Invent. math. 107, 301–322 (1992)

Inventiones mathematicae

Integral closure of modules and Whitney equisingularity

Terence Gaffney★

Northeastern University, Department of Mathematics, Boston, MA 02115, USA

Oblatum 4-II-1991 & 3-VI-1991

In this paper we introduce the notion of the integral closure of a module to singularities.

In a series of six papers [11, 12, 13, 14, 15, 16] John Mather introduced infinitesimal objects which were submodules of the set of vector fields over a map f. (These vector fields can be thought of as "tangent vectors" to f in the set of all maps, hence it is reasonable to think of submodules of them as infinitesimal objects). These submodules have been very useful in proving that families of sets and maps have the same analytic type.

In this paper, we use the integral closure operation to provide the correct infinitesimal object (again a submodule of the set of vector fields over a map) for studying the Whitney equisingularity of families of complex analytic sets.

The work of Bernard Teissier on hypersurfaces with isolated singularities [19] is one of the main inspirations of the present work. In his paper, Teissier works with the integral closure of the ideal $\left(x_i \frac{\partial f}{\partial x_j}\right)$ in $\mathcal{O}_{X_f}$ where f is the function defining the hypersurface X_f. For sets defined by p equations, $p > 1$, the analogous object is a submodule of $\mathcal{O}^p_{X,x}$ hence the present work.

A notion of the integral closure of a module related to ours has been developed by David Rees and appears in [18].

In Sect. 1 we define the integral closure of a submodule of $\mathcal{O}^p_{X,x}$, X the germ of a complex analytic set, and prove some of its basic properties. In Sect. 2, we show that a theorem of Thom-Levine type holds using the infinitesimal object associated to a deformation by the theory of Sect. 1. In Sect. 3, we use the theory of 1 and 2 to prove analogues of the main theorems on analytic equivalence of sets in the Mather school, and we also prove a theorem about Newton non-degenerate maps defining a complete intersection singularity.

In Sect. 4 we show that the main results of Sect. 3 can be extended to the real analytic and C^∞ cases. A connection between the ideas of this paper and work of Wilson, Brodersen and Wall is indicated at the end of the section.

★ Supported in part by NSF Grants DMS-8803478, INT-887075

This paper is part of two programs: to extend to ICIS (Complete intersection, isolated singularity) singularities all of the results of Teissier on hypersurfaces with isolated singularities, and to use integral closure of ideals and modules to study the equisingularity of maps and sets. The first program concludes in [6] where a formula will be proved relating the Buchsbaum-Rim multiplicity of $\frac{\mathcal{O}^p_{X_{f,x}}}{\left\{x_i\frac{\partial f}{\partial x_j}\right\}}$ and the polar multiplicities of $X_{f,x}$. The second program continues in [7] which studies the role of integral closure in $\mathscr{L}^0$-equivalence of map-germs and in a future paper which will deal with $\mathscr{A}^0$ equivalence.

Acknowledgements. The development of these ideas was supported by many people and institutions. It was during conversations at the singularities year in the fall of 1985 at the University of North Carolina that the author first realized that the notion of the integral closure of a module was valuable. These ideas grew during the author's time at the I.H.E.S. and the University of Paris VII where he was supported by the C.N.R.S. The paper was completed at the University of Warwick during the singularities year organized by David Mond, James Montaldi, Mark Roberts and Ian Stewart.

Conversations with James Damon, Lê Dung Trang, Edward Looijenga, Andrew du Plessis, David Rees, Bernard Teissier, Terry Wall and Leslie Wilson contributed to the growth of these ideas. The author thanks all of these institutions and people for their help.

1 The integral closure of a module: Basic properties

Recall the notion of the integral closure of an ideal as given in [20].

Definition 1.1 Let I be an ideal in a ring A, then $h \in A$ is in the *integral closure of* I, denoted $\bar{I}$, iff there exists a monic polynomial $P(z) = z^n + \sum_{i=0}^{n-1} a_i z^i$, $a_i \in I^{n-i}$ such that $P(h) \equiv 0$.

In the event that $A = \mathcal{O}_{X,x}$, the local ring of a complex analytic set, then $\bar{I}$ has many interesting properties.

In [20] the following equivalences are proved.

Proposition 1.2 *Suppose I is an ideal in $\mathcal{O}_{X,x}$, X a complex analytic set. Then the following statements are equivalent*

(1) $h \in \bar{I}_x$

(2) (Growth condition) *For each choice of generators (g_i) of I there exists a neighborhood U of x and a constant $C > 0$ such that*

$$\|h(z)\| \leqq C \sup_i \|g_i(z)\| \textit{ for all } z \in U .$$

(3) (Valuative criterion) *For each φ: $\mathbb{C}, 0 \to X, x$, $h \circ \varphi$ lies in $(\varphi^*(I))\mathcal{O}_1$.*

(4) *There exists a faithful $\mathcal{O}_{X,x}$ module L of finite type such that $h \cdot I \subset I \cdot L$.*

Teissier also shows that if $\mathscr{I}$ is a coherent sheaf of ideals with stalk I_x, $V(\mathscr{I})$ nowhere dense, then there is a unique coherent sheaf on X with stalk $\bar{I}_x$. This is done by relating $\mathscr{I}$ and the pullback of $\mathscr{I}$ to $NB\mathscr{I}(X)$, the normalized blow-up of X along $\mathscr{I}$.

In this section we develop the notion of the integral closure of a module and develop analogous properties for this object. In the contexts considered in this paper, it seems most convenient to define the integral closure of a module using the valuative criterion.

Definition 1.3 Suppose X, x is a complex analytic germ, M a submodule of $\mathcal{O}^p_{X,x}$. Then $h \in \mathcal{O}^p_{X,x}$ is in $\bar{M}$ iff for all φ: $\mathbb{C}, 0 \to X, x$, $h \circ \varphi$ is in $(\varphi^*(M))\mathcal{O}_1$.

Example 1.4 Suppose $X = \mathbb{C}^2$, $M \subset \mathcal{O}^2_2$ is generated by $\{(x, 0), (0, y), (y, x)\}$. Then, $\bar{M} = m_2\mathcal{O}^2_2$. For, let φ: $\mathbb{C}, 0 \to \mathbb{C}^2, 0$; then $(\varphi^*M)\mathcal{O}_1$is generated by $\{(t^n, 0), (0, t^n)\}$ where $n = \min_i(o(\varphi_i))$. But these are the generators of $(\varphi^*(m_2)\ \mathcal{O}^2_2)\mathcal{O}_1$ as well.

If X, x has several components, then it is clear that M induces a submodule M_V in $\mathcal{O}^p_{V,x}$, V any component of X, x. It is also clear from the definition that $h \in \bar{M}$ iff $h_V \in \bar{M}_V$ for all components V of X, x where h_V is the element of $\mathcal{O}^p_{V,x}$ induced by h. It is also clear from the definition that the integral closure of $\bar{M}$ is $\bar{M}$.

The following generalization of Nakayama's lemma is often useful.

Proposition 1.5 *Suppose N, M are submodules of $\mathcal{O}^p_{X,x}$*
(i) *If $\overline{(m\bar{M} + N)} = \bar{M}$ then $\bar{N} = \bar{M}$*
(ii) *If $\bar{M} \supset m\bar{M} + N \supset M$ then $\bar{N} = \bar{M}$.*

Proof. (i) Let φ: $\mathbb{C}, 0 \to X, x$, then $\varphi^*(m\bar{M} + N)\mathcal{O}_1$

$$= (\varphi^*(mM) + \varphi^*N)\mathcal{O}_1$$

$$= (\varphi^*M)\mathcal{O}_1 .$$

This implies that $\varphi^*(M)\mathcal{O}_1 = \varphi^*(N)\mathcal{O}_1 + m_1\varphi^*(M)\mathcal{O}_1$, hence the result follows form the standard from of Nakayama's lemma.

(ii) We claim $\overline{m\bar{M} + N} = \bar{M}$. We have

$$\varphi^*(M)\mathcal{O}_1 = \varphi^*(\bar{M})\mathcal{O}_1 \supset \varphi^*(m\bar{M} + N)\mathcal{O}_1 \supset (\varphi^*M)\mathcal{O}_1$$

from which the claim follows. □

The link between the integral closure of ideals and modules is very strong, and will allow us to derive the analogues of 1.2. If M is a submodule of $\mathcal{O}^p_{X,x}$, and $[M]$ is a matrix of generators of M, let $J_k(M)$ denote the ideal generated by the $k \times k$ minors of $[M]$. This is the same as the $(p - k)$-th Fitting ideal of $\mathcal{O}^p_x/M$, hence is independent of the choice of generators of M (cf. [4, 10], for a proof). If $h \in \mathcal{O}^p_{X,x}$ let (h, M) denote the submodule generated by h and M.

The following lemma which is a generalization of Cramer's rule is helpful in establishing the connection between $\bar{M}$ and $\overline{J_k(M)}$.

Lemma 1.6 *Suppose $h \in \mathcal{O}^p_{X,x}$, $M \subset \mathcal{O}^p_{X,x}$, $J_{k+1}((h, M)) = 0$, no element of $J_k(M)$ a zero divisor on $\mathcal{O}_{X,x}$. Then*

$$J_k(M)\cdot h \subset M\cdot J_k(h, M) .$$

Proof. Suppose $p = J[A]$, A a $k \times k$ submatrix of $[M]$ the matrix of generators of M. Let h_A be the k-tuple obtained by deleting the elements of h corresponding to the rows deleted from $[M]$. By Cramer's rule

$$ph_A = A\cdot q ,$$

where q is the appropriate k-tuple of elements from $J_k((h_A, A\mathcal{O}^k_x))$.

Let B be the $p \times k$ submatrix of $[M]$ obtained by deleting the same columns from $[M]$ as were deleted to get A. Consider $g = (h - Bq)$. We claim $g = 0$. Suppose not, say $g_i \neq 0$. Let $\tilde{g}$ be obtained from g by deleting the same elements that produced h_A, retaining the i-th element, B_g the submatrix obtained from B by deleting the corresponding rows.

Then $J[\tilde{g}, B_g] = \pm g_i J[A] = 0$. Since $J[A]$ is not a zero-divisor, $g_i = 0$. □

The link beween $\bar{M}$ and $\overline{J_k(M)}$ is established by

Proposition 1.7 *Suppose M is a submodule of $\mathcal{O}^p_{X,x}$, X irreducible, $h \in \mathcal{O}^p_{X,x}$. Then $h \in \bar{M}$ iff $J_k((h, M)) \subset \overline{J_k(M)}$ where k is the largest integer such that $J_k((h, M)) \neq 0$.*

Proof. ⇒ Let $\varphi\colon \mathbb{C}, 0 \dashrightarrow X, x$, then

$$\begin{aligned}\varphi^* J_k((h, M))\mathcal{O}_1 &= J_k(\varphi^*(h, M)\mathcal{O}_1)\\ &= J_k(\varphi^*(M)\mathcal{O}_1)\\ &= \varphi^*(J_k(M))\mathcal{O}_1\end{aligned}$$

hence by the valuative criterion $\overline{J_k((h, M))} = \overline{J_k(M)}$.

⇐ There are two cases.

(i) Suppose $\varphi(t)$ lies in the Z-open subset of X where the rank of (h, M) is k for $t \neq 0$.

(ii) The image of φ lies in $V(J_k((h, M)))$.

Assume we are in (i). We have $\varphi^*(J_k((h, M))) = \varphi^*(J_k(M)) = (p)$. By the previous lemma $p \cdot (h \circ \varphi) \in (\varphi^* M)\mathcal{O}_1 \cdot \varphi^* J_k((h, M))$.

Since $\varphi^*(J_k((h, M))) = (p)$, $h \circ \varphi \in \varphi^*(M)\mathcal{O}_1$.

Assume we are in (ii). Assume in addition that (X, x) is a smooth germ. Suppose $\varphi^*(M) \subsetneq \varphi^*((M, h))$. It must be the case that there exists v_0 such that for all $k > v_0$

$$\varphi^*((M, h))\mathcal{O}_1 \neq \varphi^*(M)\mathcal{O}_1 \bmod m_1^k \mathcal{O}_1^p$$

(If not, by the Artin-Rees theorem, there exists v_0 such that for $l > v_0$

$$m_1^\ell \mathcal{O}_1^p \cap \varphi^*((M, h))\mathcal{O}_1 = m_1^{\ell - v_0}(m_1^{v_0} \mathcal{O}_1^p \cap \varphi^*((h, M)))\,.$$

This would imply $\varphi^*(M)\mathcal{O}_1 + m_1(\varphi^*((M, h))\mathcal{O}_1) = \varphi^*((M, h))\mathcal{O}_1$, so $\varphi^*(M)\mathcal{O}_1 = \varphi^*((M, h))\mathcal{O}_1$, by Nakayama's lemma.)

Truncate φ at level k for $k \gg v_0$, and alter φ by adding higher order terms from m_1^{k+1} to get φ_1 such that $\varphi_1(t)$ is not in $V(J_k((h, M))$ for $t \neq 0$. Then

$$\begin{aligned}(\varphi_1^* M)\mathcal{O}_1 &\equiv \varphi^*(M)\mathcal{O}_1 \bmod m_1^k \mathcal{O}_1^p\\ \varphi_1^*((h, M))\mathcal{O}_1 &\equiv \varphi^*((h, M)) \bmod m_1^k \mathcal{O}_1^p\\ \varphi_1^*((h, M)) &= \varphi_1^*(M) \text{ by case } i\,.\end{aligned}$$

Hence $\varphi^*(M)\mathcal{O}_1 \equiv \varphi^*(h, M)\mathcal{O}_1$, $\bmod m_1^k \mathcal{O}_1^p$ which is a contradiction.

If X is singular, let $(\tilde{X}, \pi)$ be a smooth resolution of X. Since X is irreducible, the complement of $V(J_k((h, M)))$ is dense in X, so $\pi^{-1}(V(J_k(h, M)))$ is nowhere dense in $\tilde{X}$. Consider some lift $\tilde{\varphi}$ of φ to $\tilde{X}$, and approximate $\tilde{\varphi}$ by φ_1 as before. If

$\pi^*(M)\mathcal{O}_{\tilde{X},x'} = \pi^*((M,h))\mathcal{O}_{\tilde{X},x'}$, it follows that $\varphi^*(M)\mathcal{O}_1 = \tilde{\varphi}^*(\pi^*(M))\mathcal{O}_1 = \tilde{\varphi}^*(\pi^*((M,h))\mathcal{O}_1 = \varphi^*(M,h)\mathcal{O}_1$. Thus we may assume $\pi^*(M)\mathcal{O}_{\tilde{X},x'} \neq \pi^*(M,h))\mathcal{O}_{\tilde{X},x'}$, and the argument of (i) finishes the proof, for $\varphi_1^*\pi^*(J_k(M))$ is the same as $\varphi_1^*(\pi^*(J_k((h,M))))$. □

Note that the proof of the implication ⇒ shows that $h \in \bar{M}$ implies $J_k((h,M)) \subset \overline{J_k(M)}$ for all k with no assumptions on X or the rank of M.

If X is not irreducible one obtains:

Corollary 1.8 *Suppose X,x is a complex analytic germ with irreducible components (V_i). Then $h \in \bar{M}$, M a submodule of $\mathcal{O}^p_{X,x}$* iff *$J_{k_i}((h, M_i)) \subset \overline{J_{k_i}(M_i)}$, where M_i is the submodule of $\mathcal{O}^p_{V_i,x}$ induced from M and k_i is the rank of (h,M_i) on V_i. If k_i is independent of i, then $h \in \bar{M}$* iff *$J_{k_i}((h,M)) \subset \overline{J_k(M)}$.*

David Rees in [18] defined the notion of the integral closure of M in $K \otimes_R M$ where R is a Noetherian domain, K its field of fractions, M a finitely generated torsion-free R module. His definition is based on the theory of discrete valuations. However, the previous proposition and Theorem 1.2 of [18] show that $\bar{M}$ in our language is exactly the set of elements of $\mathcal{O}^p_{X,x}$ which are integral over M in Rees's language.

Using 1.7 and 1.8 we now develop the analogues of 1.2 for modules.

Proposition 1.9 *Suppose M is a submodule of $\mathcal{O}^p_{X,x}$, $h \in \mathcal{O}^p_{X,x}$. Then $h \in \bar{M}$* iff *on each component V,x of X,x there exists an ideal $I \subset \mathcal{O}_{V,x}$ such that $I \cdot h \subset I \cdot M$ in $\mathcal{O}^p_{V,x'}$, $I \neq 0$.*

Proof. We have $h \in \bar{M}$ iff $h \in \bar{M}_i$, M_i the induced submodule in $\mathcal{O}^p_{V_i,x}$, so we may assume $\mathcal{O}_{X,x}$ is a domain.

⇒ Suppose $h \in \bar{M}$, then there exists ℓ such that $\overline{(J_k(M))^{\ell+1}} = \overline{J_k(M)^{\ell}} \cdot J_k(M)$ where k is the rank of M.

Then $\overline{J_k(M)^{\ell+1}} \cdot h = \overline{J_k(M)^{\ell}} \cdot J_k(M) \cdot h$

$$\subset \overline{J_k(M)^{\ell}}(M \cdot J_k((h,M))) \qquad \text{(by 1.6)}$$

$$\subset \overline{(J_k(M))^{\ell+1}}M \qquad \text{(by 1.7)}\,.$$

⇐ Suppose there exists an ideal $K, K \neq 0$, such that $K \cdot h \subset K \cdot M$. Fix B a matrix of generators of M. Denote the $k \times k$ minors of $[h, M]$ which involve h by S. Since $k_i S$, $k_i \in K$, is the set of $k \times k$ minors of $[k_i h, M]$ involving $k_i h$, we have $k_i \cdot S \subset K \cdot J_k(M)$. Hence $S \subset \overline{J_k(M)}$ by 1.2 so $h \in \bar{M}$ by 1.7. □

There is one case where we can strengthen 1.9 to get a generalization of 1.2.

Proposition 1.10 *Suppose M is a submodule of $\mathcal{O}^p_{X,x}$, of rank p or O on each component of X, $h \in \mathcal{O}^p_{X,x}$. Then $h \in \bar{M}$* iff *there exists I a faithful submodule of $\mathcal{O}_{X,x}$ such that $I \cdot h \subset IM$.*

Proof. ⇐ follows by the same argument as 1.9.

⇒ Let $I = \overline{J_p(M)^{\ell+1}} + (g_j)$ where (g_j) is the ideal of functions vanishing on those components of X where the rank of the induced module M_i is p, ℓ chosen as in 1.9.

Then $J_p(M)\cdot h \subset M J_p((h, M))$ since Cramer's rule does not need any assumption about zero divisors. Since $h\in \bar{M}$, it follows that the component functions of h vanish on those components of X where the rank of M_i is zero. Hence $I\cdot h \subset I\cdot M$. If $g\in \mathcal{O}_{X,x}$ and $gI = 0$ then $g \equiv 0$ on X. For $g\cdot \overline{J_p(M)^{\ell+1}} = 0$ implies g is zero on all components of X where M_i has rank p, while $g\cdot(g_j) = 0$ implies g is zero on all components of X where M_i has rank 0. Hence I is faithful. □

The problem is that $J_k(M)$ may not be faithful when M_i has different ranks on the different components of X. The above proof fills in a small gap in the proof of 1.5.1 of Teissier [20].

The next proposition contains the growth condition for integral closure in the module case; the version we work with here was suggested by E.J.N. Looijenga. In what follows we let $\Gamma(E)$ denote sections of a vector bundle E.

Proposition 1.11 *Suppose $h\in \mathcal{O}^p_{X,x}$, $M \subset \mathcal{O}^p_{X,x}$ a submodule. Then $h\in \bar{M}$* iff *for each choice of generators $\{s_i\}$ of M there exists a neighborhood U of x such that for all $\varphi\in\Gamma(\mathrm{Hom}(\mathbb{C}^p,\mathbb{C}))$, $\|\varphi(z)\cdot h(z)\| \leqq C\sup_i \|\varphi(z)\cdot s_i(z)\|$ for all $z\in U$.*

Proof. It suffices to prove the result for each component of X so we may assume X is irreducible, and the rank of M is k. Choosing a set of generators $\{s_i\}$ for M gives a set of generators $\{S_i\}$ for $J_k(M)$. Choose a neighborhood U of x, $C > 0$ such that $\|g(z)\| \leqq C\sup_i \|S_i(z)\|$ for $z\in U$, iff $g\in \overline{J_k(M)}$

$\Rightarrow$ Assume $h\in\bar{M}$; for each S_i above we have

$$S_i h = \Sigma s_k a_{ik} \text{ where } a_{ik}\in J_k((h, M))$$

by Lemma 1.6.

Then $\|\varphi(z)\cdot h(z)\| = \|\sum \frac{a_{ik}}{S_i}\varphi(z)\cdot s_k\|$.

Working first at $z\in U - V(J_k(M))$, let $\|S_j(z)\| = \sup_i \|S_i(z)\|$. Then

$$\|\varphi(z)\cdot h(z)\| = \|\Sigma(a_{jk}/S_j)\varphi(z)\cdot s_k\| \leqq \Sigma(\|a_{jk}\|/\|S_j\|)\,\|\varphi(z)\cdot s_k\|$$
$$\leqq CN\sup_k \|\varphi(z)\cdot s_k\|$$

where N is the number of generators of M.

Since the inequality is between continuous functions on U and holds on an open dense subset of U, it holds on U.

Assume the above inequalities hold. Consider the set of sections of $\mathrm{Hom}(\mathbb{C}^p,\mathbb{C})$ defined as follows:

$$\varphi(h) = J[h_\varphi, S_\varphi]$$

where S_φ is some $k\times(k-1)$ submatrix of S, the matrix of generators of M, and h_φ is obtained by deleting the same rows from h as were deleted from S. Then the inequalities involving these φ imply that $J_k((h, M)) \subset \overline{J_k(M)}$ which in turn implies $h\in\bar{M}$. □

We end this section by considering the "sheafification" of our construction. The connection between M and $J_k(M)$ allows us to show that the integral closure gives rise to a coherent sheaf. We prove this using a description of $\bar{M}$ in terms of blowing up.

Proposition 1.12 *Suppose M is a coherent sheaf of submodules of $\mathcal{O}_X^p$ on X a complex analytic set. Then there exists a unique coherent sheaf $\bar{M}$ on X such that for each $x \in X, (\bar{M})_x = \bar{M}_x$ in $\mathcal{O}_{X,x}^p$.*

Proof. Suppose first that M has rank k on each component of X. Consider $NB_{J_k(M)}(X)$, the normalized blow-up of X along $J_k(M)$, with projection map π. By the proof of 1.3.1 of [20], we have that $J_k(h, M) \subset \overline{J_k(M)}$ iff $\pi^* J_k(h, M) \otimes \mathcal{O}_{NB(X)} = \pi^* J_k(M) \otimes \mathcal{O}_{NB(X)}$. Consider the sheaf on $NB(X)$ generated by $\pi^* M$.

Claim. $((\pi_*((\pi^* M) \otimes \mathcal{O}_{NB(X)})) \cap \mathcal{O}_{X,x}^p)_x = \bar{M}_x$.

If $h \in \bar{M}_x$, then $\pi^* J_k(h, M) \otimes \mathcal{O}_{NB} = \pi^* J_k(M) \otimes \mathcal{O}_{NB} = J_k(\pi^* M \otimes \mathcal{O}_{NB})$. Since on $NB(X)$, $J_{k+1}(\pi^* M \otimes \mathcal{O}_{NB}) = \pi^*(J_{k+1}(M) \otimes \mathcal{O}_{NB}) = 0$, by Lemma 1.6 we have $h \circ \pi \in (\pi^* M) \otimes \mathcal{O}_{NB}$ (i.e. at each point of $V((\pi^* J_k(M)) \otimes \mathcal{O}_{NB})$, $\pi^* J_k(M) \otimes \mathcal{O}_{NB}$ is principal, so we can divide.)

If $h \circ \pi \in \pi^* M \otimes \mathcal{O}_{NB}$, then $\pi^*(J_i((h, M))) \otimes \mathcal{O}_{NB}$

$$= J_i(\pi^*((h, M)) \otimes \mathcal{O}_{NB})$$

$$= J_i(\pi^*(M) \otimes \mathcal{O}_{NB}) \, .$$

This implies that the rank of (h, M) is k also, and $J_k(h, M) \subset \overline{J_k(M)}$. The claim then holds by Corollary 1.8. This sheaf is coherent from the properness of π.

In the general case, let V_i denote the union of the components of X on which the rank is i. If $i > 0$, on V_i, by the above construction, we have a coherent sheaf $\bar{M}_i \subset \mathcal{O}_{V_i}^p$ where $\bar{M}_{i,x} = \overline{M_{i,x}}$. (Here $M_{i,x}$ is the submodule of $\mathcal{O}_{V_i}^p$ induced from M_x.) If $i = 0$ set $\bar{M}_0 = 0$.

Let $\bar{\bar{M}}_i$ be the kernel of the sheaf morphism $\mathcal{O}_X^p \to \mathcal{O}_{V_i}^p / \bar{M}_j$, let $\bar{M} = \cap \bar{\bar{M}}_i$, then $h \in (\bar{M})_x$ iff $h \in \overline{(M_{i,x})}$ for all i iff $h \in \bar{M}_x$. □

Based on the above proof we can make the following observation.

Suppose that the rank of M_i is k on each component V_i of X. It follows from the above description of $\bar{M}$, that if $h \in \bar{M}_x$, then there exists a neighborhood U of x, and a representative h of h such that in a neighborhood of each $z \in (U - V(J_k(M))) - \operatorname{Sing} X$, $h(\tilde{z}) = \Sigma a_{i,z}(\tilde{z}) s_i(\tilde{z})$ and $\| a_{i,z}(\tilde{z}) \| \leqq C$, C depending only on U and the generators s_i. This follows because π is an equivalence over $(U - V(J_k(M))) - \operatorname{Sing} X$, and U can be chosen so that these inequalities hold on $\pi^{-1}(U)$.

2 Integral closure of a module and Whitney equisingularity

Given a set of infinitesimal objects around which one hopes to develop an infinitesimal theory of equisingularity, the first task is to prove a theorem of Thom-Levine type.

In such a theory one can associate an infinitesimal object $M(f)$ to each member f of the set of objects considered, and an infinitesimal object to a family F of such objects. If $F(t, z)$ is a family of maps parameterised by t defining a family of objects, then a theorem of Thom-Levine type says $\frac{\partial F}{\partial t} \in M(f)$ iff the family F is equisingular. For example, if the set of objects considered are function-germs, and the equisingularity relation is right equivalence of function-germs at the origin, then $M(f) = m_n J(f)$ while $M(F) = m_n\left(\frac{\partial F}{\partial z_1}, \ldots, \frac{\partial F}{\partial z_n}\right)$ and the theorem of Thom-Levine type is:

Theorem 2.1 *Suppose* $F: \mathbb{C}^{n+k}, 0 \dashrightarrow \mathbb{C}, 0$ *is a k-parameter family of function germs; then the family is right trivial* iff $\frac{\partial F}{\partial u_i} \in m_n\left(\frac{\partial F}{\partial z_j}\right)$ $1 \leqq i \leqq k$, $1 \leqq j \leqq n$.

A theorem of Thom-Levine type shows that the infinitesimal objects of your theory work well at least at the level of unfoldings or deformations. The first theorem of this type that seems to be used as a model for similar theorems is contained in Theorem 12.3 of Levine's notes of Thom's Bonn lectures [23]. This result dealt with $\mathscr{A}$-triviality of a family of map-germs. Other examples are the case of condition C by Teissier in the theory of Whitney equisingularity of hypersurfaces ([21, Sect. 2.5, p. 604] and [22, p. 589 and ff.]) and Mather's work on contact equivalence ([13] see the material around 5.3).

In this section, we prove a theorem of Thom-Levine type for Whitney equisingularity of complex analytic sets, integral closure of modules providing the relevant infinitesimal objects. A first proof of this theorem was inspired by the material of [20, III, Sect. 2.2].

We begin by recalling the following definition of the distance between linear subspaces ([20, III, Sect. 2.1]).

Definition 2.2 Suppose A, B are linear subspaces at the origin in $\mathbb{C}^n$, then

$$\operatorname{dist}(A, B) = \sup_{\substack{u \in B^\perp - \{0\} \\ v \in A - \{0\}}} \frac{|(u, v)|}{\|u\| \, \|v\|}.$$

In the applications B is the "big" space and A the "small" space.

Note that $\operatorname{dist}(A, B)$ is not in general the same as $\operatorname{dist}(B, A)$. If $B' \subset B$, then $\operatorname{dist}(A, B) \leqq \operatorname{dist}(A, B')$ because $B'^\perp \supset B^\perp$.

This allows us to talk about the Whitney conditions holding with a certain exponent.

Definition 2.3 Suppose $\bar{X} \supset Y$, X, Y strata in a stratification of a complex analytic space, and $\operatorname{dist}(TY_0, TX_x) \leqq \operatorname{dist}(x, Y)^e$. Then (X, Y) satisfy Whitney A with exponent e at $0 \in Y$.

(Here Y_0 denotes the smooth points of Y.)

If $e = 1$, and X is semi-analytic, then the hypotheses of Kuo's ratio test are satisfied, and (X, Y) satisfy both of Whitney's conditions at the origin ([K]). In this case, we say (X, Y) are w-regular.

Teissier has shown that this last condition, in the complex analytic case, is necessary as well as sufficient. Coupling the two results gives:

Theorem 2.4 ([20, p. 455]) *Let X be a complex analytic, reduced, purely d dimensional space, Y an analytic subspace of X purely of dimension t, and 0 a smoth point of Y. Then (X_0, Y) satisfy Whitney* A *with exponent* 1 *at the origin* iff *(X_0, Y) are Whitney at the origin; in particular if (X_0, Y) are Whitney at the origin, they are so in a neighborhood of* 0.

Since Teissier also showed that (X_0, Y) Whitney at the origin implies the multiplicity of X constant along Y in some neighborhood of the origin, it follows that $0 \in S(X)$ implies $Y \subset S(X)$. Further, if V is a component of X at the origin, then the pair (V_0, Y) is also Whitney at 0, so $Y \subset V$.

We can now prove the Thom-Levine type theorem for this context:

Theorem 2.5 *Let X, Y be as above, $F\colon \mathbb{C}^t \times \mathbb{C}^N \to \mathbb{C}^p$, 0 coordinates chosen so that $\mathbb{C}^t \times \{0\} = Y$, F defines X with reduced structure. Then $\frac{\partial F}{\partial s} \in \overline{\left\{ z_i \frac{\partial F}{\partial z_j} \right\}}_{\mathcal{O}_X^p}$ for all tangent vectors $\partial/\partial s$ to $\mathbb{C}^t \times \{0\}$ iff (X_0, Y) are Whitney.*

Proof. $\Rightarrow$ We are going to show that Whitney A holds with exponent 1 at 0. We do this by finding a t-dimensional subspace of TX_x which will converge to $\mathbb{C}^t \times \{0\}$ at the correct speed.

Our hypotheses imply that the module $\left\{ z_i \frac{\partial F}{\partial z_j} \right\}$ has rank $N + t - d$ on each component of X, and

$$V\left(J_{N+t-d}\left(z_i \frac{\partial F}{\partial z_j} \right) \right) \supset S(X)\,.$$

By the observation at the end of the last section, there exists a neighborhood U of 0 such that in a neighborhood of $z \in \left(U - V\left(J_{N+t-d}\left(z_i \frac{\partial F}{\partial z_j} \right) \right) \right)$

$$\frac{\partial F}{\partial s_k}(\tilde{z}) = \sum_{i,j} a^k_{i,j,z}(\tilde{z}) \tilde{z}_j \frac{\partial F}{\partial z_i}(\tilde{z}) \quad \text{and} \quad \| a^k_{i,j,z}(\tilde{z}) \| \leqq C_k,\ C_k \text{ independent of } z\,.$$

This implies that $V_{z,k}$ is tangent to TX_z where

$$V_{z,k} = \frac{\partial}{\partial s_k} - \sum_{i,j} a^k_{i,j,z}(z) z_j \frac{\partial}{\partial z_i}\,.$$

Let S_z be the t-dimensional space determined by $(V_{z,k})$. The set of vectors orthogonal to S_z is spanned by $\left\{ \frac{\partial}{\partial z_i} + \sum_{kj} \bar{a}^k_{i,j,z} \bar{z}_j \frac{\partial}{\partial s_k} \right\}$.

These are linearly independent and there are N of them. Then $\operatorname{dist}(\mathbb{C}^t \times \{0\}, S_z) \leqq C \sup_i \| \bar{z}_i \| \leqq C \operatorname{dist}(z, \mathbb{C}^t)$ where $C = \sup_k (C_k)$. Hence, $\operatorname{dist}(\mathbb{C}^t \times \{0\},\ TX_z) \leqq C \operatorname{dist}(z, \mathbb{C}^t)$.

$\Leftarrow$ Assume $(X_0, \mathbb{C}^t \times \{0\})$ are Whitney at the origin in $\mathbb{C}^t \times \mathbb{C}^N$.

Let $\pi\colon \mathbb{C}^t \times \mathbb{C}^N \to \mathbb{C}^t$ be projection onto $\mathbb{C}^t$. Since Whitney A holds at 0 with exponent 1, there exists a neighborhood U of the origin such that for all $z \in U \cap X_0$

$$\operatorname{dist}(\mathbb{C}^t \times \{0\},\ TX_z) \leqq C \operatorname{dist}(z, \mathbb{C}^t) < 1\,.$$

This implies that $\pi | TX_z$ is a submersion, hence $\pi^{-1}(0) \cap TX_z$ is a linear space of codimension t.

Let V_z denote the vector in TX_z orthogonal to $\pi^{-1}(0) \cap TX_z$ which projects to $\partial/\partial S_i$. A basis for the set of vectors orthogonal to V_z is given by a basis of

$\pi^{-1}(0) \cap TX_z$, a basis of those vectors orthogonal to TX_z and $\{e_j\} \subset \mathbb{C}^t$, $j \neq i$. Let ℓ_z denote the line determined by V_z, ℓ_i the line determined by $\partial/\partial S_i$. The above basis for vectors orthogonal to ℓ_z implies

$$\operatorname{dist}(\ell_i, \ell_z) = \operatorname{dist}(\ell_i, TX_z) \leqq \operatorname{dist}(\mathbb{C}^t, TX_z) \leqq C \operatorname{dist}(z, \mathbb{C}^t) .$$

Another basis for the vectors orthogonal to ℓ_z is $\{w_j\}$, w_j being the vector obtained from V_z by replacing 1 with $-\bar{V}_j$, V_j with 1, and the other entries with zero.

The above inequalities then imply that

$$\sup_i \| V_i \| \leqq C \operatorname{dist}(z, \mathbb{C}^t) \quad z \in U \cap X_0 .$$

Since V_z is a tangent vector we have $DF(z) \cdot V_z = 0$ which implies $\frac{\partial F}{\partial s_i}(z) = \sum_j - V_j \frac{\partial F}{\partial z_i}(z)$. Let $\varphi(z) \in \operatorname{Hom}(\mathbb{C}^p, \mathbb{C})$, then

$$\begin{aligned}\left\| \varphi(z) \cdot \frac{\partial F}{\partial S_i}(z) \right\| &= \left\| \sum_j - V_j \varphi(z) \cdot \frac{\partial F}{\partial z_i}(z) \right\| \\ &\leqq CN \operatorname{dist}(z, \mathbb{C}^t) \sup_i \left\| \varphi(z) \cdot \frac{\partial F}{\partial z_i} \right\| \\ &= CN \sup_{i,j} \left\| \varphi(z) \cdot z_i \frac{\partial F}{\partial z_i} \right\| \end{aligned}$$

hence $\frac{\partial F}{\partial s_i} \in \overline{\left\{ z_i \frac{\partial F}{\partial z_j} \right\}_{\mathcal{O}_x^p}}$ by 1.11. □

One of the keys to Teissier's work on Whitney equisingularity is his idealistic Bertini theorems ([20, II, Sect. 2]). Before moving to the applications of the next section we show these can be reformulated in terms of integral closure of modules.

We first describe the Bertini theorem with section. Suppose X is an analytic subspace of $\mathbb{C}^S \times \mathbb{C}^N$ with coordinates (t, z), containing $\mathbb{C}^S \times \{0\}$. Assume that the fibres of the projection of X to $\mathbb{C}^S$ are smooth of dimension d off a nowhere dense analytic subset of X. Suppose X is defined in an open U of $\mathbb{C}^S \times \mathbb{C}^N$ by an ideal generated by global sections $G_1, \ldots, G_p \in H^0(U, \mathcal{O}_{\mathbb{C}^S \times \mathbb{C}^N})$. If I and K are subsets of $\{1, 2, \ldots, N\}$ and $\{1, \ldots, p\}$ of i and k elements respectively, let $DG_{K,I}$ denote the submatrix of $D_z G$ with rows given by K, columns by I, G_K the map germ with component functions given by K.

Theorem 2.6 ([20, p. 372]) *For each choice of coordinates $(t_1, \ldots, t_S)$ and $(u_1, \ldots, u_N)$ on $\mathbb{C}^S$ and $\mathbb{C}^N$ respectively, for each integer ℓ, $0 \leqq \ell \leqq S$ and for each choice of subset I with $i = N - d - \ell$, there exists a nowhere dense analytically closed subset F of $\mathbb{C}^S \times \{0\}$ such that at each point $z \in \mathbb{C}^S \times \{0\} - F$ the images in $\mathcal{O}_{X,z}$ of the elements of the form*

$$J\left[\frac{\partial G_K}{\partial t_{j_1}}, \ldots, \frac{\partial G_K}{\partial t_{j_\ell}}, DG_{K,I}\right]$$

are integral in $\mathcal{O}_{X,z}$ *over the ideal generated by the images of*

$$J\left[u_{k_1}\frac{\partial G_K}{\partial u_{k_1}}, \ldots, u_{k_\ell}\frac{\partial G}{\partial u_{k_\ell}}, DG_{K,I}\right]$$

for any $K \subset \{1, \ldots, p\}$ *with* $N - d$ *elements,*

$$\{k_1, \ldots, k_\ell\} \subset \{1, \ldots, N\}, \{j_1, \ldots, j_\ell\} \subset \{1, \ldots, S\}.$$

Using only the case where $\ell = 1$ *we can derive as a corollary of this:*

Theorem 2.7 *Suppose* X, G *as above, then there exists a closed analytic set* F *of* $\mathbb{C}^S \times \{0\}$ *such that at each point* $z \in \mathbb{C}^S \times \{0\} - F$

$$\frac{\partial G}{\partial t_\ell} \in \overline{\left\{z_i\frac{\partial G}{\partial z_i}\right\}_{\mathcal{O}_{X,z}}}.$$

Proof. By the above theorem elements of the form

$$z_{n_1}\cdot \ldots \cdot z_{n_i} J\left[\frac{\partial G_K}{\partial t_\ell}, DG_{K,I}\right], n_j \in I$$

are integral over the ideal generated by $z_{n_1}\cdot \ldots \cdot z_{n_i} J\left[z_j\frac{\partial G_K}{\partial z_j}, DG_{K,I}\right]$ off some closed nowhere dense F_I. This implies that elements of the form

$$J\left[\frac{\partial G_K}{\partial t_\ell}, z_{n_1}\frac{\partial G_K}{\partial z_{n_1}}, \ldots, z_{n_i}\frac{\partial G_K}{\partial z_{n_i}}\right]$$

are integral over the ideal generated by $J\left[z_j\frac{\partial G_K}{\partial z_j}, z_{n_1}\frac{\partial G_K}{\partial z_{n_1}}, \ldots, z_{n_i}\frac{\partial G}{\partial z_{n_i}}\right]$ off F_I. Since there are only finitely many choices of I we get

$$J_k\left(\frac{\partial G}{\partial t_\ell}, \left\{z_i\frac{\partial G}{\partial z_i}\right\}\right) \subset \overline{J_k\left(z_i\frac{\partial G}{\partial z_i}\right)_z}, k = N - d$$

$$z \notin \bigcup_I F_I.$$

By 1.8 we get $\frac{\partial G}{\partial t_\ell} \in \overline{\left\{z_i\frac{\partial G}{\partial z_i}\right\}_{\mathcal{O}_{X,z}}}$, $z \notin F, F = \bigcup_I F_I$. □

We next show that if the conclusion of the above theorem holds, then the conclusion of the idealistic Bertini theorem with section holds.

Theorem 2.8 *Suppose* X, G *as above, and suppose that there exists a closed nowhere dense analytic set* F *of* $\mathbb{C}^S \times \{0\}$ *such that at each point* $z \in \mathbb{C}^S \times \{0\} - F$

$$\frac{\partial G}{\partial t_\ell} \in \overline{\left\{z_i\frac{\partial G}{\partial z_i}\right\}_{\mathcal{O}_{X,z}}} \qquad 1 \leqq \ell \leqq S.$$

Then the conclusion of 2.6 *holds for such* z.

Proof. Suppose φ: $\mathbb{C}, 0_- \to X, z$ is a path. Consider

$$\varphi^*\left(J\left[\frac{\partial G_K}{\partial t_{j_1}}, \ldots, \frac{\partial G_K}{\partial t_{j_\ell}}, DG_{K,I}\right]\right) = J$$

$$= \left[\sum_i a_{j_1,i}(t)\left(u_i \frac{\partial G_K}{\partial u_i}\right) \circ \varphi, \ldots, \sum_i a_{j_\ell,i}(t)\left(u_i \frac{\partial G_K}{\partial u_i}\right) \circ \varphi, DG_{K,I} \circ \varphi\right].$$

Clearly these elements lie in the ideal generated by

$$J\left[\left(u_{i_1} \frac{\partial G_K}{\partial u_{i_1}}\right) \circ \varphi, \ldots, \left(u_{i_\ell} \frac{\partial G_K}{\partial u_{i_\ell}}\right) \circ \varphi, DG_{K,I} \circ \varphi\right].$$

Since φ is arbitrary this implies the desired result. □

It is worth noting that 2.5 and 2.7 together imply that X_0 is Whitney over $\mathbb{C}^S \times 0 - F$ which is one of the key steps in proving the existence of Whitney stratifications. (In fact they show that one has generically Whitney with exponent 1.)

Teissier also proves a Bertini theorem for X without section, that is without assuming $\mathbb{C}^S \times \{0\} \subset X$.

Theorem 2.9 *Suppose $X \subset \mathbb{C}^S \times \mathbb{C}^N$, π: $\mathbb{C}^S \times \mathbb{C}^N \to \mathbb{C}^S$ projection on the $\mathbb{C}^S$ factor. Suppose $\pi|X$ has the same properties as in 2.6.*

Then there exists a complex analytic $B \subset X$ such that $\pi(B)$ has measure zero in $\mathbb{C}^S$ and for $z \in X - B$, elements of the form $J\left[\frac{\partial G_K}{\partial t_{j_1}}, \ldots, \frac{\partial G_K}{\partial t_{j_\ell}}, DG_{K,I}\right]$ are integral over the ideal in $\mathcal{O}_{X,z}$ generated by elements of the form $J[DG_{K,I'}]$ where $K \subset \{1, \ldots, p\}$ with $N - d$ elements, $I \subset \{1, \ldots, N\}$ fixed with $N - d - \ell$ elements, $I' \supset I$ with $N - d$ elements.

Proof. See [20, p. 375]. □

The translation of this result is straightforward.

Theorem 2.10 *Suppose X, π as above, then there exists $B \subset X$ such that $\pi(B)$ has measure zero in $\mathbb{C}^s$ and for $z \in X - B$, $\frac{\partial G}{\partial t_i} \in \overline{\left\{\frac{\partial G}{\partial u_j}\right\}}$ $1 \leqq i \leqq t$, $1 \leqq j \leqq N$.*

3 WV-equivalence, finite determinancy and Newton polygons

In this section we describe some applications of the theorems of the previous section. We first state the equivalence relation on map-germs that we study here. If f: $K^n, 0 \dashrightarrow K^p, 0$, $K = \mathbb{R}$ on $\mathbb{C}$, we denote $f^{-1}(0)$ by X_f.

Definition 3.1 Suppose f_i: $K^n, 0 \dashrightarrow K^p, 0$, $i = 0, 1$. We say f_0 and f_1 are *WV equivalent* if there exists an open set $U \subset K$, $[0, 1] \subset U$, and a map-germ F: $U \times K^n, U \times 0 \dashrightarrow K^p, 0$ such that $F(j, -) = f_j$, $j = 0, 1$, and $(X_F - U, U)$ are Whitney regular along U.

By Thom's first isotopy lemma this implies that there exists a homeomorphism of $K^n, 0$ to itself taking X_{f_0} to X_{f_1}.

Recall that two map-germs f_i: $\mathbb{C}^n, 0 \dashrightarrow \mathbb{C}^p, 0$ are contact equivalent if there exist map-germs H: $\mathbb{C}^n, 0 \dashrightarrow \mathrm{GL}(p), I$ and $r \in \mathscr{R}$, such that $(H \cdot f_0) \circ r = f_1$. If M and N are submodules of $\mathcal{O}_n^p$, then we can speak of M and N and $\mathcal{O}_n^p/M$ and $\mathcal{O}_n^p/N$ being contact equivalent as well. Our first lemma shows that integral closure is well behaved with respect to contact equivalence. We denote the pair (H, r) by A.

Recall that $T\mathscr{K}(f) = \left\{ x_i \dfrac{\partial f}{\partial x_j}, f_i e_j \right\} \mathcal{O}_n$..

Lemma 3.2 $\overline{T\mathscr{K}(A \cdot f)} = A \cdot \overline{T\mathscr{K}(f)}$.

Proof. It is easy to check that $T\mathscr{K}(A \cdot f) = A \cdot T\mathscr{K}(f)$ so it suffices to show $\overline{A \cdot T\mathscr{K}(f)} = A \cdot \overline{T\mathscr{K}(f)}$.

Suppose $h \in A \cdot \overline{T\mathscr{K}(f)}$. Let φ: $\mathbb{C}, 0 \dashrightarrow \mathbb{C}^n, 0$ be a path. Then $h \circ \varphi = (A \cdot h_1) \circ \varphi = (H \cdot h_1 \circ r) \circ \varphi$

$$= H \circ r \circ \varphi \cdot ((Df \circ r \circ \varphi) \cdot \xi_1 + B[f \circ r \circ \varphi]) .$$

This implies that $\varphi^*(A \cdot h_1)$ lies in $\varphi^*(A \cdot T\mathscr{K}(f))$, hence $h = A \cdot h_1 \in \overline{A \cdot T\mathscr{K}(f)}$.

If we assume $h \in \overline{A \cdot T\mathscr{K}(f)}$, consider $h_1 = (H^{-1} \circ r, r^{-1}) \cdot h$. An analysis similar to the above shows that $h_1 \in \overline{T\mathscr{K}(f)}$, hence $h = A \cdot h_1$ lies in $A \cdot \overline{T\mathscr{K}(f)}$. □

Proposition 3.3 *Suppose* f_i: $\mathbb{C}^n, 0 \dashrightarrow \mathbb{C}^p, 0$ $i = 0, 1$ *are two map-germs such that* $\overline{T\mathscr{K}(f_0)} = \overline{T\mathscr{K}(f_1)}$. *Then* $(X_F - S)$ (X_F), $\mathbb{C} \times 0)$ *is Whitney regular at* $(0, 0)$, $(1, 0)$ *where* $F(t, z) = f_0(z) + t(f_1(z) - f_0(z))$.

Proof. Let $G(t, z) \equiv f_0(z)$. It is clear that

$$\overline{T\mathscr{K}(f)\mathcal{O}_{n+1_{(t,0)}}} = \overline{\left\{ x_i \frac{\partial G}{\partial x_j}, G^i e_j \right\}_{(t,0)}} \equiv \bar{B}_t \text{ for all } t .$$

We claim $\bar{B}_t \equiv \overline{\left\{ x_i \dfrac{\partial F}{\partial x_j}, F_i e_j \right\}_{(t,0)}} \equiv \bar{A}_t$ for $t = 0, 1$. Consider the case where $t = 0$. We have

$$\bar{B}_0 \supset m_{n+1} \bar{B}_0 + A_0 \supset B_0 ,$$

hence by 1.5 (ii) we have $\bar{B}_0 = \bar{A}_0$.

The result for $t = 1$ is similar; rewrite F as $f_1 + (t - 1)(f_1 - f_0)$ and use the inclusion $\overline{T\mathscr{K}(f_0)} \subset \overline{T\mathscr{K}(f_1)}$.

Since $\partial F/\partial t = (f_1 - f_0) \in B_0, B_1$ it follows that

$$\frac{\partial F}{\partial t} \in \overline{\left\{ x_i \frac{\partial F}{\partial x_j}, F^i e_j \right\}_{(t,0)}} \quad \text{where } t = 0, 1 .$$

But this clearly implies

$$\frac{\partial F}{\partial t} \in \overline{\left\{ x_i \frac{\partial F}{\partial x_j} \right\}_{\mathcal{O}_{X_F,(t,0)}}}$$

where $t = 0, 1$.

The result then follows from 2.5. □

As a corollary of the method of proof we obtain:

Proposition 3.4 *Suppose* $F: \mathbb{C} \times \mathbb{C}^n, \mathbb{C} \times 0 \to \mathbb{C}^p, 0$ *where* $F(t, z) = (f_0(z) + th(z))$, *and suppose* $\overline{T\mathscr{K}(f_t)} = \overline{T\mathscr{K}(f_0)}$ *for all* $t \in \mathbb{C}$. *Then the pair* $(X_F - S(X_F), \mathbb{C} \times 0)$ *is Whitney regular along* $\mathbb{C} \times 0$.

If $\overline{T\mathscr{K}(f)} \supset m_n^k \mathcal{O}_n^p$ we say f has finite singularity type. This is equivalent, over $\mathbb{C}$, to saying that the restriction of f to its critical set is a finite map. If $n \geqq p$ this is equivalent to saying that X_f is an ICIS singularity.

Corollary 3.5 *Suppose* f *and* g *in* $\mathcal{O}_n^p$ *have finite singularity type and* $\mathcal{O}_n^p/\overline{T\mathscr{K}(f)}$, $\mathcal{O}_n^p/\overline{T\mathscr{K}(g)}$ *are contact equivalent. Then* f *and* g *are* WV *equivalent.*

Proof. The hypothesis implies that $\overline{T\mathscr{K}(f)}$ and $\overline{T\mathscr{K}(g)}$ are contact equivalent; by the previous Proposition 3.2 there exists f_1 contact equivalent with f such that $\overline{T\mathscr{K}(f_1)} = \overline{T\mathscr{K}(g)}$.

We claim f_1 and g are WV equivalent.

The proof of 3.3 shows that the stalks of $\dfrac{\mathcal{O}_{n+1}^p}{\overline{T\mathscr{K}(f_1)\mathcal{O}_{n+1}}}$ are the same as the stalks of $\dfrac{\mathcal{O}_{n+1}^p}{\overline{\left\{x_i \dfrac{\partial F}{\partial x_j}, F^i e_j\right\}}}$ on $\mathbb{C} \times \{0\}$ except at isolated points not including $(0, 0)$, $(1, 0)$, where F is the linear deformation between f_1 and g.

Since $\overline{T\mathscr{K}(f_1)\mathcal{O}_{n+1}} \supset m_n^k \mathcal{O}_{n+1}^p$ for some k, it follows that in a neighbourhood of all but a discrete set of points of $\mathbb{C} \times \{0\}$ the support of $\dfrac{\mathcal{O}_{n+1}^p}{\overline{\left\{x_i \dfrac{\partial F}{\partial x_j}, F^i e_j\right\}}}$ is $\mathbb{C} \times \{0\}$.

If $n < p$, this support is $F^{-1}(0)$. If $n \geqq p$ the support includes $S(X_F)$. Thus 3.3 implies that $(X_F - (\mathbb{C} \times \{0\}), \mathbb{C} \times \{0\})$ is Whitney except at a discrete set of points not including 0, 1. We can then map a disk of radius 2 in $\mathbb{C}$ into $\mathbb{C} \times \{0\}$ in such a way that 0 and 1 are sent to $(0, 0)$, $(1, 0)$ but the disk misses the discrete set of bad points. The induced deformation shows f_1 and g are WV equivalent. Since the set of contact equivalences are connected we map our disk of radius 2 smoothly into $\mathscr{R}$, sending 1 to id and 0 to r_0 such that $X_{f_1 \circ r_0} = X_f$. Then $F \circ (u, r_u)$ gives a deformation between X_f and X_g with the desired properties. □

Teissier proved the above theorem for the hypersurface case ($p = 1$) in [21]. (In fact, in this case it suffices for $\dfrac{\mathcal{O}_n}{T\mathscr{K}_e(f)} \approx \dfrac{\mathcal{O}_n}{T\mathscr{K}_e(g)}$ since $f \in \overline{m_n J(f)}$.)) Smooth analogues of the above result can be found in [7].

We also obtain the following estimate of WV-determinacy which is the precise analogue of Mather's [13].

Corollary 3.6 *Suppose* $\overline{T\mathscr{K}(f)} \supseteq m_n^k \mathcal{O}_n^p$, $g \equiv f \bmod m_n^{k+1} \mathcal{O}_n^p$. *Then* f *and* g *are* WV-*equivalent.*

Proof. Let $f_t = f + t(g - f)$, we claim $\overline{T\mathscr{K}(f)} = \overline{T\mathscr{K}(f_t)}$ for all t.

We have

$$x_i\left(\frac{\partial f_t}{\partial x_j} - \frac{\partial f}{\partial x_j}\right) \in m_n^{k+1}\mathcal{O}_n^p \text{ for all } t\,.$$

Hence

$$\overline{T\mathcal{K}(f)} \supset m_n\overline{T\mathcal{K}(f)} + T\mathcal{K}(f_t) \supset T\mathcal{K}(f)$$

so by 1.5 the claim holds. As in the proof of 3.5, $S(X_f) = \mathbb{C}\times\{0\}$, and the result follows by 3.4. □

One important case where a large piece of $\overline{\left\{x_i\frac{\partial f}{\partial x_j}\right\}_{\mathcal{O}_{X_f}}}$ can be computed is when f is Newton non-degenerate, a condition which we now describe.

The Newton polyhedron, $\Gamma+(g)$, where $g\in\mathcal{O}_n$ is the convex hull of $\{\alpha + \mathbb{R}_t^n \mid C_\alpha \neq 0\}$, C_α a coefficient in the Taylor expansion of g. The Newton boundary of g, $\Gamma(g)$ is the union of the compact faces of $\Gamma_+(g)$. Given $v\in\mathbb{R}_t^n\backslash 0$, we let Δ_v be the face of $\Gamma(g)$ on which the inner product with v assumes its minimum values. Let g_{Δ_v} denote the sum of the terms in the Taylor expansion of g with indices in Δ_v.

If f: $\mathbb{C}^n, 0 \dashrightarrow \mathbb{C}^p, 0$, then we associate p polyhedrons to f, namely $\Gamma(f_i)$. We assume f_i is commode for all i, which means that $\Gamma(f_i)$ intersects each coordinate axis. By f_{Δ_v} we mean $(f_{i\Delta_v})$.

Definition 3.7 We say that f is *Newton non-degenerate* if for each $v\in\mathbb{R}_t^n\backslash\{0\}$, there are no solutions in $(\mathbb{C}^*)^n$ to the system of equations

$$f_{i\Delta_v} = 0,\ 1 < i < p,\ (w, x_i\,\partial/\partial x_i(f)_{\Delta_v})) = 0$$

for some $w \neq 0$.

This definition is due to Khovanskii [26].

Theorem 3.8 *Suppose* f: $\mathbb{C}^n, 0 \dashrightarrow \mathbb{C}^p, 0$, $n > p$, f *defines an* ICIS *singularity. Suppose* f *is Newton non-degenerate and commode in the given coordinate system.*

(A) *Suppose* F: $\mathbb{C}\times\mathbb{C}^n, 0 \dashrightarrow \mathbb{C}^p, 0$ *is a deformation of* f *by terms above or on the Newton boundary of* f. *Then*

$$((X_F \cap \mathbb{C}\times\mathbb{C}^I) - \mathbb{C}\times\{0\}, \mathbb{C}\times\{0\})$$

is Whitney where $\mathbb{C}^I$ *is any coordinate plane.*

(B) *Suppose* F *is a deformation of* f *by terms above the Newton boundary of* f. *Then* $f_t = F(t, -))$ *is* WV-*equivalent to* f *for all* t.

Proof. We prove part (A) and obtain (B) as a corollary.

Suppose φ: $\mathbb{C}$, $0 \dashrightarrow X_F, 0$. Let $\mathbb{C}^I$ be the smallest dimensional coordinate plane that contains the image of $\pi_n\circ\varphi = (\varphi_1, \ldots, \varphi_n)$. The initial terms of $\pi_n\circ\varphi$ at 0 define a curve φ_{in} on $\mathbb{C}^n$ which necessarily lies in $\mathbb{C}^I$ and no smaller coordinate plane. The map φ_{in} defines a covector $p = (p_1, \ldots, p_n)$ in $\mathbb{C}^n$; p_i is the exponent of the initial term of the i-th component function of φ_{in}. If the i-th term of φ_{in} is zero, let $p_i = \infty$, or some arbitrarily high number.

We first describe the submodule of $\mathcal{O}_1^p$ generated by $\varphi_{\text{in}}^*\left\{x_i\frac{\partial f_{\Delta_p}}{\partial x_i}\right\}$, $i\in I$. To do this we note several items.

We have $f_{\Delta_p} \circ \varphi_{\text{in}} = 0$, so $\operatorname{im} \varphi_{\text{in}}$ lies in $V(f_{\Delta_p})$. This follows because the hypothesis on F implies that the lowest order terms of $F \circ \varphi$ are exactly $f_{\Delta_p} \circ \varphi_{\text{in}}$; since $F \circ \varphi = 0$, we have $f_{\Delta_p} \circ \varphi_{\text{in}} = 0$.

We also have that $f_{\Delta_p} | \mathbb{C}^I$ is Newton non-degenerate. For suppose the equations

$$\left\{ f_{j\Delta_p} = 0, \left(w, x_i \frac{\partial f_{\Delta_p}}{\partial x_i} \right) = 0 | 1 \leqq j \leqq p, i \in I \right\}$$

have a solution in $(\mathbb{C}^I)^*$. Since $f_{j\Delta_p}$, $\left(w, x_i \frac{\partial f_{\Delta_p}}{\partial x_i} \right)$ are independent of the complementary variables of $\mathbb{C}^I$, and $x_j \frac{\partial f_{\Delta_p}}{\partial x_i} = 0$ for j a complementary index of I, a solution to $f_{j,\Delta_p} = 0$, $\left(w, x_i \frac{\partial f_{\Delta_p}}{\partial x_i} \right) = 0$ on $(\mathbb{C}^I)^*$ would give a solution on $(\mathbb{C}^*)^n$.

This means that if $\mathbb{C}^I$ is a coordinate plane of dimension p or less, then

$$(\mathbb{C} \times \mathbb{C}^I) \cap X_F = \mathbb{C} \times \{0\} .$$

Finally $(f | \mathbb{C}^I)_{\Delta p} = f_{\Delta p} | \mathbb{C}^I = (f | \mathbb{C}^{I'})_{\Delta p}$ where $\mathbb{C}^I \subset \mathbb{C}^{I'}$.

This relation, though obvious, is what is needed to prove the theorem simultaneously for all the coordinate plane sections.

By the second remark, the matrix $\left[\varphi_{\text{in}, i} \frac{\partial f_{\Delta_p}}{\partial x_i} \circ \varphi_{\text{in}} \right]$, $i \in I$, has generic rank p, and the order of each entry in the j-th row is the value of $p | \Delta_p(f_j) = q_j$.

This implies that by column operations the matrix can be reduced to $[Q, O]$ where $q_{j,j} = q_j$ and the off diagonal entries of Q are zero.

We claim that $\varphi^* \left(x_i \frac{\partial F}{\partial x_i} \right) \mathcal{O}_1$ is exactly the submodule of $\mathcal{O}_1^p$ generated by the columns of Q.

This follows because the lowest order terms of $x_i \frac{\partial F}{\partial x_i} \circ \varphi$ are exactly $\varphi_{\text{in}, i} \frac{\partial f_{\Delta p}}{\partial x_i} \circ \varphi_{\text{in},}$ so Nakayama's lemma gives the desired result. Thus,

$$\frac{\partial F}{\partial t} \in \overline{\left\{ x_i \frac{\partial F}{\partial x_i} \right\}}_{\mathcal{O}_{X_F}} \quad \text{and} \quad \frac{\partial F_I}{\partial t} \in \overline{\left\{ x_i \frac{\partial F}{\partial x_i} \right\}}_{\mathcal{O}_{X_F \cap \mathbb{C}^I}} \qquad i \in I .$$

By 2.5 (A) folows.

To prove (B) we can apply the result of (A) to the germ of F_t at $(t, 0)$ for all t. The hypothesis on F implies that f_t will be Newton non-degenerate, $\Gamma(f_{t,i})$ being constant. □

From the above proof it is clear that any h above or on Newton boundary of f lies in $\overline{\left\{ x_i \frac{\partial f}{\partial x_i} \right\}}_{\mathcal{O}_{X_F}}$.

Related work in this area has been done by J. Damon ([2]) and M. Oka ([17]). Damon, using his filtration based technique of constructing vector fields, proved that any deformation of f by terms on or above the Newton boundary of f is topologically trivial.

Oka showed in [17] that such deformations have a simultaneous resolution. By results of Teissier ([20]) this implies that the deformations are Whitney equisingular along the parameter space.

4 The real analytic and C^∞ cases

Much of the theory of the preceeding sections can be extended to the real analytic case, and some to the C^∞ case.

To start with one needs a definition of the integral closure $\bar{I}_{\mathbb{R}}$ of an ideal which ignores complex phenomena. In such a theory one would have $h \in \bar{I}_{\mathbb{R}}$ iff $|h(x)| \leqq C \sup|f_i(x)|$ for all x in a neighborhood of the origin iff for all φ: $\mathbb{R}, 0 - \to \mathbb{R}^n, 0$, $h \circ \varphi \in (\varphi^* I)A_1$. (Here A_n refers to analytic germs in n variables at the origin, while C_n refers to C^∞ germs. The term $\mathcal{O}_n$ will be reserved for holomorphic germs.)

Unfortunately, the algebraic definition of the integral closure of an ideal (Definition 1.1) gives a theory sensitive to complex phenomena. For, if $h^n + \Sigma a_i h^i = 0$ holds on $\mathbb{R}^k$ with $a_i \in I^{n-i}$, then it holds on $\mathbb{C}^k$ as well, so h lies in $\bar{I}$. However, if $I = (x^2 + y^2)$ for example, we expect $\bar{I}_{\mathbb{R}}$ to be $m_2^2 \neq \bar{I}$. Work of Robson shows that a good algebraic definition can be given using monic polynomial inequalities (cf. [28, Proposition 8]).

However, in order to develop the real analytic theory simultaneously for ideals and modules in a way parallel to the complex analytic case, we again adopt the valuative criterion as a definition.

Definition 4.1 Suppose M is a submodule of $A^p_{X,x}$, X a real analytic set. Then the *real integral closure of* M, denoted $\bar{M}_{\mathbb{R}}$ in $A^p_{X,x}$ is the set of h such that for all analytic φ: $\mathbb{R}, 0 - \to X, x$, we have $h \circ \varphi \in (\varphi^* M)A_1$.

With this definition it is clear that $\overline{(x^2 + y^2)_{\mathbb{R}}} = m_2^2$. If M is an ideal, then the real integral closure of M coincides with Robson's notion of the complete hull of I ([28]). When M is a submodule of $C^p_{X,x}$, X a real analytic set, we still only use real analytic paths to check for $\bar{M}_{\mathbb{R}}$. Although this definition is likely to change as the theory of the C^∞ case is developed, it suffices for the results of this section.

Hironaka's work on the resolution of singularities provides a very nice point of view for studying $\bar{I}_{\mathbb{R}}$. Given a real analytic germ X, x and an ideal $I \subset A_{X,x}$, we can find a desingularization $(\tilde{X}, \pi)$ of (X, x) such that $\tilde{I} = (\pi^* I)A_{\tilde{X}}$ is simple; this means that at each point $\tilde{x}$ of $V(\tilde{I})$, $\tilde{I}_x$ is $(z_1^{a_1} \cdot \ldots \cdot n_n^{a_n})$, a_i non-negative integers, z_i local coordinates on $\tilde{X}$ at $\tilde{x}$. (This is the content of Desingularization I and II in Sect. 5 of [8]).

The desingularization of a singular real analytic space is constructed as follows. A real analytic space has a canonical filtration by real analytic subspaces $X^{(i)}$, $X^{(0)} = X$ and $X^{(i+1)} \supset S(X^{(i)})$; each $X^{(i)}$ is resolved by $\tilde{X}^{(i)}$ using Hironaka's resolution of singularities theorem. The desingularization is then a disjoint union of $\tilde{X}^{(i)}$. (For details see Remark 5.8.1 and the material after 5.8.2 of [8]).

If X is the Whitney umbrella with equation $X_3^2 - X_1^2 X_2 = 0$ then $X^{(0)} = X, \tilde{X}^{(0)} = \mathbb{R}^2$, $X^{(1)} = \tilde{X}^{(1)} = X_2$-axis. The problems of the real case arise when the smooth points of X are not dense in X in the metric topology.

Using these notions we can prove:

Proposition 4.2 *Suppose I, (X, x), $((\tilde{X}, \tilde{x}), \pi)$ as above, $h \in A_{X,x}$. Then, the following are equivalent.*

(i) $h \in \bar{I}_{\mathbb{R}x}$

(ii) $h \circ \pi \in \tilde{I}_{\tilde{x}}$ *for all* $\tilde{x} \in \pi^{-1}(x)$

(iii) *There exists a neighborhood U of x for each choice of generators (g_i) of I such that $|h(y)| \leqq C \sup_i |g_i(y)|$ for all $y \in U$.*

Proof. (i) $\Rightarrow$ (ii)

Suppose $h \in \bar{I}_{\mathbb{R}}$, $\tilde{x} \in \pi^{-1}(x)$. We have $h \circ \pi$ vanishes on $V(\tilde{I}_{\tilde{x}})$, for we can find a family of curves $\{\varphi_I\}$ whose image covers $V(\tilde{I}_{\tilde{x}})$ and $h \circ \pi \circ \varphi_\alpha = h \circ (\pi \circ \varphi_\alpha)$ must be zero by hypothesis for $\alpha \in I$.

This implies $(h \circ \pi)$ is divisible by x_{j_i} where $\tilde{I}_{\tilde{x}} = (x_{j_1}^{\alpha_1} \cdot \ldots \cdot x_{j_k}^{\alpha_k})$, α_i positive integers. We must show $h \circ \pi$ is divisible by $x_{j_i}^{\alpha_i}$. Write $h \circ \pi = \sum_{k=\ell}^{\infty} h_i(x_1, \ldots, \hat{x}_{j_i}, \ldots, x_n) x_{j_i}^k$, where $h_\ell(x_1, \ldots, \hat{x}_{j_i}, \ldots, x_n) \neq 0$.

We can choose φ: $\mathbb{R}, 0 \dashrightarrow \hat{X}, \tilde{x}$ such that the order of $h_\ell \circ \varphi$ is the multiplicity, m, of h_ℓ and the order of $X_{j_i} \circ \varphi$ is $N > m$.

Then $o(h \circ \pi \circ \varphi) = \ell N + m$, and $o(\varphi^* \tilde{I}) \geqq \alpha_i N$.

If $\ell < \alpha_i$ we have $\alpha_i N \geqq (\ell + 1)N > \ell N + m$ which gives a contradiction.

(ii) $\Rightarrow$ (iii) Suppose $h \circ \pi \in \tilde{I}_{\tilde{x}}$ for each $\tilde{x} \in \pi^{-1}(x)$.

This implies that in a neighborhood of $V(\tilde{I})$, $h \circ \pi$ satisfies $\| h \circ \pi(z) \| \leqq C \sup_i \| g_i \circ \pi(z) \|$, hence $\| h(z) \| \leqq C \sup \| g_i(z) \|$ holds on a neighborhood of x in X as well.

(iii) $\Rightarrow$ (i) If φ: $\mathbb{R}, 0_- \to X, x$ is real analytic the hypothesis implies that $\| h \circ \varphi(t) \| \leqq C \sup_i \| g_i \circ \varphi(t) \|$ for t close to zero.

This implies that the order of $h \circ \varphi$ is greater than or equal to the minimum of the orders of $g_i \circ \varphi$. □

(A result similar to this was proved by Risler and appears in the appendix to [29]).

With this result in place, it is easy to check that the real analytic versions of 1.5, 1.6, 1.7, 1.8 and 1.11 are true. In the hypothesis of 1.7 it is necessary to assume that the smooth points of X are dense in X for sufficiency. This assumption implies that only $\tilde{X}^{(0)}$ is necessary to resolve X and the proof proceeds as before. In the general case of 1.8 it is necessary to assume that $J_{k_i}(h, M) \subset \overline{J_k(M)}$ on $X^{(i)}$ where k_i is the generic rank of (h, M) on $X^{(i)}$. With this point of view in mind the real analytic version of 1.11 can be proved. Although the proof of 1.12 does not completely go through, (the real analytic version of Grauert's theorem would be needed) the remark after 1.12 does hold.

Examining the proof of 2.5 we see that $\frac{\partial F}{\partial s_k} \in \bar{M}$ implies that X_0 is w-regular over Y, while the converse holds provided that on each component of X the smooth points are dense on that component.

Propositions 3.3 and 3.4 go over to the real analytic case, changing Whitney regular to w-regular.

The analogue of 3.1 that we want in the real case (C^∞ or C^ω) asks that $(X_F - U, U)$ is w-regular over U. In this case, by [27], it is known that there exists a rugueuse trivialization of X_F, hence X_{f_0} and X_{f_1} are embedded homeomorphic. If such an F exists we say f_0 and f_1 are Verdier-V-equivalent which we abbreviate by VV-equivalent.

In the analytic case we have:

Proposition 4.3 *Suppose* $f\colon \mathbb{R}^n, 0 \dashrightarrow \mathbb{R}^p, 0$ *is* C^ω *and suppose* $\overline{T\mathscr{K}(f)_{\mathbb{R}}} \supseteq m_n^k \mathcal{O}_n^p$, *and* $g \equiv f \bmod m_n^{k+1}\mathcal{O}_n^p$. *Then f and g are* VV-*equivalent.*

Proof. Let $F(t, s) = f_t(x) = f(x) + t(g(x) - f(x))$.

The proof of 3.6 comes over to the analytic case, and using the analogues of 3.3 and 3.4 we get that the smooth points of X_F are w-regular over T, the parameter axis.

Since $\bar{M} = \overline{\left\{x_i \dfrac{\partial F}{\partial x_j}, F_j e_i\right\}} \supseteq m_n^k A_{n+1}^p$, it follows that $S(X_F) = T$. (If not, choose $\varphi\colon \mathbb{R}, 0 \dashrightarrow X_F, (a, 0)$ such that $\operatorname{im} \varphi \subset S(X_F)$, $\operatorname{im} \varphi \not\subset T$. Then the rank of $\varphi^* M$ is less than p in which case $\varphi^*(m_n^k A_{n+1}^p)$ does not lie in $\varphi^* M$.)

Thus, T and the components of $X_F \setminus T$ give a Whitney stratification of X_F. □

We now turn to the analogue of 3.8.

We say a real analytic $f\colon \mathbb{R}^n, 0 \dashrightarrow \mathbb{R}^p, 0$ is Newton non-degenerate over $\mathbb{R}$ if there are no solutions in $(\mathbb{R}^*)$ to the equations of Definition 3.7. The proof of 3.8 then carries over using the real versions of the propositions. Again, as in 4.3, $\overline{\left\{x_i \dfrac{\partial F}{\partial x_i}\right\}}_{X_{F,\mathbb{R}}} \supset I A_{n+1}^p$, where I is the ideal generated by the monomials lying above all the Newton boundaries, implies that $S(X_F) = T$. It is worth noting that F may be Newton non-degenerate over $\mathbb{R}$ without having finite singularity type. Thus the real integral closure is insensitive to complex phenomena as a good theory of real equisingularity should be.

We turn now to the C^∞ case. The lack of a C^∞ theory of the resolution of singularities blocks a parallel development of the theory of integral closure in this case, but we can still prove the analogue of 4.3.

Proposition 4.4 *Suppose* $f\colon \mathbb{R}^n, 0 \dashrightarrow \mathbb{R}^p, 0$ *is a* C^∞ *map germ and* $\overline{T\mathscr{K}(f)_{\mathbb{R}}} \supset m_n^k C_n^p$ *for some k,* $g \equiv f \bmod m_n^{k+1} C_n^p$, *then f and g are* VV-*equivalent.*

Proof. Consider the two parameter deformations

$$F(s, t, x) = T_0^k f(x) + s(f(x) - T_0^k f(x)) + t(g - f)$$

where $T_0^k f$ is the k-th Taylor polynomial of f at 0.

The methods used in 3.6 show

$$\varphi^* M_{a,b} = \varphi^* M \text{ for all analytic } \varphi\colon \mathbb{R}, 0 \dashrightarrow \mathbb{R}^2 \times \mathbb{R}^n, (a, b, 0)$$

where

$$M = \left\{\frac{\partial T_0^k f}{\partial x_j}, T_0^k f_i e_j\right\} C_{n+2,(a,b,0)}^p \quad \text{and} \quad M_{a,b} \left\{x_i \frac{\partial F}{\partial x_j}, F_i e_j\right\} C_{n+2,(a,b,0)}^p.$$

This implies equality of the corresponding Fitting ideals $\varphi^*(J_p(M))$ and $\varphi^*(J_p(M_{a,b}))$.

Let $\pi\colon \tilde{X} \to \mathbb{R}^2 \times \mathbb{R}^n$ be the embedded resolution of $J_p(M)$. Then we have $\pi^*(J_p(M)) C_{\tilde{X}} = \pi^*(J_p(M_{a,b})) C_{\tilde{X}}$, so $\tilde{X}$ provides the embedded resolution for $J_p(M_{a,b})$ as well. This shows that $S(X_F) = \mathbb{R}^2 \times \{0\}$ at all points (a, b) of $\mathbb{R}^2$, and allows us to rebuild the theory of integral closure in this case. □

Although we cannot prove the C^∞ version of 3.8 directly, we can use 4.4 and the real analytic version of 3.8 to prove something close.

Definition 4.5 Suppose there exists a sequence of maps f_i: $\mathbb{R}^n, 0 \to \mathbb{R}^p, 0$ $i = 1, \ldots, k$ and deformations F_i: $\mathbb{R} \times \mathbb{R}^n, \mathbb{R} \times 0 \to \mathbb{R}^p, 0$, F_i a deformation from f_i to f_{i+1}. If X_{F_i} is w-regular, then X_{f_1} and X_{f_k} are *VV-linked.*

Note that if X_{f_1} and X_{f_k} are VV-linked then they are embedded homeomorphic.

Theorem 4.6 *Suppose* f: $\mathbb{R}^n, 0 \to \mathbb{R}^p, 0$, f *Newton non-degenerate over* $\mathbb{R}$ *and commode in the given coordinate system.*

(A) *Suppose* f *is real analytic,* F: $\mathbb{R} \times \mathbb{R}^n, 0 \to \mathbb{R}^p, 0$ *a deformation of* f *by terms above or on the Newton boundary of* f. *Then* $((X_F \cap \mathbb{R} \times \mathbb{R}^I) - \mathbb{R} \times \{0\}, \mathbb{R} \times \{0\})$ *is* w*-regular where* $\mathbb{R}^I$ *is any coordinate hyperplane.*

(B) *Suppose* f *is real analytic,* F *a deformation of* f *by terms above the Newton boundary. Then* $f_t = F(t, -)$ *is VV-equivalent to* f *for all* t.

(C) *Suppose* f *is* C^∞; *if* F *satisfies the hypotheses of* (A) *then* X_f, X_{f_t} *are VV-linked for* t *small, if* F *satisfies the hypotheses of* (B) *then* X_f, X_{f_t} *are VV-linked for all* t.

Proof. Parts (A) and (B) follow from the remark after 4.3.

Part (C) follows by considering the deformation

$$G(s, t, x) = T^k f(x) + tT^k h(x) + s((f - T^k f)(x) + t(h - T^k h)(x))$$

for k large. By (B) or (A) $T^k f(x) + tT^k h(x)$ is a VV trivial deformation for all t or small t. By 4.4. the 1 parameter deformation that comes from fixing t at a "good" t-value and varying s is w-trivial for all s. Hence the result follows. □

It is worth noting that 4.6 covers cases not covered by [2] and [3]. The germ $(x^2 + y^2)^2$ is non-degenerate over $\mathbb{R}$ for example but not non-degenerate over $\mathbb{C}$.

Note that w-regular implies Whitney regular in the semi-analytic case ([9]). However, given a closed subset of Euclidean space, which has a stratification by C^∞ submanifolds, it is known that w-regular does not imply the Whitney conditions ([9]). Nonetheless, given the hypotheses of 4.4 and 4.6 it is easy to see that in the C^∞ case the map-germs are at least Whitney linked. Consider the two parameter deformation

$$F_{s,t} = T^\ell(f) + t(T^\ell(g) - T^\ell(f)) + s((f + t(g - f)) - (T^\ell f + t(T^\ell g - T^\ell f))),$$

$\ell \gg k$, k as in 4.4.

The deformation $X_{F_{0,t}}$ is w-regular over T, hence Whitney regular over T since $X_{F_{0,t}}$ is an analytic set. By choosing ℓ sufficiently large, we have $X_{F_{s,1}}$ is C^1-trivial ([24]) hence Whitney regular over S. Thus X_f and X_g are Whitney linked. If f is analytic and g is C^∞, replace $T^\ell(x)$ by f in the above deformation, and by part of the argument of 3.5 we can show f and g are VV-equivalent. (It seems absurd, but though we can prove $f + t(T^\ell g - f) = 0$ defines a Whitney equisingular deformation of X_f for all ℓ, we cannot prove that $f + t(g - f) = 0$ does.)

Finally, we note the connection between our work and that of Wilson, Brodersen and Wall on finite C^0 determinacy ([25, 1, 24]). Using the previous propositions and modifications of the previous proofs, it is possible to show that if $f: \mathbb{R}^n, 0 \dashrightarrow \mathbb{R}^p, 0$ is C^∞, then $\overline{T\mathscr{K}(f)_{\mathbb{R}}} \supseteq m_n^k C_n^p$ for some k iff $C \sup_i |g_i(x)| \geqq |x|^\ell$ for some ℓ, where g_i is a set of generators for $J(f) + f^*(m_p)C_n$, for x sufficiently close to zero. This last condition is the condition which implies f is finitely V determined in the framework of Wilson, Brodersen and Wall ([24, p. 519]).

Seen in this light, integral closure provides the right infinitesimal objects from which the relevant L-inequalities can be derived. This connection has also been checked for $\mathscr{L}$ equivalence in [7].

References

1. Brodersen, H.: On finite and infinite C^k-$\mathscr{A}$ determinacy. Preprint, University of Oslo, vol. 14 part 1 and 2, 1984
2. Damon, J.N.: Topological invariants of μ-constant deformations of complete intersection singularities. Q. J. Math., Oxf. II. Ser. **40**, 139–160 (1989)
3. Damon, J.N., Gaffney, T.: Topological triviality of deformations of functions and Newton filtrations. Invent. Math. **72**, 335–358 (1983)
4. Fitting, H.: Die Determinantenideale einer Modulo Jahresler. Jahresber. Dtsch. Math. Ver. **46**, 195–220 (1936)
5. Gaffney, T.: Polar multiplicities and equisingularity of map germs. Topology (to appear)
6. Gaffney, T.: Multiplicities and equisingularity of ICIS germs. (in preparation)
7. Gaffney, T.: Hilbert schemes, integral closure and finite $\mathscr{L}^0$-determinacy. (in preparation)
8. Hironaka, H.: Introduction to real-analytic sets and real analytic maps. Publ. Inst. Mat. "L. Tonelli" dell'Univèrsita di Pisa, Pisa 1973
9. Kuo, T.C.: The ratio test for analytic Whitney stratifications, In: Wall, C.T.C. (ed.) Proceedings of the Liverpool Singularities Symposium I. (Lect. Note Math., vol. 192, pp. 141–149) Berlin Heidelberg New York: Springer 1971
10. Looijenga, E.J.N.: Isolated singular points on complete intersections (Lond. Math. Soc. Lect. Note Ser., vol. 77) Cambridge: Cambridge University Press 1984
11. Mather, J.N.: Stability of C^∞-mappings: I The division theorem. Ann. Math, **87**, 87–104 (1968)
12. Mather, J.N.: II Infinitesimal stability implies stability. Ann. Math. **89**, 259–291 (1969)
13. Mather, J.N.: III Finitely determined map-germs. Publ. Math., Inst. Hautes Étud. Sci. **35**, 127–156 (1969)
14. Mather, J.N.: IV Classification of stable germs by $\mathbb{R}$ algebras. Publ. Math., Inst. Hautes Étud. Sci. **37**, 223–248 (1970)
15. Mather, J.N.: V Transversality. Adv. Math. **4**, 301–335 (1970)
16. Mather, J.N.: VI The nice dimensions In: Proceedings of the Liverpool Singularities Symposium I. (Lect. Notes Math., vol. 192, pp. 207–255) Berlin Heidelberg New York: Springer 1971
17. Oka, M.: Principal Zeta-Function of non-degenerate complete intersection singularity. Preprint, Tokyo Institute of Technology 1988
18. Rees, D.: Reduction of modules. Math. Proc. Camb. Philos. Soc. **101**, 431–449 (1987)
19. Teissier, B.: Cycles évanescents sections planes et conditions de Whitney. Singularités à Cargèsse. (Astérisque, vols. 7–8, pp. 285–362) Paris: Soc. Math. Fr. 1973
20. Teissier, B.: Multiplicities polaires, sections planes, et conditions de Whitney. In: Aroca, J.M. et al. (eds.) Algebraic Geometry. La Rabida 1981 (Lect. Notes Math., vol. 961, pp. 314–491) Berlin Heidelberg New York: Springer 1983
21. Teissier, B.: Introduction to equisingularity problems. In: Hartshorne, (ed.) Algebraic Geometry. Arcata 1974. (Proc. Symp. Pure Math., vol. 29) Providence, RI: Am. Math. Soc. 1975
22. Teissier, B.: The hunting of invariants in the geometry of discriminants. In: Holm, P. (ed.) Real and complex singularities. Proc. Nordic Summer School, Oslo 1976. Alphen on den Rijn: Sijthoff and Noordhoff 1977
23. Thom, R., Levine, H.: Singularities of differentiable mappings. Notes, University of Bonn. (Lect. Notes Math., vol. 192, pp. 1–89) Berlin Heidelberg New York: Springer 1971
24. Wall, C.T.C.: Finite determinancy of smooth map-germs. Bull. Lond. Math. Soc. **13**, 481–539 (1981)

25. Wilson, L.: Infinitely determined map-germs. Can. J. Math. **33** (n° 3) 671–684 (1981)
26. Khovanskii, A.G.: Newton polyhedra and toroidal varieties. Funct. Anal. Appl. **11**, 289–295 (1977)
27. Verdier, J.L.: Stratifications de Whitney et théorème de Bertini-Sard. Invent. Math. **136**, 295–312 (1976)
28. Brumfiel, G.W.: Real valuation rings and ideals. In: Géométrie Algébrique Réelle et Formes Quadratiques. Rennes 1981. (Lect. Notes Math., vol. 959, pp. 55–97) Berlin Heidelberg New York: Springer 1982
29. Lejeune-Jalabert, M., Teissier, B.: Dependence integrale sur les ideux et equisingularite. Publ. Inst. Fourier, St. Martin d'Heres F-38402, 1975