This fact and Proposition 8.8 imply that

$$\dim(\mathfrak{H}_0^t / \mathfrak{H}_{2(g-1)+1}^t) = \dim \ker \bar{\mathscr{S}} P_t A_t^{-1} \restriction_H . \tag{8.39}$$

The operator P_t is defined by

$$P_t v = \psi_\alpha(t) Z_\alpha v \theta_\alpha^{\bar{1}} .$$

The operator on the right hand side of (8.39) depends analytically on t lying in an interval about 0, thus its rank is constant on the complement of a finite set. This proves the lemma.

Through a series of lemmas we identify the $\dim(\mathfrak{H}_0^t / \mathfrak{H}_{2(g-1)+1}^t)$ with $\dim H_2$.

Lemma 8.40 *For t in a deleted interval about 0*

$$\dim(\mathfrak{H}_0^t / \mathfrak{H}_{2(g-1)+1}^t) \geqq \dim H_2 . \tag{8.41}$$

Proof. Define an analytic family of operators by

$$B_t = R A_t \mathscr{S}^t \restriction_{H_2} ,$$

R is defined in (8.36). From the definition of H_2 and the fact that $\bar{\partial}_b$ is $U(1)$-invariant it is clear that B_t maps H_2 into itself. Since B_t is analytic and $B_0 = \mathrm{Id} \restriction_{H_2}$ it follows that B_t is an isomorphism of H_2 for small values of t. If $h \in H_2$ then $\mathscr{S}^t h \in \mathfrak{H}_0^t$. We claim that for $h \neq 0$,

$$\mathscr{S}^t h \not\equiv 0 \mod \mathfrak{H}_{2(g-1)+1}^t . \tag{8.42}$$

If (8.42) were false then

$$A_t \mathscr{S}^t h \in \mathfrak{H}^0_{2(g-1)+1}$$

and therefore

$$B_t h = 0 . \tag{8.43}$$

This contradicts the fact that B_t is an isomorphism for small values of t and implies (8.41).

Lemma 8.44 *For t in a deleted interval about* 0

$$\dim(\mathfrak{H}^t_0/\mathfrak{H}^t_{2(g-1)+1}) \leqq \dim H_2 . \tag{8.45}$$

Proof. To prove this inequality we consider several analytic families of operators, for $i = 1, \ldots, m$ define

$$C^t_i = \mathbb{P}^t_i A_t^{-1} \restriction_H, \quad i = 1, \ldots, m .$$

The operator $\mathbb{P}^t_i$ is a finite rank projection operator therefore

$$\operatorname{rk} \mathbb{P}^t_i = \operatorname{tr} \mathbb{P}^t_i .$$

As $\mathbb{P}^t_i$ depends analytically on t and $C_0 = \mathbb{P}^0_i$ this implies that, for t in a neighborhood of 0,

$$\operatorname{rk} C^t_i = \operatorname{rk} \mathbb{P}^0_i, \quad i = 1, \ldots, m .$$

Using the argument employed in the proof of Proposition 8.28 we deduce that

$$\text{if } h \in H \text{ and } A_t^{-1} h \in \mathfrak{H}^t \text{ then } C^t_i h = 0, \quad \text{for } i = 1, \ldots, m . \tag{8.46}$$

Define an analytic family of operators by

$$\hat{C}^t = \mathbb{P}^t A_t^{-1} \restriction_{H_1} .$$

As above $\hat{C}^0 = \mathrm{Id}$ and therefore

$$\operatorname{rk} \hat{C}^t = \dim H_1 , \tag{8.47}$$

for t in an interval about 0. Let V_t be the linear subspace of H such that if $v \in V_t$ then $A_t^{-1} v \in \mathfrak{H}^t$. We claim that for sufficiently small t

$$V_t \cap H_1 = \{0\} . \tag{8.48}$$

If (8.48) were false then it would follow from (8.46) that we could find a non-zero function $u \in H_1$ such that

$$C^t_i u = 0 \quad \text{for } i = 1, \ldots, m . \tag{8.49}$$

Since

$$\hat{C}^t = \sum_{i=1}^{m} C^t_i \restriction_{H_1}$$

it would also follow that $\hat{C}^t u = 0$. But this contradicts (8.47). From (8.38) and (8.48) we obtain that

$$\begin{aligned} \dim(\mathfrak{H}_0^t/\mathfrak{H}_{2(g-1)+1}^t) &\leqq \dim V_t \\ &\leqq \dim H - \dim H_1 = \dim H_2 \,. \end{aligned} \tag{8.50}$$

This completes the proof of the lemma.

Putting together these inequalities we have

Theorem 8.51 *Let $\psi(t) \subset \mathscr{F}_{4(1-g)}(\mathfrak{T})$ define an analytic family of* CR-*structures on M. For t in a non-empty deleted neighborhood of* 0

$$\dim(\mathfrak{H}_0^t/\mathfrak{H}_{2(g-1)+1}^t) = \dim H_2 \,. \tag{8.52}$$

As a simple corollary we have

Corollary 8.53 *Let $\psi(t)$ be as in the previous theorem then the dimensions*

$$\dim(\mathfrak{H}_m^t/\mathfrak{H}_{m+1}^t), \quad m = 0, 1, \ldots,$$

are constant if and only if the operators $\Box_b^t$ have no small eigenvalues.

Proof. The proof of the corollary is an easy consequence of (8.11), Theorem 8.51 and the fact that

$$\dim(\mathfrak{H}_0^t/\mathfrak{H}_{2(g-1)+1}^t) = \sum_{m=0}^{2(g-1)} \dim(\mathfrak{H}_m^t/\mathfrak{H}_{m+1}^t) \,. \tag{8.54}$$

This completes the analysis in the case that M is a circle bundle of degree -1 over Σ. If M is a circle bundle of degree $-k$ over Σ then it is covered by $\tilde{M}$ a circle bundle of degree -1. The quotient map

$$p\colon \tilde{M} \to M$$

allows one to identify $C^\infty(M)$ with $\mathscr{C}^\infty(\tilde{M}) \cap \mathscr{F}^k(\tilde{M})$ and deformations of the CR-structure on M with sections lying in $\mathscr{F}^k(\tilde{\mathfrak{T}})$. Using (7.5) and (7.6) it is a simple matter to extend the results of this section to these cases. We leave the precise statements to the interested reader and content ourselves with stating two simple corollaries.

Corollary 8.55 *Suppose that M is a circle bundle of degree $-k$ over a Riemann surface of genus g. If $k \geqq g$ then any family $\psi(t)$ of perturbations of a $U(1)$-invariant structure with 'positive' Fourier coefficients is stably embeddable at* 0.

Proof. We prove this statement by showing that the operator A_t defined in (8.16) is an isomorphism between $\mathfrak{H}^t$ and $\mathfrak{H}^0$. The CR-functions on M are identified via p^* with CR-functions on $\tilde{M}$ belonging to $\mathscr{F}^k$. The deformations with 'positive' Fourier coefficients are identified with sections in $\mathscr{F}^k_{4(1-g)+k}(\tilde{\mathfrak{T}})$. To prove the corollary it suffices to prove that for such a section

$$A_t\colon \mathfrak{H}_{mk}^{\psi(t)} \cap \mathscr{F}^k \to \mathfrak{H}_{mk}^0 \cap \mathscr{F}^k \,, \tag{8.56}$$

is an isomorphism for all $m \in \mathbb{N}_0$. If $m = 1$ then (8.56) follows from (8.9) which implies that if $h \in \mathfrak{H}_k$ then

$$(\psi_\alpha(t) Z_\alpha (\mathscr{Q} \bar{\partial}_b^* \psi_\alpha(t) Z_\alpha)^l h) \theta_\alpha^{\bar{1}} \in \mathscr{F}^k_{(2+l)k-2(g-1)} \,. \tag{8.57}$$

The hypothesis implies that $(2+l)k > 2(g-1)$ for all $l \geqq 0$ and therefore

$$\mathscr{P}\psi_\alpha(t)Z_\alpha(\mathrm{Id} - \mathscr{Q}\bar{\partial}_b^*\psi_\alpha(t)Z_\alpha)^{-1}h = 0\,.$$

The case $m = 0$ follows easily from the observation that the function $u = 1$ represents $\mathfrak{H}_0^t/\mathfrak{H}_1^t$ for all values of t.

Using a similar argument we obtain stability for perturbations with 'non-negative' Fourier coefficients.

Corollary 8.58 *Let M, k, g be as in the previous corollary if $k \geqq 2g-1$ and $\psi(t)$ is a family of deformations with 'non-negative' Fourier coefficients then it is stably embeddable at* 0.

Remark. The hypothesis is equivalent to the statement that

$$\psi(t) \subset \mathscr{F}^k_{4(1-g)}(\tilde{\mathfrak{D}})\,.$$

Catlin and Lempert have constructed a family of $U(1)$-invariant perturbations of a circle bundle of degree m over a surface of genus $(m-1)(m-2)/2$, $m \geqq 5$ which is not stably embeddable. This family can easily be made real analytic and thus an equivariant version of Proposition 8.51 applies to show that the family of $\Box_b$-operators defined by this family of structures must have a finite dimensional collection of 'small eigenvalues'. This is in marked contrast to the genus zero case where Lempert has shown that this cannot occur.

One can also define a notion of stability of a given embedding relative to a class of perturbations. We say that an embedding, i, of $(M, \bar{\partial}_b)$ is stable relative to $\psi \in \mathscr{B}$, a Banach space, if there is a $\delta > 0$ such that for every $\psi \in \mathscr{B}$ with

$$\|\psi\|_{\mathscr{B}} < \delta$$

there is an embedding i^ψ of $(M, \bar{\partial}_b^\psi)$ *close* to i. We leave the notion of closeness a little vague for the present. Using the techniques of this section one can prove:

Proposition 8.59 *Let $(M, \bar{\partial}_b)$ be a strictly pseudoconvex, $U(1)$-invariant* CR-*structure on M with $M/U(1)$ a surface of genus g. If i is an embedding of $(M, \bar{\partial}_b)$ with all coordinate functions in $\mathscr{F}_k(\tilde{M})$ for a $k > 2(g-1)$ then i is stable relative to deformations in $\mathscr{F}_{4(1-g)}(\tilde{M})$ with the C^4 topology.*

A Embedding $U(1)$-invariant structures

In this appendix we present the outline of a proof that every $U(1)$-invariant CR-structure on a three manifold can be embedded into $\mathbb{C}^d$ for some d as a hypersurface in an affine algebraic variety. This result follows from theorems of Lempert and Lawson and Yau, see [LaYa, Sect. 2] and [Le1, Theorem 2.1]. We include this appendix because we require more complete information about the Fourier coefficients of the embedding functions and CR-functions in general. The only deep fact that we require is the Kodaira embedding theorem for curves.

Kodaira Embedding Theorem A1 *If Σ is a Riemann surface and L is a positive holomorphic line bundle over Σ then there exists a positive integer N such that if $n \geqq N$ then a basis of holomorphic sections of $L^{\otimes n}$ defines a projective embedding of Σ.*

For a proof see [GrHa, p. 192].

Suppose that M is a three dimensional contact manifold, with contact plane field H. Suppose further that there is a free action by $U(1)$ which preserves H and is everywhere transverse to it. We denote the action of $e^{i\phi} \in U(1)$ by

$$p \mapsto U_\phi p .$$

Since the action is free the quotient

$$\pi\colon M \to \Sigma = M/U(1)$$

is a smooth compact surface. A CR-structure on M with underlying contact field H is a one dimensional subbundle $T^{0,1}M$ of $H \otimes \mathbb{C}$. If for every $e^{i\phi} \in U(1)$ we have

$$U_{\phi *} T^{0,1}M = T^{0,1}M$$

then we say that the structure is $U(1)$-invariant.

Suppose that $T^{0,1}M$ is a $U(1)$-invariant structure, then it is easy to show that $\pi_* T^{0,1}M$ is a one dimensional subbundle of $T\Sigma \otimes \mathbb{C}$. Since there are no integrability conditions in this case, $T^{0,1}\Sigma$ defines a complex structure on the surface. Using the characters of $U(1)$ we can define a collection of line bundles L_n, $n \in \mathbb{Z}$ over Σ. Let χ_n denote the character defined by

$$\chi_n(e^{i\phi}) = e^{in\phi} .$$

The bundle L_n is the fiber product $M \times_{\chi_n} \mathbb{C}$; this is the quotient of $M \times \mathbb{C}$ by the equivalence relation:

$$(p, w) \sim (U_\phi p, \chi_n(e^{i\phi})w) .$$

Let p_n denote the projection

$$p_n\colon M \times \mathbb{C} \to M \times_{\chi_n} \mathbb{C} .$$

If we use the standard complex structure on $\mathbb{C}$ then we can define an almost complex structure on L_n by setting

$$T^{0,1}L_n = p_{n*}[T^{0,1}M \times T^{0,1}\mathbb{C}] . \tag{A2}$$

The following is proved by a local coordinate calculation:

Lemma A3 *The almost complex structure defined in* (A2) *is integrable and makes L_n into a holomorphic line bundle over Σ.*

In addition to complex structures, the bundles have canonical metrics defined on them. This follows because the action of $U(1)$ on $\mathbb{C}$ is unitary so putting the standard metric on each fiber of $M \times \mathbb{C}$ induces a metric on the quotient. The immersion of M into L_n defined by

$$i_n(p) = p_n(p, 1) ,$$

immerses M as the unit circle bundle in L_n relative to the canonical metric.

Lemma A4 *The immersions i_n are* CR*-mappings for each n with i_{-1} an orientation preserving* CR*-isomorphism.*

Proof. To prove the lemma we need to show that

$$i_{n*}T^{0,1}M = T^{0,1}i_nM . \tag{A5}$$

Introduce local coordinates (x, y, θ) on M, so that the action of $U(1)$ is given by $U_\phi(x, y, \theta) = (x, y, \theta + \phi)$. If w denotes a coordinate on $\mathbb{C}$ then the projection p_n is given by

$$p_n(x, y, \theta; w) = (x, y; e^{-in\theta} w) .$$

Let $\bar{Z} = a\partial_x + b\partial_y + c\partial_\theta$ define a local section of $T^{0,1}M$. A calculation shows that

$$p_{n*}(\bar{Z}) = a\partial_x + b\partial_y - inc(\zeta\partial_\zeta - \bar{\zeta}\partial_{\bar{\zeta}}) , \tag{A6}$$

where $\zeta = e^{-in\theta} w$ is a linear fiber coordinate. The defining function for $i_n(M)$ is $|\zeta|^2 - 1$. From (A6) it follows that $p_{n*}\bar{Z}$ is annihilates the defining function and therefore belongs to $T^{0,1} i_n M$. The equality (A5) follows from this and (A2).

If $w = e^{i\phi}$ then we have that

$$i_{-1*}(\bar{Z}) = a\partial_x + b\partial_y + c\partial_\phi .$$

This is easily seen to agree with the CR-structure induced from the embedding thus verifying the statement of the lemma for this case.

If $M \subsetneq L_{-1}$ is strictly pseudoconvex then the action of $U(1)$ is positive as defined in [Le1, S2], thus the bundles L_n, $n > 0$ are of positive degree. To reduce the CR-embedding problem to finding projective embeddings of Σ we observe that a section s of L_n defines a function f_s on M which satisfies

$$U_\phi^*(f_s) = e^{in\phi} f_s . \tag{A7}$$

This is because the pulled back bundle, $\pi^* L_n$, is canonically trivial over M. A simple coordinate calculation verifies

Proposition A8 *A holomorphic section, s of L_n, pulls back to a* CR*-function on M and conversely a* CR*-function f_s which satisfies* (A7) *defines a holomorphic section of L_n.*

As a simple corollary we have

Corollary A9 *If M is a strictly pseudoconvex* CR*-manifold with a free transverse positive $U(1)$* CR*-action, then*

$$\ker \bar{\partial}_b \subset \mathscr{F}_0 .$$

Proof. Since the Fourier components of a CR-function on M correspond to holomorphic sections of the line bundles L_n the corollary following from the fact that negative line bundles have no holomorphic sections.

In our applications it is sometimes useful to twist the bundles L_n with flat line bundles. Let $\{V_\alpha\}$ be an open cover of Σ with $\{U_\alpha\}$ their inverse images under π. Suppose that the constants, $\{e^{ig_{\alpha\beta}}\}$ define a 1-cocycle relative to the cover $\{V_\alpha\}$. Let λ denote the flat holomorphic line bundle over Σ defined by this cocycle and $\tilde{\lambda}$ its lift to M. A collection of functions $f_\alpha \in \mathscr{C}^\infty(U_\alpha)$ which satisfy

$$f_\alpha = e^{ig_{\alpha\beta}} f_\beta \quad \text{in } U_\alpha \cap U_\beta , \tag{A10}$$

defines a section of $\tilde{\lambda}$. As above we can decompose such sections into Fourier components. Let $F_n(\tilde{\lambda})$ denote sections of $\tilde{\lambda}$ such that

$$f_\alpha \in F_n(U_\alpha) \quad \text{for all } \alpha .$$

A section in $F_n(\tilde{\lambda})$ defines a section of $L_n \otimes \lambda$ and vice-versa. A section in $F_n(\tilde{\lambda})$ is locally represented by CR-functions if and only if the corresponding section of $L_n \otimes \lambda$ is holomorphic.

Proposition A11 *Let λ be a flat line bundle over Σ and $\tilde{\lambda}$ its lift to M. The pullback defines a one to one correspondence between sections of $L_n \otimes \lambda$ and sections of $\tilde{\lambda}$ belonging to $F_n(\tilde{\lambda})$. This correspondence carries holomorphic sections of $L_n \otimes \lambda$ to CR-sections in $F_n(\tilde{\lambda})$ and vice-versa.*

In fact we can extend the function f_s as a function defined on L_{-1}. The reason is the same: L_n pulled back to L_{-1} is trivial in the complement of the zero section. If we use F_s to denote a section of L_n pulled back to L_{-1} then

$$U_\lambda^* F_s = \lambda^n F_s, \quad \text{for } \lambda \in \mathbb{C}^\times . \tag{A12}$$

Thus if $n > 0$ the function F_s extends continuously to the zero section. Proposition A8 applies to the extension F_s as well. If s is holomorphic then F_s is clearly holomorphic in the complement of the zero section. If $n > 0$ then the Riemann removable singularities theorem implies that the extension across the zero section is also holomorphic.

Suppose that N is as in the statement of the Kodaira Embedding Theorem, if $n > N$ then a basis of global sections of L_n defines a embedding of Σ into $\mathbb{P}^R$ for some R. Let $s_0, \ldots, s_R$ be a basis of global holomorphic sections for L_n. Let $[z_0 : \cdots : z_R]$ denote homogeneous coordinates on $\mathbb{P}^R$. The embedding is defined by

$$z \to [s_0(z) : \cdots : s_R(z)] .$$

In order for this map to define an embedding it is necessary and sufficient that for each pair of points $z \neq \zeta$ in Σ there are vectors $a, b, c \in \mathbb{C}^{R+1} \setminus \{0\}$ so that either $c \cdot s(z) \neq 0$ or $c \cdot s(\zeta) \neq 0$ and

$$\frac{a \cdot s(z)}{c \cdot s(z)} \neq \frac{b \cdot s(\zeta)}{c \cdot s(\zeta)}, \tag{A13}$$

and for each $z \in \Sigma$ there are vectors $d, e \in \mathbb{C}^{R+1} \setminus \{0\}$ such that

$$d \cdot s(z) \neq 0 \quad \text{and}$$

$$\left(\frac{e \cdot s}{d \cdot s}\right)'(z) = \frac{e \cdot s'(z) d \cdot s(z) - e \cdot s(z) d \cdot s'(z)}{d \cdot s(z)^2} \neq 0 . \tag{A14}$$

The following proposition follows from (A12), (A13) and (A14).

Proposition A15 *If a basis of global sections of L_n, $n > 0$ defines an embedding of Σ into $\mathbb{P}^R$ then the lifted sections*

$$(F_{s_0}, \ldots, F_{s_R})$$

define a holomorphic map of $L_{-1}/\mathbb{Z}_n$ into $\mathbb{C}^{R+1}$ which is an embedding in the complement of the zero section.

As an immediate corollary of this proposition we obtain that

$$\Psi_n : p \to (f_{s_0}(p), \ldots, f_{s_R}(p))$$

defines as embedding of $M/\mathbb{Z}_n$. Since the embedding of $L_{-1}/\mathbb{Z}_n$ has polynomial growth at ∞ if follows that it is an affine algebraic variety. Thus we have M immersed as a hypersurface in an affine algebraic variety. To obtain an embedding of M we need to separate points on the $U(1)$-orbits. If p, q are relatively prime positive integers larger than N then Ψ_p, Ψ_q define immersions of $M/\mathbb{Z}_p$ and $M/\mathbb{Z}_q$ respectively into $\mathbb{C}^{P+1}$ and $\mathbb{C}^{Q+1}$. Since p, q are relatively prime it is clear that the map

$$\Psi_{p,q}: p \to (\Psi_p(p), \Psi_q(p)),$$

defines an embedding of M into $\mathbb{C}^{P+1} \times \mathbb{C}^{Q+1}$. By considering the exstensions of these maps to L_{-1} it is clear that the image of $\Psi_{p,q}$ is a hypersurface in an affine algebraic variety.

This proves the following theorem

Theorem A16 *If M is a strictly pseudoconvex three dimensional, CR-manifold with a free, transverse $U(1)$ CR-action then, for some d, M can be embedded into $\mathbb{C}^d$ as a hypersurface in an affine algebraic variety. Moreover the embedding functions can be chosen to lie in $\mathscr{F}_n$ for any fixed n. The embedding dimension may depend on n. The $U(1)$-action is realized as a linear action on $\mathbb{C}^d$ which extends to an holomorphic action of $\mathbb{C}^*$.*

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