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The Kähler cone on Calabi-Yau threefolds

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Introduction

In this paper, we develop further some of the ideas from [38] concerning Calabi-Yau threefolds – that is, complex smooth projective threefolds X with canonical class $K_X=0$ and with $h^1(\mathcal{O}_X)=0=h^2(\mathcal{O}_X)$. The second Chern class $c_2(X)$ defines a linear form on $H^2(X,\mathbb{Z})$, which by Theorem 1.5 of [19] is non-trivial unless some finite cover of X is an abelian threefold. Moreover we recall from [2] that a Calabi-Yau threefold has finite fundamental group unless some finite cover is either abelian or decomposable as a product of an elliptic curve and a K3 surface.

If the Calabi-Yau threefold X is simply connected, we know from Theorem 12.5 of [33] or the theory of [40] that its diffeomorphism class is determined up to a finite number of possibilities by the information:

(1) The cubic form $\mu: H^2(X,\mathbb{Z})\to\mathbb{Z}$ given by cup-product.
(2) The linear form $c_2: H^2(X,\mathbb{Z})\to\mathbb{Z}$ given by cup product with $c_2(X)\in H^4(X,\mathbb{Z})$.
(3) The middle cohomology $H^3(X,\mathbb{Z})$.

If moreover $H^3(X,\mathbb{Z})$ is torsion free, the above information characterizes the diffeomorphism class of X uniquely [37].

Since $H^2(X,\mathbb{Z})$ is canonically isomorphic to $\mathrm{Pic}(X)$, we shall usually think of its elements as line bundles or divisor classes on X, and denote $\mu(D)$ by D^3. In [38], we considered the real vector space $H^2(X,\mathbb{R})=H^2(X,\mathbb{Z})\otimes\mathbb{R}$ of dimension ρ. In this space we have the Kähler cone $\mathcal{K}$, and its closure $\bar{\mathcal{K}}$, the cone of *nef* classes. We also however have the cubic cone W^* defined by cup-product, and its associated cubic hypersurface $W\subset\mathbb{P}^{\rho-1}(\mathbb{R})$. In Sect. 1, we review the relationship between these two cones.

Fact 1 *Away from W^*, the cone $\bar{\mathcal{K}}$ is locally rational polyhedral, the codimension one faces corresponding to primitive birational contractions of X.*

Fact 2 *Non-singular rational points of W^* which are on $\bar{\mathcal{K}}$ but not on the linear space defined by $c_2=0$ give rise to elliptic fibre space structures on X.*

These facts are almost immediate from the theory of [38]. In Sect. 2, we study the primitive birational contractions which arise as a result of Fact. 1. The main part of this section is spent proving a general result concerning crepant primitive contraction morphisms on a smooth threefold X which contract a surface E down to a curve C of canonical singularities – we shall see (2.2) that C has to be smooth and E a conic bundle over C.

For X a Calabi-Yau threefold, it is known that the first order deformations of X are unobstructed [34, 35, 28], and so the versal deformation (Kuranishi) space of X can be regarded as an open polydisc in $H^1(X, T_X) \cong H^1(X, \Omega_X^2)$. In fact, by the theory of [36], there exists a global quasi-projective moduli space of polarized Calabi-Yau threefolds with given Hilbert polynomial (this corresponds to fixing H^3 and $H \cdot c_2$ for polarizations H). Moreover, Todorov has announced the result that the space of all deformations of a given X has the structure of a quasi-projective variety.

For a given Calabi-Yau threefold X, we know that the Kähler classes correspond bijectively with the Calabi-Yau metrics on X [39]. In [7], the local moduli space parametrizing complex structure *plus* Calabi-Yau metric is studied under the assumption that the Kähler cone is locally independent of the complex structure. This is clearly true of many examples, which for instance are embedded as complete intersections in some rigid ambient space, the Kähler cone of which restricts to the Kähler cone on X. One motivation for the present paper was to clarify any dependence the Kähler cone might have on the complex structure.

If $\pi: \mathscr{X} \to B$ is a smooth family of Calabi-Yau threefolds over say a polydisc B, then we can identify the cohomology groups $H^2(X_b, \mathbb{Z}) \cong H^2(\mathscr{X}, \mathbb{Z})$, where X_b denotes the fibre of π over $b \in B$. We can therefore consider the Kähler cones $\mathscr{K}(b)$ of $X_b \in B$, to be cones in some fixed vector space $H^2(X, \mathbb{R})$. The question raised by [7] is whether, when considering the Kuranishi family of a Calabi-Yau threefold, these cones are all the same. The answer is provided by the following result, proved in Sect. 3 and 4.

Main Theorem. *Suppose that $\pi: \mathscr{X} \to B$ is the Kuranishi family of a Calabi-Yau threefold X over a polydisc B in $H^1(X, T_X)$. The Kähler cone is invariant in the family if and only if none of the threefolds X_b contain a smooth elliptic ruled surface. More generally, the Kähler cone will be invariant over the dense subset of $b \in B$ for which X_b contains no such ruled surfaces, this subset being the complement of at most countably many codimension one submanifolds of B.*

Thus in a sense the elliptic ruled surfaces play the same role for Calabi-Yau threefolds that the (-2)-curves do for K3 surfaces. The difference is that in the case of K3 surfaces one will always have (-2)-curves on some nearby surface, whilst for Calabi-Yau threefolds it will be rare for any deformation to contain one of the above ruled surfaces – the existence of such a surface will imply that the cubic hypersurface W (an invariant under deformation) is special. In (4.6) we study an example where the Kähler cone does jump under a generic deformation. In Sect. 5, the analogy with (-2)-curves on K3 surfaces becomes even more striking when we observe that the theory of elementary transformations (or flops) goes over essentially unchanged to the case of elliptic ruled surfaces on Calabi-Yau threefolds.

In Sect. 6, we apply the above theory to the question of whether some deformation of a Calabi-Yau threefold can have an elliptic fibre space structure (compare with the case of K3 surfaces where this is always true for some small deformation). We indicate possible arithmetic obstructions (associated to the cubic cone) and geometric obstructions (determined by the cubic and Kähler cones) to there being an affirmative answer to this question in a given case. We observe that certain Calabi-Yau threefolds already in the literature cannot have a deformation with an elliptic fibre space structure because of these obstructions.

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1 The Kähler and cubic cones

Let X be a Calabi-Yau threefold, and consider the real vector space $\mathrm{H}^2(X, \mathbb{R})$ of dimension ρ. In this space we have two cones which are of particular interest.

The Kähler cone. This is the open cone $\mathcal{K} \in \mathrm{H}^2(X, \mathbb{R})$ consisting of Kähler classes; from the point of view of divisors, it is the open cone generated by the ample divisor classes. The closure $\bar{\mathcal{K}} \subset \mathrm{H}^2(X, \mathbb{R})$ is then the cone consisting of numerical classes of *nef* divisors, that is real divisor classes D with the property that $D \cdot C \geqq 0$ for all curves C on X. We shall denote the boundary of $\bar{\mathcal{K}}$ by $\partial \bar{\mathcal{K}}$. The cone $\bar{\mathcal{K}}$ is in fact just the dual of the cone $\overline{NE}(X)$ of effective 1-cycles, as studied by Mori. In our case it is more convenient to work with the cone $\bar{\mathcal{K}}$ because of its relationship to the second cone we study.

The cubic cone. The cubic cone $W^* \subset \mathrm{H}^2(X, \mathbb{R})$ is the cone defined by the cubic form given by cup-product, i.e.

$$W^* = \{D \in \mathrm{H}^2(X, \mathbb{R});\ D^3 = 0\}.$$

This in turn determines a cubic hypersurface $W \subset \mathbb{P}^{\rho-1}(\mathbb{R})$.

Remarks 1.1 (1) The rational points of $\mathcal{K}$ correspond to ample divisor classes on X.
(2) We observe that $D^3 \geqq 0$ for all D in $\bar{\mathcal{K}}$, and furthermore that the linear form c_2 is non-negative on $\bar{\mathcal{K}}$ (i.e. $c_2 \cdot D \geqq 0$ for all $D \in \bar{\mathcal{K}}$). This latter claim follows from Theorem 1.1 of [22].
(3) A non-zero element $D \in \mathrm{H}^2(X, \mathbb{R})$ has D^2 (numerically) trivial if and only if the point of $\mathbb{P}^{\rho-1}(\mathbb{R})$ corresponding to D is a *singular* point of W. To see this we remark that for any $L \in \mathrm{H}^2(X, \mathbb{R})$, the function $(D + xL)^3$ has a multiple root at $x = 0$ if and only if $D^3 = 0 = D^2 \cdot L$. In particular we note that W is singular whenever some deformation of X has the structure of a fibre space over a curve, or the structure of an elliptic fibre space over a normal surface S with Picard number $\rho(S) > 1$.

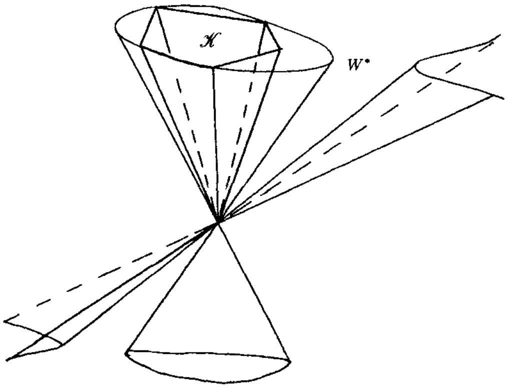

Fig. 1.

A picture of what the geometry of the above two cones in $H^2(X,\mathbb{R})\cong\mathbb{R}^\rho$ might look like (when $\rho=3$) is provided by Fig. 1 above. The hyperplane given by the linear form c_2 has not been included, but the reader should bear in mind that the cone $\bar{\mathcal{K}}$ is on the positive side of this hyperplane, although possibly also touching it.

Fact 1 *The cone $\bar{\mathcal{K}}$ is locally rational polyhedral away from W^*, the codimension one faces corresponding to primitive birational contractions on X.*

Definition 1.2 We say that a birational contraction morphism $\phi: X \to \bar{X}$ onto a normal projective variety $\bar{X}$ is *primitive* if it cannot be further factored into birational morphisms of normal varieties.

We shall always assume that the birational contraction morphisms under consideration have normal image varieties. In the terminology of [38], such morphisms are Calabi-Yau contractions, and they were studied in the cited paper without the restriction that X is smooth. For a birational contraction morphism $\phi: X\to\bar{X}$ on a Calabi-Yau threefold X, I claim that any relatively *nef* divisor $L\in \mathrm{Pic}(X)$ has a multiple which is ϕ-generated in the terminology of [16]. In other words, if L is non-negative on the relative cone $\overline{NE}(X/\bar{X})$ and $D\in\mathrm{Pic}(X)$ is the pull-back of a hyperplane section from $\bar{X}$, then $L+nD$ is semi-ample (and big) on X for n sufficiently large. The claim follows from the theory in Chap. 3 of [16], in particular Remark 3-1-2(1) when applied to L, which by assumption is both $\phi-nef$ and $\phi-$big. Using this result, the reader will easily check that the condition given in (1.2) for $\phi: X\to\bar{X}$ to be primitive is equivalent to the condition that the real vector space $N^1(X/\bar{X})$ of (numerical classes of) relative 1-cycles has dimension one, and that this in turn (cf. Lemma 3-2-5 of [16]) is equivalent to the condition that $\mathrm{Pic}_{\mathbb{R}}(\bar{X})=\mathrm{Pic}(\bar{X})\otimes\mathbb{R}$ has dimension one less than $\mathrm{Pic}_{\mathbb{R}}(X)$. In the more general context of [38], it follows also that a non-primitive Calabi-Yau contraction will factor into a composite of primitive contractions.

If $\phi: X\to\bar{X}$ is a primitive contraction, the pull-back of a very ample divisor on $\bar{X}$ defines a hyperplane in the real vector space $N^1(X)$ (dual to $H^2(X,\mathbb{R})$); this hyperplane intersects $\overline{NE}(X)\subset N^1(X)$ along a single (geometrically extremal) ray. An alternative terminology therefore is to call such contractions *extremal*, although this term is used by some authors to include negativity of the canonical class on the ray in question.

The above Fact 1 is just a restatement in more geometrical language of the results from [38]. Given a divisor D on $\partial\bar{\mathscr{K}}\setminus W^*$, we have that D is *nef* with $D^3>0$. Let H be an ample divisor on X; since D is *nef* and $D^3>0$, we deduce that $D^2\cdot H>0$. Observe that $D\cdot H^2\geqq 0$ and that the Hodge index theorem applied to rational approximations to D implies that $(D^2\cdot H)(H^3)\leqq(D\cdot H^2)^2$. Hence $D\cdot H^2>0$.

We can now find arbitrarily close to D, *rational* divisor classes D_0 not in $\bar{\mathscr{K}}$ but with $D_0^3>0$, $D_0^2\cdot H>0$ and $D_0\cdot H^2>0$. This however is the circumstance in which the Key Lemma from [38] applies; its proof shows that for some positive $\lambda\in\mathbb{Q}$ (in fact $\lambda=\sup\{x\in\mathbb{R};\ D_0+xH\notin\bar{\mathscr{K}}\}$), we have that $D_\lambda=D_0+\lambda H$ is a rational divisor in $\partial\bar{\mathscr{K}}$, and that some multiple of D_λ defines a birational contraction morphism.

By slightly moving the divisor D_0, we may assume that the birational contraction $\phi\colon X\to\bar{X}$ defined by D_λ is primitive. This says that $\phi^*\mathrm{Pic}_{\mathbb{R}}(\bar{X})$ is a rational hyperplane of $\mathrm{H}^2(X,\mathbb{R})$, and that the pull-back of the *nef* cone on $\bar{X}$ corresponds to the intersection of this hyperplane with $\bar{\mathscr{K}}$; this intersection is then clearly a codimension one face of $\bar{\mathscr{K}}$ which has D_λ in its interior.

An analogous argument tells us that if D_λ yields a birational contraction morphism of an r-dimensional space of numerical classes of 1-cycles, then D_λ is in the interior of a codimension r rational face of $\bar{\mathscr{K}}$.

Since the above argument can be applied in a neighbourhood of any $D\in\partial\bar{\mathscr{K}}\setminus W^*$, we see that $\bar{\mathscr{K}}$ is indeed locally rational polyhedral as claimed.

Remarks 1.3 (1) Fact 1 tells us that the picture in Fig. 1 is not too misleading. It may however happen that as one approaches W^*, one obtains infinitely many faces of $\bar{\mathscr{K}}$ (for an example of this, see [4]). This should be compared with the structure of the cone $\overline{NE}(V)$ for a projective manifold V, which cone is locally rational polyhedral on the negative half-space defined by the canonical class K_V, but may have infinitely many faces as one approaches the hyperplane $K_V\cdot Z=0$.

(2) The codimension one faces of $\bar{\mathscr{K}}$ correspond as we have seen to primitive birational contraction morphisms $\phi\colon X\to\bar{X}$. This morphism may be small, in which case ϕ contracts finitely many curves isomorphic to $\mathbb{P}^1$. If however ϕ is a divisorial contraction, a standard argument proves that the exceptional locus E has to be irreducible (since $E_i\cdot Z<0$ for any component E_i of E and any curve Z on E); moreover, when E is contracted to a curve C, the generic fibre of E over C is irreducible. In Sect. 3, we shall see that ϕ is either a contraction of E down to a point or to a non-singular curve C of genus g. In the first case, we note that $E\cdot Z\geqq 0$ for all irreducible curves other than those on E, i.e. in the numerical class contracted. Thus for any ample divisor H on X, the line $\{E+xH;\ x\in\mathbb{R}\}$ cuts $\partial\bar{\mathscr{K}}$ along the same face *independent* of H. Thus our original divisor D was in the interior of a rather large face of the cone $\bar{\mathscr{K}}$. The same statement is true if E is a ruled surface over a curve of genus $g>0$, since if $E+\lambda H\in\partial\bar{\mathscr{K}}$ gives rise to a contraction of *either* the whole of E to a point *or* curves on E not contained in the ruling, then we would have $g=0$.

For the part of $\partial\bar{\mathscr{K}}$ which lies in W^*, we have in general no rationality statement corresponding to that in Fact 1. We can however say something if we are given rational points on that part of the boundary. Here we shall need the assumption that X is not the étale quotient of an abelian threefold.

Fact 2 *Non-singular rational points of W^* which are on $\mathscr{K}$ but not on the linear space defined by $c_2=0$ give rise to elliptic fibre space structures on X.*

If D is a divisor whose class represents such a non-singular point of W^*, the fibre space map will be $\phi_{nD}: X \to S$ where S is a normal surface and n is sufficiently large. If H is ample on X, then D is *nef* with $D^2 \cdot H > 0$. Moreover, since $c_2 \cdot D \neq 0$, we have by (1.1) that $D \cdot c_2 > 0$. Therefore we can apply (3.2)′ of [38] to obtain the claimed result.

2 Primitive birational contractions

We have seen in Sect. 1 that the boundary $\partial\mathscr{K}$ of the Kähler cone of a Calabi-Yau threefold X consists of a part which lies in W^* and a locally rational polyhedral part corresponding to birational contractions of X (with image a normal threefold $\bar{X}$). In this section, we study this latter part. Since a class in $\partial\mathscr{K} \setminus W^*$ can be approximated arbitrarily closely by classes of divisors in $\partial\mathscr{K} \setminus W^*$ for which the corresponding contraction ϕ_{nD} (n sufficiently large) is primitive (i.e. D is in the interior of a codimension one face of $\mathscr{K}$), the important case to look at is that of primitive contractions.

Assume now that $D \in \mathrm{Pic}(X)$ represents a point in $\partial\mathscr{K} \setminus W^*$ in the interior of a codimension one face of $\mathscr{K}$, and that the corresponding primitive contraction is $\phi_D = \phi: X \to \bar{X}$. We noted in Sect. 1 that if the exceptional locus E has codimension one, then E will be irreducible. In the case when the exceptional locus consists of curves, this will no longer in general be true, since distinct curves may well be numerically equivalent from the global point of view.

Definition 2.1 We say that a primitive contraction is of *Type I* if it contracts only finitely many curves, of *Type II* if it contracts an irreducible surface down to a point, and of *Type III* if it contracts an irreducible surface down to a curve.

If the primitive birational contraction ϕ_D is of Type I, the image of the exceptional locus consists of a finite number of isolated singularities, each with a small resolution. The singularities are clearly terminal of index 1, and so by the theory of [30] are compound Du Val (cDV in the now standard notation) singularities. These singularities have been studied by a number of authors (§8 of [27], §5 of [30], [25, 9]). The basic technique is to work in the analytic category and consider a cDV point $(V, 0)$ with small resolution $g: X \to V$; the exceptional locus of g will consist of a finite number of curves isomorphic to $\mathbb{P}^1$. We can however consider V as the total space of a 1-parameter family of deformations of its generic hyperplane section H (a Du Val singularity), and the small resolution X as the total space of a 1-parameter family of deformations of a partial resolution $\tilde{H}$ of H. We are therefore in a position to apply the theory of simultaneous partial resolutions of Du Val singularities, and this is the method employed in the papers cited above.

If the primitive birational contraction ϕ_D is of Type II, then the exceptional locus is a (generalized) Del Pezzo surface. The reader is referred to [29] for a discussion of this case. We note that $nD - E$ is ample on X for n sufficiently large, and so from the exact sequence

$$0 \to \mathcal{O}(nD-E) \to \mathcal{O}(nD) \to \mathcal{O}_E \to 0$$

and vanishing theorems we obtain $h^1(E, \mathcal{O}_E)=0$ – in fact as commented on p. 149 of [24], this is true for any irreducible generalized Del Pezzo surface.

For much of this paper, it will however turn out that the interest lies in primitive contractions $\phi_D: X \to \bar{X}$ of Type III, and it is these that we now study in this section. The image $\bar{X}$ will have a curve of canonical singularities corresponding to the surface contracted, and otherwise be non-singular. Since the results may be of independent interest, we shall in fact classify (for an arbitrary smooth complex projective threefold X) the possible exceptional surfaces for primitive crepant contractions $\phi_D: X \to \bar{X}$, contracting a surface down to a curve of canonical singularities on $\bar{X}$.

Theorem 2.2 *Suppose that X is a smooth complex projective threefold, and that $\phi_D: X \to \bar{X}$ is a crepant contraction morphism, contracting some irreducible surface E down to a curve C of canonical singularities on $\bar{X}$, and an isomorphism outside E. The curve C is then smooth and E is a conic bundle over C. The possible singular fibres of the morphism $h: E \to C$ are either two lines or a line taken twice. The possible singularities of E are A_n singularities at the point where distinct components of a singular fibre meet, or A_1 singularities appearing as a pair on some double fibre.*

We comment that by the theory of [29], C is generically a curve of A_1 singularities, but there may well be certain 'dissident' points at which the threefold $\bar{X}$ has rather more complicated cDV singularities. The fact that the singularities of $\bar{X}$ will be cDV follows from Theorem 2.2 of [29] and the observation that E is essentially the only possible crepant exceptional divisor on a resolution of $\bar{X}$. We prove first a general lemma concerning curves of cDV singularities.

Lemma 2.3 *Suppose that $C \subset V$ is a curve of* cDV *singularities (generically* cA_1*) on a threefold V with $P \in C$ a singularity of the curve of multiplicity r. If $\phi: X \to V$ is a crepant partial resolution of V with X having at worst isolated singularities, then $\phi^{-1}(P)$ consists of at least r components.*

Proof. The statement being a local one on V, we can take V to be a small neighbourhood of the singularity P. We use the technique mentioned earlier in the context of isolated cDV singularities with small resolution (described in detail in [30]), and consider V as the total space of a 1-parameter deformation $V \to \Delta$ over the open disc, with simultaneous partial resolution $X \to \Delta$. Let us change notation so as to denote the first family by $\mathcal{Z} \to \Delta$ (where the fibre Z_0 over 0 is a general hyperplane section of the singularity), and the second family by $\mathcal{Y} \to \Delta$. We have a diagram

$$\begin{array}{ccc} \mathcal{Y} & \xrightarrow{\ \phi\ } & \mathcal{Z} \\ & \searrow \quad \swarrow & \\ & \Delta & \end{array}$$

and we wish to show that $\phi^{-1}(P) \subset Y_0$ consists of at least r components. Observe that the general fibre Y_t $(t \neq 0)$ contains r exceptional (-2)-curves.

We may assume that Y_0 is the only singular fibre of the family $\mathcal{Y} \to \Delta$; by Theorem 1.14 of [30], Y_0 is a partial resolution of Z_0 obtained by contracting some set of (-2)-curves on the minimal desingularization. In particular, Y_0

has only Du Val singularities. After taking a finite cover $\psi: \tilde{\Delta} \to \Delta$ ramified over the origin, we may assume that the monodromy action is trivial on the r exceptional (-2)-curves of the general fibre, and furthermore that there exists a simultaneous minimal resolution of singularities in the family. We obtain therefore a family of smooth surfaces $\tilde{\mathscr{Y}} \to \tilde{\Delta}$ and a corresponding diagram

$$\begin{array}{ccccc} \tilde{\mathscr{Y}} & \to & \mathscr{Y} & \to & \mathscr{X} \\ \downarrow & & \downarrow & & \downarrow \\ \tilde{\Delta} & \to & \Delta & = & \Delta \end{array}$$

The exceptional locus of the map $\tilde{\mathscr{Y}} \to \mathscr{X}$ contains divisors $\tilde{\mathscr{E}}_1, \ldots, \tilde{\mathscr{E}}_r$ whose fibres over $t \neq 0$ contract down to the A_1-singularities on $Z_{\phi(t)}$.

The family $\tilde{\mathscr{Y}} \to \tilde{\Delta}$ is however by [14] a pullback of the simultaneous resolution of the versal deformation of the singularity Z_0. Using the standard theory of such resolutions (see [9, p. 673] for a brief summary), we deduce that the $\tilde{\mathscr{E}}_i$ will restrict to distinct (-2)-curves in the exceptional locus on $\tilde{Y}_0$, which we recall is the minimal desingularization of Z_0.

If now we take the images $\mathscr{E}_1, \ldots, \mathscr{E}_r$ of these divisors on $\mathscr{Y}$, these restrict to give the r exceptional (-2)-curves on the fibres Y_s for $s \neq 0$. On the central fibre Y_0, they restrict to give distinct curves $E_1, \ldots, E_r$ in the exceptional locus of $Y_0 \to Z_0$, and the lemma is proved.

Proof of (2.2) Let D be the pull-back of a very ample divisor on $\bar{X}$, and let $\phi = \phi_D$. Since $nD - E$ is ample on X for n sufficiently large, it follows from vanishing and the exact sequence

$$0 \to \mathcal{O}_X(nD - E) \to \mathcal{O}_X(nD) \to \mathcal{O}_E(nD) \to 0$$

that the map $H^0(\mathcal{O}_X(nD)) \to H^0(\mathcal{O}_E(nD))$ is surjective. The morphism $\phi: E \to C$ therefore exhibits E as a fibre space over C. From vanishing and the exact sequence of sheaves

$$0 \to \mathcal{O}_X(nD - 2E) \to \mathcal{O}_X(nD - E) \to \mathcal{O}_E(nD - E) \to 0$$

we deduce that $nD - E$ is very ample on a neighbourhood of E for n sufficiently large.

We now show that C has to be smooth; suppose that $P \in C$ is a singularity of C of multiplicity $r > 1$. Since $-E \cdot l = 2$ for the general fibre l of E over C, it follows that $\phi^{-1}(P)$ can have at most two components, and hence by (2.3) we have $r = 2$ and the fibre of E over P contains exactly two components.

Now choose a general element S in the linear system $|nD - E|$; the surface S intersects the fibre of E over P in two distinct points, say Q_1 and Q_2, at which it is smooth. Thus on some neighbourhood of Q_1, the morphism $\phi|_S$ is unramified, and hence induces an analytic isomorphism from a neighbourhood of Q_1 in $E \cap S$ to a neighbourhood of P on C. Hence we conclude that the curve C has embedding dimension at most two at P.

We observe now from the fact that ϕ is a small morphism, that $\phi_* \mathcal{O}_X(-E) = \mathscr{I}_C$, the ideal sheaf of C in $\bar{X}$ (cf. the argument of (2.14) on p. 148 of [30]). Moreover, the sheaf $\mathcal{O}_X(-E)$ is 'relatively generated by its sections', i.e. the map $\phi^* \phi_* \mathcal{O}_X(-E) \to \mathcal{O}_X(-E)$ is surjective. Following the same line of argument as in [30], we deduce that the map $\phi^* \mathscr{I}_C \to \mathcal{O}_X(-E)$ is surjective, and in particular we have $\mathscr{I}_C \cdot \mathcal{O}_X = \mathcal{O}_X(-E)$ is invertible. The universal property of blowing up implies that the morphism ϕ factors through the blow-up X_1 of $\bar{X}$ in C. The threefold X_1 is clearly non-singular in codimension one. Since C is locally a complete intersection, the same is true of X_1 (see calculation given below), which then by Serre's criterion must therefore be normal. Since ϕ was assumed to be primitive, we deduce that the blow-up of $\bar{X}$ in C will just be X.

Apart from assuming that E is Cartier, we have not really used yet the non-singularity of X, the above arguments going through equally well if X say had isolated cDV singularities – it is now that we use the fact that X is smooth. Recall that $P \in \bar{X}$ is a cDV singularity and so can be (analytically) embedded in $\mathbb{C}^4$. Our deduction concerning the curve C at P says that we can choose local analytic coordinates x, y, z, t for $\mathbb{C}^4$ with $\mathscr{I}_{C,P} = \langle x, y, h \rangle$, where P is the origin and $h \in \mathbb{C}[[z, t]]$. Since P is a singular point of C, h has no linear or constant parts.

The blow-up of $\bar{X}$ in C is the strict transform of $\bar{X}$ in the blow-up $\tilde{\mathbb{C}}^4$ of $\mathbb{C}^4$ in C, and both as we have noted are locally just X. The blow-up of $\mathbb{C}^4$ in C lies in $\mathbb{C}^4 \times \mathbb{P}^2$ (where $\mathbb{C}^4$ has coordinates (x, y, z, t), and $\mathbb{P}^2$ has homogeneous coordinates $(u:v:w)$ say), and is given by equations

$$\operatorname{rank}\begin{pmatrix} x & y & h \\ u & v & w \end{pmatrix} \leqq 1.$$

Take the affine piece given by $u=1$; we therefore have affine equations in $\mathbb{C}^6$, $y = xv$, $xw = h$. We can embed this affine piece of $\tilde{\mathbb{C}}^4$ in $\mathbb{C}^5$ (coordinates (x, z, t, v, w)), with affine equation $xw = h$. This affine variety is singular above P at points where $w=0$. Hence $\tilde{\mathbb{C}}^4 \supset \mathbb{P}^2 \supset L$, where the $\mathbb{P}^2$ is the exceptional locus above P, and L is the line given by $w=0$, which as we have seen is the singular locus of $\tilde{\mathbb{C}}^4$. Now consider the intersection of X with the exceptional locus $\mathbb{P}^2$; this is just the fibre of E above P, a conic in $\mathbb{P}^2$. We observe that X has singularities at points where this conic intersects the line L, contradicting the assumption that X was smooth. Thus we have shown that P cannot be a singular point of C in this case.

It now follows immediately that E is a conic bundle over C (since $-E \cdot l = 2$ for each fibre l), and the final sentence of the theorem concerning the possible singularities of the surface E follows by standard combinatorics.

Remark. From the local point of view, all the above singularities on E can occur. For the case of one A_m singularity ($m \geqq 1$) on a fibre, we can take $\bar{X}$ locally given by the equation

$$x^2 + y^2 + z^3 + t^{m-1} z^2$$

and X the blow-up along the curve C: $x = y = z = 0$. For the case of two A_1 singularities on a double fibre, we can take $\bar{X}$ locally given by the equation

$$x^2 + z y^2 + z^3 + t(y^2 - z^2)$$

and X the blow-up along the curve C: $x = y = z = 0$. In both the examples given, X is in fact smooth above the singular locus C of $\bar{X}$. If however we take a higher power than three in the z term, we then obtain a threefold X with cDV singularities.

3 Behaviour of the Kähler cone under deformations

Let us consider the case of $\pi: \mathscr{X} \to B$ is a complex analytic family of Calabi-Yau threefolds over a polydisc B, with fibre $X = X_0$ over $0 \in B$ (often we shall be dealing with the Kuranishi family for the threefold X, with B a polydisc therefore in $\mathrm{H}^1(T_X)$). We can identify the cohomology groups in the family, with $\mathrm{H}^2(X_b, \mathbb{Z}) \cong \mathrm{H}^2(\mathscr{X}, \mathbb{Z}) \cong \mathrm{H}^2(X, \mathbb{Z})$ for $b \in B$, the isomorphisms being induced from the inclusion maps. Any class L in $\mathrm{H}^2(X, \mathbb{Z}) \cong \mathrm{Pic}(X)$ therefore induces an invertible sheaf $\mathscr{L}$ on $\mathscr{X}$ with $c_1(\mathscr{L}|_{X_b}) = L$ for all $b \in B$. We shall usually denote the sheaf $\mathscr{L}|_{X_b}$ by $\mathscr{O}_{X_b}(L)$. Since the map $\pi: \mathscr{X} \to B$ is smooth, we have that $\mathscr{L}$ is flat over B, and so the Grauert semi-continuity theorem ([31], [1, p. 134]) may be applied.

To study the behaviour of the Kähler cone under deformations, we shall typically consider the intersection of $\partial \bar{\mathscr{K}}$ with a rational line of the form $\{M + xH; x \in \mathbb{R}\}$, where both M and H are rational, and H is ample. By shrinking B if necessary, we may assume [21] that H remains ample on X_b for all $b \in B$. As described above, we can identify the cohomology groups in the family and thus consider the Kähler cone $\mathscr{K}(b)$ of X_b as a cone in $\mathrm{H}^2(X, \mathbb{R})$. We set

$$\lambda(b) = \sup\{x \in \mathbb{R}; M + xH \notin \bar{\mathscr{K}}(b)\}.$$

If $M + \lambda(b) H \notin W^*$, then by Fact 1 from Sect. 1, it is a $\mathbb{Q}$-Cartier divisor class, some multiple of which defines a birational contraction morphism on X_b.

We shall see below that to understand how the Kähler cone varies, we shall need to understand the case when $M + \lambda(0) H \notin W^*$, and so is a $\mathbb{Q}$-Cartier divisor class, determining a birational contraction morphism on $X = X_0$. In particular, we need to discover whether any exceptional curves on X deform in the family.

In the case for instance when the contraction $\phi: X \to \bar{X}$ determined by $M + \lambda(0) H$ is small (Type I), the singularities of $\bar{X}$ are cDV and the components of the exceptional locus of ϕ have normal bundles $\mathscr{O}(-1) \oplus \mathscr{O}(-1)$, $\mathscr{O}(-2) \oplus \mathscr{O}$ or $\mathscr{O}(-3) \oplus \mathscr{O}(1)$; in the first of these cases, the curve is a stable submanifold in the sense of Kodaira [20], and so it will deform to give a similar curve on any small deformation of X. Friedman observed on p. 678-9 of [9] that in general a weak stability property holds for the exceptional locus of a small resolution of a cDV singularity. We adopt a different approach.

Proposition 3.1 *Let X be a Calabi-Yau threefold and Z a curve on X isomorphic to $\mathbb{P}^1$ which does not deform in X. For any smooth family $\pi: \mathscr{X} \to B$ of deformations of $X = X_0$ over a base manifold B, the curve Z will deform in the family.*

Proof. We consider the relative Douady space of $\mathscr{X}$ over B, and let S be the component containing the point z corresponding to the curve Z. Let $g: S \to B$ be the corresponding proper map of complex manifolds.

The dimension d of S is just the dimension of the complete local ring $\hat{\mathscr{O}}_{S,z}$, and this ring is approximated by Artinian rings. To calculate d, we express $\hat{\mathscr{O}}_{S,z}$ as A/I, where A is a formal power series ring over $\mathbb{C}$ with maximal ideal M, where $\dim(M/M^2)$ is the local embedding dimension of S at z. The argument of Proposition 3 from [23] therefore applies (since we need only consider (algebraic) deformations over Artinian rings) to show that $d \geq \mathrm{h}^0(N_{Z/\mathscr{X}}) - \mathrm{h}^1(N_{Z/\mathscr{X}})$.

This inequality can also be obtained by analytic means and is implicit in the theory of [26].

The right hand side of the above inequality is $\chi(N_{Z/\mathscr{X}})=\dim(B)$, by an easy application of Riemann-Roch. Thus $\dim(S)\geqq\dim(B)$. Furthermore, since Z by assumption does not deform in X_0, the point $z\in S$ is isolated in the fibre $g^{-1}(0)$. Therefore we deduce that the proper map $g\colon S\to B$ is surjective, and the result is proved.

Remark. In (3.1), the parameter space for the deformations of Z in the family is not claimed to be B; the point is that it maps surjectively to B.

We consider now the Kuranishi family $\pi\colon \mathscr{X}\to B$ (with B a polydisc in $\mathrm{H}^1(T_X)$) for the Calabi-Yau threefold $X=X_0$, and suppose $M, H\in \mathrm{H}^2(X,\mathbb{Q})$ with H ample on X_b for all $b\in B$. We study the behaviour of the function $\lambda(b)$, which was defined above, in a neighbourhood of $0\in B$.

Proposition 3.2 (i) *If* $M+\lambda(0)H\notin W^*$, *then* $\lambda(b)\leqq\lambda(0)$ *for all b in some neighbourhood of* $0\in B$.
(ii) *In* (i), *if moreover* $M+\lambda(0)H$ *lies on a codimension one closed face of* $\bar{\mathscr{K}}$ *whose interior rational points do not correspond to the contraction of a smooth minimal ruled surface to a curve of positive genus, then* $\lambda(b)$ *is constant on some neighbourhood of the origin.*
(iii) *If* $M+\lambda(0)H\in W^*$, *then* $\lambda(b)$ *is continuous at* $b=0$.

Proof. (i) Let $D\in\mathrm{Pic}(X)$ be a multiple of $M+\lambda(0)H$ with $\phi_D\colon X\to\bar{X}$ the corresponding birational contraction morphism. Let $\mathscr{D}$ be the invertible sheaf on $\mathscr{X}$ corresponding to the class D, as explained above. Since D is *nef* and big, it follows that $\mathrm{h}^1(\mathscr{O}_{X_0}(D))=0$. Applying Corollary 3.9 from p. 122 of [1], we deduce that the sheaf $\pi_*\mathscr{D}$ is free on some neighbourhood of $0\in B$, with the natural map

$$\pi_*\mathscr{D}\otimes\mathbb{C}\to \mathrm{H}^0(X_0,\mathscr{O}_{X_0}(D))$$

an isomorphism. Moreover, the Proper Coherence Theorem ([8, p. 64]) implies that $\mathrm{H}^0(\mathscr{X},\mathscr{D})\cong\mathrm{H}^0(B,\pi_*\mathscr{D})$. We may assume therefore (shrinking B if necessary) that the map

$$\mathrm{H}^0(\mathscr{X},\mathscr{D})\to\mathrm{H}^0(X,\mathscr{O}_X(D))$$

is a surjection. Since the linear system $|D|$ on $X=X_0$ is base point free, we see that the linear system $|\mathscr{D}|$ has no base points above an appropriate neighbourhood of 0. For all b in such a neighbourhood, we know that D is *nef*, and hence $\lambda(b)\leqq\lambda(0)$.
(ii) The condition in (ii) implies that ϕ_D factors through a primitive contraction $\psi\colon X\to X_1$ of Type I or II, or of Type III with the surface E contracted *either* containing a singular fibre *or* being ruled over a rational curve. In the light of (i), we need to show that in all these cases, there is a neighbourhood of $0\in B$ over which the class D is not ample on the threefolds X_b.

In the case when ψ is of Type I, let Z denote any irreducible curve in the exceptional locus of ψ. By (3.1), it deforms in the family, and so we may assume that for each $b\in B$, the threefold X_b contains a deformation Z_b of Z. The fact that $(M+\lambda(0)H)\cdot Z=0$ implies that $(M+\lambda(0)H)\cdot Z_b=0$, and hence that $M+\lambda(0)H$ is not ample on X_b. Therefore $\lambda(b)\geqq\lambda(0)$ for all $b\in B$. This argument however works equally well if ψ is of Type III with the surface E contracted having either a reducible or a double fibre, since in both cases we have a curve Z isomorphic to $\mathbb{P}^1$ which does not deform in X, and so (3.1) applies.

When the primitive contraction ψ is of Type II, then ψ contracts down some irreducible generalized Del Pezzo surface E to a point. We noted before that $h^1(E, \mathcal{O}_E)=0$, and so from the exact sequence

$$0 \to \mathcal{O}(-E) \to \mathcal{O} \to \mathcal{O}_E \to 0$$

we deduce that $h^i(E, \mathcal{O}(E))=0$ for $i>0$, and $h^0(E, \mathcal{O}(E))=1$. If $\mathscr{E}$ is the invertible sheaf on $\mathscr{X}$ corresponding to the class E, the argument as in (i) shows (shrinking B if necessary) that the map $H^0(\mathscr{X}, \mathscr{E}) \to H^0(X_0, \mathcal{O}_{X_0}(E))$ is surjective, and also that $h^0(\mathcal{O}_{X_b}(E))=1$ for all b. This says that there is a (uniquely determined) flat family of effective divisors E_b for $b \in B$ with $E_0=E$. In passing, we observe that if E is smooth, it is in fact a stable submanifold in the sense of Kodaira.

On $X_0=X$, we have $(M+\lambda(0)E)\cdot E\cdot H=0$, and thus on X_b it follows that $(M+\lambda(0)E)\cdot E_b\cdot H=0$. Since E_b is effective and H is ample, we deduce again that $M+\lambda(0)H$ is not ample on X_b, and hence $\lambda(b) \geqq \lambda(0)$ for all b in some neighbourhood of $0 \in B$.

This argument however works equally well if ψ is of Type III with the surface E having $h^1(E, \mathcal{O}_E)=0$ (i.e. E being ruled over a curve of genus 0). Part (ii) of the proposition has therefore now been demonstrated.

(iii) If $M+\lambda(0)H \in W^*$, then it is clear that $\lambda(b) \geqq \lambda(0)$ for all $b \in B$. Given any $\varepsilon>0$, we observe that $M+(\lambda(0)+\varepsilon)H$ is ample on $X=X_0$, and hence remains ample on X_b for b in some neighbourhood of 0 [21]. Thus in this case $\lambda(b)$ is continuous at $b=0$.

Corollary 3.3 *Suppose that $\pi\colon \mathscr{X} \to B$ is a smooth family of Calabi-Yau threefolds, and that for all $b \in B$ the threefold X_b contains no smooth minimal ruled surfaces over a curve of positive genus; then the Kähler cone is locally constant in the family.*

Proof. We can reduce first down to the local case, and then to the case of the Kuranishi family of a threefold $X=X_0$. We claim then that the Kähler cone is (locally) constant on B. For this, we need to show that given classes $M, H \in \mathrm{Pic}(X_0)$ with H ample on X_b for $b \in B$, the function $\lambda(b)$ defined above is constant.

If $\mu=\sup\{x \in \mathbb{R}; (M+xH)^3 \leqq 0\}$, we know that $\lambda(b) \geqq \mu$ for all $b \in B$. From (3.2), we know that $\lambda(b)$ is continuous everywhere, and locally constant when $\lambda(b)>\mu$. Thus $\lambda(b)$ must be constant as claimed.

4 Deformation properties of ruled surfaces in the fibres

For the purposes of understanding the behaviour of the Kähler cone in families, we saw in Sect. 3 that it is necessary to study the case when the Calabi-Yau threefold X contains a smooth surface E ruled over a curve C of positive genus. We observe first that if X contains such a surface, then there is a primitive Type III contraction of it. To see this choose an ample divisor H on X and consider the line $\{E+xH; x \in \mathbb{R}\}$. Clearly this line does meet $\partial\bar{\mathscr{K}}$ at a point of W^*, and so there is a corresponding divisor class D which defines a contraction morphism $\phi_D: X \to \bar{X}$, an isomorphism outside E. We noted in (1.3) that ϕ_D cannot contract E to a point and cannot be a Type I contraction of curves on E; thus ϕ_D must be a Type III contraction of the fibres of E. We now study the behaviour of such ruled surfaces under deformations of the threefold.

Proposition 4.1 *Suppose that $\pi: \mathscr{X} \to B$ is the Kuranishi family for $X = X_0$ (where B is taken to be a polydisc in $\mathrm{H}^1(T_X)$), and that X contains a smooth ruled surface E over a curve C of genus g. The locus $\Gamma \subset B$ of deformations for which E deforms in the family is a complex submanifold of codimension g. In particular, $g \leqq \mathrm{h}^{1,2}(X)$.*

Proof. The easiest way to see this is to apply the theory of [28]. By Theorem 2.1 of [28], the result will follow if we can show that the natural map $\mathrm{H}^1(T_X) \to \mathrm{H}^1(N_{E/X})$ is always surjective, where in our case $T_X \cong \Omega_X^2$ and $N_{E/X} \cong \omega_E$. We need to show therefore that the natural map $\mathrm{H}^{2,1}(X) \to \mathrm{H}^{2,1}(E)$ is surjective, the corresponding map or alternatively by Hodge Theory that $\mathrm{H}^{1,2}(X) \to \mathrm{H}^{1,2}(E)$ is surjective. This last map corresponds to the map on sheaf cohomology $\mathrm{H}^2(X, \Omega_X^1) \to \mathrm{H}^2(E, \Omega_E^1)$ obtained from the exact sequences of sheaves

$$0 \to \Omega_X^1(-E) \to \Omega_X^1 \to \Omega_X^1|_E \to 0$$

and

$$0 \to \mathcal{O}_E(-E) \to \Omega_X^1|_E \to \Omega_E^1 \to 0.$$

We denote by $\phi: X \to \bar{X}$ the primitive morphism corresponding to E (where by the previous section we can assume $g > 0$). We note that $\bar{X}$ has only quotient singularities (except perhaps at the dissident points of the curve of singularities to which E is contracted – as E is also minimal over C, these do not occur). This then ensures that $\mathrm{H}^0(\Omega_X^2(E)) = \mathrm{H}^0(\Omega_X^2) = 0$. Thus $\mathrm{H}^3(\Omega_X^1(-E)) = 0$, and so the map $\mathrm{H}^2(\Omega_X^1) \to \mathrm{H}^2(\Omega_X^1|_E)$ is surjective. The map $\mathrm{H}^2(\Omega_X^1|_E) \to \mathrm{H}^2(\Omega_E^1)$ is clearly surjective, and so the result follows (since $\mathrm{H}^2(\Omega_E^1)$ has dimension g).

Remark. This result should extend to the case when E has singular fibres, since by (2.2) E has only Du Val singularities, and is therefore an orbifold. Thus Hodge Theory still works, provided we work with the reflexive sheaves $\tilde{\Omega}_E^i$ rather than the sheaves Ω_E^i [32]. The sheaf $\tilde{\Omega}_E^2$ is of course just the dualizing sheaf ω_E. The argument of (4.1) goes through essentially unchanged to show that the natural map $\mathrm{H}^1(\Omega_X^2) \to \mathrm{H}^1(\omega_E)$ is surjective, and in particular that $g \leqq \mathrm{h}^{1,2}$. For the statement concerning the codimension of Γ in the general case, a recent preprint of Kawamata [17] is clearly highly relevant.

From now on, we consider the case of E a smooth surface, ruled over a curve C of genus $g > 0$, which lies on a Calabi-Yau threefold X. The arguments from Sect. 3 illustrate the fact that for the purposes of investigating the behaviour of the Kähler cone under deformations, we do not need to know whether the whole surface E deforms, but only whether some fibre does. Let $h: E \to C$ be the structure map for E and $Z = h^{-1}(P)$ a fibre. If $\phi: X \to \bar{X}$ is the primitive contraction morphism contracting E to C, we have that Z is locally a complete intersection of E and a divisor D, the pull-back under ϕ of a general hyperplane section through P. Since by assumption Z is a smooth fibre, the point P will just be a cA_1 singularity of $\bar{X}$ and the divisor D will be smooth along Z.

We now consider the question of whether Z extends 'sideways' to first order in a 1-parameter family $\pi: \mathscr{X} \to \Delta$. Working to first order, we have an induced morphism of schemes $\pi_1\colon X_1 \to \Delta_1 = \operatorname{Spec}(\mathbb{C}[\varepsilon]/(\varepsilon^2))$, and so Z extends sideways to first order in the family $\pi\colon \mathscr{X} \to \Delta$ if and only if it extends to a subscheme Z_1 of X_1 over Δ_1, with Z_1 not a subscheme of X. We have a diagram of natural sheaf morphisms

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T_X & \longrightarrow & T_{X_1}|_X & \longrightarrow & \mathcal{O}_X & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & N_{Z/X} & \longrightarrow & N_{Z/X_1} & \longrightarrow & \mathcal{O}_Z & \longrightarrow & 0 \end{array}$$

where $N_{Z/X} \cong \mathcal{O}_Z \oplus \mathcal{O}_Z(E)$. We will have an extension sideways Z_1 of Z in the family $\pi_1\colon X_1 \to \Delta_1$ if and only if the group $\mathrm{H}^0(N_{Z/X_1})/\mathrm{H}^0(N_{Z/X})$ is non-zero.

Corresponding to the above diagram, we have a diagram of cohomology groups

$$\begin{array}{ccc} \mathrm{H}^0(\mathcal{O}_X) & \delta_1 & \mathrm{H}^1(T_X) \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} \\ \mathrm{H}^2(\mathcal{O}_Z) & \delta_2 & \mathrm{H}^1(N_{Z/X}) \end{array}$$

and $\delta_1(1) = \theta \in \mathrm{H}^1(T_X)$, the class corresponding to the given 1-parameter deformation via the Kodaira-Spencer map. Whether Z extends sideways in X_1 is asking whether $\alpha(1)$ is in the image of $\mathrm{H}^0(N_{Z/X_1})$, i.e. whether $\delta_2\alpha(1) = 0$. Since the diagram commutes, this is just the condition that $\beta(\theta) = 0$ under the natural map $\mathrm{H}^1(T_X) \to \mathrm{H}^1(N_{Z/X})$.

In our case however, $\mathrm{H}^1(\mathcal{O}_Z) = 0$, and so there is a natural isomorphism $\mathrm{H}^1(N_{Z/X}) \cong \mathrm{H}^1(\omega_E|_Z)$. Therefore Z extends sideways to first order in the family $\pi\colon \mathscr{X} \to \Delta$ if and only if the image of the class θ under the natural map $\mathrm{H}^1(T_X) \to \mathrm{H}^1(\omega_E|_Z)$ is zero. Now $\mathrm{H}^1(\omega_E|_Z) \cong \mathrm{H}^1(\omega_Z) \cong \mathbb{C}$, and thus for a fibre Z to extend sideways to first order, we have a codimension one condition on $\mathrm{H}^1(T_X)$.

The map $\mathrm{H}^1(T_X) \to \mathrm{H}^1(\omega_E|_Z)$ factors as $\mathrm{H}^1(T_X) \to \mathrm{H}^1(\omega_E) \to \mathrm{H}^1(\omega_E|_Z)$ where as we noted in (4.1) the first of these maps is surjective.

With $h\colon E \to C$ the structure map giving the ruling on E, we have

$$R^1 h_* \omega_E \cong (R^1 h_* \omega_{E/C}) \otimes \omega_C$$

where duality theory implies that $R^1 h_* \omega_{E/C} \cong h_* \mathcal{O}_E \cong \mathcal{O}_C$. Thus $R^1 h_* \omega_E \cong \omega_C$, and the Leray spectral sequence identifies $\mathrm{H}^1(\omega_E)$ with $\mathrm{H}^0(\omega_C)$. Moreover, if $Z = h^{-1}(P)$, the maps $\mathrm{H}^1(\omega_E) \to \mathrm{H}^1(\omega_E|_Z) \cong \mathrm{H}^1(\omega_Z)$ may be identified with the maps $\mathrm{H}^0(\omega_C) \to \mathrm{H}^0(\omega_C|_P) \cong \mathbb{C}$. If $\pi\colon \mathscr{X} \to \Delta$ is a 1-parameter deformation of $X = X_0$ with Kodaira-Spencer class θ which has non-zero image in $\mathrm{H}^0(\omega_C)$, then the general fibre of E does not extend sideways in the family, consistent with the previous result (4.1).

We see from the above discussion that the case $g=1$ should be very different from the case $g>1$. For $g=1$, the general $\theta \in \mathrm{H}^1(T_X)$ will have a nowhere vanishing image in $\mathrm{H}^0(\omega_C)$, and so there will be a first order obstruction to extension sideways for any fibre of E. For $g>1$, the general $\theta \in \mathrm{H}^1(T_X)$ will have non-zero image vanishing at $2g-2$ points of C. For such a point, the corresponding fibre will therefore extend sideways to first order in the family.

As pointed out to the author by Prof. Fujiki, the case when $g>1$ can in fact be dealt with by a simple dimension counting argument, on which argument the proof below is based.

Proposition 4.2 *Suppose that* $\pi: \mathscr{X} \to B$ *is the Kuranishi family for* $X = X_0$ *and that* X *contains a smooth surface* E *ruled over a curve* C *of genus* $g>1$. *Let* Z *be any fibre of* E *over* C*; then (shrinking* B *if necessary) we have that every threefold* X_b *in the family contains a curve which is a deformation of* Z.

Proof. Let $D \in \mathrm{Pic}(X)$ be a divisor with $\phi_D: X \to \bar{X}$ the primitive contraction morphism for E. If we take $\mathscr{D}$ to be the invertible sheaf on $\mathscr{X}$ corresponding to the class D, we saw in (3.2 i) that by shrinking B if necessary, we could assume that $\phi_D: \mathscr{X} \to \bar{\mathscr{X}}$ is a birational contraction morphism over B.

Let $P \in C$ be the point corresponding to the given fibre Z. The general hyperplane section H of X through P has an A_1 singularity at P, and locally X can be thought of as the total space of a 1-parameter family of A_1 surface singularities. Since however the family $\pi: \mathscr{X} \to B$ may be locally embedded by the linear system on $\mathscr{X}$ corresponding to the class H (cf. argument of (3.2 i)), we can take a neighbourhood U of P in $\bar{\mathscr{X}}$, and consider it as a family of deformations of the A_1 singularity H over a parameter space $B \times \Delta$, for some open disc $\Delta \subset \mathbb{C}$. On $\mathscr{X}$, we therefore have a family $\tilde{U}$ of deformations over $B \times \Delta$ of a neighbourhood of Z in the pullback of the surface H.

If now S denotes the versal deformation space for the A_1 surface singularity, we can think of S as an open disc around $0 \in \mathbb{C}$, where the origin corresponds to the singular fibre of the versal family. The family $U \to B \times \Delta$ induces a holomorphic map (of germs) $\theta: B \times \Delta \to S$, and the fibres of U therefore have A_1 singularities over the (non-empty) hypersurface $\theta^{-1}(0) = W$ in $B \times \Delta$ (as usual, having allowed ourselves to shrink B and Δ if necessary). Thus if we consider the family $\tilde{U} \to B \times \Delta$, the fibres contain a curve isomorphic to $\mathbb{P}^1$ which is a deformation of Z precisely over the hypersurface $W \subset B \times \Delta$.

Considering the projection $B \times \Delta \to B$ and family $\tilde{U} \to B$, we observe that the image of W must be dense in B. This follows from (4.1), since the threefold $\tilde{U}_b$ only contains a positive dimensional family of deformations of Z for b in the locus on B over which the whole of E deforms, a closed analytic subvariety of codimension $g>1$. Consideration of dimensions then implies immediately that the morphism $W \to B$ must be dominating.

By shrinking B, we may assume that the family $\pi: \mathscr{X} \to B$ is projectively embedded, and so may consider the Hilbert scheme $\mathrm{Hilb}(\mathscr{X}/B)$ [12, 11]. If T is the component which contains the point corresponding to the curve Z, we have a proper map $T \to B$, whose image contains the image of W, a dense subset of B. Thus the map $T \to B$ is surjective, and so for every $b \in B$, the threefold X_b contains some deformation of Z.

Remark 4.3 We deduce from (4.2) that both in (3.2 ii) and (3.3), we may now replace the phrase 'curve of positive genus' by 'elliptic curve'. In other words, we have seen that it is only the existence on our threefolds of *elliptic* ruled surfaces that can cause the Kähler cone to fail to be invariant under small deformations. We now show that if X contains such a surface E, then the Kähler cone will jump under the general small deformation; this then will provide a precise answer to the question raised in [7].

Proposition 4.4 *Suppose $\pi: \mathscr{X} \to \Delta$ is a 1-parameter family of deformations of $X = X_0$ with Kodaira-Spencer class $\theta \in H^1(T_X)$, and that X contains a smooth surface E ruled over an elliptic curve C. If E does not deform (locally at 0) in the family, then the Kähler cone is not invariant under the deformation. More precisely, for any $D \in \mathrm{Pic}(X)$ with $\phi_D: X \to \bar{X}$ the primitive contraction morphism for E, the class D will be ample on X_t for $t \in \Delta^*$, some punctured neighbourhood of zero.*

Proof. Since $H^0(N_{E/X}) = 0$, we observe first that E not deforming in a 1-parameter family is the same as the condition that E does not extend sideways in the family.

Taking $\mathscr{D}$ to be the invertible sheaf on $\mathscr{X}$ corresponding to the class D, we have seen that by shrinking Δ if necessary, we can assume that $\phi_{\mathscr{D}}: \mathscr{X} \to \bar{\mathscr{X}}$ is a birational contraction morphism over Δ. If D is not ample on X_t for t in some punctured neighbourhood, we may assume that $\phi_{\mathscr{D}}$ is not an isomorphism on X_t for $t \in \Delta$. Since moreover E does not deform in the family, we may assume that the contraction morphisms $X_t \to \bar{X}_t$ induced by $\phi_{\mathscr{D}}$ are all of Type I for $t \in \Delta^*$, the punctured disc.

Take an irreducible (2-dimensional) component $\mathscr{Z}$ of the exceptional locus of $\phi_{\mathscr{D}}$ which maps surjectively to Δ. By assumption its fibre Z_0 over zero lies on E, and moreover $D \cdot Z_0 = 0$; therefore Z_0 is concentrated on the fibres of E. Let Z be a component of Z_0, a fibre of E, and let $P \in C$ be the corresponding point of C.

Since the general hyperplane section of $\bar{X}$ through P is an A_1 surface singularity, we may consider $\bar{\mathscr{X}}$ as locally a 2-parameter deformation of such a singularity, and $\mathscr{X}$ locally as a 2-parameter deformation of a neighbourhood of Z in the pullback D of a general hyperplane section of $\bar{X}$ through P (cf. proof of (4.2)). We deduce therefore that Z deforms in the 1-parameter family $\pi: \mathscr{X} \to \Delta$. In particular, by taking an n-fold cyclic covering $\tilde{\Delta} \to \Delta$ for suitable $n > 0$, we have that Z extends sideways in the induced 1-parameter deformation $\tilde{\pi}: \tilde{\mathscr{X}} \to \tilde{\Delta}$.

The ruled surface E will of course now extend sideways to at least order $n-1$, and so our previous first order argument needs to be refined to deal with higher order obstructions. Our conclusion will be that the first non-vanishing obstruction to E extending sideways in $\tilde{\pi}: \tilde{\mathscr{X}} \to \tilde{\Delta}$ will provide a non-vanishing obstruction to Z extending sideways, thus yielding the required contradiction.

Although working with the family $\tilde{\pi}: \tilde{\mathscr{X}} \to \tilde{\Delta}$, for ease of notation we shall now omit the tildes. Our family $\pi: \mathscr{X} \to \Delta$ is now therefore by assumption trivial up to order $(n-1)$. For $r > 0$, set $\Delta_r = \mathrm{Spec}(\mathbb{C}[\varepsilon]/(\varepsilon^{r+1}))$, the rth order neighbourhood of 0 in Δ, and $\pi_r: X_r \to \Delta_r$ the induced rth order deformation of $X = X_0$. By assumption, E does not extend sideways in the family; we suppose that the mth order obstruction is the first one which is non-zero. Thus E extends to a scheme E_{m-1} over Δ_{m-1}, but has a non-vanishing obstruction at the next step. The extension sideways of Z restricts to an infinitesimal extension $Z_{m-1} \subset X_{m-1}$ over Δ_{m-1}. It is left to the reader to check that Z_{m-1} is a closed subscheme of E_{m-1} – indeed this is true of any extension sideways of Z to order $m-1$, and follows from the fact that $h^0(Z_r, \mathcal{O}_{Z_r}(E_r)) = 0$ for $1 \leqq r \leqq m-2$.

Consider now the mth order infinitesimal deformation $\pi_m: X_m \to \Delta_m$ induced by the family; this corresponds to an extension of sheaves

$$0 \to T_{X_{m-1}/\Delta_{m-1}} \to T_{X_m}|_{X_{m-1}} \to \mathcal{O}_{X_{m-1}} \to 0.$$

Moreover we have a diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & T_{X_{m-1}/\Delta_{m-1}} & \longrightarrow & T_{X_m}|_{X_{m-1}} & \longrightarrow & \mathcal{O}_{X_{m-1}} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & N_{E_{m-1}/X_{m-1}} & \longrightarrow & N_{E_{m-1}/X_m} & \longrightarrow & \mathcal{O}_{E_{m-1}} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & N_{E_{m-2}/X_{m-2}} & \longrightarrow & N_{E_{m-2}/X_{m-1}} & \longrightarrow & \mathcal{O}_{E_{m-2}} & \longrightarrow & 0
\end{array}$$

(where the horizontal rows are exact and the vertical maps are surjective), and a similar diagram with Z substituted for E everywhere. Taking cohomology, we obtain a diagram

$$\begin{array}{ccc}
\mathrm{H}^0(\mathcal{O}_{X_{m-1}}) & \longrightarrow & \mathrm{H}^1(T_{X_{m-1}/\Delta_{m-1}}) \\
\downarrow & & \downarrow \\
\mathrm{H}^0(\mathcal{O}_{E_{m-1}}) & \longrightarrow & \mathrm{H}^1(N_{E_{m-1}/X_{m-1}}) \\
\downarrow & & \downarrow \\
\mathrm{H}^0(\mathcal{O}_{E_{m-2}}) & \longrightarrow & \mathrm{H}^1(N_{E_{m-2}/X_{m-2}})
\end{array}$$

and again a similar diagram with Z substituted for E. We denote by θ_m the image of $1 \in \mathrm{H}^0(\mathcal{O}_{X_{m-1}})$ in $\mathrm{H}^1(T_{X_{m-1}/\Delta_{m-1}})$. The image $\bar{\theta}_m$ of θ_m in $\mathrm{H}^1(N_{E_{m-1}/X_{m-1}})$ is the obstruction to E_{m-1} extending at the next step, and so by assumption is non-zero, whilst the image in $\mathrm{H}^1(N_{E_{m-2}/X_{m-2}})$ does vanish.

From the exact sequence

$$0 \to N_{E/X} \to N_{E_{m-1}/X_{m-1}} \to N_{E_{m-2}/X_{m-2}} \to 0$$

and the corresponding cohomology sequence, we see that the obstruction $\bar{\theta}_m$ may therefore be considered as an element of $\mathrm{H}^1(N_{E/X}) = \mathrm{H}^1(\omega_E)$. Similarly, the obstruction to Z_{m-1} extending at the next step (an element of $\mathrm{H}^1(N_{Z_{m-1}/X_{m-1}})$) may be considered as an element of $\mathrm{H}^1(N_{Z/X})$, and is then just the image of $\bar{\theta}_m$ under the map $\mathrm{H}^1(N_{E/X}) \to \mathrm{H}^1(N_{Z/X})$. If, as in the first order calculation, we identify $\mathrm{H}^1(\omega_E)$ with $\mathrm{H}^0(\omega_C)$, we have that Z extends to mth order if and only if the element $\bar{\theta}_m \in \mathrm{H}^0(\omega_C)$ vanishes at P. By assumption however $\bar{\theta}_m$ is non-zero, and so (since $g=1$) is nowhere vanishing. Hence there is a non-vanishing mth order obstruction to Z extending sideways in the family contrary to expectation, and the required contradiction has been obtained.

Proof of Main Theorem. We have now proved all but the last sentence of the theorem. For this we consider the (at most countably many) classes $E_i \in \mathrm{H}^2(\mathcal{X}, \mathbb{Z})$

which can represent an elliptic ruled surface on some threefold of the family. Using (4.1), we see that each class E_i corresponds to a codimension one submanifold Γ_i of B. We observe that $B_0 = B \setminus \bigcup \Gamma_i$ is a dense subset of B by Baire's category theorem. I claim that the Kähler cone is constant on B_0. By connectivity, we need only show this in a neighbourhood of each point of B.

We suppose therefore that we have classes M, $H \in \operatorname{Pic}(X)$ with H ample on X_b for all $b \in B$, and consider the behaviour of the function $\lambda(b)$ defined in Sect. 3. By (3.2) and its extension noted in (4.3), we observe that for $b \in B_0$, the function has the following properties:

(i) $\lambda(b) \geqq \mu$;
(ii) if $\lambda(b) > \mu$, it is a rational number, and the function is locally constant at b;
(iii) if $\lambda(b) = \mu$, it is continuous at b.

As in (3.3), we deduce that $\lambda(b)$ is constant on the connected set B_0. Since this holds for all $M \in \operatorname{Pic}(X)$, the Kähler cone in constant over B_0.

Corollary 4.5 *For any connected analytic family $\pi: \mathscr{X} \to B$ of Calabi-Yau threefolds, the Kähler cone is locally constant over some dense subset B_0 of B (locally the complement of at most countably many analytic subvarieties).*

Proof. Details here will be left to the reader. We reduce immediately to the local case by connectivity of the family. Any class E which represents an elliptic ruled surface on *all* threefolds in the family will then correspond to a codimension one face of the Kähler cone which is invariant, and so can be ignored. The result can therefore be deduced from the above theory using the versality the Kuranishi family.

Example 4.6 For an example in which the Kähler cone does jump under a general deformation, we consider a simple example taken from [18]. We let $\bar{X}$ be the normal projective threefold

$$\left(\begin{array}{c||ccc} \mathbb{P}^2 & 2 & 0 & 1 \\ \mathbb{P}^2 & 0 & 2 & 1 \\ \mathbb{P}(1,1,2) & 0 & 0 & 4 \end{array}\right).$$

The notation here indicates a complete intersection of three hypersurfaces in the product $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}(1,1,2)$, the hypersurfaces having degrees (in each set of coordinates) (2, 0, 0), respectively (0, 2, 0), respectively (1, 1, 4). As calculated in [18], $\bar{X}$ has an elliptic curve C of A_1 singularities; when we blow C up, we obtain a Calabi-Yau threefold X with $b_2(X) = 4$ and $h^{1,2}(X) = 68$, and containing a smooth ruled surface over C.

As pointed out to the author by Prof. Borcea, both $\bar{X}$ and X have a small deformation X_1 which is a hypersurface in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ defined by a polynomial which is homogeneous of degree two in each set of variables. This follows since both the quadratic cone $\mathbb{P}(1,1,2)$ and its minimal desingularization F_2 have $\mathbb{P}^1 \times \mathbb{P}^1$ as a small deformation. Thus X has as a small deformation

$$\left(\begin{array}{c||ccc} \mathbb{P}^2 & 2 & 0 & 1 \\ \mathbb{P}^2 & 0 & 2 & 1 \\ \mathbb{P}^1 & 0 & 0 & 2 \\ \mathbb{P}^1 & 0 & 0 & 2 \end{array}\right).$$

which is isomorphic to the one claimed.

The closed Kähler cone $\bar{\mathcal{K}}$ on X_1 is generated by the divisors D_i ($i=1, \ldots, 4$) obtained by pulling back a point from one of the $\mathbb{P}^1$ factors and restricting to X_1. The cubic form on X_1 (and hence on X) is therefore just a multiple of

$$x_1 x_2 x_3 + x_2 x_3 x_4 + x_3 x_4 x_1 + x_4 x_1 x_2.$$

The corresponding hypersurface $W \subset \mathbb{P}^{\rho-1}(\mathbb{R})$ is a cubic surface with four nodes at the points corresponding to the D_i.

If one considers the deformation of F_2 to $\mathbb{P}^1 \times \mathbb{P}^1$, one observes that one of the rulings of $\mathbb{P}^1 \times \mathbb{P}^1$ corresponds to the ruling on F_2, and therefore represents a *nef* class, whilst the other ruling corresponds to a *non-nef* class on F_2. Thus three of our classes, say D_1, D_2, D_3, remain *nef* on $X = X_0$, whilst the remaining one D_4 is not *nef* on X. The closed Kähler cone of X is therefore the cone on the tetrahedron generated by D_1, D_2, D_3 and say $D_3 + D_4$, and under a general small deformation this jumps to the cone on the tetrahedron generated by all four D_i.

Remarks 4.7 The codimension one submanifolds Γ_i in (4.5) form a stratification of the base B, the strata consisting of sets of points which lie on the same subcollection of the Γ_i. It follows that the Kähler cone is in fact constant on each stratum, and differs on different strata. The general picture would however be very much more pleasant if we knew that the Γ_i were locally finite in number, for the stratification would be an 'open stratification' with each stratum being open in its closure, and B_0 would be the complement of an analytic subvariety of B. It is an open question whether this is always true. If not, then the examples where it fails will exhibit very special features. It can be shown for instance that such examples will have the cubic hypersurface W containing a real singularity at which the tangent cone is two hyperplanes.

The above question seems also to be related to the theory of intermediate Jacobians as developed in [10]. If a Calabi-Yau threefold X contains a ruled surface E over a curve C of genus $g > 0$, then we have a cylinder homomorphism

$$\Phi\colon \mathrm{H}_1(C, \mathbb{Z}) \to \mathrm{H}_3(X, \mathbb{Z}),$$

which induces the Abel-Jacobi map from the Jacobian $J(C)$ to the intermediate Jacobian

$$J(X) = (\mathrm{H}^{3,0}(X) \oplus \mathrm{H}^{2,1}(X))^* / (\mathrm{H}_3(X, \mathbb{Z}) \otimes \mathbb{Z}).$$

If γ_1, γ_2 denote two 1-cycles on C, we check easily that $(\Phi(\gamma_1), \Phi(\gamma_2))_X = -2(\gamma_1, \gamma_2)_C$, where the pairings are the obvious ones induced from cup-product. Thus the image of $J(C)$ under the Abel-Jacobi map is a g-dimensional abelian subvariety of the principally polarized analytic torus $J(X)$. In particular, if we know that $J(X)$ contains no elliptic curves, then X contains no elliptic ruled surfaces.

5 Remarks on elementary transformations

Suppose that X is a Calabi-Yau threefold containing an elliptic ruled surface E. We observe that $E^3 = 0$ and $E \cdot c_2 = 0$, and so the corresponding point in

$\mathbb{P}^{\rho-1}(\mathbb{R})$ lies both on W and the hyperplane $c_2=0$. However, we also know that the quadratic form $D \mapsto E \cdot D^2$ on Pic(X) has rank 2; explicitly, it is the product of linear forms $(l \cdot D)(m \cdot D)$, where l is a fibre of E and m is the 1-cycle $-E|_E$. From this observation it is easy to see that E represents an *Eckardt* point of the cubic hypersurface W (i.e. a smooth point for which the tangent hyperplane intersects the cubic in a cubic cone). In general we would not expect any points of the hyperplane (even with *real* coefficients) to be Eckardt points of W, and the larger the value of ρ, the stronger the statement being made about the cubic becomes. Thus in most cases we shall be able to deduce that the Kähler cone is locally constant in familes containing X merely because there are no rational points of the hyperplane $c_2=0$ which are Eckardt points of W.

In the case however when we do have a ruled surface E over an elliptic curve C on our Calabi-Yau threefold X, the situation is very analogous to that of a (-2)-curve on a K3 surface as described in [6]. Suppose $\pi: \mathscr{X} \to \Delta$ is a 1-parameter family of deformations of $X = X_0$ under which E does not extend sideways. I claim that there is an *elementary transformation* or *flop* which results in a different family $\pi': \mathscr{X}' \to \Delta$, albeit with the same fibres.

In the case when E does not extend sideways even to first order, we saw in Sect. 4 that the same was true of all fibres of E. Therefore every fibre l of E has normal bundle $\mathcal{O}_l \oplus \mathcal{O}_l(-1) \oplus \mathcal{O}_l(-1)$ in $\mathscr{X}$. We can blow up $\mathscr{X}$ in E, obtaining a fibre bundle over C with fibres $\mathbb{P}^1 \times \mathbb{P}^1$, which can then be contracted along the other ruling to form a new family $\pi': \mathscr{X}' \to \Delta$. The operation described here is just the elementary transformation of [6] performed on the fibres of E. In the case when E deforms to order $r-1$ but not to order r, the argument of (4.4) showed that this remains true for all fibres of E. In this case we need to blow up r times before obtaining a family of surfaces $\mathbb{P}^1 \times \mathbb{P}^1$ which can be contracted in a different direction. Contracting all the way down to E again, we have the required flop – this is just the elementary transformation of [30] performed on the fibres of E.

If we fix a fibre $X_1 \neq X_0$ of $\pi: \mathscr{X} \to \Delta$, we obtain an identification $\alpha: \mathrm{H}^2(X_0, \mathbb{Z}) \cong \mathrm{H}^2(X_1, \mathbb{Z})$. From the flopped family, we obtain a different identification $\alpha': \mathrm{H}^2(X_0, \mathbb{Z}) \cong \mathrm{H}^2(X_1, \mathbb{Z})$. The composite $\alpha' \alpha^{-1}$ defines an involution σ on $\mathrm{H}^2(X_0, \mathbb{Z})$ which preserves both the cone W^* and the hyperplane $c_2=0$. As in the case of K3 surfaces [6], the involution σ is just a reflection; in our case it fixes divisors D with $D \cdot l=0$ and sends E to $-E$. From this description, it is clear that it does indeed preserve both W^* and the hyperplane $c_2=0$. For an arbitrary element $D \in \mathrm{H}^2(X, \mathbb{R})$, we have

$$\sigma(D) = D + (D \cdot l) E.$$

In passing we remark that the induced involution on the cubic hypersurface $W \subset \mathbb{P}^{\rho-1}(\mathbb{R})$ is just the canonical involution associated with the Eckardt point defined by E.

The theory of root systems now applies as in the case of K3 surfaces [6]. For instance, it follows that if $\pi: \mathscr{X} \to \Delta$ is any 1-parameter family of deformations of $X = X_0$ and $H \in \mathrm{Pic}(X_1) \cong \mathrm{Pic}(\mathscr{X})$ is ample on each fibre X_t for $t \neq 0$ (where X_1 is some fixed fibre different from X_0), then by making elementary transformations on elliptic ruled surfaces on X_0 we may obtain a family

$\pi': \mathscr{X}' \to \Delta$ for which $H \in \mathrm{Pic}(X_1) \cong \mathrm{Pic}(\mathscr{X}')$ is semi-ample on all fibres (cf. Sect. 7 of [30]). Details of this will be given in a subsequent paper.

If we turn to the example described in (4.6), the flop on $\mathrm{H}^2(X, \mathbb{Z})$ is very easy to see. Here the Eckardt point of W is the point $E = D_3 - D_4$ and the corresponding involution on $\mathbb{P}^{\rho-1}(\mathbb{R})$ reflects the tetrahedron generated by D_1, D_2, D_3 and $D_3 + D_4$ to that generated by D_1, D_2, D_4 and $D_3 + D_4$.

6 Remarks on elliptic fibrations

It is well-known that any K3 surface is the deformation of one with an elliptic structure, and one might hope that the same might be true for Calabi-Yau threefolds. It has been observed in [15] that the condition $b_2(X) > 1$ is clearly necessary for this to be true. The next example shows that this condition is not sufficient.

Example 6.1 Consider the weighted complete intersection $\bar{X}_{6,6} \subset \mathbb{P}(1, 2, 2, 2, 2, 3)$ cut out by two general equations of degree six. As observed in [18] this has a curve C of A_1 singularities, and on blowing up this curve we obtain a Calabi-Yau threefold X with $b_2(X) = 2$. The curve C has genus 10. If one calculates the cubic form on $\mathrm{H}^2(X, \mathbb{Z})$, one discovers that the cubic hypersurface $W \subset \mathbb{P}^{\rho-1}(\mathbb{R})$ (in this case just three points) has no rational points. If some deformation of X had an elliptic fibre space structure, this would clearly provide a point of W (given by pulling back a hyperplane from the base). Therefore, in this example, no deformation of X can have an elliptic structure.

This kind of arithmetic obstruction to some deformation of X having an elliptic structure ceases to be relevant for large enough $b_2(X)$ (for instance $b_2(X) > 19$, when the rational points of W are always dense [38]). One might therefore hope that although for small values of $b_2(X)$ there will be examples of Calabi-Yau threefolds with no elliptic deformations, this would not be the case for $b_2(X)$ sufficiently large. This too however seems not to be true, since obstructions to there existing elliptic deformations are provided also by the geometry of the two cones $\bar{\mathscr{K}}$ and W^* in $\mathrm{H}^2(X, \mathbb{R})$.

Proposition 6.2 *If X is a Calabi-Yau threefold not containing any elliptic ruled surfaces and for which the two cones $\bar{\mathscr{K}}$ and W^* are disjoint (and hence by Fact 1 from Sect. 1, $\bar{\mathscr{K}}$ is rational polyhedral), then no deformation of X can have an elliptic structure.*

Proof. For any connected family $\pi: \mathscr{X} \to B$ containing X, (4.5) implies that the Kähler cone will be locally constant on a dense subset B_0 of B, over which the threefolds X_b contain no elliptic ruled surfaces and have $\bar{\mathscr{K}}(b)$ disjoint from W^*. The result has now been reduced to the local case, with B say a polydisc. For any point $b \in B$, we deduce from (3.2) that the cone $\bar{\mathscr{K}}(b)$ is always contained in the cone $\bar{\mathscr{K}}$ at the general point, and hence is also disjoint from W^*. Hence no X_b in the family possesses an elliptic fibre space structure.

One suspects that the situation of (6.2) (for which there seems to be no obvious restriction on $b_2(X)$ for it to occur) should not be too uncommon.

Example 6.3 An example of an interesting Calabi-Yau threefold X with $b_2(X) = 12$ and $\mathrm{h}^{1,2}(X) = 15$ is constructed by Borcea in [3], where in his notation the threefold is $Z=F_1(V_4)$. It follows from the geometry of this example and the first part of Theorem 6.2 from [3] that the codimension one faces of $\bar{\mathcal{K}}$ in this case all correspond to small contractions, and hence that X contains no ruled surfaces of positive genus. In the second part of his Theorem 6.2, Borcea writes down the generators for the cone $\bar{\mathcal{K}}$; it is easily checked that all these generators D have $D^3>0$, and hence $\bar{\mathcal{K}}$ is disjoint from W^*.

It might be noted here that Kollár has recently shown that this latter property also holds for any Calabi-Yau threefold X lying on a smooth Fano fourfold (see Appendix to [5]). If we consider such threefolds X containing no elliptic ruled surfaces, then in these cases also, no deformation of X can have an elliptic fibre space structure.

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