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A note on complex projective threefolds admitting holomorphic contact structures

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Let X be a $(2n+1)$-dimensional complex manifold. A (holomorphic) contact structure on X is a $2n$-dimensional *non-integrable* holomorphic distribution on X. Dually, we can think of a contact structure on X as a holomorphic line subbundle L of Ω^1_X (the holomorphic cotangent bundle of X) such that if θ is a local section of L, then $\theta \wedge (d\theta)^n$ is *everywhere non-zero*. This, in particular, implies that the canonical bundle $K_X \cong (n+1)L$, where $K_X = \wedge^{2n+1}\Omega^1_X$ is the canonical line bundle of X.

The purpose of this paper is to give a complete classification of complex *projective* contact threefolds. Specifically, we show the following:

Theorem 2 *If X is a complex projective contact threefold, then X is isomorphic to either $\mathbb{CP}^3$, or $\mathbb{P}(T_M)$ for some smooth projective surface M.*

From now on, unless specifically stated, X will be a complex contact 3-fold and L will be the contact line bundle. Sometimes we use the pair (X, L) to mean the same thing. In particular, we have $K_X = 2L$. Before we prove the above theorem, let us look at some examples of complex contact threefolds.

Example 1 It is well-known that there is a holomorphic contact structure on $\mathbb{P}(T_M)$ for any complex manifold M. By $\mathbb{P}(T_M)$ we mean $T^*_M\backslash\{0\}/\mathbb{C}^*$ instead of $T_M\backslash\{0\}/\mathbb{C}^*$. In particular, $\mathbb{P}(T_M)$ has a holomorphic contact structure if M is a complex surface.

Example 2 The old-dimensional complex projective space $\mathbb{CP}^{2n+1}$ has a contact structure. The contact structure is induced from the natural symplectic structure on $\mathbb{C}^{2n+2}$. The associated contact line bundle is $\mathcal{O}_{\mathbb{CP}^{2n+1}}(-2)$. In particular, $\mathbb{CP}^3$ has a contact structure with contact line bundle $\mathcal{O}_{\mathbb{CP}^3}(-2)$.

Example 3 More generally, Salamon [Sa] showed that the twistor space Z of a quaternionic-Kähler manifold (M^{4n}, g) $(n \geqq 2)$ is a complex $(2n+1)$-dimensional

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contact manifold if the scalar curvature of M is non-zero. Recall that (M^{4n}, g) is called a quaternionic-Kähler manifold if its holonomy group is contained in $\mathrm{Sp}(n)\mathrm{Sp}(1)$. For example, $\mathbb{HP}^n$ is a quaternionic-Kähler manifold and its twistor space is just $\mathbb{CP}^{2n+1}$. Salamon's argument can be easily extended to the case of *self-dual* Einstein Riemannian four-manifolds (see the argument on p. 416 in [Bes]). Therefore if (M^4, g) is a self-dual Einstein Riemannian four-manifold with non-zero scalar curvature, then its twistor space Z is a complex contact threefold. Moreover, if Z is Kähler, then $c_1(Z) > 0$ (see Hi]), hence, Z is necessarily projective.

Before we prove Theorem 2, we need the following lemma.

Lemma 1 *Let X be a compact Kähler manifold with $c_1(X) = 0$. Then X has no holomorphic contact structure.*

Following a comment by the referee, the original proof of the above lemma can be simplified. Therefore we will adopt the simplified proof, which is kindly provided by the referee.

Proof. Suppose that X admits a holomorphic contact structure with L as its contact line bundle. By Calabi-Yau theorem, X admits a Ricci-flat Kähler metric. Since $K_X = (n+1)L$, L is a flat line bundle with the induced hermitian metric. We can think of the contact structure on X as a holomorphic section θ of $\Omega^1_X \otimes L^*$. Then θ is a covariant constant section by the standard Bochner technique. Let us choose an arbitrary locally defined covariant constant contact 1-form, which is also denoted by θ. Since the metric connection is torsion free, $d\theta = 0$, which clearly contradicts the fact that θ defines a holomorphic contact structure. Hence we are done. □

Recall that a line bundle is called *nef* if its intersection number with every effective curve is non-negative. Now we can start proving the main result of this paper.

Theorem 2 *If (X, L) is a complex projective contact threefold, then X is either isomorphic to $\mathbb{CP}^3$ or $X \cong \mathbb{P}(T_M)$ for some smooth complex projective surface M.*

Proof. We first claim generally that if (X, L) is a complex projective contact manifold of any dimension n, then its Kodaira dimension $\kappa(X) \leqq 0$. This was essentially proved by Bogomolov. Since $K_X = (n+1)L$ and $L \subset \Omega^1_X$ is a line subbundle, we have $\kappa(X) \leqq 1$ by Theorem 4 of §12 in [Bo]. Moreover if $\kappa(X) = 1$, then by example 12.9 of [Bo] there is a morphism $f\colon X \to S$, where S is a smooth curve, and a general fiber of f is an integral submanifold of $L \subset \Omega^1_X$. This contradicts the fact that (X, L) is a contact manifold. Hence $\kappa(X) \leqq 0$.

Now let us come back to the three-dimensional situation. We claim that if (X, L) is a complex projective contact threefold, then K_X is *not nef*. If K_X is nef, then Theorem 1.1 of [Mi] implies that $\kappa(X) \geqq 0$. Therefore $\kappa(X)$ has to be zero. By the Abundance Theorem (see [Ka]), K_X is numerically trivial, i.e., $c_1(X) = 0$. However, Lemma 1 above implies that this is impossible. Hence K_X is not nef.

Since K_X is not nef, Mori's theory of extremal ray (see [Mo]) implies that there is a rational curve $C \subset X$ such that $4 \geqq -K_X \cdot C > 0$ and C generates an extremal ray $R = \mathbb{R}_+[C]$. Let us recall the *length* (denoted by $l(R)$) of the extremal ray R is the minimal intersection number with $-K_X$ among all rational curves whose numerical equivalence classes belong to the ray R. It is clear that $0 < l(R) \leqq 4$. Since $K_X = 2L$ in our case, $l(R) = 4$ or 2. If $l(R) = 4$, then Corollary 2.6 of [Wi] *implies that* $X \cong \mathbb{CP}^3$. If $l(R) = 2$, then we can assume that there is a *smooth* rational curve C such that $-K_X\cdot C=2$, and whose numerical equivalence class belongs to R. This is clear from the proofs of Theorem 3.3 of [Mo] and Proposition 2.3 of [Wi]. Since $K_X=2L$, we have $L\cdot C=-1$.

Let $\varphi\colon L \subsetneq \Omega^1_X$ be the natural bundle inclusion. Consider the restriction of the contact sequence to C:

$$0\to L^{\perp}|_C\to T_X|_C \xrightarrow{\varphi^*|_C} L^*|_C\to 0 \tag{1}$$

where φ^* is the dual of φ, and $L^{\perp}\subset T_X$ is the two-dimensional holomorphic distribution coming from the given contact structure. Because T_C is a subbundle of $T_X|_C$ we have a $\mathcal{O}_C$-module homomorphism $\alpha_C\colon T_C\to L^*|_C$. Since $T_C\cong\mathcal{O}_C(2)$ and $L^*|_C\cong\mathcal{O}_C(1)$, we conclude that α_C has to be zero, i.e., C is a contact curve. Then by a theorem of Bryant [Br], $N_{C/X}\cong\mathcal{O}_C\oplus\mathcal{O}_C$. By Theorem 3.5 of [Mo], X is isomorphic to a conic bundle in this case, i.e., there is a smooth projective surface M and a morphism $\pi\colon X\to M$ such that a *general* fibre of π is isomorphic to $\mathbb{CP}^1$. It is clear from the proof of Theorem 3.5 of [Mo] that the morphism π is the contraction morphism associated with the extremal ray $R=\mathbb{R}_+[C]$. We claim that X is in fact a $\mathbb{CP}^1$-bundle over M. To show this, it suffices to show that every fibre of π is reduced and irreducible. Note that the smooth rational curve C above is a fiber for π. If π has a non-reduced, or a reducible fiber, then $[C]=[C_1]+[C_2]$ for two effective curves C_1 and C_2 on X, where $[C]$ means the numerical equivalence class of the curve C. Since $R=\mathbb{R}_+[C]$ is an extremal ray, $[C_i]\in R$ for $i=1,2$. Hence $K_X\cdot C_i<0$ for $i=1,2$. Since $K_X=2L$, $L\cdot C_i\leqq -1$ for $i=1,2$. Therefore $L\cdot C=L\cdot C_1+L\cdot C_2\leqq -2$. But this is a contradiction since $L\cdot C=-1$. Hence every fibre of π is reduced and irreducible. So X must be a $\mathbb{CP}^1$-bundle over the smooth projective surface M.

Therefore we can write X as $\mathbb{P}(E)$ for some rank-two vector bundle over M. Let $X\xrightarrow{\pi}M$ be the natural projection. Note that any fiber of π generates the extremal ray R. Let $\mathcal{O}_X(1)$ be the tautological line bundle of X. It is easy to see that $K_X=-2\mathcal{O}_X(1)+\pi^*(K_M+\wedge^2E)$. However $K_X=2L$. Hence $\pi^*(K_M+\wedge^2E)=-2L_0$, where $L_0=\mathcal{O}_X(1)-L$. Since $L_0\cdot C=-1+1=0$, $L_0=\pi^*L_1$ for some line bundle L_1 on M. Therefore $K_M+\wedge^2E=-2L_1$, i.e., $\wedge^2(E\otimes L_1)=\wedge^2T_M$. Hence if we replace E by $E\otimes L_1$, then we can assume that $\wedge^2E\cong\wedge^2T_M$ and $\mathcal{O}_X(-1)$ is the contact line bundle. Then there is a natural bundle injection $\lambda\colon\mathcal{O}_X(-1)\to\Omega^1_X$, which defines the given contact structure.

We will show that $E\cong T_M$. We first prove the following claim:

Claim. Let $\Omega^1_{X/M}$ be the relative contangent bundle. Then

$$H^0(\Omega^1_{X/M}(1))=0,\qquad H^1(\Omega^1_{X/M})\cong\mathbb{C}.$$

Proof of the claim. Consider the relative Euler sequence:

$$0\to\Omega^1_{X/M}(1)\to\pi^*E\to\mathcal{O}_X(1)\to 0. \tag{2}$$

Since $H^0(\pi^*E)\cong H^0(E)\cong H^0(\mathcal{O}_X(1))$, we have $H^0(\Omega^1_{X/M}(1))=0$. Since π is a $\mathbb{CP}^1$-bundle, we have $\pi_*\Omega^1_{X/M}=0$. Hence Leray spectral sequence for π implies that $H^1(\Omega^1_{X/M})\cong H^0(R^1\pi_*\Omega^1_{X/M})$. However by the relative duality, $R^1\pi_*\Omega^1_{X/M}\cong(\pi_*\mathcal{O}_X)^*\cong\mathcal{O}_M$. Hence $H^1(\Omega^1_{X/M})\cong\mathbb{C}$. Hence the claim is proved.

Now consider the tangential sequence:

$$0\to T_{X/M}\to T_X\to\pi^*T_M\to 0 \tag{3}$$

Since $H^0(\Omega^1_{X/M}(1)) = 0$, the bundle injection (from the contact structure on X) $\lambda: \mathcal{O}_X(-1) \to \Omega^1_X$ induces a surjective bundle map $\sigma: \pi^* T_M \to \mathcal{O}_X(1)$. Let $\mathcal{N}$ be the kernel of σ. Then $\mathcal{N}$ is a line bundle. Moreover $\mathcal{N} \cong \pi^*(\wedge^2 T_M) \otimes \mathcal{O}_X(-1)$. However by our assumption, $\wedge^2 E \cong \wedge^2 T_M$. Therefore sequence (2) implies that $\mathcal{N} \cong \Omega^1_{X/M}(1)$. Thus we obtain an exact sequence:

$$0 \to \Omega^1_{X/M}(1) \to \pi^* T_M \xrightarrow{\sigma} \mathcal{O}_X(1) \to 0. \tag{4}$$

Let e_1, respectively e_2 be the extension class of (2), respectively (4). They are elements in $H^1(\Omega^1_{X/M}) \cong \mathbb{C}$. They are non-zero since their corresponding sequences do not split (because their restrictions to a fiber of π do not split for trivial reasons). Hence they differ only by a non-zero scalar. This last fact implies that $\pi^* E \cong \pi^* T_M$. Since $\pi_* \mathcal{O}_X = \mathcal{O}_M$, we have $E \cong T_M$. Hence we are done. □

Corollary 3 *If X is a contact threefold such that $c_1(X)$ is positive (i.e. X is Fano), then X is isomorphic either to* $\mathbb{CP}^3$ or $\mathbb{P}(T_{\mathbb{CP}^2})$.

Proof. Suppose that X is not isomorphic to $\mathbb{CP}^3$. Then Theorem 2 implies that $(X, L) \cong (\mathbb{P}(T_M), \mathcal{O}_{\mathbb{P}(T_M)}(-1))$ for some smooth complex projective surface M. Hence $-K_X = \mathcal{O}_{\mathbb{P}(T_M)}(2)$, which is ample by our assumption. Therefore T_M is also ample by definition of ampleness of vector bundles. This clearly implies that $M \cong \mathbb{CP}^2$ by Mori's proof of Hartshorne's conjecture. Hence we are done. □

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