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Mating quadratic maps with the modular group

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1 Introduction

Consider the one complex parameter family of 2:2 correspondences on the Riemann sphere, defined by $z \mapsto w$, where

$$\left(\frac{az+1}{z+1}\right)^2 + \left(\frac{az+1}{z+1}\right)\left(\frac{aw-1}{w-1}\right) + \left(\frac{aw-1}{w-1}\right)^2 = 3. \tag{1.1}$$

We show that there is a non-empty set M of values of the parameter a for which the dynamics of this correspondence is that of a *mating* of the modular group PSL(2, Z) with a quadratic map $q_c: z \mapsto z^2 + c$, in the following sense. For these values of a the Riemann sphere is partitioned into two subsets, each fully invariant under the correspondence: a *regular domain* Ω, homeomorphic to an open disc, on which the action of the correspondence resembles that of PSL (2, Z) on the complex upper half-plane, and a *global attractor* Λ, the one-point union of two subsets Λ_+, Λ_-, each resembling the filled-in Julia set K_c of q_c and on each of which the actions of appropriate backward or forward branches of the correspondence resemble that of q_c on K_c. The precise nature of these 'resemblences' is made clear in Theorems 1 and 2 below.

Computer experiments indicate that every connected filled-in Julia set K_c of the quadratic family can be realised in this way (see Figs. 8, 10, 11 and 13) and that the set M of values of the parameter a such that the correspondence is a mating of this type is homeomorphic to the Mandelbrot set (see Fig. 14). However, the situation is most easily described for *real* values of a and c.

Theorem 1 *For a real,* $4 \leqq a \leqq 7$, *the Riemann sphere is partitioned into two sets,* Ω *and* Λ, *each fully-invariant under* $f: z \mapsto w$ *and* $f^{-1}: w \mapsto z$, *defined by* (1.1), *and such that*

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(i) *Ω is open and simply-connected, and the action of the correspondence* (1.1) *on Ω is conformally conjugate to that of* PSL(2, Z) *acting by Möbius transformations* $z \mapsto z+1$, $z \mapsto z/(z+1)$ *on the complex upper half-plane;*

(ii) *Λ is the one-point union of two connected closed subsets Λ_+, Λ_-, which are forward and backward invariant respectively. The correspondence* (1.1) *has a forward branch carrying Λ_- onto itself with degree* 2 *and a backward branch carrying Λ_+ onto itself with degree* 2*; the remaining forward branch on Λ carries Λ_- homeomorphically onto Λ_+.*

Our computer plots suggest that the sets Λ_+, Λ_- of Theorem 1 are *homeomorphic* to filled-in Julia sets K_c of the quadratic family $q_c: z \mapsto z^2 + c$. The homeomorphisms appear to 'pull out cusps' from K_c, changing the conformal structure on the boundary J_c. To prove that our Λ_+ and Λ_- are homeomorphic to K_c we should have to extend the Douady-Hubbard theory of *polynomial-like mappings* [8] to a theory of *pinched polynomial-like mappings* (see Sect. 6 for a discussion). This is a difficult technical problem, and we content ourselves here with proving that for each of the correspondences of the form (1.1), with a real and $4 < a < 7$, there exist arbitrarily small perturbations of the form

$$\left(\frac{az+1}{z+1}\right)^2 + \left(\frac{az+1}{z+1}\right)\left(\frac{aw-1}{w-1}\right) + \left(\frac{aw-1}{w-1}\right)^2 = 3k \tag{1.2}$$

such that Λ_+ and Λ_- become disjoint, but such that each can be proved *quasi-conformally equivalent* to some K_c, using the standard Douady-Hubbard theory. These perturbations have the unfortunate effect of allowing critical points of the correspondence to enter the regular domain Ω, with the consequence that the action of the correspondence there is no longer that of PSL(2, Z), but they are the best we can hope for if we are to obtain *quasi-conformal* conjugacies to quadratic maps on Λ_+ and Λ_-.

Theorem 2 *There exist continuous functions $a_{\min}, a_{\max}: [\frac{1}{4}, 1] \to [1, 7]$ satisfying $a_{\min}(1) = 4$, $a_{\max}(1) = 7$ and $1 < a_{\min}(k) < a_{\max}(k)$ (except for $1 = a_{\min}(\frac{1}{4}) = a_{\max}(\frac{1}{4})$) such that for $\frac{1}{4} < k < 1$ and $a_{\min}(k) \leqq a \leqq a_{\max}(k)$, the Riemann sphere is partitioned into two sets, Ω and Λ, each fully-invariant under $f: z \mapsto w$ and $f^{-1}: w \mapsto z$ defined by* (1.2), *and such that*

(i) *Ω is open, homeomorphic to an annulus, and the action of the correspondence f defined by* (1.2) *on it is discontinuous;*

(ii) *Λ is the disjoint union of two closed connected subsets Λ_+, Λ_-, forward and backward invariant respectively. On a neighbourhood of Λ_- there is a branch of f which is hybrid equivalent to a quadratic map q_c ($z \mapsto z^2 + c$), with a connected filled-in Julia set K_c; this hybrid equivalence sends Λ_- to K_c by a quasi-conformal bijection. On a neighbourhood of Λ_+ there is a branch of the inverse correspondence f^{-1} with the same property* (*for the same c*)*. The remaining forward branch of f on Λ carries Λ_- onto Λ_+ by a conformal bijection.*

Comments. 1. We say that an action of a correspondence is *discontinuous* at z if there exists a connected neighbourhood U_z of z such that on U_z there are only a finite number of *branches* F of the iterated correspondence (forwards, backwards or mixed) such that $F(z) = z$, and if all other *branches* have $F(U_z) \cap U_z = \emptyset$. (See Sect. 3 for the definition of a *branch* of an iterated correspondence.)

2. Recall that a *hybrid equivalence* between a quadratic map q_c and a holomorphic map g is a quasi-conformal equivalence ϕ such that $\bar{\partial}\phi = 0$ on K_c [8], and that if the Julia set J_c has measure zero (which is the case in all known examples) then $\bar{\partial}\phi = 0$ on K_c if and only if ϕ is conformal on the interior of K_c.

The term 'mating' was introduced by Douady and Hubbard to describe (in the simplest case) a map manufactured from two polynomial maps p_1, p_2 (of the same degree) by gluing together their filled-in Julia sets (along boundaries) to form a sphere, the gluing being performed in such a way that p_1 and p_2 match on the boundary and so together define a single self-map of the sphere. The general definition of a mating of polynomials (for example [15]) is rather more technical. but the simplified description above applies at least in the case that p_1 and p_2 are hyperbolic and their Julia sets connected. Douady, Hubbard, Shishikura, Rees and Tan Lei have considered conditions under which matings of polynomials are realisable by rational maps (see [15] for details and further references). 'Matings' also occur in the category of Kleinian groups: for example one can consider a quasi-Fuchsian group as a 'mating' of two (group-theoretically isomorphic) Fuchsian groups. The building blocks for our 'matings' between quadratic maps and the modular group are:

(i) the standard action of PSL(2, Z) on the upper half-plane H via $z \mapsto z + 1$, $z \mapsto z/(z + 1)$;

(ii) two copies K_c^+, K_c^- of a filled-in Julia set K_c for a quadratic map $q_c: z \mapsto z^2 + c$ (c in the Mandelbrot set); we equip these with coordinates z^+ and z^- respectively.

We assume that we have a parametrisation of the boundary J_c of K_c as a quotient of the circle R/Z, with q_c acting as a quotient of the binary shift $\theta \mapsto 2\theta \bmod 1$ (as is the case, for example, if q_c is hyperbolic), and we 'mate' PSL(2, Z) with q_c as follows. We divide the boundary $R \cup \{\infty\}$ of H into two parts, $[0, \infty]$ and $[-\infty, 0]$, and homeomorphically map

$$[0, \infty]/(0 \sim \infty) \to R/Z(\to J_c^+)$$

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots}} \mapsto 0\cdot\underbrace{00\ldots0}_{a_0}\ \underbrace{11\ldots1}_{a_1}\ \underbrace{00\ldots0}_{a_2}\ \ldots \tag{1.3}$$

This is the unique continuous (orientation-reversing) map which sends 0 to 1 and carries the action of $z \mapsto z + 1$, $z \mapsto z/(z + 1)$ on $[0, \infty]$ across to that of $\theta \mapsto \theta/2$, $\theta \mapsto \theta/2 + 1/2$ on R/Z. Similarly we map $[-\infty, 0]/(-\infty \sim 0) \to R/Z(\to J_c^-)$ by pre-composing the above map with $z \mapsto -z$ and post-composing with $\theta \mapsto (1 - \theta)$. We glue together K_c^+, H and K_c^- (along boundaries) using these maps, and define a 2:2 correspondence on the union by fitting together $q_c^{-1}: z^+ \mapsto \pm\sqrt{z^+ - c}$ on K_c^+, $z \mapsto z + 1$, $z \mapsto z/(z + 1)$ on H, $q_c: z^- \mapsto (z^-)^2 + c$ on K_c^-, and the map $K_c^- \to K_c^+$, $z^- \mapsto -z^+$. The result we call an *abstract mating* of q_c with PSL(2, Z).

Our belief is that the family (1.1), for a in a certain subset of parameter space, are all realisations of such abstract matings. If we could prove that in the limiting case ($k = 1$) of Theorem 2(ii) the branches of f mapping Λ_- two-to-one onto itself, and of the inverse correspondence mapping Λ_+ two-to-one onto itself, described in Theorem 1(ii), remain topologically conjugate to q_c (on K_c), then (at least in the hyperbolic case) we should have proved that (1.1) is a realisation of the corresponding abstract mating. We examine in detail three examples, $a = 4$, 5 and 7, which correspond to $c = -2$, 0, and 1/4 respectively. We prove that we have a realisation of a mating when $a = 4$, and we produce strong heuristic arguments in the case $a = 7$.

The layout of this paper is as follows. In Sect. 2 we investigate *diagram conditions* on correspondences, with the objective of identifying a class of correspondences having actions resembling that of the modular group. The class of *reversible maps of triples*, having *graph a sphere* (see Sect. 2 for definitions) gives us (1.2). Imposing the *contact condition* (again see Sect. 2) reduces us to (1.1). In Sect. 3 we develop a theory of *limit sets*, *regular sets* and *fundamental domains* for these correspondences, and in Sect. 4 we apply this theory to prove Theorems 1 and 2. In Sect. 5 we consider examples, and we conclude, in Sect. 6, with remarks on the problems involved in generalising our proofs to the full situation evident in computer plots.

Our strategy throughout the article is to concentrate attention on the algebraic conditions that correspondences must satisfy in order to have actions resembling modular group actions, and on the dynamical behaviour of these particular correspondences. At almost every stage in the development of our analysis there are generalisations that can be made, but in order to keep the paper to reasonable length, and (we hope) to keep it readable, we confine ourselves here to the theory needed for the examples in question. A more general setting will be presented in [4].

The work described here forms part of an on-going research programme to investigate correspondences which behave dynamically like rational maps or Kleinian groups. In [13] Sullivan constructed a partial 'dictionary' between the theories of such maps and groups, and showed how quasi-conformal deformation theory could be applied very productively in different ways in the two fields. Correspondences form a category in which rational maps and Kleinian groups are particular examples, a natural category in which to attempt to complete Sullivan's programme. Our examples in this paper exhibit 'rational-map-like' and 'Kleinian-group-like' behaviour in a single system for the first time, and it also seems likely that using the techniques of [12], [13] and [14] it will be possible to prove that (generically) they exhibit *structural stability* for appropriate classes of perturbations. Sullivan [14] proved that structural stability is generic for holomorphic families of rational maps, but the corresponding question for discrete representations of Kleinian groups remains a major unsolved question [13].

A survey of our results on limit sets of correspondences can be found in [3], and technical details in [4]. For background on quadratic maps see [1, 6, 7], and for Kleinian groups see [10].

2 Diagram conditions on correspondences

All our correspondences will be 2 : 2 'maps' $z \mapsto w$ defined by equations $p(z, w) = 0$, where p is polynomial of degree 2 in each of z and w, with complex coefficients. We say that two such correspondences p_1 and p_2 are *equivalent* if there exists a Möbius transformation M such that $p_2(z, w) = 0 \Leftrightarrow p_1(Mz, Mw) = 0$. Equivalent correspondences have conformally conjugate dynamics.

We say that a correspondence $z \mapsto w$ is a *map of pairs* if whenever w_1, w_2 are images of z_1, and w_1 is also an image of z_2, then w_2 is an image of z_2. Diagramatically we may represent this condition by:

$$\begin{tikzcd} z_1 \arrow[r] \arrow[dr] & w_1 \\ z_2 \arrow[r] \arrow[ur] & w_2 \end{tikzcd} \tag{2.1}$$

It is an elementary observation that a 2:2 correspondence is a map of pairs if and only if it can be expressed in the form

$$q_1(z) = q_2(w) \tag{2.2}$$

with q_1 and q_2 rational functions of degree two. A necessary and sufficient condition for such a separation of the variables z and w to occur was given in [2]. It is that the 3×3 matrix of coefficients of $p(z, w)$ have zero determinant. Note also that a map of pairs comes equipped with two involutions, that interchanging z_1 and z_2 if they have the same images, and that interchanging w_1 and w_2 if they have the same pre-images, in other words the covering involutions for the degree two maps q_1 and q_2.

The next stage of complication is that of a *map of triples*. Here we are still dealing with 2:2 correspondences, as we are throughout this article, but we impose the diagram condition:

$$\begin{tikzcd} z_1 \arrow[dr] \arrow[ddr] & w_1 \\ z_2 \arrow[ur] \arrow[dr] & w_2 \\ z_3 \arrow[uur] \arrow[ur] & w_3 \end{tikzcd} \tag{2.3}$$

Formal definitions of *diagram conditions* will be given in [4], but for our purposes here it will suffice to define a 2:2 correspondence to be a *map of triples* if and only if there exists a fractional cubic map C, (i.e. a degree three rational map), and a fractional linear map M, such that the correspondence sends each z to the two solutions w of $C(Mz) = C(w)$ other than $w = Mz$. For such a correspondence the map C is unique up to post-composition by a Möbius transformation, and the map M is unique. We shall say that the correspondence is a *reversible map of triples* if M is an *involution*, ie. $M = M^{-1} = J$ (say): then $z \to w$ if and only if $Jw \to Jz$.

We can investigate the dynamics of a correspondence either by considering it as $z \mapsto w$ on the *dynamical plane* (the Riemann sphere), or by lifting it to a 2:2 correspondence on its *graph* $\Gamma(f) = \{(z, w): p(z, w) = 0\}$. Abstractly, the *objects* of the latter (pairs (z, w)), are the *morphisms* of the former (arrows $z \mapsto w$). Both points of view are useful in different situations.

When we lift a correspondence to its graph, we always obtain a map of pairs: a point (z, w) on the graph maps to (u, v) under the lift if and only if $w = u$, that is $\pi_1(z, w) = \pi_2(u, v)$. Here we have adopted the somewhat unusual convention that π_1 denote projection onto the second factor and π_2 that onto the first. The reader should think of these as projections 'forgetting the first factor' and 'forgetting the second' respectively. Let I_1 and I_2 denote the covering involutions for π_1 and π_2. Thus I_1 interchanges pairs of points on the graph which have the same image under the (lifted) correspondence, and I_2 interchanges pairs of points with the same pre-image. Our *diagram conditions* on arrows $z \mapsto w$ translate directly into *algebraic conditions* on these involutions:

$z \mapsto w$ is a *map of pairs* if and only if $I_1 I_2 = I_2 I_1$;

$z \mapsto w$ is a *map of triples* if and only if $I_1 I_2 I_1 = I_2 I_1 I_2$;

$z \mapsto w$ is a *reversible map of triples* if and only if there exists an involution J of the sphere such that $I_1 I_2 I_1(z, w) = I_2 I_1 I_2(z, w) = (Jw, Jz)$ for all $(z, w) \in \Gamma$.

We shall concentrate attention on maps of triples, indeed on reversible maps of triples, since these turn out to be the correspondences which act like the modular group in appropriate circumstances. However it is worth remarking that there is a natural generalisation of the notions of maps of pairs and maps of triples: we define *maps of n-tuples* to be 2:2 correspondences with the property that the covering involutions I_1 and I_2 (defined as above) generate a dihedral group of order 2n. This generalisation, and other 'diagram conditions', will be considered in [4].

The *global orbit* of a point z_0 under a correspondence (allowing mixed sequences of forward and backward iteration) can be quite a complicated object to describe algebraically in general, but for a reversible map of triples it has a particularly simple structure, as we now show.

For any correspondence $f: z \mapsto w$ let $O_+(z_0)$ denote the set of all forward images of z_0. Thus

$$O_+(z_0) = \bigcup_{n>0} f^n(z_0) \tag{2.4}$$

where $f(z_0)$ denotes the set of all w such that f maps z_0 to w. Similarly let

$$O_-(z_0) = \bigcup_{n>0} f^{-n}(z_0) \tag{2.5}$$

where $f^{-1}(z_0)$ denotes the set of all w such that f maps w to z_0, and let

$$O_\pm(z_0) = O_+(z_0) \cup O_-(z_0) \cup \{z_0\}. \tag{2.6}$$

Thus $O_\pm(z_0)$ denotes the set of all points accessible from z_0 by purely forward or purely backward iteration of the correspondence.

Lemma 1 *If f is a reversible map of triples, with time-reversing involution J, then $O_\pm(z_0) \cup O_\pm(Jz_0)$ is the global orbit of z_0 under f.*

Proof. Since the (at most three) values of $ff^{-1}f(z_0)$ are the (at most two) values of $f(z_0)$ together with Jz_0, we see that Jz_0, and hence $O_\pm(Jz_0)$, is contained in the global orbit of z_0.

It remains to show that every point of the global orbit of z_0 lies in either $O_\pm(z_0)$ or $O_\pm(Jz_0)$. Consider an arbitrary point $z \in O_\pm(z_0) \cup O_\pm(Jz_0)$. It will suffice to show that all (at most four) points in $f(z) \cup f^{-1}(z)$ are also in $O_\pm(z_0) \cup O_\pm(Jz_0)$. This is trivial for $z = z_0$, so suppose, without loss of generality that $z \in O_+(z_0)$. The (at most two) values of $f(z)$ are then automatically in $O_+(z_0)$ and one value of $f^{-1}(z)$ is in $O_+(z_0) \cup \{z_0\}$. Denote this value by x. We now have a diagram of triples

$$\begin{tikzcd} x \arrow[dr] \arrow[ddr] & Jx \\ Jz \arrow[ur] \arrow[dr] & z \\ y \arrow[uur] \arrow[ur] & Jy \end{tikzcd} \tag{2.7}$$

for some y. Observe, from (2.7), that $f(x) = \{z, Jy\}$. Hence $Jy \in O_+(z_0)$ (since $x \in \{z_0\} \cup O_+(z_0)$). But J sends $O_+(z_0)$ to $O_-(Jz_0)$, (J being a time-reversing involution) so that $y \in O_-(Jz_0)$ as required. □

Not every point z for an n:n correspondence $f: z \to w$ need have n *distinct* images w. We shall say that z is a *forward singular point* if it has fewer than n images w, and that w is a *backward singular point* if it has fewer than n pre-images z. Care must be taken over the distinction between *singular points* and *critical points*. A *forward critical point* is a point z with the property that dw/dz vanishes for at least one of the n branches $f: z \to w$. The corresponding w is a *forward critical value*. There are analogous definitions of *backward critical points* and *backward critical values*. Note that while a backward critical value is necessarily a forward singular point, a forward singular point may be a backward critical value, or a *multiple point* (where the graph of the correspondence has two or more intersecting sheets), or a *degenerate point* (where critical points for the two graph projections coincide). In what follows we shall employ both terminologies: generally speaking we shall consider singular points when examining the topology of the graph of a correspondence, and critical points when we look at dynamics under iteration.

The graph Γ of one of our maps of triples $p(z, w) = 0$ is a Riemann surface. To compute the genus of Γ we consider the projections π_1 and π_2. For a map of triples the group generated by the two covering involutions is finite (D_6). It follows that (once double points of Γ have been resolved) the fixed points of I_2 on Γ are distinct from those of I_1, so that there are no *degenerate* singular points in the sense described above. Hence the (forward) singular points z of the correspondence are the branch points and double points for the double covering π_2. As p is quadratic in each of z and w there are 4 such points z (counted one for a branch point, two for a double point). The possibilities for double and branch points for π_2 are therefore:

(i) 2 double points: Γ two (intersecting) spheres;
(ii) 1 double point and 2 branch points: Γ a (self-intersecting) sphere;
(iii) 4 branch points: Γ a torus.

While (iii) is the generic situation, our primary concern in this paper will be with correspondences of type (ii). Note that a correspondence is of type (i) if and only if it factorises into a pair of Möbius transformations, i.e. $p(z, w)$ is equivalent to some $(w - Az)(w - Bz)$ where A and B are fractional linear.

We now specialise to *reversible maps of triples*. If z_1 is a singular point, having unique image Jz_2, then Jz_1 has unique pre-image z_2 (by the reversibility condition). Thus branch points have diagram

$$\begin{tikzcd}[row sep=tiny]
{} \arrow[dr] & & & {} \\
& z_1 \arrow[dddr] & Jz_1 \arrow[ur] \arrow[dr] & \\
{} \arrow[ur] & & & {} \\
{} \arrow[dr] & & & {} \\
& z_2 \arrow[uuur] \arrow[r] & Jz_2 \arrow[ur] \arrow[dr] & \\
{} \arrow[ur] & & & {}
\end{tikzcd} \tag{2.8}$$

and double points have diagram

$$\begin{tikzcd}[row sep=tiny]
{} \arrow[dr] & & & {} \\
& z_1 \arrow[r] & Jz_1 \arrow[ur] \arrow[dr] & \\
{} \arrow[ur] & & & {}
\end{tikzcd} \tag{2.9}$$

If our reversible map of triples has exactly one double point, we may place it at $z_1 = J^{-1}(\infty)$, and choose our fractional cubic C such that $C(\infty) = \infty$, indeed such that C is a polynomial. We may further normalise the two critical points of C to lie at $z = \pm 1$ (since if the critical points of C are identical the correspondence has *two* double points), in other words we may assume that $C(z) = z^3 - 3z$, and deduce

Lemma 2 *Every reversible map of triples with graph a single sphere is equivalent to a correspondence of the form*

$$(Jz)^2 + (Jz)w + w^2 = 3 \tag{2.10}$$

for some involution J. This J is unique up to conjugacy by $z \mapsto -z$.

Proof. By the remarks above, our correspondence is equivalent to $z \mapsto w$, where

$$(Jz)^3 - 3Jz = w^3 - 3w \tag{2.11}$$

but $w \neq Jz$. The expression in the statement of the lemma follows by dividing through by $w - Jz$. Uniqueness up to conjugacy by $z \mapsto -z$ follows from the fact that we have normalized points to lie at ∞ and ± 1. □

Since an involution is uniquely determined by its two fixed points, Lemma 2 gives us a two (complex) dimensional parametrisation of the moduli space of reversible maps of triples having graph a single sphere.

Returning to condition (i) on the graph Γ, if our reversible map of triples has *two* double points we can place them at 0 and ∞, and assume that $C(z) = z^3$. We deduce

Lemma 3 *Every reversible map of triples with graph a pair of spheres is equivalent to a correspondence of the form*

$$(Jz)^2 + (Jz)w + w^2 = 0 \tag{2.12}$$

for some involution J. This involution is unique up to conjugation by scalars.

The correspondence in Lemma 3 can be more easily expressed as

$$w = e^{\pm 2\pi i/3} Jz \tag{2.13}$$

and it follows at once that

Lemma 4 *There is a bijection between equivalence classes of reversible maps of triples with graph a pair of spheres, and equivalence classes of representations of the free product $C_2 * C_3$ in* PSL(2, C).

Here C_p denotes the cyclic group of order p. Of course $C_2 * C_3 = \mathrm{PSL}(2, Z)$, the modular group, the standard isomorphism being given by taking generators

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}. \tag{2.14}$$

If we write τ for the generator of order 3, the correspondence (2.13) has the form $z \to \tau^{\pm 1} Jz$. The first few terms in the (directed) orbits $O_\pm(z_0)$, $O_\pm(Jz_0)$ discussed in Lemma 1 now have the form

$$
\begin{tikzcd}[column sep=small]
{} \arrow[r, shift left] \arrow[r, shift right] & J\tau z_0 \arrow[dr] & & \tau J z_0 \arrow[r, shift left] \arrow[r, shift right] & {} \\
& & z_0 \arrow[ur] \arrow[dr] & & \\
{} \arrow[r, shift left] \arrow[r, shift right] & J\tau^{-1} z_0 \arrow[ur] & & \tau^{-1} J z_0 \arrow[r, shift left] \arrow[r, shift right] & {} \\
{} \arrow[r, shift left] \arrow[r, shift right] & J\tau J z_0 \arrow[dr] & & \tau z_0 \arrow[r, shift left] \arrow[r, shift right] & {} \\
& & J z_0 \arrow[ur] \arrow[dr] & & \\
{} \arrow[r, shift left] \arrow[r, shift right] & J\tau^{-1} J z_0 \arrow[ur] & & \tau^{-1} z_0 \arrow[r, shift left] \arrow[r, shift right] & {}
\end{tikzcd}
\tag{2.15}
$$

and it is easily seen that $O_{\pm}(z_0)$ is made up of Wz_0 where W runs through all reduced words in $\tau^{\pm 1}$ and J having an even number of letters, and $O_{\pm}(Jz_0)$ is made up of Wz_0 where W runs through all reduced words having an odd number of letters. Together, as guaranteed by Lemma 1, $O_{\pm}(z_0) \cup O_{\pm}(Jz_0)$ contains Wz_0 for all W in the group $C_2 * C_3$.

We remark that the moduli space of representations of $C_2 * C_3$ is itself a space of considerable interest, and should have features analogous to those of the Maskit embedding of the moduli space of punctured tori, and the Riley slice of representations of $C_\infty * C_\infty$ having two parabolic generators [9]. In particular we would expect there to be an open region $\mathscr{D}$ of parameter space, homeomorphic to a punctured disc, where the action of $C_2 * C_3$ is discrete, faithful and has limit set a Cantor set. This region $\mathscr{D}$ should be thought of as analogous to the complement of the Mandelbrot set for quadratic maps $q_c\colon z \mapsto z^2 + c$, an analogy which is developed for other free products of cyclic groups in [5]. However our concern here will only be with the classical action of the modular group on the complex upper half-plane.

The correspondences described in Lemma 2 can be expressed as the two-parameter family

$$\left(\frac{az+1}{z+1}\right)^2 + \left(\frac{az+1}{z+1}\right)\left(\frac{aw-1}{w-1}\right) + \left(\frac{aw-1}{w-1}\right)^2 = 3k \tag{2.16}$$

that is to say the family (1.2) of the Introduction. Here we have normalized J to be the involution $z \mapsto -z$.

It is also useful to derive the equation of the lift of such a correspondence to its graph Γ. This we now do by considering the properties which characterise the lift. Given a reversible map of triples with graph a sphere, we know that the covering involutions I_2, I_1 for the projections from Γ generate a dihedral group of order 6, and that $I_2 I_1 I_2 = I_1 I_2 I_1 = \tilde{J}$, the lift of a time-reversing involution J on the dynamical plane. Since there is only one faithful action of D_6 on the sphere (up to conjugacy by Möbius transformations), we may choose a co-ordinate Z on Γ such that I_1, I_2 and $\tilde{J}$ are given by

$$I_1\colon Z \mapsto -1-Z \quad I_2\colon Z \mapsto -Z/(Z+1) \quad \tilde{J}\colon Z \mapsto 1/Z. \tag{2.17}$$

The correspondence on Γ, which lifts our given correspondence on the dynamical plane, now has the form $Z \to W$ where

$$M(Z(Z+1)) = W^2/(W+1) \tag{2.18}$$

for some Möbius transformation M (since $Z \mapsto Z(Z+1)$ identifies pairs related by I_1, and $W \mapsto W^2/(W+1)$ identifies pairs related by I_2). But in order that the time-reversal symmetry be respected we require that

$$M\left(\frac{1}{W}\left(\frac{1}{W}+1\right)\right) = \frac{1}{Z^2}\Big/\left(\frac{1}{Z}+1\right) \tag{2.19}$$

be the *same* correspondence, and hence that

$$M^{-1}(1/\zeta) = 1/M\zeta \tag{2.20}$$

for all ζ. A necessary and sufficient condition is that M have the form

$$\begin{pmatrix} \lambda & 1 \\ -1 & \mu \end{pmatrix} \tag{2.21}$$

(where λ and μ may both be ∞, in which case their ratio is a parameter). Our family of graph correspondences can therefore be written

$$\frac{\lambda Z(Z+1)+1}{-Z(Z+1)+\mu} = \frac{W^2}{W+1}. \tag{2.22}$$

We stress that (2.22) are just the graph correspondences of the family (2.16). It is merely an exercise in algebra to relate λ and μ to a and k. However it is an exercise which is easier to carry out once we have made an examination of the dynamics of both families, and once we have imposed a final constraint, the *contact condition*, which will restrict both families to one parameter. This is the condition that one of the fixed points of J, or equivalently of $\tilde{J}$, be a *fixed point of the correspondence*. Our motivation for imposing this condition comes from dynamical considerations (see (3.3)). However, in the present context, our interest in the contact condition is purely algebraic. On the dynamical plane, requiring the origin to be a fixed point of (2.16) yields $k = 1$, that is

$$\left(\frac{az+1}{z+1}\right)^2 + \left(\frac{az+1}{z+1}\right)\left(\frac{aw-1}{w-1}\right) + \left(\frac{aw-1}{w-1}\right)^2 = 3. \tag{2.23}$$

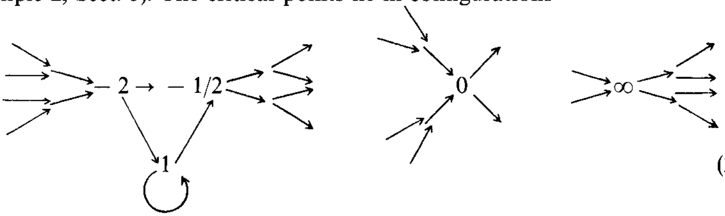
On the graph, requiring the point 1 to be a fixed point of (2.22) yields $\mu = 4(\lambda+1)$, that is

$$\frac{\lambda Z(Z+1)+1}{-Z(Z+1)+4(\lambda+1)} = \frac{W^2}{W+1}. \tag{2.24}$$

To begin to understand the dynamics of (2.23) and (2.24), we first identify their critical and double points. As a by-product we shall obtain the relation between λ and a.

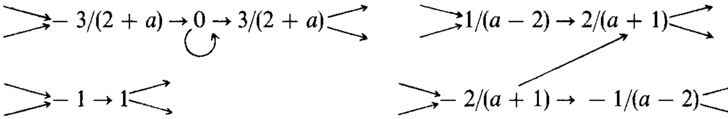
For (2.24) critical points of the *forward* map ($Z \mapsto W$) are $Z = -1/2, \infty$, and those for the *backward* map ($W \mapsto Z$) are $W = 0, -2$. There are no double points (except in the special cases $\lambda = 0$, $\lambda = -8/7$, and in the case $\lambda = 4$ which has two double points $-1/2 \mapsto 0$ and $\infty \mapsto -2$ and which corresponds to $a = 4$ – see

Example 2, Sect. 5). The critical points lie in configurations



(2.25)

These critical points are just the zeros of the derivatives of each side of (2.24). For (2.23), which is not a map of pairs, it is easier to compute the *singular* points (points w with unique pre-image z, or points z with unique image w): these are the forward and backward critical *values* of the correspondence, and its double points. A simple calculation for (2.23) yields



(2.26)

The projection π_1, which identifies Z with $-1-Z$, has the form

$$\pi_1(Z)=\frac{p(Z(Z+1)+q}{rZ(Z+1)+s} \tag{2.27}$$

for some $p, q, r, s \in C$, and the projection π_2 is given by $\pi_2(Z)=J\pi_1(\tilde{J}Z)$, that this

$$\pi_2(Z)=\frac{-qZ^2-p(Z+1)}{sZ^2+r(Z+1)}. \tag{2.28}$$

However, from the critical point diagrams (2.25) and (2.26) we know that $\pi_2(-2)=-3/(2+a)$, $\pi_1(-2)=\pi_2(1)=\pi_1(1)=\pi_2(-1/2)=0$, $\pi_1(-1/2)=3/(2+a)$, $\pi_2(\infty)=-2/(a+1)$ and $\pi_1(\infty)=-1/(a-2)$. It follows that $p=1$, $q=-2$, $r=2-a$ and $s=-(a+1)$. Thus

$$\pi_1(Z)=\frac{Z(Z+1)-2}{(2-a)Z(Z+1)-(a+1)} \tag{2.29}$$

and

$$\pi_2(Z)=\frac{2Z^2-(1+Z)}{(2-a)(1+Z)-(a+1)Z^2}. \tag{2.30}$$

The graph correspondence is given by $\pi_1(Z)=\pi_2(W)$ and the parameters λ and μ are obtained from the formula

$$\begin{pmatrix}\lambda & 1\\ -1 & \mu\end{pmatrix}=\begin{pmatrix}-q & -p\\ s & r\end{pmatrix}^{-1}\begin{pmatrix}p & q\\ r & s\end{pmatrix} \tag{2.31}$$

whence

$$\lambda=\frac{2(a-2)}{5-a}. \tag{2.32}$$

The diligent reader is invited to perform the analogous calculation for the two-parameter families (2.16) and (2.22): the computations in Lemma 5 (Sect. 4) provide the necessary hints.

3 Directionalities and limit sets

Attractors for arbitrary closed relations on compact Hausdorff spaces have been considered by McGehee [11]. We adapt and develop some of his ideas, for our particular correspondences. A much more complete development, for rational correspondences in general, will be presented in [4]. We modify McGehee's notation, in that $f^{-1}(S)$ will denote the set of all points z for which *there exists* $w \in S$ with $p(z, w) = 0$ (where p is the relation defining f), rather than the set of z such that *all* w with $p(z, w) = 0$ lie in S. Our $f^{-1}(S)$ is McGehee's $f^*(S)$. In particular f^{-1} will denote the correspondence inverse to f.

We say that a subset S of the Riemann sphere defines a *directionality* for the correspondence $f: z \mapsto w$ if

$$f(\bar{S}) \subset S^{\circ}. \tag{3.1}$$

Here $f(\bar{S})$ denotes $\{w: p(z, w) = 0 \text{ for some } z \in \bar{S}\}$, where p is the relation defining f, in other words $f(\bar{S})$ is the set of *all* images of points $z \in \bar{S}$. Associated to such a directionality is an *attractor* $\bigcap_{n \geqq 0} f^n(\bar{S})$. Moreover, since (3.1) can be written in the symmetric form

$$\Gamma \cap (\bar{S} \times \overline{S^c}) = \emptyset \tag{3.2}$$

(where Γ is the graph of the correspondence), associated to the same directionality there is a *repeller* $\bigcap_{n \geqq 0} f^{-n}(\overline{S^c})$.

We shall say that S defines a *contact directionality* if there exists a *contact point* $z_f \in \partial S$ such that

$$\Gamma \cap (\bar{S} \times \overline{S^c}) = \{(z_f, z_f)\}. \tag{3.3}$$

A consequence is that the associated attractor and repeller meet at the single point z_f instead of being disjoint.

In the case of either a directionality or a contact directionality the attractor Λ_+ is forward invariant ($f(\Lambda_+) = \Lambda_+$) and the repeller Λ_- is backward invariant ($f^{-1}(\Lambda_-) = \Lambda_-$). Moreover $S - f(S)$ behaves rather like a *fundamental domain* in that no forward orbit started in $S - f(S)$ returns to it (except the contact point, if there is one). Similarly, $S^c - f^{-1}(S^c)$ behaves like a fundamental domain for f^{-1}. Thus we have the beginnings of the ideas of *limit sets* and *regular sets*. However what we really seek are limit sets and regular sets for arbitrary 'zig-zag' combination of f and f^{-1}, not just 'unidirectional' orbits. The general theory is considered in [4]: all we need here is a special case which can be applied when f is a *reversible map of triples*. Recall that in this case I_1 and I_2 generate a dihedral group of order 6, acting on the graph Γ. We say that a subset D of the Riemann sphere defines an *equivariant* (*contact*) *directionality* for such an f if

(i) D is the projection $\pi_2(\bar{\Delta})$ of the closure of a fundamental domain Δ for the action of $\langle I_1, I_2 \rangle$ on Γ, and

(ii) D defines a (contact) directionality for f.

To make the meaning of this definition quite clear we must define the term 'fundamental domain' for the action of a group G on the Riemann sphere. We shall do this for a discrete group in general, not just for a finite group such as D_6, as the general case will motivate the analogous concept for an iterated correspondence.

We say that the action of a discrete group G on the Riemann sphere is *discontinuous* at z if there exists a neighbourhood U of z such that there are only finitely many $g \in G$ with $gU \cap U \neq \emptyset$, and if $gz = z$ for each such g. A *fundamental domain* for the action of G on some invariant open set Ω, on which G acts discontinuously, is an open subset Δ of Ω such that every G-orbit on Ω contains at least one point of $\bar{\Delta}$ and at most one point of Δ°.

We next define what we mean by a *discontinuous* action, and a *fundamental domain*, for an iterated correspondence f. For this we must first clarify exactly what we mean by a *branch* F of the iterated correspondence. Given a path $z_0, \ldots, z_n$ with specified transitions $e_1, \ldots, e_n$ ($e_i \in \{\pm 1\}$), (i.e. where $(z_{i-1}, z_i) \in \Gamma(f^{e_i})$, and given a connected neighbourhood U_0 of z_0, there is defined a sequence of connected neighbourhoods U_i, $i = 1, \ldots n$, given recursively by: U_i is the w-projection of the connected component (in $\Gamma(f^{e_i})$) containing (z_{i-1}, z_i) of the inverse image of U_{i-1} under the z-projection.

The composite correspondence $U_0 \to U_n$ in the above situation is called a *branch* F, along $z_0 \ldots z_n$, of the *global* iteration (that is, forwards backwards and mixed) of f on U_0. Note that U_n will be a connected neighbourhood of z_n, but may be ramified or folded copy of the original U_0 if critical points are encountered along the path. Note also that at a double point of a correspondence we have a single 2:2 branch rather than two separate 1:1 branches: the definition of 'branch' could be amended to avoid this conflict with the usual terminology, but there seems to be no particular benefit in doing so, as the results below hold with either version.

We say that the global action of the correspondence f is *discontinuous* at z if there exists a connected neighbourhood U of z such that there are only a finite set of distinct branches F on U of the globally iterated correspondence f having $F(\{z\}) = \{z\}$, and if all other branches have $F(U) \cap U = \emptyset$.

A *fundamental domain* for the global action of f on an open set Ω on which f acts discontinuously, is an open subset Δ of Ω such that every full orbit of f on Ω contains at least one point of $\bar{\Delta}$ and at most one point of Δ°.

Theorem 3 *If D defines an equivariant (contact) directionality for a reversible map of triples f, then the action of f on the union Ω of the full orbits of points in $D \cap J(D)(\setminus\{z_f\})$ is discontinuous and any fundamental domain for the action of J on $D \cap J(D)$ is a fundamental domain for the action of f on Ω. Moreover the complement of Ω consists of the union of the attractor $\Lambda_+ = \bigcap_{n \geq 0} f^n(D)$ and the repeller $\Lambda_- = \bigcap_{n \geq 0} f^{-n}(JD)$.*

Before we prove this theorem we illustrate it with an example of a *real* 2:2 correspondence with an equivariant directionality (Fig. 1). For simplicity we have drawn a piecewise-linear graph Γ. The covering involutions I_2 and I_1, for the projection of Γ onto the z-axis and w-axis respectively, generate a dihedral group D_6. In terms of coordinates chosen such that the point $z = 0$, $w = 0$ is at the centre of the figure, the involution J is the map $z \mapsto -z$, which lifts to $\tilde{J} = I_1 I_2 I_1 = I_2 I_1 I_2 : (z, w) \mapsto (-w, -z)$, in other words reflection in the anti-diagonal. The group D_6 permutes the six straight line segments of Γ. We take as fundamental domain Δ one of the two straight line segments of Γ meeting on the right-hand edge of figure 1 and let $D = \pi_2(\Delta)$, the projection of Δ onto the z-axis. Then D defines a directionality for the correspondence if and only if the point marked P in the

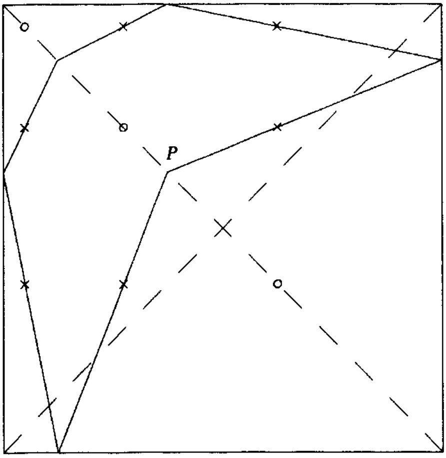


Fig. 1. Graph of a piecewise-linear real 2:2 correspondence with equivariant directionality. The crosses mark a typical D_6-orbit $\{(z_i, w_j): i, j = 1, 2, 3, i \neq j\}$. The 'missing points' $(z_i, w_i = Jz_i)$ are marked by circles

figure is above the diagonal, and a contact directionality if and only if P is on the diagonal. There is an attractor Λ_+ (not shown) within D, and a repeller $\Lambda_- = J\Lambda_+$ within JD. On Γ itself there are three copies of the lift of Λ_+, namely an attractor $\pi_2^{-1}(\Lambda_+)$, a repeller $\pi_1^{-1}(\Lambda_-)$ and an intermediate copy $I_1(\pi_2^{-1}(\Lambda_+)) = I_2(\pi_1^{-1}(\Lambda_-))$: these three sets are invariant under I_2, I_1 and $\tilde{J}$ respectively.

Proof of Theorem 3. Since $D = \pi_2(\bar{\Delta})$, we know that $f(D) = \pi_1(\bar{\Delta} \cup I_2(\bar{\Delta}))$, and since Δ is a fundamental domain for the action of $\langle I_1, I_2 \rangle = D_6$ on Γ we deduce that

$$(f(D))^c = (\pi_1(\bar{\Delta} \cup I_2\bar{\Delta}))^c = (\pi_1(\tilde{J}\bar{\Delta}))^\circ = JD^\circ \tag{3.4}$$

whence, modulo points in ∂D or $\partial(JD)$,

$$D - f(D) = D \cap JD = JD - f^{-1}(JD). \tag{3.5}$$

Now the fact that D defines a (contact) directionality for f ensures that the sets $f^n(D) - f^{n+1}(D), n \geq 0$ and $f^{-n}(JD) - f^{-(n+1)}(JD), n \geq 1$, form a disjoint (modulo points in ∂D or $\partial(JD)$) cover of $\Omega := \bar{C} - (\bigcap_{n \geq 0} f^n(D) \cup \bigcap_{n \geq 0} f^{-n}(JD))$. The injectivity of π_1 restricted to $\pi_2^{-1}(D)$ and of π_2 restricted to $\pi_1^{-1}(JD)$ guarantees that these differences are in fact the whole images $f^n(D - f(D))$ (or $f^{-n}(JD - f^{-1}(JD))$), and that the branch of f^{-1} mapping $f^{n+1}(D - f(D))$ back onto $f^n(D - f(D))$ is a 2:1 surjection, branched only at critical points of f^{-1}. It follows that for a generic point $z_0 \in D - \Lambda_+$ the forward orbit $O_+(z_0)$ of z_0 is a free binary tree, as is the backward orbit $O_-(Jz_0)$ of Jz_0. But by Lemma 1 (Sect. 2) the global orbit of any z_0 under a reversible map of triples is $O_\pm(z_0) \cup O_\pm(Jz_0)$. We deduce that for generic $z_0 \in \Omega$ there is only one branch F of the (globally) iterated correspondence such that $F(z_0) = z_0$, namely the identity branch. 'Non-generic' points $z_0 \in \Omega$ are those which have global orbits containing fixed points of J or critical or double points of f. But if z_0 is a critical point there are two branches F of the iterated correspondence having $F(z_0) = z_0$ (namely the identity and the 2:2 correspondence $f^{-1}f$), at a double point there are three branches (the identity, $f^{-1}f$, and Jf), and at a fixed point of J there are two branches. Thus for any $z_0 \in \Omega$

there are only finitely many distinct branches having $F(z_0) = z_0$, and the action of f on Ω is therefore discontinuous. Moreover if D_0 is a fundamental domain for the action of J on $D \cap JD$, then for each $z_0 \in \Omega$ the global orbit $O_{\pm}(z_0) \cup O_{\pm}(Jz_0)$ contains at least one point of the closure of D_0, and this point is unique if it lies in the interior of D_0. Hence D_0 is a fundamental domain for f in our generalized sense for correspondences. □

Theorem 4 *Suppose that $D = \pi_2(\bar{\Delta})$ is an equivariant (contact) directionality for a reversible map of triples f with graph a sphere and that D is a topological disc with boundary a Jordan curve. Then D contains at most one backward critical value. The complement Ω of the attractor and repeller is connected. The attractor and repeller are themselves connected if and only if there is a backward critical value contained in $f^n(D)$ for all $p \geqq 0$.*

Note. To say that the complement Ω is connected is equivalent to saying that the attractor and repeller are both *full* sets, that is, their individual complements are each connected.

Proof. We equip the graph Γ of f with the coordinate Z introduced in Sect. 2, so that actions of I_1, I_2 and $\tilde{J}$ are as given in (2.17). We first see that D contains at most one backward critical value because otherwise an arc through D° connecting the backward critical values lifts to a loop in $\pi_2^{-1}(D^\circ) \cup \{0, -2\}$ which passes through the critical points $Z = 0$ and $Z = -2$ of the projection π_2, but which separates the two copies of the double point ($Z = (-1 \pm i\sqrt{3})/2$). This gives a contradiction because $\pi_2^{-1}(D^\circ)$ (and hence any such loop) is disjoint from its image under $I_1 I_2$, a rotation with fixed points $Z = (-1 \pm i\sqrt{3})/2$). Thus the inverse-image under π_2 of D (or any simply-connected or full subset) is full. Now $\pi_2^{-1}(D^\circ)$ (or $\pi_2^{-1}(D^\circ \cup \{z_f\})$ in the contact case) is injectively mapped by π_1. Hence the inverse image under π_2 of any compact full subset of D° (for of $D^\circ \cup \{z_f\}$) is homeomorphically mapped by π_1 to a compact full set. Since $f(D) = J((D^\circ)^c)$ is homeomorphic to a closed disc it follows that the sets $f^n(D)$, $n \geqq 0$ (and hence also their intersection) are all compact and full. Now for each $n \geqq 1, f^{n+1}(D)$ is connected if and only if $f^n(D)$ is both connected and contains a backward critical value. Since the intersection of a decreasing chain of compact connected sets is connected the result follows. In the case that $f^n(D)$ contains no backward critical value for some n, each component of $f^m(D)$, $m \geqq n$, contains two components of $f^{m+1}(D)$ and hence the attractor has uncountably many components. □

It should be mentioned that there is some dependence of attractor Λ_+ and repeller Λ_- on the choice of directionality D, and that for some choices of D the complement Ω of $\Lambda_+ \cup \Lambda_-$ may only be part of the (maximal) *regular set* $R(f)$, the set of *all* points where f acts discontinuously.

When D is a *Jordan* directionality (a topological disc, with Jordan curve boundary, as in Theorem 4), the set Ω is always a connected fully-invariant subset of $R(f)$, and is hence contained in a fully-invariant component $R_0(f)$ of $R(f)$. Any other component of $R(f)$ cannot be fully-invariant because it is contained in D^c or JD^c and hence not fixed by J. In the non-contact case, the fact that the appropriate branches of iteration of the correspondence fail to be equicontinuous on the boundaries of Λ_+ and Λ_- guarantees that $\Omega = R_0(f)$, and that Λ_+, Λ_-, and Ω are independent of the choice of D (among Jordan directionalities). In the contact case, however, it is possible to find Jordan D such that Λ_+ and Λ_- are "decorated" by pieces (e.g. "horodiscs") from $R_0(f)$, tacked on at the orbit of the contact point, so that Ω becomes a proper subset of $R_0(f)$.

If we drop the restriction that D be a *Jordan* directionality, we can find the opposite situation where now $R_0(f)$ becomes a proper subset of Ω. For example, starting with a Jordan directionality D for which the repeller, Λ_-, contains an attracting periodic orbit which is not super-attracting (under forward iteration of a branch of f), we can modify D by adding to it small disc neighbourhoods along the cycle (mapping to subdiscs under the one branch of f) and subtracting from D the images under J of these subdiscs. Thus $D \cap JD$ is enlarged by a factor of several small annuli. In this case we see that Ω contains a fully-invariant system of components in addition to $R_0(f)$—namely the basin of attraction of the periodic cycle, together with its image under J, but minus the cycle itself (now belonging to the attractor) and its global orbit.

We are now in a position to define the "connectedness locus" $\mathscr{C}$ for reversible maps of triples f with graph a sphere. We say $f \in \mathscr{C}$ if the regular set $R(f)$ contains a fully invariant component which is either annular or simply-connected. If we define $\mathscr{D}$ to be the set of f having an equivariant Jordan directionality or contact directionality then for $f \in \mathscr{C} \cap \mathscr{D}$ every such directionality gives rise to a connected attractor and repeller We conjecture that the set of a such that the correspondence defined by (1.1) lies in $\mathscr{C} \cap \mathscr{D}$ is (or at least contains a component which is) a homeomorphic copy M of the Mandelbrot set $\{c : q_c^n(0) \not\to \infty$ as $n \to \infty\}$. At the end of this paper we display a computer plot (Fig. 14) which indicates that there is such a copy of the Mandelbrot set contained in the disc $|a-4| \leqq 3$.

4 The proofs of Theorems 1 and 2

Let Δ be the closed subset of the complex plane bounded by the unit circle, the real axis from $-1/2$ to $+1$, and the line $\Re e(z) = -1/2$, $\Im m(z) \leqq 0$ see (Fig. 2). Then Δ is a fundamental domain for the action of $\langle I_1, I_2 \rangle$ on Γ, where I_1 and I_2 are the involutions $Z \mapsto -1-Z$ and $Z \mapsto -Z/(Z+1)$ respectively (as in Sect. 2). The projected image of Δ onto the dynamical plane, $D = \pi_2(\bar{\Delta})$, is as illustrated in Fig. 3 (for a real), namely the complement of the left-hand heart-shaped region. The precise boundary of D can be found by applying (2.30) (for the case $k=1$) to the boundary of Δ: it crosses the real axis at $z = -1$ at an angle of $\pi/3$ and at $z=0$ orthogonally. The correspondence maps D one-to-two onto the right-hand heart-shaped region (shown cross hatched in Fig. 3). The (double) point -1 has unique image 1, the point 0 has images 0 and $3/(2+a)$, and the two images of the upper half of the boundary ∂D are the upper half of the boundary of the cross-hatched region and the straight line segment from $3/(2+a)$ to 1. Similarly the images of the lower half of ∂D are the lower half of the boundary of the cross-hatched region and the same straight line segment.

Lemma 5 *For* $0 < k \leqq 1$ *and* $a \neq 1$ *satisfying* $|a-4| \leqq 3$, *the region* $D = \pi_2(\bar{\Delta})$ *defines an equivariant (contact) directionality for the correspondence* $f: z \mapsto w$ *given by* (1.2) *and has a contact point* ($z_f = 0$) *if and only if* $k = 1$.

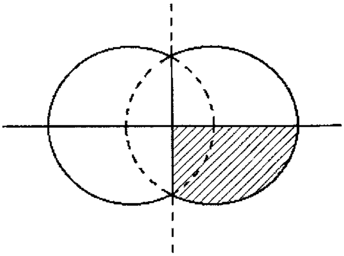


Fig. 2. A fundamental domain Δ for the action of $\langle I_1, I_2 \rangle$ on Γ

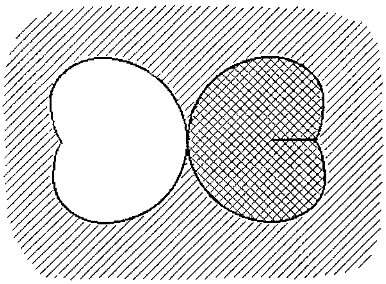


Fig. 3. $D = \pi_2(\bar{\Delta})$ and $f(D) \subset D$

Proof. To compute $\pi_2(\bar{\Delta})$ we decompose the projection $z = \pi_2(Z)$ into a chain of three maps:

$$u = \frac{1 - e^{2\pi i/3}Z}{1 - e^{-2\pi i/3}Z},$$

$$\zeta = \sqrt{k}\left(u + \frac{1}{u}\right)\left(= \sqrt{k}\,\frac{2 + 2Z - Z^2}{1 + Z + Z^2}\right), \qquad \text{and} \quad z = \frac{\zeta - 1}{a - \zeta}$$

(so that $\zeta = (az + 1)/(z + 1)$).

Thus the set of z in D is obtained by the criteria

$$Z \in \bar{\Delta} \cup I_2(\bar{\Delta}) \Leftrightarrow -\pi/3 \leqq \arg(u) \leqq \pi/3$$

$$\Leftrightarrow \Re e(\zeta) \geqq \sqrt{k + (\Im m(\zeta))^2/3}.$$

Since $f(D) = (JD^\circ)^c$ it follows that D is a (contact) directionality if and only if D° contains a fundamental domain for J (less contact point in ∂D). Now J has fixed points $z = 0$ and $z = \infty$ so this is true if and only the region

$$\{\zeta : \Re e(\zeta) > \sqrt{k + (\Im m(\zeta))^2/3}\}$$

contains a fundamental domain for the involution with fixed points $\zeta = 1$ and $\zeta = a$.

Finally observe that we can find such a fundamental domain (a round disc with points 1 and a on its rim) provided that a lies in the disc $|\zeta - 4| \leqq 3$ which is itself contained in the above region (when $k < 1$, and modulo the boundary point $\zeta = 1$ when $k = 1$). $\square$

Lemma 6 *For* $\frac{1}{4} < k \leqq 1$ *and* a *real,* $2\sqrt{k} < a \leqq a_{\max}(k)$, *the correspondence* $f: z \mapsto w$ *defined by* (1.2) *has an inverse branch defined on a real interval* $[+z_0, +z_1]$, *which is a unimodal map (i.e. satisfying* $z_1 \mapsto z_0 \mapsto z_0$ *and with a unique maximum). For* $a_{\min}(k) \leqq a \leqq a_{\max}(k)$ *this interval is invariant and contained in the attractor* $\bigcap_{n \geqq 0} f^n(D)$ *which itself is connected. The corresponding repeller* $\bigcap_{n \geqq 0} f^{-n}(JD)$ *is connected and contains the interval* $[-z_1, -z_0]$.
The formulae for $a_{\max}(k)$, z_0, z_1 *and* $a_{\min}(k)$ *are:*

$$a_{\min}(k) = 7 - 4\sqrt{3(1-k)},$$

$$z_0 = \sqrt{\frac{-a^2 + 8a - 1 - 6k - (a-1)\sqrt{a^2 - 14a + 1 + 48k}}{6(a^2 - k)}},$$

$$z_1 = \frac{2a^2 + 5a - 1 + 12k + (2a+1)\sqrt{a^2 - 14a + 1 + 48k}}{9a^2 - 3a + 12k + 3a\sqrt{a^2 - 14a + 1 + 48k}},$$

$$a_{\min}(k) = \frac{9}{2}\sqrt{k} - 2 + \sqrt{3\left(1 - 3\sqrt{k} + \frac{11}{4}k\right)}.$$

Proof. The unimodal branch of f^{-1} comes from the real segment $[-1/2, +1]$ in the graph Γ. This segment is mapped homeomorphically by π_1 to the interval

$$[\pi_1(+1), \pi_1(-1/2)] = \left[\frac{1-\sqrt{k}}{a-\sqrt{k}}, \frac{1+2\sqrt{k}}{a+2\sqrt{k}}\right]$$

and is mapped $2:1$ by π_2 (with critical point $Z = 0$) onto the interval

$$[\pi_2(+1), \pi_2(0)] = \left[\frac{\sqrt{k}-1}{a-\sqrt{k}}, \frac{2\sqrt{k}-1}{a-2\sqrt{k}}\right].$$

When $k = 1$ (and $a > 2$) the resulting branch of $f^{-1} = \pi_2 \circ \pi_1^{-1}$ is a unimodal map on the former interval $[0, 3/(a+2)]$, but when $k < 1$ there is a sub-interval $[+z_0, +z_1]$ on which this branch of f^{-1} is unimodal provided that its graph intersects the diagonal. The condition for this is $a \leqq a_{\max}(k)$, which is the condition that equation (1.2) have all real fixed points—the least positive one being $+z_0$. The value of z_1 can be computed using the fact that $\{+z_0, -z_0, +z_1\}$ is a backward triple, whence

$$\frac{az_0 - 1}{z_0 - 1} + \frac{a(-z_0) - 1}{(-z_0) - 1} + \frac{az_1 - 1}{z_1 - 1} = 0,$$

giving

$$z_1 = \frac{3 - (2a+1)z_0^2}{(a+2) - 3az_0^2}.$$

The condition for this interval to be mapped into itself by our branch of f^{-1} is that the maximum value $\pi_2(0) = (2\sqrt{k} - 1)/(a - 2\sqrt{k})$ not exceed z_1. An arduous calculation yields the precise condition to be $a \geqq a_{\min}(k)$. However when $k = 1$ the inequality $1/(a-2) \leqq 3/(a+2)$ easily gives us $a \geqq 4$ as the condition.

Finally observe that $a_{\min}(k) \leqq a \leqq a_{\max}(k)$ gives us an interval $[+z_0, +z_1]$ satisfying $f([+z_0, +z_1]) \supseteq [+z_0, +z_1]$ and therefore, being contained in D, is contained in $f^n(D)$ for all $n \geqq 0$. The result follows by Lemma 5, Theorem 4 and the fact that $[+z_0, +z_1]$ contains a backward critical value—namely $\pi_2(0)$. $\square$

Of interest is the observation that solving when the maximum value equals the original right-hand end point $\pi_1(-1/2) = (1 + 2\sqrt{k})/(a + 2\sqrt{k})$ gives $a = 4k$. This is the condition for f to be a *real* 2:2 correspondence of the interval $[-\pi_1(-1/2), \pi_1(-1/2)]$.

Figure 4 illustrates real sections of the graphs of the correspondences considered in the lemma above. In parts (d) to (f) of the figure the right-hand 'unimodal box' is the region $[z_0, z_1] \times [z_0, z_1]$ discussed in the proof. In parts (a) to (c) of the figure the right-hand and left-hand 'unimodal boxes' meet at the origin, the central point of each picture.

Proof of Theorem 1 Lemma 6 and Theorem 4 establish that the attractor and repeller are connected with connected complement. The contact condition $k = 1$ guarantees they meet in a single point whence their union Λ is also connected with connected complement. Hence the complement Ω is homeomorphic to a disc – indeed conformally equivalent to the upper half-plane via some Riemann map. Since all critical points are in Λ, the action of the correspondence on Ω is that of a group, and moreover the group is PSL(2, Z), by the methods of Sect. 2. By Theorem 3, any fundamental domain for the action of J on $D \cap JD$ is a fundamental domain for this group action on Ω. The hatched region in Fig. 5 is such a fundamental domain (for D as illustrated in Fig. 3). We shall show that the Riemann map ϕ from Ω to the complex upper half-plane H carries the action of the correspondence to the *standard* action of PSL(2, Z) (which has fundamental domain as illustrated in Fig. 6). We first observe that ϕ conjugates the two branches of f to two conformal homeomorphisms of the upper half-plane which, by Caratheodory, extend to the boundary, and hence, by the Schwarz reflection principle, extend to two Möbius transformations of the sphere which generate a Fuchsian group. The only possible actions are from a one-real parameter subfamily of the moduli space of representations of $C_3 * C_2$. Furthermore we can rule out the representations where the two branches of f correspond to elliptic Möbius transformations since any fixed points of f belong to the attractor or repeller. We normalize ϕ so that the regular fixed point (∞) of J is sent to i (thus $\phi J \phi^{-1}$ is $z \mapsto -1/z$) and so that the double point -1 has as image under ϕ a point on the unit circle with negative real part.

Using the fact a is real, the uniqueness of the Riemann map subject to this normalisation guarantees that complex conjugation in the dynamical plane is carried by ϕ to the inversion $z \mapsto 1/\bar{z}$. The images of the positive quadrant segment of the unit circle under the two branches of $\phi f^{-1} \phi^{-1}$ are geodesic arcs which cross at the double point and bound the region $\phi(\Omega \cap D)$ in the upper half-plane H, where $D = \pi_2(\Delta)$. Its image under the map $z \mapsto -1/z$ (and hence also under the reflection $z \mapsto -\bar{z}$) is $\phi(\Omega \cap JD)$. See figure 7, where the region $\phi(D \cap JD \setminus \{z_f\})$ is shown hatched. To show that our group action is genuinely that of the modular group it only remains to show that four bounding geodesics of this region strike the boundary of the upper half-plane at two points of coincidence – namely 0 and ∞.

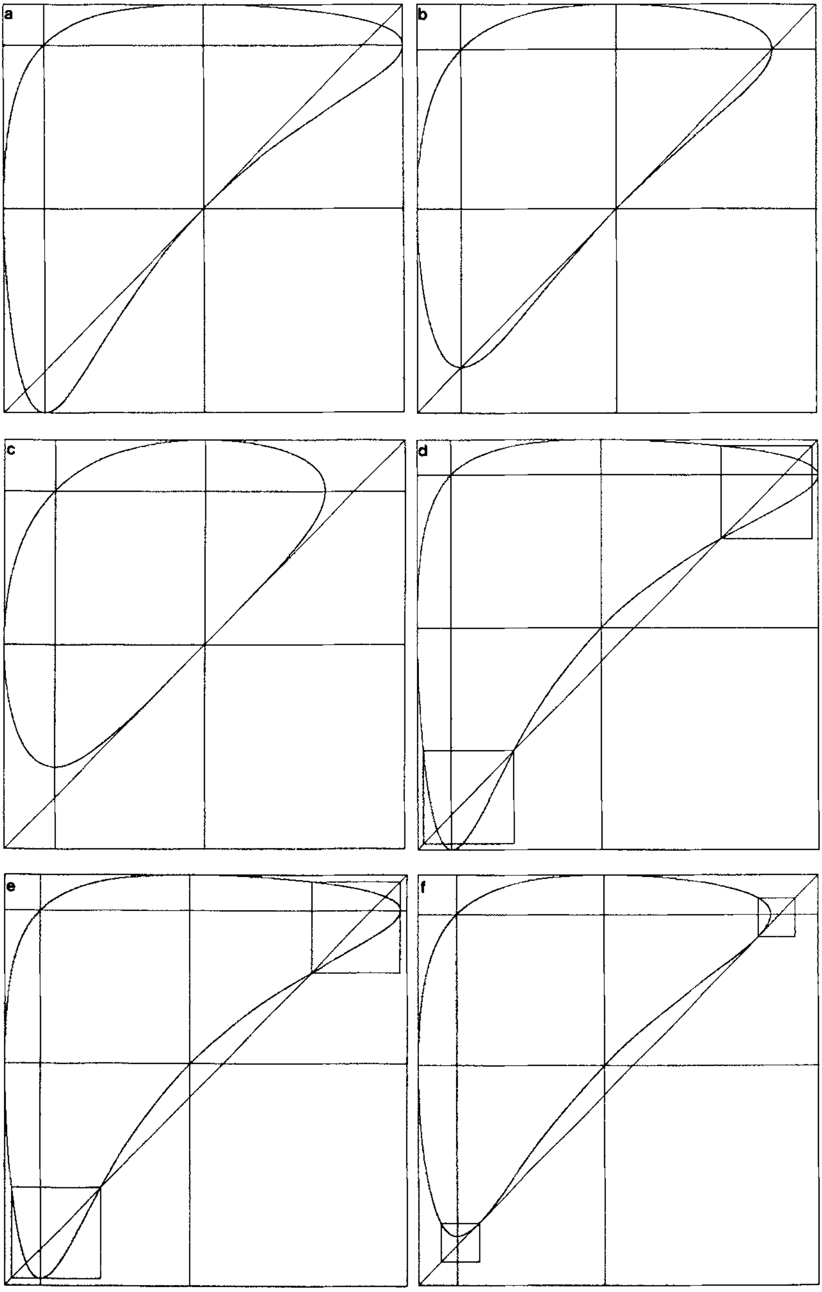


Fig. 4a–f. Real sections of graphs of correspondences in families (1.1) and (1.2): **a–c**: $k = 1$: $a = 4$, 5 and 7 *respectively*; **d–f**: $k = 0.8$: $a = 4k$, $a_{\min}(k)$ and $a_{\max}(k)$ respectively

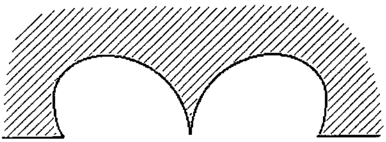


Fig. 5. A fundamental domain for the action of J on $D \cap JD$

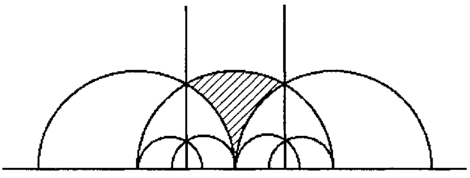


Fig. 6. A fundamental domain for the standard action of PSL(2, Z)

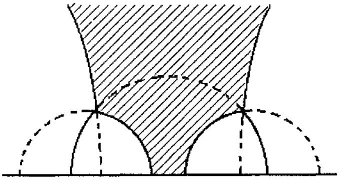


Fig. 7. $\phi(D \cap JD)$

Were this not the case then ϕ^{-1} would send the central hexagonal region, bounded by the four geodesics and the (extended) real line ∂H, to the quadrateral region $D \cap JD$. We obtain a contradiction by seeing that ϕ^{-1} converges to a constant limiting value (z_f) along the two real intervals bounding the hexagon. This follows because removing any open neighbourhood U of the contact point z_f from $D \cap JD$ gives a compact subset of Ω whose image under ϕ is a compact subset of H and therefore clear of some neighbourhood of the two real bounding intervals. The intersection of this neighbourhood with the hexagon therefore maps entirely inside U under ϕ^{-1}. Standard theory of analytic maps says that any holomorphic function of H having an interval along ∂H where it converges to a constant limiting value must itself be constant.

Finally, we must show that the dynamics of f on Λ are as claimed in Theorem 1(ii). But f, restricted to Λ_+ is a 1-to-2 surjection, and similarly f^{-1}, restricted to Λ_- is a 1-to-2 surjection. Since f is 2-to-2 and $\Lambda_+ \cap \Lambda_-$ is invariant (Theorem 3) the result follows. □

Remark. For the values of a in Theorem 1 the lift $\tilde{\Omega}$ of Ω to the graph of f is a pair of open discs, and the action of the lifted correspondence $\tilde{f}$ on $\tilde{\Omega}$ is that of the group PGL(2, Z). (See Example 2: $a = 4$ in Sect. 5).

Proof of Theorem 2. In the situation of Theorem 4 (Sect. 3), in the non-contact case, the branch of f^{-1} sending $f(D^\circ)$ to D° is *polynomial-like* in the sense of Douady and Hubbard [8]. It follows from their *Straightening Theorem* that this branch of f^{-1} is hybrid equivalent to a quadratic map $q_c : z \mapsto z^2 + c$, acting on a neighbourhood of its filled-in Julia set K_c, and that the hybrid equivalence sends Λ_+ to K_c by a quasi-conformal bijection. Since Λ_+ remains connected for $a_{\min}(k) \leqq a \leqq a_{\max}(k)$

by Lemma 6 we are done. It only remains to observe that Λ_- carries the same dynamics for f as does Λ_+ for f^{-1} (by the time-reversal symmetry J) and that Ω is homeomorphic to an annulus, since it is the complement of two disjoint connected and full sets Λ_+ and Λ_-. The correspondence acts discontinuously on Ω by Theorem 3 (Sect. 3). □

5 Three examples: $a = 4$, 5 and 7

These three values correspond to $c = -2, 0$ and $1/4$ on the Mandelbrot set. (Real sections were displayed in Fig. 4a–c.)We start with the example where the dynamics on K_c are the simplest, namely $c = 0$. For this we need the critical point $-2/(a+1)$ of (1.1) to be a fixed point, that is $-2/(a+1) = -1/(a-2)$, i.e. $a = 5$.

Example 1 $a = 5$

The critical and double points for (1.1) now have orbits

$$\begin{array}{cc} -1/3 \to 1/3 & -3/7 \to 0 \to 3/7 \\ -1 \to 1 & \end{array} \tag{5.1}$$

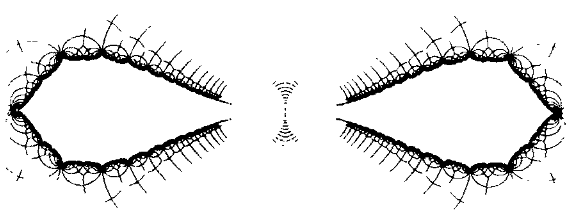


Fig. 8. Ω and Λ for (1.1), with $a = 5$. A tessellation of Ω by copies of a fundamental domain (not that of Fig. 5) is shown. The blank area at the centre of this plot (and subsequent plots) is due to inefficiency in the computer program near parabolic points

If we lift the correspondence (1.1) to its graph Γ, we obtain (by (2.32)) $\lambda = \infty$, and thus (by (2.24))

$$\frac{Z(Z+1)}{4} = \frac{W^2}{W+1} \tag{5.2}$$

as the equation of the lifted correspondence. The critical orbits of (5.2) are

$$\begin{tikzcd} \infty \arrow[loop left] \arrow[r] & -1 \arrow[r] & 0 \arrow[loop right] & {} \arrow[r] & -2 \arrow[r] \arrow[dr] & -1/2 \arrow[r] & {} \\ & & & {} \arrow[ur] & & 1 \arrow[u] \arrow[loop below] & {} \arrow[from=1-6, to=2-7] \end{tikzcd} \tag{5.3}$$

Under arbitrary iteration (forward, backward and mixed) the correspondences (1.1) with $a = 5$, and (5.2) exhibit the limit sets illustrated in Figs. 8 and 9. Figure 8 is the quotient of Fig. 9 under the projection π_2. In Fig. 8 we see a *regular domain* Ω, an *attractor* Λ_+ (the right-hand lobe), and a *repeller* Λ_- (the left-hand lobe). In Fig. 9 we see the lifts of these regions to Γ: here the attractor is the right-hand lobe, the repeller is outer region (containing ∞) and the left-hand lobe is the 'intermediate limit set' arising from the action of the dihedral group $\langle I_1, I_2 \rangle$ of order 6 (recall the discussion of the piecewise-linear example displayed in Fig. 1).

Observe that while the dynamics of f on Λ_- appears to be that of $q_0 : z \mapsto z^2$,the boundary of Λ_- is most certainly not *conformally* equivalent to a circle. Indeed it appears to be a circle R/Z with cusps 'pulled out' at the points 0, 1/2, 1/4, 3/4, . . . $p/2^n$, . . . ; we shall have more to say about this 'pulling out' process in Sect. 6.

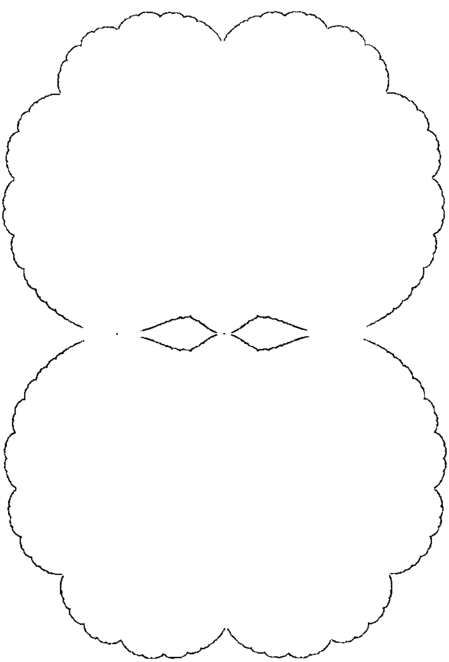


Fig. 9. $\tilde{\Omega}$ and $\tilde{\Lambda}$ for (5.2)

Example 2 $a = 4$

When $a = 4$, the graph correspondence (2.24) has $\lambda = 4$ (by (2.32)), and hence equation

$$\frac{4Z(Z+1)+1}{-Z(Z+1)+20} = \frac{W^2}{W+1}. \tag{5.4}$$

This has no critical points, only double points

$$\rightrightarrows -1/2 \to 0 \rightrightarrows \qquad \rightrightarrows \infty \to -2 \rightrightarrows \tag{5.5}$$

and (5.4) therefore factorises into two Möbius transformations

$$\left(W - \frac{2Z+1}{4-Z}\right)\left(W + \frac{2Z+1}{5+Z}\right) = 0. \tag{5.6}$$

Conjugating the matrices

$$A = \begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix} \quad B = \begin{pmatrix} -2 & -1 \\ 1 & 5 \end{pmatrix} \tag{5.7}$$

by

$$\begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix} \tag{5.8}$$

normalises them to

$$A' = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad B' = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \tag{5.9}$$

respectively. Thus (5.4) is an action of $\mathrm{PGL}(2, Z)$ on the Riemann sphere. the *forward limit set* (that of the semigroup generated by A and B) is $[-1/2, 1] \subset R$, the *backward limit set* is $[1, -2]$ (including ∞), and the *intermediate limit set* is $[-2, -1/2]$.

Passing to the dynamical plane, we obtain the correspondence (1.1) with $a = 4$. This has critical points and double points

$$\rightrightarrows -2/5 \to -1/2 \to 0 \to 1/2 \to 2/5 \rightrightarrows$$

$$\rightrightarrows -1 \to 1 \rightrightarrows \tag{5.10}$$

Note that it is *critically resolvable*, in the language of [3], in other words all forward critical values are also backward critical values. As remarked in [3], this is a necessary and sufficient condition for a 2:2 correspondence to lift to a pair of maps on its graph.

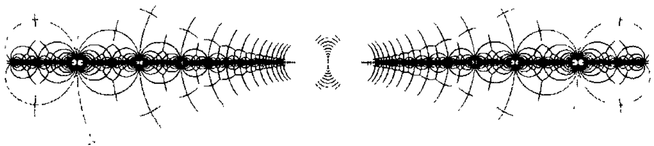


Fig. 10. Ω and Λ for (1.1), with $a = 4$

From our knowledge of the lift (5.4) we deduce that on the dynamical plane the attractor Λ_+ is the interval $[0, 1/2] \subset R$, and the repeller Λ_- is $[-1/2, 0]$. The regular domain Ω is $\bar{C} \setminus [-1/2, 1/2]$. A computer plot is displayed in Fig. 10. Restricted to Λ_- the correspondence has the dynamics of $z \mapsto z^2 - 2$ on $K_{-2} = [-2, 2]$: this follows at once from the fact that $[-1/2, 0]$ is mapped onto itself two-to-one, with the critical point $-2/5$ being sent to the end point $-1/2$.

Example 3 $a = 7$

At the value $a = 7$ the correspondence (1.1) has just one fixed point, the point $z = 0$, and the dynamics of f on Λ_- is like that of $z \mapsto z^2 + 1/4$ (which has a unique fixed point at $z = 1/2$). A computer picture (Fig. 11) suggests that in this case the boundary of Λ_- is a *quasi-circle* with angles $2\pi/3$ rather than the cusps of the $a = 5$ case. We are grateful to Curt McMullen for suggesting the following explanation.

The fixed point of the Blaschke product corresponding to $z \mapsto z^2 + 1/4$ has the local dynamics of $z \mapsto z - z^3$, that is to say the fixed point has neighbourhood a 'flower' with two petals, one on either side of the Julia set. For $z \mapsto z - z^3$ an orbit approaching the fixed point does so like $1/\sqrt{n}$, whereas for $z \mapsto z/(z + 1)$ it does so like $1/n$. It follows that the dynamics of $z \mapsto z - z^3$ and the modular group can be mated along a boundary which at each orbit point has angle $4\pi/3$ on one side (that corresponding to $z \mapsto z - z^3$) and $2\pi/3$ on the other side (that corresponding to the modular group).

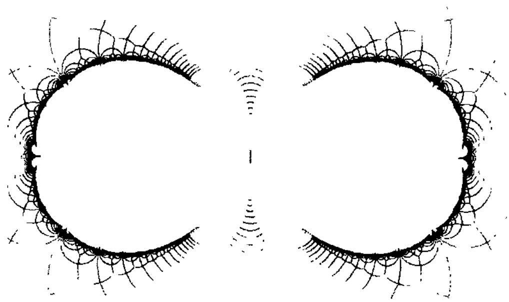


Fig. 11. Ω and Λ for (1.1), with $a = 7$

6 Possible generalisations

There are two main questions to consider. These are whether it is possible to prove that Λ_+ and Λ_- are homeomorphic to K_c in the situation of Theorem 1 (not just in the perturbed situation of Theorem 2), and what we can say about the set of *complex* values of a for which analogous results to these theorems hold.

Any approach to the first question would seem to require a generalisation of the Douady-Hubbard theory of *polynomial-like mappings* [8] to a theory of *pinched polynomial-like mappings*. We define a map f to be *pinched quadratic-like* if f is a map of degree 2 from a closed disc D onto a disc $f(D)$ containing D, such that $\partial f(D)$ meets ∂D at a single point P, which is a parabolic fixed point of f, and if f is holomorphic on D except at the other inverse image Q of P, where it has a square root singularity (see Fig. 12(a)). This is exactly the situation we have with our equivariant contact directionalities of Sect. 4. Given a pinched quadratic-like map, the inverse images of the outer pinched annulus fit together as illustrated in Fig. 12(b). We conjecture that if we blow up each pinch point in this figure to a line segment, and blow up f appropriately, it should be possible to find a complex structure for which the blown-up f is holomorphic, and hence is a genuine quadratic-like map (Fig. 12(c)). It would follow that our 'cusped' filled-in Julia sets Λ_+ and Λ_- would be obtained by taking filled-in quadratic Julia sets K_c and 'pulling out cusps' by contracting appropriate segments of external rays to points. However the construction of such a theory seems quite a difficult technical exercise. A variation on this approach would be to excise an orbit of horodiscs from the

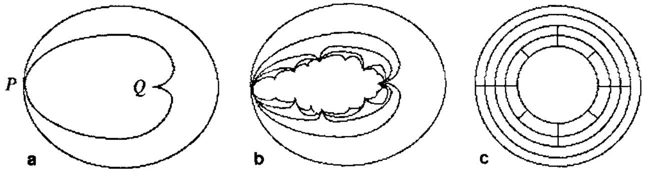


Fig. 12a–c. A pinched quadratic-like map: **a** $D \subset f(D)$; **b** a sequence of inverse images of D; **c** the conjectured 'blown-up' version

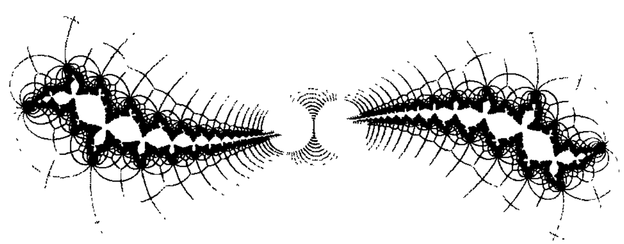


Fig. 13. Ω and Λ for (1.1), with $a = 4.54 + 0.44i$

regular region Ω for a correspondence of the type considered in Theorem 1, and, by making appropriate identifications of pairs of points on the resulting boundary, construct a topological $2:2$ corresponding which is a reversible map of triples, with graph a sphere, and having an equivariant *non-contact* directionality, but with unchanged dynamics on the (now disjoint) attractor and repeller. The new Ω could then be equipped with a conformal structure preserved by the new correspondence. If one could overcome the remaining (difficult) problem of extending this conformal structure to the new $\Lambda_+ \cup \Lambda_-$ one would have conjugacy to a holomorphic $2:2$ correspondence and could then use Theorem 2 to deduce that the original Λ_+ and Λ_- were homeomorphic to filled-in Julia sets of quadratic maps.

As to the second question, that of realising the rest of the 'Mandelbrot set' for matings, we first remark that it is not too hard to compute 'landmarks', both on and off the real axis. For example, the next point to examine after $a = 4$, 5, and 7, is

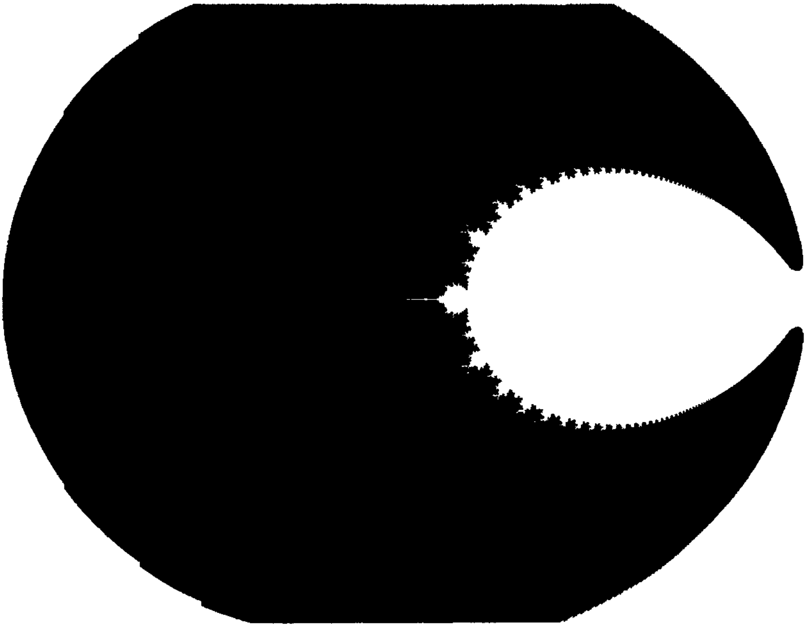


Fig. 14. The 'Mandelbrot set' M, for matings of quadratic maps with the modular group: the black disc containing M is the region $|a - 4| < 3$ in parameter space

$a = (3 + \sqrt{33})/2$, when the critical point has period 2, which corresponds to $c = -1$. Experiment suggests that every filled-in Julia set K_c, for c in the Mandelbrot set, can be realised in the family (1.1) (Fig. 13 displays another example, a mating of 'Douady's rabbit' with the modular group). All our proofs are adapted primarily to the case a real and the search for the best choice of fundamental domain Δ for the action of $\langle I_1, I_2 \rangle$ on Γ with the property that $\pi_2(\Delta)$ defines a directionality for the correspondences (1.1) and (1.2) is not attempted for a non-real. Nor do we have a proof that the action of $\mathrm{PSL}(2, Z)$ on Ω remains the *standard* action for such values of a (though Fig. 13 suggests that it does).

Our final illustration, Fig. 14, shows (in black) the region for the parameter a of (1.1) for which (using the co-ordinate ζ of Lemma 5) every branch of the forward orbit of the critical point $\zeta = -1$ eventually enters the disc $|\zeta - 4| \leq 3$. By Lemma 5, the disc $|a - 4| < 3$ is in $\mathscr{D}$, and by Theorem 4, the black region in Fig. 14 is certainly outside the 'connectedness locus' $\mathscr{C}$.

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