Invent math. 115, 405–429 (1994)

Inventiones mathematicae

Symplectic packings and algebraic geometry

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Oblatum 9-XI-1992 & 17-VI-1993

1 Introduction and main results

1.1 The symplectic packing problem

Consider all possible symplectic embeddings of k disjoint standard balls of equal radii into a given symplectic manifold M of the same dimension. Denote by $\hat{v}(M, k)$ the supremum of volumes which can be filled by such embeddings. If the volume of M is finite, set

$$v(M, k) = \hat{v}(M, k)/\text{Volume}(M)\,.$$

A basic aspect of the symplectic packing problem is to distinguish between the following two cases:

- $v(M, k) = 1$, that is there exists a *full filling*;
- $v(M, k) < 1$, that is there is a *packing obstruction.*

The history of this problem goes back to Fefferman and Phong [F-P], who raised a somewhat similar question in connection with the uncertainty principle in quantum mechanics. Our formulation is basically due to Gromov (see [G1] and also the discussion below).

Assume for a moment that M is the standard ball B^{2n}, $n \geqq 2$. Then the packing problem also makes sense if one replaces symplectic embeddings by volume-preserving or isometric ones. The symplectic version is somehow intermediate, and it is not clear apriori what answer one should expect: existence of full fillings as in the volume-preserving case, or packing obstructions as in the isometric case. Quite amazingly, the answer depends on the dimension and the number of balls. It was shown by Gromov [G1] that $v(B^{2n}, k) \leqq \frac{k}{2^n}$ when $k > 1$, that is for $1 < k < 2^n$ balls there are packing obstructions. In the present paper we prove that $v(B^{2n}, p^n) = 1$ for all positive integers p. Thus, for *infinite number of values of k full fillings do exist.* We present below (see 2.2.A and 3.1.A) two different approaches leading to existence of full fillings. Both of them are based on constructions of algebraic-geometric nature, namely symplectic blowing-up and symplectic branched covering. These constructions were invented by Gromov [G2] and described in more detail by Guillemin and Sternberg in [G-S]. A deep connection between the symplectic packing problem and blowing-up was established by the first author in [McD1, McD3].

Another way in which algebraic geometry comes into play is Gromov's theory of pseudo-holomorphic curves in symplectic manifolds [G1]. Note that Gromov's packing obstructions appeared as an application of this theory. Combining recent results of the first author on pseudo-holomorphic curves in 4-manifolds [McD2] with the theory of symplectic blowing-up one can obtain a detailed picture of interrelations between the symplectic packing problem and algebraic geometry of rational surfaces. It turns out that every *rational exceptional curve* on the blow-up of $\mathbb{C}P^2$ at k points gives an obstruction for symplectic packing of B^4 by k balls with suitable ratio of radii (see 1.3.C below). Moreover this correspondence is one-to-one if $k \leqq 9$ (see 1.3.E below). Using the classification of exceptional curves on del-Pezzo surfaces (see e.g. [D]) we give an exact solution of the symplectic packing problem by $k \leqq 8$ balls in dimension 4. In particular, we compute the values of $v(B^4, k)$ for $k \leqq 9$ (see 1.4 and the table below).

k	2	3	4	5	6	7	8	9
$v(B^4, k)$	$\frac{1}{2}$	$\frac{3}{4}$	1	$\frac{20}{25}$	$\frac{24}{25}$	$\frac{63}{64}$	$\frac{288}{289}$	1

Note that the estimates $v(B^4, 2) \leqq 1/2$, $v(B^4, 3) \leqq 3/4$, $v(B^4, 5) \leqq 20/25$ and $v(B^4, 6) \leqq \frac{24}{25}$ are due to Gromov [G1].

The problem of computation of $v(B^4, k)$ for $k \geqq 10$ seems to be very difficult. As we shall see in 1.4 the question of whether there are packing obstructions in this case is related to an old conjecture by Nagata [N] which was raised in connection with the 14-th Hilbert problem.

1.2 Basic notions

1.2.A. Preliminary definitions and notations. A *symplectic structure* on an even-dimensional manifold M^{2n} is a closed differential 2-form, say Ω, whose top power Ω^n is a volume form. Given a symplectic manifold (M, Ω) there always exists an almost-complex structure J on M such that $\Omega(\xi, J\xi) > 0$ for every non-zero $\xi \in TM$. In this situation we shall say that Ω *tames* J. If in addition J is integrable, and the form $\Omega(\xi, J\eta)$ is symmetric then Ω is called *Kähler* with respect to J.

Let $\mathbb{C}^n$ be the linear complex space with coordinates $z_k = x_k + iy_k$, $k = 1, \ldots, n$. The form $\omega = \sum_{k=1}^{n} dx_k \wedge dy_k$ is called the standard symplectic form on $\mathbb{C}^n$. Obviously ω is Kähler with respect to the complex structure on $\mathbb{C}^n$. Set $B^{2n}(\lambda) = \{z \in \mathbb{C}^n \mid |z| \leqq \lambda\}$. We shall refer to $(B^{2n}(\lambda), \omega)$ as to the *standard ball of radius* λ. We denote by $\coprod_{q=1}^{k} B^{2n}(\lambda_q)$ the disjoint union of k balls, and by $\varphi = \coprod_{q=1}^{k} \varphi_q : \coprod_{q=1}^{k} B^{2n}(\lambda_q) \to M$ a map whose restriction to the q-th ball coincides with $\varphi_q : B^{2n}(\lambda_q) \to M$. If φ is an embedding and $\varphi_q^* \Omega = \omega$ for all q then φ is called *a symplectic embedding*, or *a symplectic packing*.

Finally, we denote by σ_n the uniqe $U(n+1)$-invariant Kähler form on $\mathbb{C}P^n$ whose integral over $\mathbb{C}P^1$ is equal to π. It is not hard to show that the complement to a hyperplane endowed with such a form is symplectomorphic to the standard open unit ball in $\mathbb{C}^n$ (see [McD3] or Appendix below). In particular, the volume of $\mathbb{C}P^n$ with respect to the corresponding volume form $(\sigma_n)^n/n!$ is $\pi^n/n!$. This, of course, follows from de Rham theory, as well as by direct calculation. A geometric proof is given in 3.1.C below.

We refer the reader to [A-G] for the elements of symplectic geometry.

1.2.B. Symplectic blowing up and down. Basic to many of our constructions is the idea of symplectic blowing up and down. The process of blowing up and down in the complex category is well-known. Blowing up replaces a point x in V by the set Σ of all lines through this point. This set of lines is biholomorphic $\mathbb{C}P^{n-1}$ and is called an *exceptional divisor.* Conversely, blowing down replaces the exceptional divisor by a point.

In the symplectic category one must consider what happens to the symplectic form, and it turns out that the role of a point is played by a symplectically embedded standard ball. In fact, blowing up amounts to removing the interior of a symplectic ball and collapsing the bounding sphere to the exceptional divisor by the Hopf map. Similarly, to blow down one removes the exceptional sphere and glues in a ball. The radius λ of the ball corresponds to the cohomology class of the restriction of the blown-up form to the exceptional divisor Σ. Thus a large ball corresponds to a large exceptional divisor, and a small ball to a small divisor. Note also that, since balls are contractible, the symplectic blow up is diffeomorphic to the usual complex blow-up.

More details may be found in 2.1 below.

1.3 Symplectic packings of B^4 and exceptional curves on rational surfaces

By definition, a *rational exceptional curve* on a complex surface is a holomorphically embedded 2-sphere with self-intersection index equal to -1.

Denote by V_k a complex surface which is obtained from $\mathbb{C}P^2$ by complex blowing-up at k distinct points. We think of V_k as a fixed manifold equipped with one of a family of possible complex structures, corresponding to different choices of the blown-up points. Denote by $A, E_1, \ldots, E_k$ the standard basis in $H_2(V_k; \mathbb{Z})$ where $A = [\mathbb{C}P^1]$ and $E_1, \ldots, E_k$ are classes of exceptional divisors. Let $a, e_1, \ldots, e_k$ be the Poincaré-dual basis in $H_2(V_k; \mathbb{Z})$. (Thus $a(A) = 1$, $e_i(E_j) = -\delta_{ij}$.) Define $\mathbb{Z}_+$ (resp. $\mathbb{Z}_{\geq 0}$) as the set of all positive (resp. non-negative) integers.

Definition 1.3.A. A vector $(d, m_1, \ldots, m_k) \in \mathbb{Z}_+ \times \mathbb{Z}^k_{\geq 0}$ is called *exceptional* if for some V_k the class $dA - \sum_{q=1}^k m_q E_q$ is represented by a rational exceptional curve.

Definition 1.3.B. A vector $(\alpha, \mu_1, \ldots, \mu_k) \in \mathbb{Z}^{k+1}_+$ is called *Kähler* if for some V_k the class $\alpha a - \sum_{q=1}^k \mu_q e_q$ is represented by a Kähler form.

The next two results establish relations between these notions and symplectic packings.

Theorem 1.3.C. *Suppose that $B^4(1)$ admits a symplectic packing by k standard balls of radii $\lambda_1, \ldots, \lambda_k$. Then for every exceptional vector $(d, m_1, \ldots, m_k)$ the following inequality holds*:

$$\sum_{q=1}^{k} m_q \lambda_q^2 < d .$$

This theorem is proved below in 2.3. The idea behind the proof is very simple. The process of symplectic blowing-up constructs from each packing of $B^4(1)$ by balls of radii $\lambda_1, \ldots, \lambda_k$ a symplectic form on V_k in the class $\pi(a - \sum_{q=1}^k \lambda_q^2 e_q)$. Gromov's theory of pseudo-holomorphic curves implies that every exceptional vector may be represented by an embedded 2-sphere C in V_k which is symplectic with respect to this form, and the given inequality just expresses the fact that the integral of the symplectic form over C is positive. Thus the packing obstruction comes from the existence of this 2-sphere C. Since C will be realised as a pseudo-holomorphic curve (this just means that it is holomorphic with respect to some almost complex structure on V_k), we often use the language of complex geometry and call it a curve.

Theorem 1.3.D. (see [McD1], and 2.1.D below) *For each Kähler vector* $(\alpha, \mu_1, \ldots, \mu_k)$ *there exists a symplectic packing of* $B^4(1)$ *by k standard balls of radii* $\sqrt{\mu_1/\alpha}, \ldots, \sqrt{\mu_k/\alpha}$.

This follows immediately from symplectic blowing down. The following result shows that Theorem 1.3.C has a converse provided that $k \leqq 9$.

Theorem 1.3.E. *Let* $k \leqq 9$ *and let be* $\lambda_1, \ldots, \lambda_k$ *be positive real numbers such that*

$$\sum_{q=1}^{k} m_q \lambda_q^2 < d \tag{a1}$$

for every exceptional vector $(d, \mathrm{m}_1, \ldots, \mathrm{m}_k)$ *and*

$$\sum_{q=1}^{k} m_q \lambda_q^4 < 1\,. \tag{a2}$$

Then $B^4(1)$ *admits a packing by k balls of radii* $\lambda_1, \ldots, \lambda_k$.

Proof. By Nakai's criterion (see e.g. [F-M]), a cohomology class ρ on a complex surface V is represented by a Kähler form iff $\rho^2 > 0$ and $\langle \rho, [C] \rangle > 0$ for all complex curves $C \subset V$. For general V, it is very difficult to understand which homology classes are represented by complex curves. However, if $V = V_k$ is the blow-up of $\mathbb{C}P^2$ at k generic points for some $k \leqq 9$, then V_k is a *good and generic* surface in the sense of Friedman and Morgan [F-M], and, as they show [F-M, 3.4], it suffices to check that $\rho(C) > 0$ on the exceptional divisors and on the anti-canonical divisor. More precisely, one has to verify the following inequalities:

- $\langle \rho, [C] \rangle > 0$ for every rational exceptional curve C on V_k;
- $\rho \cdot \rho > 0$;
- $\rho \cdot c_1 > 0$, where $c_1 = 3a - \sum_{q=1}^k e_q$ is the first Chern class of V.

Observe that, if $\rho = a - \sum_{q=1}^k \lambda_q^2 e_q$, the two first inequalities are just reformulations of (a1) and (a2). In order to check the last one notice that

$$\left(\sum_{q=1}^{k} \lambda_q^2 \right)^2 < k \leqq 9$$

due to (a2). Thus any class ρ which satisfies the given inequalities is represented by a Kähler form, and so the needed packing exists by 1.3.D. □

Remark 1.3.F. The assumption (a2) has a straightforward geometric meaning. It states that the common volume filled by the images of the balls $B^4(\lambda_1), \ldots, B^4(\lambda_k)$ is less than the volume of $B^4(1)$.

It turns out that when $k \leqq 8$ the exceptional vectors form a finite set which can be exactly computed (see [D, p. 35]). Summing up the results of 1.3.C and 1.3.E, we obtain the following

Corollary 1.3.G. *There exists a symplectic packing of $B^4(1)$ by $2 \leqq k \leqq 8$ standard balls of radii $\lambda_1 \geqq \lambda_2 \geqq \cdots \geqq \lambda_k$ if and only if the following inequalities hold*:

(v) $\sum_{q=1}^{k} \lambda_q^4 < 1$ (*volume inequality*);
(c1) $\lambda_1^2 + \lambda_2^2 < 1$, *if* $k \geqq 2$
(c2) $\lambda_1^2 + \cdots + \lambda_5^2 < 2$, *if* $k \geqq 5$
(c3) $2\lambda_1^2 + \sum_{q=2}^{7} \lambda_q^2 < 3$, *if* $k \geqq 7$
(c4) $2\lambda_1^2 + 2\lambda_2^2 + 2\lambda_3^2 + \lambda_4^2 + \cdots + \lambda_8^2 < 4$, *if* $k = 8$
(c5) $2\sum_{q=1}^{6} \lambda_q^2 + \lambda_7^2 + \lambda_8^2 < 5$, *if* $k = 8$
(c6) $3\lambda_1^2 + 2\sum_{q=2}^{8} \lambda_q^2 < 6$, *if* $k = 8$.

1.4 Packings by equal balls in dimension 4 and Nagata's conjecture

The results in the case of $k \leqq 8$ *equal* balls are presented in the following table:

k	Volume estimate	The best curves-assisted estimate	Inequality giving the best estimate
2	$\lambda < \frac{1}{\sqrt[4]{2}}$	$\lambda < 1/\sqrt{2}$	(c1)
3	$\lambda < \frac{1}{\sqrt[4]{3}}$	$\lambda < 1/\sqrt{2}$	(c1)
4	$\lambda < 1/\sqrt{2}$	$\lambda < 1/\sqrt{2}$	(v) + (c1)
5	$\lambda < 1/\sqrt[4]{5}$	$\lambda < \sqrt{2/5}$	(c2)
6	$\lambda < 1/\sqrt[4]{6}$	$\lambda < \sqrt{2/5}$	(c2)
7	$\lambda < 1/\sqrt[4]{7}$	$\lambda < \sqrt{3/8}$	(c3)
8	$\lambda < 1/\sqrt[4]{8}$	$\lambda < \sqrt{6/17}$	(c6)

It turns out that the exceptional curves technique of 1.3.C does not generate packing obstructions for $k \geqq 9$ equal balls (or when $k = 4$; this will be discussed below in 1.4.C). More precisely we claim that for every exceptional vector $(d, m_1, \ldots, m_k)$ the inequality

$$\sum_{q=1}^{k} m_q \lambda^2 < d$$

is *worse* than the obvious volume inequality $k\lambda^4 < 1$ provided $k \geqq 9$. Indeed, we have to show that

$$\frac{d}{\sum_{q=1}^{k} m_q} > \frac{1}{\sqrt{k}}.$$

Recall that the first Chern class of V evaluated on a rational exceptional curve is equal to 1 (see [D]). In other words

$$3d - \sum_{q=1}^{k} m_q = 1 .$$

Therefore

$$\frac{d}{\sum_{q=1}^{k} m_q} > \frac{1}{3} \geqq \frac{1}{\sqrt{k}},$$

and our claim follows.

As we shall see in 2.3.B, one cannot get new packing obstructions for equal balls when $k \geqq 9$ by considering other curves in V_k, because none of the other curves which exist generically (in the sense of Fredholm theory) are in classes which generate new packing obstructions. Because it is not known whether the complex curves which exist for generic complex structures on V_k must in fact be generic in the sense of Fredholm theory, the symplectic packing problem is open for k equal balls, $k \geqq 9$, except in the special case when $k = p^2$ for some integer p. In this last case, a full filling does exist (see 1.4.C below). Interestingly enough, the existence of a full filling for $k \geqq 10$ would follow from an old *conjecture* of Nagata, which was formulated in connection with his construction of a counterexample to the 14-th problem of Hilbert (see [N]). In a slightly modified and weakened form, this conjecture states the following.

Conjecture [N] For every $k \geqq 9$ there exist k points on $\mathbb{C}P^2$ such that for every irreducible curve C on the corresponding blow-up V_k the following inequality holds:

$$d \geqq \frac{\sum_{q=1}^{k} m_q}{\sqrt{k}},$$

where $[C] = dA - \sum_{q=1}^{k} m_q E_q$.

Remark 1.4.A. Nagata's conjecture can be easily proved for the case $k = p^2$ where p is an integer (see [N] for a more precise result). Indeed, one just has to take the k points to lie on an irreducible smooth curve X of degree p. Then, if $\tilde{X}$ denotes the lift (or proper transform) of X in V_k, the desired inequality is equivalent to the requirement that $C \cdot \tilde{X} \geqq 0$. For $k = 2, 3, 5, 6, 7, 8$ the assertion of the conjecture is wrong (see [N]).

Theorem 1.4.B. *Assume that Nagata's conjecture is true for some k. Then there exists a full symplectic packing of B^4 by k equal standard balls.*

Proof. Let V_k be the blow-up of $\mathbb{C}P^2$ described in Nagata's conjecture. Take positive integers μ, α such that $1/\sqrt{k} > \mu/\alpha$ and μ/α is arbitrarily close to $1/\sqrt{k}$. Set $\rho = \alpha a - \sum_{q=1}^{k} \mu e_q \in H^2(V_k; \mathbb{Z})$. We claim that the class ρ is represented by a Kähler form. Indeed, due to a version of Nakai's criteria (see [F-M, 3.4]) it is sufficient to check that $\rho \cdot \rho > 0$ and $\langle \rho, [C] \rangle > 0$ for every irreducible curve C on V_k.

The first inequality is obvious. Let us verify the second one. Let C be an irreducible curve with $[C] = dA - \sum_{q=1}^{k} m_q e_q$. Then

$$\langle \rho, [C] \rangle = \alpha d - \mu \sum m_q = \mu d \left(\frac{\alpha}{\mu} - \frac{\sum m_q}{d} \right).$$

Notice that $\alpha/\mu > \sqrt{k}$ by definition and $(\sum m_q)/d \leqq \sqrt{k}$ according to Nagata's conjecture. Therefore $\langle \rho, [C] \rangle > 0$ and our claim follows.

Now the assertion of the theorem follows from 1.3.D. □

Remark 1.4.C. Combining 1.4.A with 1.4.B we obtain that, for any $p \geqq 1$, B^4 *admits a full filling by* p^2 *equal symplectic balls.*

1.5 Existence of full symplectic fillings by equals balls

It turns out that the phenomenon described in 1.4.C is a particular case of the following general fact. Set $V = \mathbb{C}P^{m_1} \times \cdots \times \mathbb{C}P^{m_d}$. Endow V with a symplectic structure $\Omega_\mu = \mu_1 \sigma_{m_1} \oplus \cdots \oplus \mu_d \sigma_{m_d}$, where $\mu_1, \ldots, \mu_d$ are positive numbers and σ_q is the standard symplectic form on $\mathbb{C}P^q$.

Theorem 1.5.A. *Let* $k_1, \ldots, k_d$ *be positive integers such that* $[k_1 : \ldots : k_d] = [\mu_1 : \ldots : \mu_d]$. *Then for every closed complex submanifold* $\Gamma \subset V$ *there exists a full filling of* $(V - \Gamma, \Omega_\mu)$ *by*

$$\frac{(m_1 + \cdots + m_d)!}{m_1! \cdot \ldots \cdot m_d!} k_1^{m_1} \cdot \ldots \cdot k_d^{m_d}$$

standard symplectic balls of equal radius.

There are two ways of proving this theorem, via fibrations as in 2.2 and via branched covers as in §3.

Corollary 1.5.B. *For every complex submanifold* $\Gamma \subset \mathbb{C}P^m$ *the manifold* $(\mathbb{C}P^m - \Gamma, \sigma)$ *admits a full filling by one standard symplectic ball.*

Corollary 1.5.C (cf. 3.1.B) *For every positive integer k the standard ball* B^{2m} *admits a full filling by* k^m *standard equal symplectic balls.*

Proof. Recall that if $\Gamma \subset \mathbb{C}P^m$ is a hyperplane then $(\mathbb{C}P^m - \Gamma, \sigma)$ is symplectomorphic to $(\operatorname{Int} B^{2m}, \omega)$. The needed assertion follows from 1.5.A. □

Corollary 1.5.D. (cf. 3.1.C) *For every positive integer k the product of m standard symplectic 2-spheres* $S^2 \times \cdots \times S^2$ *admits a full filling by* $m!k^m$ *standard equal symplectic balls.*

We will see in 2.1.E. below that Theorem 1.5.A continues to hold for certain singular submanifolds Γ. In particular, it holds when $\Gamma \subset \mathbb{C}P^m \times \mathbb{C}P^1$ is the union of the submanifolds $\mathbb{C}P^{m-1} \times \mathbb{C}P^1$ and $\mathbb{C}P^m \times \{\text{pt.}\}$. Since the rescaling $\sigma \mapsto \mu\sigma$ of the symplectic form on projective space corresponds to multiplying the radius of the ball by $\sqrt{\mu}$, we find:

Proposition 1.5.E. *For each positive integer k, the product* $B^{2m}(\lambda) \times B^2(\sqrt{k}\lambda)$ *may be fully filled by* $(m+1)k$ *standard equal symplectic balls.*

Corollary 1.5.F. *For each positive integer k, the product* $B^2 \times \cdots \times B^2$ *of m copies of the unit 2-disc admits a full filling by* $m!k^m$ *standard equal symplectic balls.*

Proof. This is proved by induction on m. Recall that the volume of the $2m$-ball $B^{2m}(1)$ is $\pi^m/m!$. (A proof is given in 3.1.C.) Thus the radius of the $m!k^m$ balls which fill $\times_1^m B^2$ is almost $1/\sqrt{k}$, and so the inductive step is accomplished by applying 1.5.E with $\lambda \to 1/\sqrt{k}$. Further details are left to the reader. □

Remark 1.5.G. Corollary 1.5.F implies that any compact symplectic manifold M may be asymptotically fully filled by equal balls, in the sense that

$$\lim_{k \to \infty} v(M, k) = 1 .$$

In fact, by using Darboux charts and the fact that the interior of $B^2(1)$ is symplectomorphic to the interior of a square, it is not hard to see that one can fill up as much of the volume of M as one wants by a finite number of polydiscs of the form $B^2(\lambda) \times \cdots B^2(\lambda)$. Thus it is enough to prove the statement for a polydisc of this kind. But this follows easily from 1.5.F.

1.6 Discussion

1.6.A. Maximal packings. A set of k symplectically embedded open balls of equal radii into (M, Ω) is called a *maximal packing* if the balls are pairwise disjoint and the volume filled by the balls is maximal, i.e. equal to $\hat{v}(M, k)$.

Question. Does a maximal packing exist when M is compact?

Direct geometric constructions of maximal packings for $M = \mathbb{C}P^n$ and $k \leqq n + 1$ are given in the Appendix by Yael Karshon. Other explicit constructions for full fillings of $M = \mathbb{C}P^n$ by $k = p^n$ balls have recently been found by Lisa Traynor and the authors (see [T]).

1.6.B. Symplectic packings in higher dimensions. Gromov's proof of the estimate $v(B^{2n}, k) \leqq k/2^n$ is based on the study of pseudo-holomorphic curves of degree 1. There is no obvious way to get better packing inequalities using curves of higher degree. It is also hard to generalise our indirect methods for constructing packings, since these rely on Nakai's criterion which is valid only in complex dimension 2. However, it is true that

$$v(B^{2n}, k) = k/2^n$$

for $1 < k \leqq 2^n$. This holds because B^{2n} can be fully filled by 2^n balls, and so it is enough to consider k of these balls.

1.6.C. Symplectic packings and symplectic capacities. Note that the symplectic packing problem is quite important even in case of $k = 1$ ball. Indeed, the quantity

$$c(M) = (\hat{v}(M^{2n}, 1))^{1/n}$$

is a *symplectic capacity* in the sense of Ekeland and Hofer [E-H]. In case $M = B^2(\lambda_1) \times \cdots \times B^2(\lambda_n)$ the value of $c(M)$ was computed in [G1] by pseudo-holomorphic curves methods as well as in [E-H] by quite different variational methods. It would be interesting to apply variational methods to symplectic packings by more than one ball.

1.7 Organisation of the paper

The rest of the paper is organised as follows. In §2 we formulate our general results concerning blowing up and down and prove 1.3.C, 1.3.D and 1.5.A. These results are based on an explicit local construction which is presented in §5. In §3 we describe how to produce symplectic packings using branched coverings. In particular we give another proof of 1.5.C and 1.5.D. In order to treat 1.5.E, we need a version of the Moser stability theorem which is valid for symplectic subvarieties. This problem is discussed in §4.

2 Blowing up and down and symplectic packings

In this section, we prove the results in §1 which involve blowing up and down, modulo certain technicalities which are relegated to §5. Our first aim is to make precise the relation between embedded balls and forms on the blow-up manifold. We then show how to construct packings using fibrations, and show how exceptional curves on blow-ups of $\mathbb{C}P^2$ lead to packing obstructions for the unit ball $B^4(1)$.

As we explained in 1.2.B, in the symplectic version of blowing up the role of a point is played by a symplectically embedded ball. Thus, given a symplectic embedding $\phi: B^{2n}(\lambda) \to V$, one constructs a symplectic manifold $(\bar{V}_\phi, \omega_\phi)$, which is diffeomorphic to the usual complex blow-up $\bar{V}$, by cutting out the interior of the ball $\operatorname{Im} \phi$ and collapsing its bounding sphere to the exceptional divisor via the Hopf map. In our present situation, we need to compare these manifolds $(\bar{V}_\phi, \omega_\phi)$ as ϕ varies. Although these manifolds are all diffeomorphic to the complex blow-up, it is hard to choose a diffeomorphism in a canonical way. Therefore, we will proceed a little differently, by standardizing the construction of the blow-up manifold and then isotoping the symplectic form to fit. The details of this construction are somewhat technical and so are postponed until §5. However, the main results are easily described.

As always there is an interplay between the complex and symplectic points of view. This reflects itself in the fact that we often consider families of symplectic forms with varying cohomology class. (Such families are called *deformations* or *pseudo-isotopies*.) Most often, these families arise by fixing the almost complex structure J, and considering a family of closed 2-forms which tame J. It is easy to check that the taming condition implies that these forms are non-degenerate, and hence symplectic. This might be thought of as the complex point of view. In contrast the symplectic point of view corresponds to families of forms ω_t, $0 \leqq t \leqq 1$, in a fixed cohomology class. By Moser's theorem, such families are *isotopies*, that is, there is a family of diffeomorphisms $\{f_t\}$ of the underlying manifold, with $f_0 = \mathbb{1}$, such that $f_t^* \omega_t = \omega_0$ for all t.

In the situation considered here, we will consider families of forms of varying cohomology class on blow-up manifolds. What will vary is the size of the forms on the exceptional divisors. This corresponds to changing the radius of the embedded balls. Thus the geometric point of view, in which the size of the balls is fixed, corresponds to the symplectic point of view in which we consider isotopies of forms, not general families.

2.1 Packings and deformations

Let (V, Ω, J) be a symplectic manifold such that Ω tames J. Suppose that J is integrable near the points $x_1, \ldots, x_k$, and let $(\bar{V}, \bar{J})$ be its complex blow-up at these points. Thus there is a holomorphic map $\Theta: (\bar{V}, \bar{J}) \to (V, J)$ which is a bijection over $V - \{x_1, \ldots, x_k\}$ and maps the exceptional divisors to the points x_q. As in 1.3, we write $e_1, \ldots, e_k$ for the cohomology classes on $\bar{V}$ which are Poincaré dual to the exceptional divisors.

Let us now consider what happens to the symplectic form. The symplectic form $\bar{\Omega}$ on the blow-up is constructed so that it restricts to a multiple $\mu\sigma_{n-1}$ of the standard form on $\Sigma \equiv \mathbb{C}P^{n-1}$. The crucial fact is that the Hopf map $\pi: S^{2n-1} \to \mathbb{C}P^{n-1}$ pulls σ_{n-1} back to a form on the unit sphere which extends to the standard form ω on $\mathbb{C}^n$. It follows that the exceptional divisor $(\Sigma, \mu\sigma_{n-1})$ has a deleted neighborhood $N(\Sigma) - \Sigma$ which is symplectomorphic to an annulus in $\mathbb{C}^n$ of the form

$$\{z \in \mathbb{C}^n | \sqrt{\mu} < |z| < \sqrt{\mu} + \delta\}.$$

Thus the correct way to blow down $(\Sigma, \mu\sigma_{n-1})$ is to remove a neighborhood of Σ and replace it by an embedded ball of radius $\sqrt{\mu} + \varepsilon$, for small ε. Conversely, one blows up a symplectic manifold by removing a symplectically embedded standard ball of radius $\lambda + \varepsilon$ and replacing it by a neighborhood of an exceptional divisor equipped with the form $\lambda^2\sigma_{n-1}$.

It follows that, if

$$\varphi = \coprod_{q=1}^{k} \varphi_q : \coprod_{q=1}^{k} (B(\delta_q + \varepsilon), \omega, i) \to (V, \Omega, J)$$

is a symplectic and holomorphic embedding of k small balls into V whose centers are mapped to the points $x_1, \ldots, x_k$, one can construct a symplectic form $\bar{\Omega}$ on $\bar{V}$ which is tamed by $\bar{J}$ and lies in the class

$$[\Theta^*(\Omega)] - \sum_{q=1}^{k} \pi\delta_q^2 e_q .$$

The point is that φ maps into a region of V on which J is integrable and Ω is Kähler. Thus one can take $\bar{\Omega}$ to be Kähler near the exceptional divisors, and equal to $\Theta^*(\Omega)$ away from them. This is the basic local construction, and it is described in detail in 5.1–5.4. In this situation, we say that $\bar{\Omega}$ *is constructed by symplectic blow-up from a symplectic, holomorphic embedding.*

We will see that this construction is reversible. In other words, when one blows down the exceptional divisor in $(\bar{V}, \bar{\Omega})$, one obtains a form on V which tames J and so is isotopic to Ω. (In fact, the blow-down form equals Ω outside a neighborhood of the exceptional divisor.) Thus this construction works very nicely for embeddings φ which are symplectic and holomorphic. Of course, these exist only if Ω is Kähler near the x_q and if the associated metric is flat near the x_q. In this situation, we will say that Ω is *J-standard* near the x_q. Our first result shows that this restriction is not important.

Proposition 2.1.A. *If the symplectic form Ω on V tames an almost complex structure J which is integrable near the points $x_1, \ldots, x_k$, then Ω is isotopic to a form Ω' which also tames J and is J-standard near the x_q.*

This is Proposition 5.5.A. Our next result shows that one can construct a symplectic form on the blow-up $\bar{V}$ given *any* symplectic embedding of balls.

Proposition 2.1.B. *Let*

$$\varphi = \coprod \varphi_q : (B(\lambda_q), \omega) \to (V, \Omega)$$

be a symplectic embedding of k balls into the symplectic manifold (V, Ω). Choose an almost complex structure J on V which is tamed by Ω and is integrable near k distinct *points* $x_1, \ldots, x_k$, *and let* $(\bar{V}, \bar{J})$ *be the corresponding complex blow-up. Then there exists a family* $\bar{\Omega}_t$ *of symplectic forms on* $\bar{V}$ *such that* $\bar{\Omega}_0$ *tames* $\bar{J}$ *and*

$$[\bar{\Omega}_1] = [\Theta^*\Omega] - \sum_{q=1}^{k} \pi\lambda_q^2 e_q .$$

To see this, consider the family of symplectic embeddings obtained by restricting φ to balls of radii $t\lambda_q$, for $\varepsilon \leqq t \leqq 1$. Blowing up this family, we obtain a deformation of symplectic forms on $\bar{V}$ which satisfies the desired cohomological condition. Moreover, it follows from Proposition 2.1.A that the initial form $\bar{\Omega}_0$, which corresponds to the "smallest" packing, can be constructed by symplectic blow-up from a symplectic, holomorphic embedding as described above. Therefore, it may be chosen so that it tames $\bar{J}$. Full details of the proof may be found in 5.5.

We now state the corresponding results about blowing down. As before, we assume that J is an almost complex structure on (V, Ω) which is integrable near the points $x_1, \ldots, x_k$ and that Ω tames J everywhere. We write $(\bar{V}, \bar{J})$ for the complex blow-up at the x_q.

Proposition 2.1.C. *Suppose that there exists a family* $\bar{\Omega}_t$ *of symplectic forms on* $\bar{V}$, *such that* $\bar{\Omega}_0$ *tames* $\bar{J}$, *the restrictions of all the forms* $\bar{\Omega}_t$ *to the exceptional divisors tame* $\bar{J}$, *and*

$$[\bar{\Omega}_t] = [\Theta^*\Omega] - \sum_{q=1}^{k} \pi\lambda_q^2(t) e_q ,$$

for suitable positive constants $\lambda_q(t)$, $0 \leqq t \leqq 1$. *Then the manifold* (V, Ω) *admits a symplectic embedding of k disjoint standard symplectic balls of radii* $\lambda_1, \ldots, \lambda_k$, *where* $\lambda_q = \lambda_q(1)$, *for all q.*

The idea of the proof is as follows. In view of 2.1.A we can assume that Ω is J-standard near the points $x_1, \ldots, x_k$. Let $\bar{\Omega}$ be the symplectic form on $\bar{V}$ which is constructed by the basic local construction described above. Since $\bar{\Omega}_0$ and $\bar{\Omega}$ both tame $\bar{J}$, we may, by extending the family $\bar{\Omega}_t$ by the linear family $s\bar{\Omega}_0 + (1-s)\bar{\Omega}$, suppose that $\bar{\Omega}_0 = \bar{\Omega}$. Then, the family of forms $\bar{\Omega}_t$ on $\bar{V}$ blows down to a family of forms Ω_t on V which, by the reversibility of blowing up and blowing down, starts at $\Omega_0 = \Omega$. Further, for each t, the form Ω_t admits a symplectically embedded set of k disjoint standard balls of radii $\lambda_1(t), \ldots, \lambda_k(t)$. The assumption on the cohomology class of the $\bar{\Omega}_t$ implies that all the forms Ω_t are cohomologous to the initial form $\Omega_0 = \Omega$. Thus Ω_1 is isotopic to Ω, and the desired result follows easily. For more details see 5.5.

As an immediate consequence of the previous result we obtain the following

Corollary 2.1.D. *Suppose that there exists a symplectic form* $\tilde{\Omega}$ *on* $\bar{V}$ *such that* $\tilde{\Omega}$ *tames* $\bar{J}$ *and*

$$[\tilde{\Omega}] = [\Theta^*\Omega] - \sum_{q=1}^{k} \pi\lambda_q^2 e_q .$$

Then (V, Ω) *admits a symplectic embedding of k disjoint standard symplectic balls of radii* $\lambda_1, \ldots, \lambda_k$.

Remark 2.1.E. In the applications of this corollary, (V, Ω, J) will be a Kähler manifold, so that J is integrable everywhere. Then the blow-up $(\bar{V}, \bar{J})$ is also a complex manifold, and the hypotheses will be satisfied by any Kähler form $\tilde{\Omega}$. Observe also that if in this situation Γ is a closed complex submanifold of

$V - \{x_1, \ldots, x_k\}$, then *the manifold* $(V - \Gamma, \Omega)$ *admits a symplectic embedding of k disjoint standard symplectic balls of radii* $\lambda_1, \ldots, \lambda_q$.

Indeed, we may assume that Γ does not meet the small holomorphic balls used to construct $\bar{\Omega}$. Then, if $\bar{\Omega}_t = t\bar{\Omega} + (1-t)\tilde{\Omega}$; the restriction of $\bar{\Omega}_t$ to the proper transform $\bar{\Gamma}$ of Γ tames $\bar{J}|_{\bar{\Gamma}}$. This implies that the restriction of the blow-down forms Ω_t to Γ tames $J|_\Gamma$ for all t. Hence $\Omega_t|_\Gamma$ is symplectic for all t, with constant cohomology class. Therefore the Moser stability theorem (see 4.1.B below), when applied to the pair (V, Γ), implies that $(V - \Gamma, \Omega_1)$ is symplectomorphic to $(V - \Gamma, \Omega)$. Take now $\lambda_q(t)$ as in 2.1.C. Then our assumption on Γ guarantees that for all t the form Ω_t admits a symplectic packing by k balls of radii $\lambda_1(t), \ldots, \lambda_k(t)$ and each of these balls does not meet Γ. Hence our claim follows.

As an example, one can take the pair (V, Γ) to be $(\mathbb{C}P^2, \mathbb{C}P^1)$, thereby proving Theorem 1.3.D.

If Γ is a singular complex subvariety of V, one cannot expect $(V - \Gamma, \Omega_1)$ to be symplectomorphic to $(V - \Gamma, \Omega)$. However, given any compact subset K of $V - \Gamma$, one can hope to show that (K, Ω_1) is symplectomorphic to a subset of $(V - \Gamma, \Omega)$, which is all one needs in order to construct full fillings of $(V - \Gamma, \Omega)$. In Proposition 4.1.C, we show that this is indeed the case when $V = \mathbb{C}P^n \times \mathbb{C}P^1$ and $\Gamma = (\mathbb{C}P^{n-1} \times \mathbb{C}P^1) \cup \mathbb{C}P^n \times \{z_0\}$. This proves Proposition 1.5.E.

2.2 Fillings via fibrations

In this section we apply the above results to prove Theorem 1.5.A on the existence of full fillings of certain products of projective spaces. The first result describes the general method. It will be convenient to write $\mathscr{E}$ for the subspace of $H^2(\bar{V}, \mathbb{R})$ which is general by the classes $e_1, \ldots, e_k$.

Theorem 2.2.A. *Let* (V, J) *be a closed n-dimensional complex manifold and let* $(\bar{V}, \bar{J})$ *be its blow-up at k distinct points* $x_1, \ldots, x_k$. *Let* Ω *be a symplectic form on V which tames J. Suppose that there exists a holomorphic map* $f\colon \bar{V} \to \mathbb{C}P^{n-1}$, *which induces a biholomorphism on each exceptional divisor, and is such that*

$$[\Theta^*\Omega] - [f^*\sigma] \in \mathscr{E}.$$

Then, for every closed complex submanifold $\Gamma \subset V - \{x_1, \ldots, x_k\}$, *the manifold* $(V - \Gamma, \Omega)$ *admits a full filling by k equal symplectic balls.*

Proof. Fix a real positive R, and set

$$\tilde{\Omega}(R) = (1 + R)^{-1}(\Theta^*\Omega + Rf^*\sigma).$$

Obviously, $\tilde{\Omega}(R)$ is symplectic form for each R which satisfies the assumptions of 2.1.D, 2.1.E with $\lambda_q^2 = (1 + R)^{-1}R$. Note that $\lambda_q \to 1$ when $R \to \infty$. Therefore $(V - \Gamma, \Omega)$ admits a packing by k equal standard balls of radius λ for all $\lambda < 1$.

It remains to check that this packing is arbitrary full, that is

$$\mathrm{Volume}(V, \Omega) = k \cdot \mathrm{Volume}(B(1), \omega) = k\pi^n/n!.$$

Geometrically, this assertion is quite clear since the volume which is not filled by the balls is equal to the volume of $\bar{V}$ with respect to $\tilde{\Omega}(R)$. As one can easily verify, the last quantity tends to 0 when R goes to infinity, because when R is very large the form is essentially a pull-back from $\mathbb{C}P^{n-1}$.

More formally, note that

$$n!\,\mathrm{Volume}(V,\Omega)=\langle[\Omega]^n,[V]\rangle=\langle[\Theta^*\Omega]^n,[\bar V]\rangle\,.$$

On the other hand, because f is a biholomorphism on each exceptional divisor, and because σ integrates to π on each projective line,

$$[f^*(\sigma)]=[\Theta^*\Omega]-\sum_{q=1}^{k}\pi e_q\,.$$

Therefore

$$0=\langle[f^*\sigma]^n,[\bar V]\rangle=\left\langle\left([\Theta^*\Omega]-\pi\sum_{q=1}^{k}e_q\right)^n,[\bar V]\right\rangle$$
$$=\langle[\Theta^*\Omega]^n,[\bar V]\rangle+k\langle(-\pi e_q)^n,[\bar V]\rangle\,.$$

Since e_q is Poincaré dual to an exceptional divisor,

$$\langle(-\pi e_q)^n,[\bar V]\rangle=-\pi^n=-n!\,\mathrm{Volume}(B(1),\omega)\,.$$

Therefore

$$\mathrm{Volume}(V,\Omega)=k\cdot\mathrm{Volume}(B(1),\omega)\,.$$

This completes the proof of the theorem. □

Proof of Theorem 1.5.A. Set $m=\sum_{q=1}^{d}m_q$. Let $x_q=[x_{q0}:\ldots:x_{qm_q}]$ be projective coordinates on $\mathbb{C}P^{m_q}$. Let $k_1,\ldots,k_d$ be positive integers such that $[k_1:\ldots:k_d]=[\mu_1:\ldots:\mu_d]$. For every $1\leqq q\leqq d$ take polynomials $P_{q1}(x_q),\ldots,P_{qm}(x_q)$ which are homogeneous of degree k_q.

Consider a map $F:V\to\mathbb{C}P^{m-1}$, $(x_1,\ldots,x_d)\to[F_1:\ldots:F_m]$, where

$$F_r(x_1,\ldots,x_d)=\prod_{q=1}^{d}P_{qr}(x_q)\,.$$

A simple argument shows that if the polynomials $\{P_{qr}\}$ are *generic* then the map F is well defined on the complement of exactly $\dfrac{(m_1+\cdots+m_d)!}{m_1!\cdots m_d!}k_1^{m_1}\cdot\ldots\cdot k_d^{m_d}$ points, and moreover F can be lifted to a map $f:\bar V\to\mathbb{C}P^{m-1}$, where $\bar V$ is the blow-up of V at these points. Note also that f induces a biholomorphism on each exceptional divisor.

Without loss of genericity we may assume that $\mu_q=k_q$ for the $q=1,\ldots,d$. We claim that in this case

$$[\Theta^*\Omega_\mu]=[f^*\sigma_{m-1}]+\sum_{q=1}^{k}\pi e_q\,.$$

Indeed, set $Q_r=\{pt\}\times\cdots\times\{pt\}\times\mathbb{C}P^{m_r}\times\{\mathrm{pt}\}\times\cdots\times\{\mathrm{pt}\}$. Choosing Q_r generically we can see that the restriction of F to Q_r is a polynomial map of degree k_r. Therefore $[(F|_{Q_r})^*\sigma_{m-1}]=k_r\sigma_{m_r}$. Our claim follows immediately.

The result now follows from 2.2.A. □

2.3 Deformations and exceptional curves

In this section we investigate the correspondence between curves on the blow-up of $\mathbb{C}P^2$ and packing inequalities for the unit ball. As in the complex case we shall define a *symplectic rational exceptional curve* in a symplectic 4-manifold as a symplectically embedded 2-sphere with self-intersection number equal to -1.

We start with the following result which is proved in [McD2, Lemma 3.1]. We shall sketch the proof here in order to illustrate certain properties of exceptional curves.

Proposition 2.3.A. *Let (M^4, Ω_0) be a closed symplectic manifold admitting a symplectic rational exceptional curve in a class $E \subset H_2(M; \mathbb{Z})$. Suppose that Ω_0 is included into a family of symplectic forms Ω_t, $t \in [0;1]$. Then (M, Ω_1) admits a symplectic rational exceptional curve in the class E. In particular, $\langle [\Omega_1], E\rangle > 0$.*

Sketch of the proof (see [McD2] for the details) Let $\mathscr{T}$ be the space of smooth almost complex structures on M. Let $\mathscr{F}$ be the space of smooth paths

$$\{J: [0;1] \to \mathscr{T} \mid J_t \text{ is tamed by } \Omega_t\}.$$

Note that $\mathscr{F}$ is open in $C^\infty([0;1], \mathscr{T})$, $\mathscr{F}$ is non-empty, and for every almost complex structure $\tilde{J}$ on M which is tamed by Ω_0 there exists a path $\{\tilde{J}_t\} \in \mathscr{F}$ with $\tilde{J}_0 = \tilde{J}$. Denote by C a Ω_0-symplectic exceptional sphere in the class E. Choose $\tilde{J}$ such that:

- $\tilde{J}$ is tamed by Ω_0;
- $\tilde{J}$ is *generic*;
- C is $\tilde{J}$-holomorphic (see [McD2, p. 690]).

Now choose a generic path $\{\tilde{J}_t\} \in \mathscr{F}$ with $\tilde{J}_0 = \tilde{J}$. Because $E \cdot E = -1$, there is, by Positivity of intersections, at most one J-holomorphic E curve for each J. Therefore, if there is no $\tilde{J}_1$-holomorphic sphere in class E, Gromov's compactness theorem (see [G1]) implies that for some $t^* \in [0;1]$ there exists a $\tilde{J}_{t*}$-holomorphic cusp-curve in the class $A_1 + \cdots + A_d = E$. Denote by c_1 the first Chern class of M^4. Since E is represented by an exceptional curve then $c_1(E) = 1$. Since $d \geqq 2$, there exists ℓ such that $c_1(A_\ell) \leqq 0$. Note that A_ℓ is represented by a holomorphic sphere which appears in a 1-parametric generic family. Thus a dimension counting argument (see [McD2]) shows that

$$1 + 2(c_1(A_\ell) + 2) - 6 \geqq 0, \quad \text{i.e.} \quad c_1(A_\ell) > 0.$$

This contradiction implies that there exists a $\tilde{J}_1$-holomorphic sphere in the class E. Note finally that every $\tilde{J}_1$-holomorphic sphere in the class E is embedded [McD2, (2.6)]. Thus it is a Ω_1-symplectic rational exceptional curve. This completes the proof. □

Proof of Theorem 1.3.C. Assume that $B(1)$, and therefore $(\mathbb{C}P^2, \sigma)$, admits a symplectic packing by k standard balls of radii $\lambda_1, \ldots, \lambda_k$. Let $\bar{V}$ be the blow-up of $\mathbb{C}P^2$ at k points, which are chosen so that the exceptional vector $(d, m_1, \ldots, m_k)$ is represented by a holomorphic rational exceptional curve C in $\bar{V}$. According to 2.1.B there exists a family $\bar{\Omega}_t$, $t \in [0;1]$, of symplectic forms on $\bar{V}$ such that $\bar{\Omega}_0$ tames the complex structure on $\bar{V}$ and

$$[\bar{\Omega}_1] = \pi a - \sum_{q=1}^{k} \pi \lambda_q^2 e_q.$$

Since $\bar{\Omega}_0$ tames the complex structure on $\bar{V}$, C is a *symplectic exceptional curve* with respect to $\bar{\Omega}_0$. Applying 2.3.A we have that $\langle [\bar{\Omega}_1], [C]\rangle > 0$, that is $d - \sum_{q=1}^{k} m_q \lambda_q^2 > 0$. This completes the proof. □

Remark 2.3.B. A similar argument works for curves C of higher genus. However, one cannot get any new packing inequalities for k equal balls in $B^4(1)$ or $\mathbb{C}P^2$ in this way. To see this, note first that we may suppose that $k > 9$ since our previous results show that exceptional curves detect all packing inequalities for $k \leqq 9$. We must consider the moduli space of parametrized (J, j)-holomorphic curves $f:(\Sigma_g, j) \to (V_k, J)$ as j varies in Teichmüller space and J varies in some suitable space of almost complex structures on V_k. In order for a curve C to have such a parametrization for a *generic* almost complex structure $\tilde{J}$, the formal dimension of the moduli space of all unparametrized $\tilde{J}$-curves must be non-negative. This dimension is just the Fredholm index of a suitable operator, and has the formula $2(c_1(C) + g - 1)$. (See [McD2, §2] for more details.) Thus, if $[C] = dA - \sum_{q=1}^{k} m_q E_q$, and $N = \sum_{q=1}^{k} m_q$ one has:

- $3d - N \geqq 1 - g$;
- $(d^2 - \sum m_q^2) - (3d - N) \geqq 2(g - 1)$;

where the second inequality comes from the adjunction formula

$$g \leqq 1 + 1/2(C \cdot C - c_1(C)) .$$

Using this, one easily finds that

$$\frac{d}{N} \geqq \frac{1}{\sqrt{k}},$$

which, as we saw in 1.4, corresponds to a packing inequality which is worse than the volume inequality when $k > 9$.

3 Branched coverings and symplectic packings

3.1 Symplectic packings via branched coverings

Recall that a covering $\alpha: X \to Y$ which is branched over a subset $Q \subset Y$ is called *regular* if the action of the group of deck transformations associated with the covering $\alpha: X - \alpha^{-1}(Q) \to Y - Q$ is free and discrete.

In this section, we construct full fillings using the following result.

Theorem 3.1.A. *Let (X, Ω, J) be a closed complex symplectic manifold of complex dimension n such that Ω tames J. Let $\alpha: X \to \mathbb{C}P^n$ be a regular holomorphic branched covering of order k. Assume that $[\alpha^*\sigma] = [\Omega]$. Then for every closed complex submanifold Γ, the manifold $(X - \Gamma, \Omega)$ admits a full symplectic filling by k equal standard balls.*

The basic idea of the proof is very simple. If $Q \subset \mathbb{C}P^n$ is the branching locus of α, one first shows that $\mathbb{C}P^n - (Q \cup \Gamma)$ can be fully filled by one ball. Since

$$\alpha: X - \alpha^{-1}(Q) \to \mathbb{C}P^n - Q$$

is a k-fold cover, this ball lifts to k disjoint balls in X. The only problem is that α does not preserve the symplectic form, so that the lifted balls need not be symplectically embedded. However, because α is holomorphic, it preserves the symplectic forms up to isotopy, and so one can isotop the lifted balls to make them symplectic.

Full details of this argument are given in 3.2. The rest of this section is devoted to examples.

Example 3.1.B (cf.1.5.C) Consider a holomorphic map

$$\alpha : \mathbb{C}P^m \to \mathbb{C}P^m ,$$

$$[x_0 : \ldots : x_m] \mapsto [x_0^k : \ldots : x_m^k] .$$

Obviously α is a regular branched covering of order k^m. Hence $\mathbb{C}P^m - \mathbb{C}P^{m-1}$, and therefore $B^{2m}(1)$, admits a full filling by k^m equal symplectic balls.

Example 3.1.C (cf.1.5.D) We shall identify $\mathbb{C}P^m$ with the projectivization of the space of homogeneous polynomials of two variables z, w of degree m. Define a map

$$\alpha : \times_1^m \mathbb{C}P^1 \to \mathbb{C}P^m ,$$

$$([x_{10} : x_{11}], \ldots , [x_{m0} : x_{m1}]) \mapsto \prod_{d=1}^{m} (x_{d0} z - x_{d1} w) .$$

Obviously α is a regular holomorphic branched covering of order $m!$. Therefore the product of m copies of $(\mathbb{C}P^1, \sigma)$ admits a full filling by $m!$ equal symplectic balls. It is clear from the proof of 3.1.A that each of these balls gives a full filling of $\mathbb{C}P^m$. Hence the volume of $\mathbb{C}P^m$ is $1/m!$ times the volume of the product, and so must equal $\pi^m/m!$.

3.2 Symplectic branched coverings (cf. [G2, G-S])

Given a branched covering, the pull-back of a symplectic form degenerates near the branching locus. The next result shows that such a degeneration disappears after a suitably chosen perturbation.

Proposition 3.2.A. *Let $(X, \tilde{\Omega}, \tilde{J})$ be closed symplectic complex manifolds of the same dimension such that $\Omega, \tilde{\Omega}$ tame J and $\tilde{J}$ respectively. Let $\alpha : X \to Y$ be a holomorphic covering which is branched over a subset $Q \subset Y$. Suppose that $[\alpha^*\Omega] = [\tilde{\Omega}]$. Then for every neighborhood $\mathscr{U}$ of Q there exists a symplectic form τ on X with the following properties*:

1) $\tau|_{X - \alpha^{-1}(\mathscr{U})} = \alpha^*\Omega|_{X - \alpha^{-1}(\mathscr{U})}$;
2) τ *tames* $\tilde{J}$;
3) $[\tau] = [\tilde{\Omega}]$.

Proof. Set $\mathscr{V} = \alpha^{-1}(\mathscr{U})$. Let $\mathscr{V}'$ be a neighborhood of $\alpha^{-1}(Q)$ such that $\bar{\mathscr{V}}' \subset \mathscr{V}$. Fix a Riemannian metric on X. Choose $\delta > 0$ such that

$$\alpha^*\Omega(\xi, \tilde{J}\xi) \geqq \delta |\xi|^2$$

for all $\xi \in T(X - \mathscr{V}')$. Let f be a bump function on X such that $f \equiv 0$ on $X - \mathscr{V}$ and $f \equiv 1$ on $\mathscr{V}'$. Note that $\tilde{\Omega} - \alpha^*\Omega = d\lambda$ for some one-form λ on X. Set

$$\tau = \alpha^*\Omega + \kappa d(f\lambda), \quad \text{where } \kappa > 0 .$$

We claim that τ is the needed symplectic form provided κ is sufficiently small. Let us divide the proof in several steps.

1) Observe that τ is closed and cohomologous to $\tilde{\Omega}$. Also it coincides with $\alpha^*\Omega$ on $X - \mathscr{V}$.
2) *Outside* $\mathscr{V}'$ we have

$$\tau(\xi, \tilde{J}\xi) \geqq \delta|\xi|^2 - \kappa c|\xi|^2 ,$$

where c depends only on f and λ. Therefore choosing κ sufficiently small we get that τ tames $\tilde{J}$ outside $\mathscr{V}'$.
3) *Inside* $\mathscr{V}'$, $\tau = \alpha^*\Omega + \kappa(\tilde{\Omega} - \alpha^*\Omega) = (1-\kappa)\alpha^*\Omega + \kappa\tilde{\Omega}$. Therefore τ tames $\tilde{J}$ inside $\mathscr{V}'$ if $\kappa < 1$.

Combining the results of (1–3), we obtain the assertion of the proposition. □

Proof of Theorem 3.1.A. Fix a hyperplane $S \subset \mathbb{C}P^n$ and $\varepsilon > 0$. There exists a neighborhood $\mathscr{W}$ of S such that $\mathbb{C}P^n - \mathscr{W}$ admits a symplectic embedding φ of the ball $B^{2n}(1-\varepsilon)$.

Suppose that the covering $\alpha: X \to \mathbb{C}P^n$ is branched over a set $Q \subset \mathbb{C}P^n$. Since $\alpha(\Gamma) \cup Q \neq \mathbb{C}P^n$ there exists a complex automorphism, say F of $\mathbb{C}P^n$ such that $F(S) = S$ and $F(\alpha(\Gamma) \cup Q) \subset \mathscr{W}$. Set $R = F(Q)$, $\beta = F \circ \alpha$. Then $\beta: X \to \mathbb{C}P^n$ is a holomorphic regular covering branched over R. Let $\mathscr{U}$ be a neighborhood of R such that $\mathscr{U} \subset \mathscr{W}$. Choose a symplectic form τ on X associated with β and $\mathscr{U}$ according to 3.2.A. Denote by G the group of deck transformations acting on $X - \beta^{-1}(R)$. Obviously G acts by τ-preserving diffeomorphisms on $X - \beta^{-1}(\mathscr{W})$. Let $\bar{\varphi}: B^{2n}(1-\varepsilon) \to X - \beta^{-1}(\mathscr{W})$ be the lift of φ. Then $\bar{\varphi}$ is a symplectic embedding with respect to τ. Moreover,

$$\mathrm{Image}(\bar{\varphi}) \cap \mathrm{Image}(g \circ \bar{\varphi}) = \emptyset \quad \text{for every } g \in G$$

because of the regularity of β. Thus $\coprod_{g \in G} g \circ \bar{\varphi}$ is a symplectic embedding of $k = \#(G)$ balls of radii $(1-\varepsilon)$ into $(X - \beta^{-1}(\mathscr{W}), \tau)$. Since $\Gamma \subset \beta^{-1}(\mathscr{W})$ by construction, we have the desired embedding into $(X - \Gamma, \tau)$.

It remains to show that we may replace τ here by Ω. By construction, the forms τ and Ω are cohomologous and both tame J. Thus they may be joined by the isotopy $\Omega_t = (1-t)\Omega + t\tau$, $t \in [0; 1]$. Because Γ is complex, the restriction of each form Ω_t to Γ is *symplectic*. Therefore the Moser stability theorem of 4.1.B implies that (X, τ) and (X, Ω) are symplectomorphic by a symplectomorphism which preserves Γ. Hence $(X - \Gamma, \Omega)$ admits a symplectic filling by k symplectic balls of equal radius $(1-\varepsilon)$.

Note that $\mathrm{Volume}(X, \Omega) = k\,\mathrm{Volume}(\mathbb{C}P^n, \sigma) = k\,\mathrm{Volume}(B(1), \omega)$. Therefore our filling can be made arbitrarily full. This completes the proof. □

4 Symplectic isotopies of subvarieties

If we want to fill open manifolds such as products of balls, it is useful to generalise results like 2.1.E and 3.1.A above to the case when Γ is a complex subvariety rather than a submanifold. The missing step is the analog of the symplectic neighborhood theorem for such subvarieties. In this section we first discuss the usual theorem, and then discuss an extension to subvarieties. Our main result is Proposition 4.1.C below which is used for the proof of 1.5.E.

Theorem 4.1.A (The symplectic neighborhood theorem) *Suppose given two symplectic embeddings $\phi_i: (\Gamma, \omega) \to (V_i, \Omega_i)$, $i = 0, 1$ and an isomorphism Φ between the pull-back symplectic vector bundles $(\phi_i^*(TV_i), \phi^*(\Omega_i))$. Then, any diffeomorphism from a neighborhood N_0 of $\phi_0(\Gamma)$ in V_0 onto a neighborhood N_1 of $\phi_1(\Gamma)$ in V_1 which induces the given bundle isomorphism Φ is isotopic to a symplectomorphism, through maps of pairs $(N_0, \phi_0(\Gamma)) \to (V, \phi_1(\Gamma))$ which induce Φ.*

This is the well-known Darboux–Weinstein theorem. It may be proved by Moser's argument (which is called the homotopic method in [A-G]): see [A-G, 2.1.5].

Corollary 4.1.B (Moser stability for pairs) *Let Ω_t, $0 \leqq t \leqq 1$, be an isotopy of symplectic forms on a compact manifold V which are all non-degenerate on the submanifold Γ. Then there is a family of diffeomorphisms $F_t: (V, \Gamma) \to (V, \Gamma)$ such that $F_0 = \mathbb{1}$ and $F_1^* \Omega_1 = \Omega_0$.*

Sketch of proof. When $\Gamma = \emptyset$, this is the usual Moser stability theorem, and may be proved by the homotopic method quoted above. Therefore, by applying it to the restrictions $\Omega_t|_{T\Gamma}$, we obtain an isotopy g_t of Γ such that $g_t^*(\Omega_t) = \Omega_0$ for all t. Let $\tilde{g}_t: (V, \Gamma) \to (V, \Gamma)$, $0 \leqq t \leqq 1$ be an extension of g_t. It is not hard to see that we may choose $\tilde{g}_t$ in the directions normal to Γ so that, for each t, $\tilde{g}_t$ also induces an isomorphism between the symplectic vector bundles $(TV|_\Gamma, \Omega_0) \to (TV|_\Gamma, \Omega_t)$. We are now essentially reduced to the situation considered in the symplectic neighborhood theorem, and the proof may be completed by the homotopy argument of [A-G]. □

We would like an analog of this corollary which holds when Γ has singularities. The part of the above proof which causes difficulties is the construction of a suitable extension $\tilde{g}_t$. We will consider the simplest case here, assuming that Γ is the union of two complex submanifolds **A, B** of V which intersect transversally. In this case, it is easy to see that 4.1.B cannot hold as stated. The problem occurs along the intersection $K = A \cap B$. For example, consider the Ω_t-symplectic orthogonal of TK in TA which we will denote by $TK^{\perp_t} \cap TA$, and the similar bundle $TK^{\perp_t} \cap TB$. If these are symplectically orthogonal when $t = 0$, they must remain so under any isotopy of the type $(F_t^{-1})^*\Omega_0$, while an arbitrary isotopy Ω_t need not have this property.

However, for our purposes we do not need the full strength of 4.1.B. We are given an isotopy Ω_t, $0 \leqq t \leqq 1$, of Kähler forms where Ω_0 is the given form and there is a packing of $(V - \Gamma, \Omega_1)$. In order to get a corresponding packing of $(V - \Gamma, \Omega_0)$, it would clearly suffice to find a family of diffeomorphisms F_t, $0 \leqq t \leqq 1$, such that $F_t^*(\Omega_t) = \Omega_0$ outside a neighborhood of Γ which is so small that it is disjoint from the balls in the packing.

A geometric approach to this problem is given in Lemma 3.11 of [McD4]. Here we present a more explicit argument which works in the special case of interest to us, and exploits the fact that B has a trivial normal bundle. (In fact, all we need is that K have a trivial normal bundle in A.)

Proposition 4.1.C. *Let $V = \mathbb{C}P^n \times S^2$, $A = \mathbb{C}P^{n-1} \times S^2$, $B = \mathbb{C}P^n \times \{z_0\}$ and $\Gamma = A \cup B$, and suppose that $\Omega_t, 0 \leqq t \leqq 1$; is an isotopy of symplectic forms starting at the standard form $\Omega_0 = \sigma_n \oplus \sigma_1$. If Ω_t is non-degenerate on A and B for all t, then, for every neighborhood $\mathscr{U}$ of $K = A \cap B$ there are diffeomorphisms*

$F_t : V \to V \to V, 0 \leqq t \leqq 1$ *such that*

- $F_0 = \mathbb{1}$;
- $F_t(\Gamma) = \Gamma$;
- $F_t^*(\Omega_0) = \Omega_t$ *outside* $\mathscr{U}$.

Proof. We divide the proof into several steps. We will call a deformation F_t which satisfies the first two conditions above an *allowable deformation.*

1) Let p, q be Darboux coordinates in a neighborhood D of the point z_0 in S^2, and let σ denote the standard symplectic form on B. If f, g are functions on B which vanish on K, consider the 2-form

$$\omega(f, g) = dp \wedge dq + \sigma + dp \wedge df + dg \wedge dq,$$

on $\mathbb{C}P^n \times D$. It is easy to check that this is symplectic with the correct orientation iff

$$1 + \{f, g\} > 0,$$

where $\{\,,\}$ denotes the Poisson bracket (see [A-G]) on B. We will say that pairs of functions (f, g) which satisfy these conditions are "good".

2) We next claim that there is an allowable deformation which takes the path Ω_t into a path Ω_t' which equals the normalized path $\omega(f_t, g_t)$ near K. To see this, first observe that by successive application of Corollary 4.1.B to K, A and B, we may arrange that the restriction of the family Ω_t to A and to B is constant. Then, there are closed 1-forms α_t and β_t on B such that

$$\Omega_t = dp \wedge dq + \sigma + dp \wedge \alpha_t + \beta_t \wedge dq$$

on $TV|_K$. Because Ω_t is standard on A, the restrictions of α_t and β_t to K must vanish, and hence these forms may be written as df_t, dg_t for suitable functions f_t, g_t on B which vanish on K. Thus $\Omega_t = \omega(f_t, g_t)$ on T_xV at all points $x \in K$.

We claim that, for each t, there is an allowable isotopy $G_{t,s}$ of V such that $G_{t,1}^*(\Omega_t) = \omega(f_t, g_t)$ near K. To see this, consider the path of symplectic forms

$$\tau_{t,s} = s\Omega_t + (1-s)\omega(f_t, g_t), \quad 0 \leqq s \leqq 1 .$$

The relative Poincare lemma* implies that there are 1-forms $\rho_{t,s}$ which vanish on $A \cup B$ and are such that

$$\frac{\partial \tau_{t,s}}{\partial s} = d\rho_{t,s} .$$

The usual Moser argument now constructs the desired isotopy near K. Finally, one extends this to an allowable ambient isotopy of V.

3) Let Σ_t be an isotopy of symplectic forms which starts at Ω_0 and such that all Σ_t are standard (that is coincide with $\omega(0,0)$) near K. Then Moser's argument implies that there exists an allowable deformation F_t such that $F_t^*\Omega_0 = \Sigma_t$.

Assume for a moment that we have a deformation of the original path Ω_t to Σ_t which is supported in a neighborhood $\mathscr{U}$. Then outside $\mathscr{U}$ each form $F_t^*\Omega_0$ coincides with Ω_t, and hence the diffeomorphisms F_t satisfy all the requirements of the proposition.

As we shall see in the next step such a deformation exists.

* The proof of this is not so obvious: for details and further discussion see [McR]. In fact, all we need here is that the deformation from Ω_t to Ω_t' has support near K, which will be achieved if $\rho_{s,t} = 0$ on K.

4) In view of step 2 we can assume without loss of generality that Ω_t coincides with some $\omega(f_t, g_t)$ near K. Hence in order to find the deformation of step 3 it suffices to show that *each good pair (f, g) can be deformed through good pairs to a good pair which vanishes near K*. Moreover, we need this deformation to depend smoothly on f, g and to have support in the given neighborhood $\mathscr{U}$ of K.

Let R be a smooth function on B which measures the square of the distance from K. Thus R vanishes on K together with its first derivatives, and $R > 0$ outside K. We may suppose that the set where $R < 1$ is contained in $\mathscr{U}$. We shall look for a deformation of the form $(H_s(R)f,\ H_s(R)g)$, $0 \leqq s \leqq 1$, where $H_s = (1-s) + sH$ for a suitable cut-off function H which vanishes near 0 and equals 1 for $R \geqq 1$.

The main observation is that the function

$$E = g\{f, R\} + f\{R, g\}$$

vanishes on K together with its first derivatives. Therefore, for some $c > 0$ and u, $0 < u < 1$, there is an estimate

$$E(x) \geqq -cR(x),$$

which holds when $R(x) \leqq u$. Choose δ between 0 and 1 so that $1 + \{f, g\} > \delta$ when $R \leqq 1$, and then choose H so that $H = 1$ for $R \geqq u$, and

$$H' \leqq \delta/2cR.$$

(This is possible since $\int_0^u 1/R = \infty$.)

We claim that H is as needed, that is, it gives good pairs $(H_s f, H_s g)$ for all s. To see this, observe that

$$1 + \{H_s f, H_s g\} = 1 + (H_s)^2\{f, g\} + H_s H_s' E.$$

Since $0 \leqq H_s \leqq 1$, and $H_s' = 0$ when $R > u$,

$$1 + (H_s)^2\{f, g\} \geqslant \delta$$

and

$$H_s H_s' E \geqq -cRsH_s H' \geqq -cRH' \geqq -\delta/2,$$

by construction. Our claim follows. This completes the proof. □

5 Blowing-up and down

This section proves the results stated in 2.1.

5.1 Local models

Let $(\mathbb{C}^n, \omega, i)$ and $(\mathbb{C}P^{n-1}, \sigma, j)$ be linear and projective complex spaces respectively endowed with the standard Kähler structures. We shall assume that $\langle[\sigma], [\mathbb{C}P^1]\rangle = \pi$. Let $\mathscr{L} \subset \mathbb{C}^n \times \mathbb{C}P^{n-1}$ be the incidence relation, that is $\mathscr{L} = \{(z, \ell) | z \in \ell\}$. Denote by pr_1 and pr_2 the natural projections of $\mathscr{L}$ to $\mathbb{C}^n$ and $\mathbb{C}P^{n-1}$ respectively. Note that pr_1 induces a biholomorphism $\mathscr{L} - \mathrm{pr}_1^{-1}(0) \to \mathbb{C}^n - \{0\}$, and pr_2 is a complex line bundle whose zero section is identified with $\mathrm{pr}_1^{-1}(0)$. Thus $\mathscr{L}$ is the blow up of $\mathbb{C}^n$ at the origin. We write $B(\lambda)$ for the ball $\{z \in \mathbb{C}^n | |z| \leqq \lambda\}$, and set $\mathscr{L}(\lambda) = \mathrm{pr}_1^{-1}(B(\lambda))$. For κ, $\lambda > 0$ define a Kähler form $\rho(\kappa, \lambda)$ on $\mathscr{L}$ as $\kappa^2 \mathrm{pr}_1^* \omega + \lambda^2 \mathrm{pr}_2^* \sigma$.

Proposition 5.1.A. *For every $\varepsilon > 0$, $\lambda > 0$ there exists a Kähler form $\bar{\tau} = \bar{\tau}(\varepsilon, \lambda)$ on $\mathscr{L}$ such that the following holds:*

- $\bar{\tau} = \mathrm{pr}_1^*(\lambda^2\omega)$ *on* $\mathscr{L} - \mathscr{L}(1+\varepsilon)$;
- $\bar{\tau} = \rho(1, \lambda)$ *on* $\mathscr{L}(\delta)$ *for some* $\delta > 0$.

Moreover $\bar{\tau}$ and δ can be chosen smooth with respect to ε and λ.

Proposition 5.1.B. *For every $\varepsilon > 0$, $\delta > 0$, $\lambda > 0$ there exists a Kähler form $\tau = \tau(\varepsilon, \delta, \lambda)$ on $\mathbb{C}^n$ such that the following holds:*

- $\mathrm{pr}_1^*(\tau) = \rho(\delta, \lambda)$ *on* $\mathscr{L} - \mathscr{L}(1+\varepsilon)$;
- $\tau = \lambda^2\omega$ *on* $B(1)$.

Moreover, τ can be chosen smooth with respect to ε, δ, λ.

Before proving the propositions, let us introduce the following notion. An embedding $F : \mathbb{C}^n - \{0\} \to \mathbb{C}^n$ is called *monotone* if in spherical coordinates $(u, r) \in S^{2n-1} \times (0; +\infty)$ it can be written as $(u, r) \mapsto (u, f(r))$ where f is a strictly increasing function. The following two observations are important for our purposes.

- *For every monotone embedding F the form $F^*\omega$ is Kähler (with respect to the usual complex structure i).*

To see this, observe that $F^*\omega$ is Kähler at the point x iff the tangent space to $x = (u, r)$ has a basis $v_1, iv_1, \ldots, v_n, iv_n$ such that $F^*\omega$ vanishes on all pairs except those of the form (v_q, iv_q). Such a basis may be found by taking v_1 to point in the radial direction, and $v_2, \ldots, v_n$ to be a basis for the complex vector space $T_u S^{2n-1} \cap iT_u S^{2n-1}$.

- *There exists a smooth family of monotone embeddings $h_\lambda : \mathbb{C}^n - \{0\} \to \mathbb{C}^n - B(\lambda)$ such that $\mathrm{pr}_1^* h_\lambda^* \omega = \rho(1, \lambda)$.*

This remark is due to [G-S]. In polar coordinates, we may take h_λ to be the map $h_\lambda(u, r) = (u, (r^2 + \lambda^2)^{1/2})$.

Proof of Proposition 5.1.A. Take $\delta > 0$ such that $h_\lambda(B(\delta)) \subset B(\lambda + \lambda\varepsilon/2)$. Using a suitable smoothing procedure, one can find a monotone embedding F such that

$$F(z) = \lambda z \quad \text{for} \quad |z| > 1 + \varepsilon, \quad \text{and}$$

$$F(z) = h_\lambda(z) \quad \text{for} \quad |z| \leqq \delta .$$

Obviously the form $\bar{\tau} = \mathrm{pr}_1^*(F^*\omega)$ has the needed properties. □

Proof of Proposition 5.1.B. Note that $\rho(\delta, \lambda) = \delta^2 \rho(1, v)$ for $v = \lambda/\delta$. Using a suitable smoothing procedure, one can find a monotone embedding G such that

$$G(z) = vz \quad \text{for } |z| \leqq 1, \quad \text{and}$$

$$G(z) = h_v \quad \text{for } |z| \leqq 1 + \varepsilon .$$

Obviously, the form $\tau = \delta^2 G^*\omega$ has the needed properties. □

5.2 Global setting

Let (V, J) be an almost complex manifold such that J is integrable near the k points $x_1, \ldots, x_k$, and let $\Theta : (\bar{V}, \bar{J}) \to (V, J)$ be its blow-up at these points. Throughout the following discussion, we will assume that J and the points x_q are fixed. Consider a holomorphic embedding $\varphi = \coprod_{q=1}^k \varphi_q : \coprod (B(1 + 2\varepsilon_q), i) \to (V, J)$

such that $\varphi_q(0) = x_q$ for $1 \leqq q \leqq k$ and $\varepsilon_q > 0$. (Note that J has to be integrable on the image of φ.) Denote by $\bar{\varphi}_q$ the lifting of φ_q to a holomorphic map $\mathscr{L}(1 + 2\varepsilon_q) \to \bar{V}$ such that the following diagram is commutative:

$$\begin{array}{ccc} \mathscr{L}(1 + 2\varepsilon_q) & \xrightarrow{\bar{\varphi}_q} & \bar{V} \\ \downarrow \mathrm{pr}_1 & & \downarrow \Theta \\ B(1 + 2\varepsilon_q) & \xrightarrow{\varphi_q} & V. \end{array}$$

5.3 Symplectic blowing-up

Suppose that V is endowed with a symplectic form Ω such that $\varphi_q^* \Omega = \lambda_q^2 \omega$ for some $\lambda_q > 0$. Define a symplectic form $\bar{\Omega}$ on $\bar{V}$ as follows:

$$\bar{\Omega} = \begin{cases} \Theta^* \Omega & \text{on } \bar{V} - \bigcup_{q=1}^{k} \bar{\varphi}_q(\mathscr{L}(1 + \varepsilon_q)) \\ (\bar{\varphi}_q^*)^{-1} \bar{\tau}(\varepsilon_q, \lambda_q) & \text{on } \bar{\varphi}_q(\mathscr{L}(1 + 2\varepsilon_q)) \quad \text{for all } q = 1, \dots, n, \end{cases}$$

where $\bar{\tau}$ is defined in 5.1.A. We shall say that $\bar{\Omega}$ is obtained from Ω by the *symplectic blowing-up* associated with φ. Let us describe several properties of the symplectic blow-up which immediately follow from 5.1.A.

5.3.A. If Ω tames J (resp. Ω is Kähler) then $\bar{\Omega}$ tames $\bar{J}$ (resp. $\bar{\Omega}$ is Kähler).

5.3.B. Recall that $H^2(\bar{V}, \mathbb{R}) = H^2(V, \mathbb{R}) \oplus \mathscr{E}$, where $\mathscr{E}$ is a real linear vector space generated by the classes $e_1, \dots, e_k$ which are Poincaré dual to the classes of exceptional divisors $\Theta^{-1}(x_1), \dots, \Theta^{-1}(x_k)$. With these notations

$$[\bar{\Omega}] = [\Theta^* \Omega] - \sum \pi \lambda_q^2 e_q .$$

5.3.C. For each $q = 1, \dots, k$ there exists $\delta_q > 0$ such that $\bar{\varphi}_q^* \bar{\Omega} = \rho(1, \lambda_q)$ on $\mathscr{L}(\delta_q)$.

5.3.D. If Ω and φ are included in smooth families Ω_t and φ_t, which satisfy the given hypotheses for all t, then $\bar{\Omega}$ can be included into a smooth family $\bar{\Omega}_t$ of symplectic forms on $\bar{V}$ such that for every t properties 5.3.A-5.3.C hold.

5.4 Symplectic blowing-down

Suppose now that $\bar{V}$ is endowed with a symplectic form, say $\bar{\Omega}$ such that $\bar{\varphi}_q^* \bar{\Omega} = \rho(\delta_q, \lambda_q)$ for some $\lambda_q > 0$, $\delta_q > 0$ $(q = 1 \dots k)$.

Define a symplectic form Ω on V as follows:

$$\Omega = \begin{cases} (\Theta^*)^{-1} \bar{\Omega} & \text{on } V - \bigcup_{q=1}^{k} \varphi_q(B(1 + \varepsilon_q)) ; \\ (\varphi^*)^{-1} \tau(\varepsilon_q, \delta_q, \lambda_q) & \text{on } \varphi_q(B(1 + 2\varepsilon_q)) \quad \text{for all } q = 1, \dots, k , \end{cases}$$

where τ is defined in 5.1.B. We shall say that Ω is obtained from $\bar{\Omega}$ by the *symplectic blowing down* associated with the embedding φ. Let us describe several properties of the symplectic blow-down which immediately follow from 5.1.B.

5.4.A. If $\bar{\Omega}$ tames $\bar{J}$ (resp. $\bar{\Omega}$ is Kähler) then Ω tames J (resp. Ω is Kähler);

5.4.B. The cohomology class of Ω satisfies the following relation:

$$[\bar{\Omega}] - [\Theta^*\Omega] \in \mathscr{E};$$

5.4.C. For each $q = 1, \ldots, k$, $\varphi_q^*\Omega = \lambda_q^2\omega$ on $B(1)$. In other words (V, Ω) admits a symplectic embedding of k disjoint standard symplectic balls of radii $\lambda_1, \ldots, \lambda_k$.

5.4.D. If $\bar{\Omega}$ and φ are included into smooth families $\bar{\Omega}_t$ and φ_t, which satisfy the given hypotheses for all t, then Ω can be included into a smooth family Ω_t of symplectic forms on V such that for every t properties 5.4.A-5.4.C hold.

5.5 *Normalization*

In order to carry out the blowing-up construction described above we have to choose an auxiliary almost complex structure J on V, and start with a symplectic and holomorphic embedding of a ball into V. The next result shows how to transform an arbitrary symplectic embedding into one with the required properties by a suitable perturbation of the symplectic structure. It is a slightly sharper form of Proposition 2.1.A.

Proposition 5.5.A. *Let $\varphi : (B(\delta), \omega) \to (V, \Omega)$ be a symplectic embedding, and let J be an almost complex structure on V which is tamed by Ω and is integrable near the point $\varphi(0)$. Then for every compact subset $K \subset V - \varphi(0)$ there exist a number $\delta' \in (0, \delta)$, a symplectic form Ω' on V which is isotopic to Ω, and a symplectic embedding $\varphi' : (B(\delta), \omega) \to (V, \Omega')$ with the following properties:*

- *$\varphi'|_{B(\delta')}$ is holomorphic;*
- *Ω' tames J and coincides with Ω on K.*

For the proof of the proposition we need the following

Lemma 5.5.B. *Let Ω be a symplectic form on $B(1)$ which tames the standard complex structure i. Then there exists a new symplectic form, say Ω' on $B(1)$ with the following properties:*

- *Ω' coincides with Ω near the boundary of the ball;*
- *Ω' tames i;*
- *Ω' is i-standard near 0, i.e. it is Kähler, and the associated metric is flat.*

Proof. We divide the proof into two steps.

1) We claim that for every $\kappa > 1$ and every $1 > \varepsilon > 0$ there exists a Kähler form, say τ_κ on $B(1)$ which is equal to $\kappa^2\omega$ in $B(\varepsilon/2\kappa)$ and coincides with $\varepsilon^2\omega$ near the boundary. Indeed, take a map h, which is monotone in the sense of 5.1.B, such that $h(z) = (\kappa/\varepsilon)z$ for $z \in B(\varepsilon/2\kappa)$ and h is equal to the identity map near the boundary. Then the form $\tau_\kappa = h^*(\varepsilon^2\omega)$ is as needed.

2) Let ρ be a bump function on $B(1)$ which is equal to 1 near the origin and vanishes near the boundary. Choose $\varepsilon > 0$ so that $\Omega - \varepsilon^2\omega$ tames i, and set $\rho_k(z) = \rho(2(\kappa/\varepsilon)z)$. Finally, denote by β a primitive of Ω, that is $\Omega = d\beta$, and consider

$$\Omega' = \Omega + \tau_\kappa - \varepsilon^2\omega - d(\rho_\kappa\beta).$$

We claim that Ω' has the desired properties provided κ is sufficiently large. Indeed, Ω' coincides with Ω near the boundary, and, near the origin, is equal to

$(\kappa^2 - \varepsilon^2)\omega$, and hence J-standard. Moreover, Ω' tames i outside $B(\varepsilon/2\kappa)$ due to our assumption. It remains to check that this is true also inside $B(\varepsilon/2\kappa)$.

Note that inside $B(\varepsilon/2\kappa)$

$$\Omega' = (\kappa^2 - \varepsilon^2)\omega + (1 - \rho_\kappa)\Omega - 2(\kappa/\varepsilon) d\rho \wedge \beta .$$

Therefore for every non-zero vector ξ we have

$$\Omega'(\xi, i\xi) > ((\kappa^2 - \varepsilon^2) - c(\kappa/\varepsilon))|\xi|^2 ,$$

where c is a positive constant which depends only on κ and ρ. Thus Ω' tames i provided κ is sufficiently large. This completes the proof. □

Proof of Proposition 5.5.A. In view of 5.5.B we can assume that Ω is J-standard near the point $\varphi(0)$ and therefore by composing φ with a suitable symplectomorphism of $B(\delta)$ we can achieve that $\varphi_* \circ i = J \circ \varphi_*$ at the origin $\{0\}$. Since J is integrable near $\varphi(0)$, there exists a diffeomorphism ψ of V such that $\psi(\varphi(0)) = \varphi(0)$ and

$$\psi_* \circ \varphi_* \circ i \circ \varphi_*^{-1} = J \circ \psi_*$$

in a sufficiently small neighborhood of $\varphi(0)$. Take $\Omega' = (\psi^*)^{-1}\Omega$, $\varphi' = \psi \circ \varphi$. Obviously, φ' is symplectic on $B(\delta)$ and holomorphic on $B(\delta')$ for some $\delta' \in (0, \delta)$. Moreover, ψ can be chosen to have support in $V - K$ and to be arbitrarily C^1-close to the identity. This implies the last assertion of the proposition. □

Proof of Proposition 2.1.B. By the symplectic neighborhood theorem for hypersurfaces (see 4.1.A), we can extend the original packing φ to a symplectic embedding of balls of radii $\lambda_q + \varepsilon$, for some small $\varepsilon > 0$. The normalization procedure described in 5.5.A above allows us to assume without loss of generality that φ_q is holomorphic on $B(\delta)$ for some $\delta > 0$ and each $q = 1, \ldots, k$.

Let $S_{t,q} : B(\lambda_q + \varepsilon) \to B(\lambda_q + \varepsilon)$ be a family of diffeomorphisms with the following properties:

- $S_0 = \mathbb{1}$:
- $S_{t,q}$ is equal to the identity near $\partial B(\lambda_q + \varepsilon)$;
- $S_{t,q}^* \omega = \mu_q(t)\omega$ on $B(\delta)$, where $\mu_q(1) = \lambda_q^2(1 + \gamma)^2 \delta^{-2}$ for some $\gamma > 0$.

Let $F_t : V \to V$ *be the extension of* $\coprod_{q=1}^k \varphi_q \circ S_{t,q} \circ \varphi_q^{-1}$ by the identity map, and set $\Omega_t = F_t^* \Omega$. Set $\psi_q(z) = \varphi_q\left(\frac{\delta}{1+\gamma} z\right) : B(1 + \gamma) \to V$. Obviously, ψ_q is holomorphic and

$$\psi_q^* \Omega_t = \frac{\delta^2}{(1+\gamma)^2} \mu_q(t)\omega .$$

Taking the blow-up, say $\bar{\Omega}_t$ of the family Ω_t associated to $\psi = \coprod \psi_q$ we obtain a pseudo-isotopy which satisfies the required conditions. □

Proof of Proposition 2.1.C. We will assume that $\bar{\Omega}$ is constructed from a symplectic and holomorphic embedding:

$$\varphi = \coprod_{q=1}^{k} \varphi_q : \coprod (B(1 + 2\varepsilon_q), \kappa_q^2 \omega, i) \to (V, \Omega, J)$$

as in 5.3. According to 5.3.C there exists $\delta > 0$ such that for each $q = 1, \ldots, k$ we have

$$\bar{\varphi}_q^* \bar{\Omega} = \rho(1, \kappa_q) \quad \text{on } \mathscr{L}(\delta) .$$

The symplectic neighborhood theorem implies that there exist $\delta' > 0$ and a family of diffeomorphisms $F_t: \bar{V} \to \bar{V}$, $t \in [0; 1]$ with the following properties:

- $F_0 = \mathbb{1}$;
- F_t preserves the exceptional divisors for all t;
- For all q and t, $\bar{\varphi}_q^* F_t^* \bar{\Omega}_t = \rho(1, \lambda_q(t))$ on $\mathscr{L}(\delta')$. (Note that $\lambda_q(0) = \kappa_q$.)

Set $R(z) = \frac{\delta'}{1+\gamma} z$, and set $\psi_q = \varphi_q \circ R: B(1+\gamma) \to V$. Let Ω_t be the symplectic blow-down of the family $F_t^* \bar{\Omega}_t$ associated with $\psi = \coprod_{q=1}^k \psi_q$. Note that each form of this family is cohomologous to Ω_0 in view of 5.4.B. According to 5.4, Ω_0 tames J and $[\Omega_0] = [\Omega]$. Therefore Ω and Ω_1 are isotopic. Moreover, (V, Ω_1) admits a symplectic embedding of $\coprod_{q=1}^k (B(\lambda_q), \omega)$ due to 5.4.C. Combining these facts we obtain the needed assertion. □

Acknowledgements. The work on this paper was initiated during our stay at Ruhr-Universität, Bochum in July, 1991 (this visit was partially supported by DSF-SFB 237). We would like to express our deep gratitude to Helmut Hofer for the hospitality and for extremely stimulating discussions.

Special thanks to Ron Liviné for giving us the reference to Nagata's paper [N]. We are grateful also to Michele Audin, Yael Karshon, Andrey Reznikov, Eugenii Shustin, and Alexey Skorobogatov for various useful consultations.

And we thank Mikhael Gromov for critical remarks improving the quality of the exposition.

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