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Kodaira dimension of moduli space of vector bundles on surfaces

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The purpose of this paper is to study the geometry of the moduli space of rank two vector bundles of fixed Chern classes over a smooth algebraic surface X. Of the two topics which we will concentrate on, one is to determine the dimension and the singularity of the moduli space and the other is to calculate the Kodaira dimension of this moduli space.

For X a smooth algebraic surface over $\mathbb{C}$ and H, I two line bundles on X, where H is ample, let $\mathfrak{M}_H(d, I)$ be the moduli space of rank two H-semistable sheaves E over X with $\det E = I$ and $c_2(E) = d$. In late 70's, Gieseker showed that $\mathfrak{M}_H(d, I)$ is projective and later, Maruyama, Gieseker and Taubes showed that $\mathfrak{M}_H(d, I)$ is non-empty when d is large. In the mean time, a lot of work has been done in understanding the geometry of $\mathfrak{M}_H(d, I)$ for special surfaces. However, not much is known about the geometry of $\mathfrak{M}_H(d, I)$ for general X. One basic result along this direction is the Donaldson's generic smoothness theorem. Namely, Donaldson [4] (later generalized by Friedman [6] and Zuo [26]) showed that when d is sufficiently large, the (open) subset $\mathscr{V}_H(d, I) \subseteq \mathfrak{M}_H(d, I)$ of locally free H-stable sheaves has the expected dimension and further,

$$\operatorname{codim}(\operatorname{Sing} \mathscr{V}_H(d, I), \mathscr{V}_H(d, I)) \geqq \tfrac{1}{2}d, \quad d \gg 0 .$$

It is this generic smoothness result that allows Donaldson to show that his polynomial invariants of the underlining smooth four manifold X is non-trivial. In this paper, we will demonstrate that a stronger result holds for the whole moduli space $\mathfrak{M}_H(d, I)$. More precisely, we prove

Theorem 0.1 *For any polarized smooth algebraic surface (X, H) and any fixed line bundle I on X, there is a constant C depending on (X, H, I) such that whenever $d \geqq C$, then*

(1) *$\mathfrak{M}_H(d, I)$ has pure dimension $4d - I^2 - 3\chi(\mathcal{O}_X)$ as expected by* R.R.;
(2) *$\mathfrak{M}_H(d, I)$ is normal and further, for any closed $s \in \mathfrak{M}_H(d, I)$ that corresponds to a stable sheaf over X, $\mathfrak{M}_H(d, I)$ is a local complete intersection at s.*

One should view this theorem in contrast to the fact that the moduli space $\mathfrak{M}_H(d, I)$ is not in general smooth. As to the proof, it is based on Donaldson's generic smoothness theorem (of $\mathscr{V}_H(d, I)$) and the deformation theory. Loosely speaking, the deformation theory [14] shows that the complete local ring $\hat{R}$ of any closed point $z \in \mathfrak{M}_H(d, I)$ associated to a stable sheaf E is determined by a morphism of complete local algebras $o: T^2 \to T^1$, where $T^i = \mathbb{C}[[t_1, \ldots, t_{m_i}]]$ and $m_i = \dim \mathrm{Ext}^i(E, E)^0$. (Here and later, we always use the superscript 0 to denote the traceless part of the corresponding groups or sheaves). More precisely, $\hat{R} = T^1 \otimes_{T_2} \mathbb{C}$. On the other hand, a generalization of Donaldson's generic smoothness result implies that R has dimension $m_1 - m_2$ when d is sufficiently large. Thus $\hat{R}$ must be of the form $\mathbb{C}[[t_1, \ldots, t_{m_1}]]/(f_1, \ldots . f_{m_2})$ and consequently, $\mathfrak{M}_H(d, I)$ is a l.c.i. at z.

Next we turn our attention to the study of Kodaira dimension of the moduli space $\mathfrak{M}_H(d, I)$. When $X = \mathbf{P}^2$, work of Barth, Hulek, Ellingsrud, Strømme and Maruyama shows that the moduli space $\mathfrak{M}_H(d, I)$ is rational when either $\deg I$ or d is odd. In case both $\deg I$ and d are even, they also gave quite a description of $\mathfrak{M}_H(d, I)$. In short, $\mathfrak{M}_H(d, I)$ always has Kodaira dimension $\kappa = -\infty$. For some ruled surfaces, Qin also showed that $\mathfrak{M}_H(d, I)$ has Kodaira dimension $-\infty$ [24]. As to K3 surfaces, a consequence of Mukai's work [18] shows that $\mathfrak{M}_H(d, I)$ has Kodaira dimension $\kappa = 0$. Recently, O'Grady has proved that when X is a surface of general type (satisfying some extra conditions), then $\mathfrak{M}_H(d, I)$ has Kodaira dimension $\kappa \geqq 0$ [22]. All these indicate strongly that the Kodaira dimension of $\mathfrak{M}_H(d, I)$ is very closely related to the Kodaira dimension of X. To this end, one ponders what should be the Kodaira dimension of $\mathfrak{M}_H(d, I)$ when X is a surface of general type. In this paper, we will prove

Theorem 0.2 *Let (X, H) be any minimal polarized smooth algebraic surface of general type and let I be any line bundle over X so that $c_1(I) \cdot Z$ is even for any (-2)-exceptional curve of $Z \subseteq X$. Suppose $\chi(\mathcal{O}_X) + I \cdot I$ is even and that there is a reduced canonical divisor $D \in |K_X|$, then there is a constant C depending on (X, H, I) such that whenever $d \geqq C$, then $\mathfrak{M}_H(d, I)$ is of general type.*

The proof of this theorem is inspired by Donaldson's work on polynomial invariants of smooth four manifolds [4]. There are two main ingredients in the establishment of this result. The first is to express the dualizing sheaf ω of $\mathfrak{M}_H(d, I)$ in terms of some line bundles of which we know how to construct global sections. This can best be explained by looking at the fiber $\omega \otimes k(s)$, where $s \in \mathfrak{M}_H(d, I)$ is a smooth point associated to a locally free sheaf E. Since the Zariski tangent space $\mathbf{T}_s\mathfrak{M}_H(d, I)$ is $\mathrm{Ext}^1(E, E)^0$, $\omega \otimes k(s)$ can canonically be identified to $\left(\bigwedge^{\mathrm{top}} \mathrm{Ext}^1(E, E)^0\right)^{-1}$. For simplicity, we assume that there is a $\theta \in H^0(K_X)$ with $D = \theta^{-1}(0)$ smooth. Then there is an exact sequence

$$0 \to H^0_D(\mathscr{E}nd^0(E_{|D}) \otimes K_X) \to \mathrm{Ext}^1_X(E, E)^0 \to$$

$$\xrightarrow{\otimes\theta} \mathrm{Ext}^1_X(E, E \otimes K_X)^0 \to H^1_D(\mathscr{E}nd^0(E_{|D}) \otimes K_X) \to 0$$

induced by the exact sequence

$$0 \to E \xrightarrow{\otimes\theta} E \otimes K_X \to (E \otimes K_X)_{|D} \to 0 .$$

By using Serre duality, we get

$$\bigotimes_{i=0}^{1}\left(\bigwedge^{\mathrm{top}} H^i_D(\mathscr{E}nd^0(E_{|D})\otimes K_X)\right)^{(-1)^i} = \left(\bigwedge^{\mathrm{top}}\mathrm{Ext}^1(E,E)^0\right)\otimes\left(\bigwedge^{\mathrm{top}}\mathrm{Ext}^1(E,E\otimes K_X)^0\right)^{-1}$$

$$= \left(\bigwedge^{\mathrm{top}}\mathrm{Ext}^1(E,E)^0\right)^{\otimes 2}.$$

Now if we let $\mathfrak{M}_D(2,I)$ be the moduli space of rank two vector bundles on D with $c_1 = I_{|X}$, then when $E_{|D}$ is stable,

$$\left(\bigotimes_{i=0}^{1}\left(\bigwedge^{\mathrm{top}} H^i_D(\mathscr{E}nd^0(E_{|D})\otimes K_X)\right)^{(-1)^i}\right)^{-1}$$

is the fiber of an ample line bundle on $\mathfrak{M}_D(2,I)$ over the closed point associated to the vector bundle $E_{|D}$. More precisely, there is an ample line bundle $\mathscr{L}_D$ on $\mathfrak{M}_D(2,I)$ so that under the rational map

$$\Psi:\mathfrak{M}_H(d,I) \dashrightarrow \mathfrak{M}_D(2,I),\quad \Psi(E)=E_{|D} \text{ when it is semistable,}$$

$\Psi^*(\mathscr{L}_D)=\omega^{\otimes 2}$. Therefore, the pullback sections of $H^0(\mathfrak{M}_D(2,I),\mathscr{L}_D^{\otimes n})$ give rise to a group of meromorphic pluricanonical sections of $\mathfrak{M}_H(d,I)$. In fact, all of them turn out to be regular. In this way, we obtain a lot of pluricanonical sections of $\mathfrak{M}_H(d,I)$. In this paper, we will use the fact that X is of general type in an essentially similar way to show that

$$\dim H^0(\mathfrak{M}_H(d,I),\omega^{\otimes n}) = \beta\cdot n^{c(d)} + O(n^{c(d)-1}),\quad \beta>0,\ d\gg 0$$

when I satisfy the condition of the theorem, where $c(d)=\dim\mathfrak{M}_H(d,I)$.

In order to determine the Kodaira dimension of $\mathfrak{M}_H(d,I)$, we need to look at the dualizing sheaf of a desingularization $\widetilde{\mathfrak{M}}_H(d,I)$ of $\mathfrak{M}_H(d,I)$. Let $T_1,\ldots,T_m$ be the exceptional divisors of $\pi:\widetilde{\mathfrak{M}}_H(d,I)\to\mathfrak{M}_H(d,I)$. Then there are integers α_i such that the dualizing sheaf $\tilde\omega$ of $\widetilde{\mathfrak{M}}_H(d,I)$ satisfies

$$\tilde\omega = \pi^*\omega\left(\sum_{i=1}^{m}\alpha_i T_i\right).$$

(For simplicity, we assume ω is locally free.) Note that when some of the α_i's are negative, which is possible in general, the question when a section $\eta_n\in H^0(\mathfrak{M}_H(d,I),\omega^{\otimes n})$ can be lifted to a section in $H^0(\widetilde{\mathfrak{M}}_H(d,I),\tilde\omega^{\otimes n})$ is quite delicate. In our situation, this has been made easier by the existence of a 2-canonical section η_2 (of $\widetilde{\mathfrak{M}}_H(d,I)$), constructed by O'Grady [23], that vanishes along the exceptional divisor of $\pi:\widetilde{\mathfrak{M}}_H(d,I)\to\mathfrak{M}_H(d,I)$. Namely, $\eta_2\in H^0(\widetilde{\mathfrak{M}}_H(d,I),\tilde\omega^{\otimes 2}(-\sum^m T_i))$. Therefore, if we let α be a positive integer so that $\alpha+\alpha_i\geqq 0$, then for any $\eta_n\in H^0(\mathfrak{M}_H(d,I),\omega^{\otimes n})$, $\pi^*\eta_n\otimes\eta_2^{\otimes\alpha n}$ is a regular section of the line bundle

$$\pi^*\omega^{\otimes n}\otimes\tilde\omega^{\otimes 2\alpha n}\left(-n\alpha\sum T_i\right) = \tilde\omega^{\otimes(1+2\alpha)n}\left(-n\sum(\alpha+\alpha_i)T_i\right)\subsetneq\tilde\omega^{\otimes(1+2\alpha)n}.$$

Thus we will have

$$\dim H^0(\tilde{\mathfrak{M}}_H(d,I),\tilde{\omega}^{\otimes(1+2\alpha)n}) \geqq \dim H^0(\mathfrak{M}_H(d,I),\omega^{\otimes n}) = \beta\cdot n^{c(d)} + O(n^{c(d)-1}), \quad \beta > 0$$

which is what we need to establish the theorem.

The paper is organized as follows: Theorem 0.1 is proved in §1 and §2. Also in §2, we shall explicitly construct the desingularization $\tilde{\mathfrak{M}}_H(d,I)$ and discuss the existence of local universal family on $\tilde{\mathfrak{M}}_H(d,I)$. §3 is devoted to construct symplectic forms of $\mathfrak{M}_H(d,I)$ and its associated two-canonical forms. We shall finish the proof of Theorem 0.2 in §4.

Notations

Throughout this paper, X will be a smooth algebraic surface over $\mathbb{C}$, H will be a fixed ample divisor and I will be a fixed line bundle on X. We recall the definition of stability of torsion free sheaves. A rank r sheaf E on X is said to be stable (resp. semistable) provided that E is coherent, torsion free and that for any proper subsheaf $L \subset E$,

$$\frac{1}{\operatorname{rank}(L)}\chi_L(n) < \frac{1}{\operatorname{rank}(E)}\chi_E(n) \quad (\text{resp. } \leqq)$$

holds for n sufficiently large. Here $\chi_E(n) = \chi(E\otimes H^{\otimes n})$ is the Hilbert polynomial of E. E is said to be μ-semistable if whenever $L \subset E$ is a proper subsheaf with $\operatorname{rank}(L) < \operatorname{rank}(E)$, then

$$\frac{1}{\operatorname{rank}(L)}\deg(L) \leqq \frac{1}{\operatorname{rank}(E)}\deg(E)$$

where $\deg(E) = c_1(E)\cdot H$.

From now on, all schemes considered in this paper are over $\mathbb{C}$. Suppose S is a quasi-projective scheme and that E is a family of sheaves on $X\times S$ flat over S, then for any closed $s\in S$, we use E_s to denote the restriction of E to the fiber $X\times\{s\}$ of $X\times S$ over $s\in S$. We also use p_X and p_S to denote the projection of $X\times S$ to X and S respectively. Occasionally, we will use p_1 and p_2 instead. For any sheaf E and line bundle L on X, we denote by $\operatorname{Ext}^i(E, E\otimes L)^0$ the traceless part of $\operatorname{Ext}^i(E, E\otimes L)$. That is, the kernel of $\operatorname{Ext}^i(E,E\otimes L)\xrightarrow{\operatorname{tr}} H^i(L)$.

1 Singularity of Grothendieck's Quot-scheme

In this section, we will study the singularity of the Grothendieck's Quot-scheme. First of all, let us recall the definition of Quot-scheme introduced in [9]. For any integer d and any component $\Sigma \subseteq \operatorname{Pic}(X)$, let $\mathscr{E}(d,\Sigma)$ be the set of rank two coherent sheaves E over X with $\det E\in\Sigma$ and $c_2(E) = d$. For technical reason, we will work with the set $\mathscr{E}(d,\Sigma,n) = \{E(n)\mid E\in\mathscr{E}(d,\Sigma)\}$, where $E(n) = E\otimes H^{\otimes n}$. Clearly, tensoring $H^{\otimes n}$ gives a canonical identification between $\mathscr{E}(d,\Sigma)$ and $\mathscr{E}(d,\Sigma,n)$. Let $\mathscr{S}$ be the category of all separable schemes of finite type over $\mathbb{C}$. We fix a (coherent) locally free sheaf W. For any $S\in\mathscr{S}$, we let $\mathfrak{Quot}_n(d,\Sigma)(S)$ be the set of all quotient sheaves $p_X^* W \to E$ on $X\times S$ flat over S such that $E_s\in\mathscr{E}(d,\Sigma,n)$ for any closed $s \in S$, where E_s is the restriction of E to the fiber $X \times \{s\}$ over $s \in S$. Clearly, $\mathfrak{Quot}_n(d, \Sigma)$ is a contravariant functor on the category $\mathscr{S}$.

Theorem 1.1 (Grothendieck [9]) *$\mathfrak{Quot}_n(d, \Sigma)$ is represented by a projective scheme $\mathscr{Q}_n(d, \Sigma)$. Namely, there is a quotient sheaf $p_X^* W \to \mathscr{F}$ on $X \times \mathscr{Q}_n(d, \Sigma)$ flat over $\mathscr{Q}_n(d, \Sigma)$ such that for any $S \in \mathscr{S}$ and any flat quotient sheaf $p_X^* W \to F$ on $X \times S$ in $\mathfrak{Quot}_n(d, \Sigma)(S)$, there is a unique morphism $\varphi: S \to \mathscr{Q}_n(d, \Sigma)$ such that the quotient sheaf $p_X^* W \to F$ is isomorphic to the pullback quotient sheaf $p_X^* W \to (1_X \times \varphi)^* \mathscr{F}$.*

For the purpose of studying the moduli of semistable sheaves, we are interested in the subset $\mathscr{E}(d, \Sigma)^s \subseteq \mathscr{E}(d, \Sigma)$ of H-semistable sheaves. We also need to choose a special W. For any pair (d, Σ), let n be a large integer so that for any $E \in \mathscr{E}(d, \Sigma)^s$, $h^i(E(n)) = 0$ for $i > 0$ and $H^0(E(n)) = \mathbb{C}^{\oplus N}$ generates E[7]. We then choose W of Theorem 1.1 to be $\oplus^N \mathscr{O}_X$ and form the corresponding functor $\mathfrak{Quot}_n(d, \Sigma)$ and the Quot-scheme $\mathscr{Q}_n(d, \Sigma)$. We further let $\mathscr{Q}_n(d, \Sigma)^{ss} \subseteq \mathscr{Q}_n(d, \Sigma)$ be the (open) subset of H-semistable quotient sheaves. The goal of this section is to prove

Theorem 1.2 *There is a constant C depending on (X, H, Σ) such that whenever $d \geqq C$, then*

(1) *$\mathscr{Q}_n(d, \Sigma)^{ss}$ has pure dimension*

$$\eta(d, \Sigma, n) = 4d - I^2 - 3\chi(\mathscr{O}_X) + N^2 - 1 + h^1(\mathscr{O}_X); \tag{1.1}$$

(2) *$\mathscr{Q}_n(d, \Sigma)^{ss}$ is normal and is a* l.c.i. *everywhere.*

As explained in the introduction, the proof of the theorem consists of two parts. The first is to show that under the assumption, $\mathscr{Q}_n(d, \Sigma)^{ss}$ has the expected dimension $\eta(d, \Sigma, n)$. The second is to apply the general deformation theory to the functor $\mathfrak{Quot}_n(d, \Sigma)$ to show that $\mathscr{Q}_n(d, \Sigma)^{ss}$ is indeed a l.c.i. everywhere.

We first show that there is a constant C depending on (X, H, Σ) so that

$$\dim \mathscr{Q}_n(d, \Sigma)^{ss} \leqq \eta(d, \Sigma, n), \quad \forall d \geqq C. \tag{1.2}$$

First of all, for any closed point $z \in \mathscr{Q}_n(d, \Sigma)^{ss}$ which corresponds to the quotient sheaf $W \to E$, the Zariski tangent space of $\mathscr{Q}_n(d, \Sigma)^{ss}$ at z is

$$\mathrm{T}_z \mathscr{Q}_n(d, \Sigma)^{ss} = \mathrm{Hom}(F, E),$$

where F is the kernel of $W \to E$ [9]. By using R.R., Maruyama [16, p. 596] calculated that

$$\dim \mathrm{T}_z \mathscr{Q}_n(d, \Sigma)^{ss} = \eta(d, \Sigma, n) + \dim \mathrm{Ext}^2(E, E)^0. \tag{1.3}$$

Further, he showed that $\mathscr{Q}_n(d, \Sigma)^{ss}$ is in fact smooth at the quotient sheaf E when the connecting homomorphism $\mathrm{Ext}^1(F, E) \to \mathrm{Ext}^2(E, E)$ is trivial. On the other hand, Mukai tells us that the image of this homomorphism is always contained in $\mathrm{Ext}^2(E, E)^0$ [18]. Thus $\mathscr{Q}_n(d, \Sigma)^{ss}$ is smooth at z whenever $\mathrm{Ext}^2(E, E)^0$ is trivial.

Now let

$$\Psi: \mathscr{Q}_n(d, \Sigma)^{ss} \to \mathscr{E}(d, \Sigma)^{ss}$$

be the obvious map sending quotient sheaf $W \to E$ to $E(-n)$. If we further divide the set $\mathscr{E}(d, \Sigma)^{ss}$ into two subsets $\mathscr{A}_0(d, \Sigma)$ and $\mathscr{A}_1(d, \Sigma)$, where $\mathscr{A}_0(d, \Sigma) \subseteq \mathscr{E}(D, \Sigma)^{ss}$ consists of stable sheaves E with vanishing $\mathrm{Ext}^2(E, E)^0$ and $\mathscr{A}_1(d, \Sigma) = \mathscr{E}(d, \Sigma, n)^{ss} \backslash \mathscr{A}_0(d, \Sigma)$, then the previous argument shows that

$\mathcal{Q}_n(d, \Sigma)^{ss}$ is smooth along $\Psi^{-1}(\mathcal{A}_0(d, \Sigma))$. In particular, we will have

$$\dim \Psi^{-1}(\mathcal{A}_0(d, \Sigma)) = \eta(d, \Sigma, n) \tag{1.4}$$

when it is non-empty. Next, we will show that when d is large,

$$\dim \Psi^{-1}(\mathcal{A}_1(d, \Sigma)) \leqq \eta(d, \Sigma, n) - 2. \tag{1.5}$$

Clearly, (1.4) and (1.5) together imply (1.2). Since $h^0(E) = N$, for any $E \in \mathscr{E}(d, \Sigma, n)^{ss}$, $\dim \Psi^{-1}(E) \leqq N^2 - 1$. Thus (1.5) follows from the following lemma.

Lemma 1.3 *There is a constant C_0 depending on (X, H, Σ) such that the number of moduli of $\mathcal{A}_1(d, \Sigma)$ is no more than $3d + C_0(\sqrt{d - I^2/4} + 1)$.*

Proof. We first recall the Donaldson's generic smoothness theorem. Let $\mathscr{B}(d, \Sigma)$ be the set of all rank two locally free μ-semistable sheaves E with $\det E \in \Sigma$ and $c_2(E) = d$ that have non-vanishing extension groups $\operatorname{Ext}^2(E, E)^0$. By [4, 6, 26], there is a constant C_0 depending on (X, H, Σ) such that the number of moduli (abbreviated $\#_{\text{mod}}$)

$$\#_{\text{mod}} \mathscr{B}(d, \Sigma) \leqq 3d + C_0(\sqrt{d - I^2/4} + 1). \tag{1.6}$$

Note that by Bogomolov inequality, $\mathscr{B}(d, \Sigma)$ is empty when $d < I^2/4$.

As to the proof of the lemma, we use the double dual operation to relate the set $\mathcal{A}_1(d, \Sigma)$ to $\mathscr{B}(d, \Sigma)$. More precisely, the operation that sends E to its double dual $E^{\vee\vee}$, $E^{\vee\vee}$ is always locally free because X is a smooth surface, defines a map

$$\mathbf{F} : \mathcal{A}_1(d, \Sigma) \to \bigcup_{d' \leq d} \mathscr{B}(d', I).$$

We claim that for any $V \in \mathscr{B}(d', \Sigma)$,

$$\#_{\text{mod}} \mathbf{F}^{-1}(V) \leqq 3(d - d'). \tag{1.7}$$

Suppose we have already established (1.7), then

$$\begin{aligned}
\#_{\text{mod}} \mathcal{A}_1(d, \Sigma) &\leqq \sup_{I^2/4 \leqq d' \leqq d} \{ \#_{\text{mod}} \mathscr{B}(d', \Sigma) + \sup\{ \#_{\text{mod}} \mathbf{F}^{-1}(V) | V \in \mathscr{B}(d', \Sigma)\}\} \\
&\leqq \sup_{I^2/4 \leqq d' \leqq d} \{3d' + C_0(\sqrt{d' - I^2/4} + 1) + 3(d - d')\} \\
&\leqq 3d + C_0(\sqrt{d - I^2/4} + 1).
\end{aligned}$$

Thus Lemma 1.3 will be established if we have (1.7). Indeed, for any $E \in \mathbf{F}^{-1}(V)$, where $V \in \mathscr{B}(d', \Sigma)$, E is a subsheaf of V whose quotient V/E is a sheaf (has length $\ell(V/E) = d - d'$) supported on discrete point set. Thus the question is to determine the number of moduli of the set of all quotient sheaves $V \to Q$ with $\ell(Q) = d - d'$. In [15], it is shown that this set has dimension exactly $3(d - d')$. Thus $\#_{\text{mod}} \mathbf{F}^{-1}(V) \leqq 3(d - d')$. This proves Lemma 1.3 and thus (1.2). □

As to the local structure of $\mathcal{Q}_n(d, \Sigma)^{ss}$, we will use the obstruction theory to show that $\mathcal{Q}_n(d, \Sigma)^{ss}$ is a l.c.i. scheme. This is accomplished by first showing that locally, $\mathcal{Q}_n(d, \Sigma)^{ss}$ is defined by an ideal $J \subset \mathbb{C}[t_1, \ldots, t_k]$ generated by at most $k - \eta(d, \Sigma, n)$ elements. Because we have already proved that $\dim \mathcal{Q}_n(d, \Sigma)^{ss} \leqq \eta(d, \Sigma, n)$, J is generated by exactly $k - \eta(d, \Sigma, n)$ elements and consequently,

$\mathcal{Q}_n(d, \Sigma)^{ss}$ is a l.c.i. at z. To do this, we need to study the completion of local rings of points on $\mathcal{Q}_n(d, \Sigma)^{ss}$. First, let us fix some notations. Let $z \in \mathcal{Q}_n(d, \Sigma)^{ss}$ be a closed point corresponding to the quotient sheaf $W \to E$. Let R be the local ring of z on $\mathcal{Q}_n(d, \Sigma)^{ss}$ and let $\hat{R}$ be the completion of R. Let $m_1 = \dim \operatorname{Hom}(F, E)$, where F is the kernel of $W \to E$, and let $m_2 = \operatorname{Ext}^2(E, E)^0$. Then because $\operatorname{Hom}(F, E)$ is the Zariski tangent space of $\mathcal{Q}_n(d, \Sigma)^{ss}$ at z,

$$\hat{R} = \mathbb{C}[[t_1, \ldots, t_{m_1}]]/\hat{J}, \quad \hat{J} \subseteq (t_1, \ldots, t_{m_1})^2.$$

Clearly, if Theorem 1.2 is true, then $\hat{J}$ is generated by exactly m_2 elements. Here, we first prove

Lemma 1.4 *With the notation as before, then the ideal $\hat{J}$ can be generated by at most m_2 elements.*

In order to prove Lemma 1.4, it is natural to look at a subfunctor of $\mathfrak{Quot}_n(d, \Sigma)$ which dictates the local moduli of the quotient sheaf E. Let $\mathscr{C}$ be the category of all Artin local $\mathbb{C}$-algebras. For any $A \in \mathscr{C}$, we define $\mathfrak{Quot}_E(A) \subset \mathfrak{Quot}_n(d, \Sigma)(\operatorname{Spec} A)$ to be the subset of all quotient sheaves E_A on $X \times \operatorname{Spec} A$ such that $E_A \otimes k(z_0) = E$ as quotient sheaves of W, where z_0 is the only closed point of $\operatorname{Spec} A$. Clearly, $\mathfrak{Quot}_E$ is a covariant functor of the category $\mathscr{C}$. We say $\mathfrak{Quot}_E$ is pro-represented by a complete local Noetherian $\mathbb{C}$-algebra S if

$$\mathfrak{Quot}_E(A) = \operatorname{Hom}_{\text{local } \mathbb{C}\text{-algebra}}(S, A), \quad \forall A \in \mathscr{C}.$$

Because $\mathfrak{Quot}_n(d, \Sigma)$ is represented by the scheme $\mathcal{Q}_n(d, \Sigma)^{ss}$ and R is the local ring of $\mathcal{Q}_n(d, \Sigma)^{ss}$ at z, $\mathfrak{Quot}_E$ is pro-represented by the complete local ring $\hat{R}$.

Now we recall the obstruction theory of $\mathfrak{Quot}_E$ that is needed in relating the structure of the $\hat{R}$ to the dimension of cohomology groups of E. We recall the following definition:

Definition 1.5 The functor $\mathfrak{Quot}_E$ is said *to have an obstruction theory with coefficients in V*, where V is a finite dimensional $\mathbb{C}$-linear space, if the following holds: For any triple $(A, I, E_{A/I})$, where $A \in \mathscr{C}$ is an Artin ring, $I \subseteq A$ is an ideal annihilated by the maximal ideal $\mathfrak{m} \subset A$ and $E_{A/I}$ is a quotient sheaf of $W \otimes_{\mathbb{C}} A/I$ on $X \times \operatorname{Spec} A/I$ in $\mathfrak{Quot}_E(A/I)$, there is a function $\operatorname{ob}(A, I, E_{A/I}) \in V \otimes_{\mathbb{C}} I$ has the following properties:

(1) $\operatorname{ob}(A, I, E_{A/I}) = 0$ if and only if there is a quotient sheaf E_A of $W \otimes_{\mathbb{C}} A$ on $X \times \operatorname{Spec} A$ flat over A that induces the quotient sheaf $E_{A/I}$ when restricted to $X \times \operatorname{Spec} A/I$.
(2) Suppose there is another triple $(A', I', E'_{A'/I'})$ as before has the property that $f: A' \to A$ is a surjective morphism satisfying $f(I') \subseteq I$. let $f_*: V \otimes I' \to V \otimes I$ and $f^*: \operatorname{Spec} A/I \to \operatorname{Spec} A'/I'$ be the induced maps. Then

$$f_*(\operatorname{ob}(A', I', E'_{A'/I'})) = \operatorname{ob}(A, I, (\mathrm{id} \times f^*)^* E'_{A'/I'}).$$

The power of the existence of an obstruction theory for $\mathfrak{Quot}_E$ is best illustrated by the following proposition. The proof of which can be found in [6, IV]. (See also [17, §1].)

Proposition 1.6 *Assume $\mathfrak{Quot}_E$ is prorepresented by the complete local ring $\mathbb{C}[[t_1, \ldots, t_{m_1}]]/\hat{J}$, where $m_1 = \dim \mathrm{T}_E \mathcal{Q}_n(d, \Sigma)^{ss}$ and $\hat{J} \subseteq (t_1, \ldots, t_{m_1})^2$. Suppose $\mathfrak{Quot}_E$ has an obstruction theory with coefficients in V, then $\hat{J}$ can be generated by at most* dim *V elements.*

Back to the proof of Lemma 1.4, we see that what need to be checked is that there is an obstruction theory for the functor $\mathfrak{Quot}_E$ with coefficients in $\mathrm{Ext}^2(E, E)^0$.

Lemma 1.7 *There is an obstruction theory for the Artin functor* $\mathfrak{Quot}_E$ *whose coefficients lie in* $\mathrm{Ext}^2(E, E)^0$.

Proof. The proposition is a variant of Mukai–Artamkin theorem about the deformation of arbitrary coherent sheaves on X. In the following, we give the necessary modification needed for our situation. The argument we proceed follows closely to that of [6, IV]. (See also [2, 17].) Let A be a local Artin ring, $I \subseteq A$ be an ideal annihilated by the maximal ideal $\mathfrak{m}$. Suppose $W \otimes_{\mathbb{C}} A/I \to E_{A/I}$ is a quotient sheaf in $\mathfrak{Quot}_E(A/I)$. In [2]. [6] and [18], they defined an obstruction function $\mathrm{ob}(A, I, E_{A/I}) \in \mathrm{Ext}^2(E, E) \otimes_{\mathbb{C}} I$ whose vanishing is equivalent to the existence of a sheaf E_A on $X \times \operatorname{Spec} A$ flat over $\operatorname{Spec} A$ which induces the sheaf $E_{A/I}$ when restricted to $X \times \operatorname{Spec} A/I$. Therefore, to show that $\mathrm{ob}(A, I, E_{A/I})$ is the obstruction function for the functor $\mathfrak{Quot}_E$, we only need to show that the quotient sheaf $W \otimes_{\mathbb{C}} A/I \to E_{A/I}$ on $X \times \operatorname{Spec} A/I$ extends to a quotient sheaf $W \otimes_{\mathbb{C}} A \to E_A$ on $X \times \operatorname{Spec} A$ flat over $\operatorname{Spec} A$ if and only if $E_{A/I}$ extends to a sheaf on $X \times \operatorname{Spec} A$ flat over $\operatorname{Spec} A$. One direction is obvious. The other direction is also true because of the following reason: Since $W \otimes_{\mathbb{C}} A$ is locally free,

$$0 \to I \to A \to A/I \to 0$$

induces the following exact sequence

$$\mathrm{Hom}(W \otimes_{\mathbb{C}} A, E_A) \to \mathrm{Hom}(W \otimes_{\mathbb{C}} A/I, E_{A/I}) \to H^1(\mathcal{H}om(W, E) \otimes_{\mathbb{C}} I).$$

By assumption, $H^1(E) = 0$. Thus the quotient homomorphism $W \otimes_{\mathbb{C}} A/I \to E_{A/I}$ lifts to a homomorphism $\varphi_A : W \otimes_{\mathbb{C}} A \to E_A$. Because $\varphi_A \otimes k(0)$: $W \to E$ is a quotient homomorphism and E_A is flat over A, φ_A must be surjective. Thus $\{\varphi_A : W \otimes A \to E_A\} \in \mathfrak{Quot}_E(A)$ is the desired extension. This completes the proof of the Lemma 1.7. □

To derive a similar result for the local ring R, we need a lemma that relates the structure of a local ring to its completion. First, we fix some terminology. Let $P = \mathbb{C}[t_1, \ldots, t_n]$ be a polynomial ring, $\mathfrak{m} = (t_1, \ldots, t_n)$ be the maximal ideal and let $J \subseteq (t_1, \ldots, t_n)$ be an ideal of P. If we view J and $R = P/J$ as P-modules, we can form the $\mathfrak{m}$-adic completion $\hat{J}$, $\hat{P}$ and $\hat{R}$ of modules J, P and R respectively. By [1, §10], $\hat{J}$ is the same as the completion of J with respect to $J \cap \mathfrak{m}$. We also denote by $J_{\mathfrak{m}}$, $P_{\mathfrak{m}}$ and $R_{\mathfrak{m}}$ the localization of J, P and R at $\mathfrak{m}$ respectively. We have

Lemma 1.8 *Suppose* $\hat{R}$ *has the form*

$$\hat{R} = \mathbb{C}[[t_1, \ldots, t_n]]/(f_1, \ldots, t_m), \quad f_1, \ldots, f_m \in (t_1, \ldots, t_m),$$

then the local ring $J_{\mathfrak{m}}$ *can also be generated by m elements.*

Proof. Since we have an exact sequence of P-modules

$$0 \to J \to P \to R \to 0,$$

by [1, 10.12], their $\mathfrak{m}$-adic completions also fits into the exact sequence

$$0 \to \hat{J} \to \hat{P} \to \hat{R} \to 0. \tag{1.9}$$

By assumption, $\hat{R} = \hat{P}/(f_1, \ldots, f_m)$. Thus $\hat{J} = (f_1, \ldots, f_m)$.

Now we consider the natural homomorphism $J_{\mathfrak{m}} \to \hat{J}$. Because $J_{\mathfrak{m}}/J_{\mathfrak{m}} \cdot \mathfrak{m} = \hat{J}/\hat{J} \cdot \hat{\mathfrak{m}}$, where $\hat{\mathfrak{m}}$ is the $\mathfrak{m}$-adic completion of the module $\mathfrak{m}$, we can find $\bar{f}_1, \ldots, \bar{f}_m \in J_{\mathfrak{m}}$ such that $\rho(\bar{f}_i) - f_i \in \hat{J} \cdot \hat{\mathfrak{m}}$, where $\rho: J_{\mathfrak{m}} \to \hat{J}$. We claim that $J_{\mathfrak{m}} = (\bar{f}_1, \ldots, \bar{f}_m)$. Indeed, for any $f \in J_m$, there is an $f' \in (\bar{f}_1, \ldots, \bar{f}_m)$ such that $\rho(f) - \rho(f') \in \hat{J} \cdot \hat{\mathfrak{m}}$. Thus $f - f' \in J_{\mathfrak{m}} \cdot \mathfrak{m}$. Therefore, we have

$$J_{\mathfrak{m}} \subseteq (\bar{f}_1, \ldots, \bar{f}_m) + J_{\mathfrak{m}} \cdot \mathfrak{m}.$$

Because $(\bar{f}_1, \ldots, \bar{f}_m) \subseteq J_{\mathfrak{m}}$, by Nakayama's lemma, $J_{\mathfrak{m}} = (\bar{f}_1, \ldots, \bar{f}_m)$. This completes the proof of Lemma 1.8. □

Now we are ready to prove Theorem 1.2.

Proof of theorem 1.2 Let $z \in \mathcal{Q}_n(d, \Sigma)^{ss}$ be any closed point corresponding to the quotient sheaf E. Let F be the kernel of $W \to E$. We know that $T_z \mathcal{Q}_n(d, \Sigma)^{ss} = \mathrm{Hom}(F, E)$. Assume $m_1 = \dim \mathrm{Hom}(F, E)$ and $m_2 = \dim \mathrm{Ext}^2(E, E)^0$. By R.R., $m_1 - m_2 = \eta(d, \Sigma, n)$. Thus by our previous arguement, there is a constant C depending on (X, H, Σ) such that when $d \geqq C$, then $\dim \mathcal{Q}_n(d, \Sigma)^{ss} \leqq m_1 - m_2$.

Next, because $\mathcal{Q}_n(d, \Sigma)^{ss}$ is quasi-projective, we can assume that locally near z, $\mathcal{Q}_n(d, \Sigma)^{ss}$ is defined by an ideal $J \subset \mathbb{C}[t_1, \ldots, t_k]$ with z defined by the maximal ideal $\mathfrak{m} = (t_1, \ldots, t_k)$. Then lemma 1.4 and lemma 1.7 imply that the $\mathfrak{m}$-adic completion $\hat{J}$ of J is generated by at most $k - (m_1 - m_2) = k - \eta(d, \Sigma, n)$ elements. Thus, by applying Lemma 1.8, we conclude that the localization of J at $\mathfrak{m}$, $J_{\mathfrak{m}}$, is generated by at most $k - \eta(d, \Sigma, n)$ elements. But since we know that $\dim \mathcal{Q}_n(d, \Sigma)^{ss} \leqq \eta(d, \Sigma, n)$, we must then have

$$\dim(\mathcal{Q}_n(d, \Sigma)^{ss} \text{ at } z) = \eta(d, \Sigma, n)$$

and that $\mathcal{Q}_n(d, \Sigma)^{ss}$ is defined by exactly $k - \eta(d, \Sigma, n)$ polynomials in $\mathbb{C}[t_1, \ldots, t_k]$ near z. Thus, $\mathcal{Q}_n(d, \Sigma)^{ss}$ is a l.c.i. at z.

It remains to prove that $\mathcal{Q}_n(d, \Sigma)^{ss}$ is normal everywhere. But this is obvious because when d is large, $\mathcal{Q}_n(d, \Sigma)^{ss}$ is smooth at codimension one points thanks to (1.2). Thus the theorem has been established. □

2 The moduli schemes of semistable sheaves

In this section, we shall first apply Theorem 1.2 to study the singularities of the moduli scheme $\mathfrak{M}_H(d, I)$ of rank two semistable sheaves E on X with $\det E = I$ and $c_2(E) = d$. We will then study resolutions of $\mathfrak{M}_H(d, I)$ and study the question whether there exist local (in the classical or étale topology) universal families on the resolutions. The bulk of this section is devoted to answer this question at closed points correspond to strictly semistable sheaves. We remark that when all sheaves in $\mathfrak{M}_H(d, I)$ are stable, that is the case when d is odd, then the second half of this section is not needed for further study in §3 and §4.

In light of Theorem 1.2, we will first work on the moduli space $\mathfrak{M}_H(d, \Sigma)$ of H-semistable sheaves E with $\det E \in \Sigma$ and $c_2(E) = d$. We will show that when d is large, $\mathfrak{M}_H(d, \Sigma)$ is normal and is a local complete intersection at the closed points that correspond to stable sheaves. As to the proof, we will realize $\mathfrak{M}_H(d, \Sigma)$ as geometric invariant theory quotient of $\mathcal{Q}_n(d, \Sigma)^{ss}$ by reductive group

$G = \mathrm{PGL}(N, \mathbb{C})$ [7] and then deduce the properties of $\mathfrak{M}_H(d, \Sigma)$ from that of $\mathcal{Q}_n(d, \Sigma)^{ss}$.

To work out the detail of this argument, a quick review of the construction of the moduli shceme $\mathfrak{M}_H(d, \Sigma)$ is in order. For any (d, Σ), let n be large as in Theorem 1.2 so that for any $E \in \mathscr{E}(d, \Sigma, n)$, we have $h^i(E) = 0$, $i \geq 1$, and $H^0(E)$ generates E. Thus for any semistable sheaf $E \in \mathscr{E}(d, \Sigma, n)$, an identification $\mathbb{C}^N \cong H^0(E)$ corresponds to a unique closed point in $\mathcal{Q}_n(d, \Sigma)^{ss}$ and E itself corresponds to a unique G orbit in $\mathcal{Q}_n(d, \Sigma)^{ss}$. Certainly, $\mathcal{Q}_n(d, \Sigma)^{ss}$ is a G-scheme. In [7], Gieseker showed that when n is sufficiently large, a good quotient of $\mathcal{Q}_n(d, \Sigma)^{ss}$ by G exists which is exactly the moduli space $\mathfrak{M}_H(d, \Sigma)$. To characterize all closed points of $\mathfrak{M}_H(d, \Sigma)$, we recall the concept of S-equivalence class of semistable sheaves: For any semistable sheaf E, there is a filtration

$$0 = F_0 \subset F_1 \subset \cdots \subset F_t = E$$

so that F_i/F_{i-1} are stable and $\chi_{F_i/F_{i-1}}$ are proportional to χ_E. Set $\mathrm{gr}(E) = \oplus_{i=1}^t F_i/F_{i-1}$. Two sheaves E_1 and E_2 are said to be S-equivalent if $\mathrm{gr}(E_1) = \mathrm{gr}(E_2)$. By abuse of notation, we call a closed point $z \in \mathfrak{M}_H(d, \Sigma)$ (or $z \in \mathcal{Q}_n(d, \Sigma)^{ss}$) a stable point if the corresponding sheaf E is stable. Otherwise, we call if strictly semistable. In the following, we will use $\mathfrak{M}_H(d, \Sigma)^s$ (resp. $\mathcal{Q}_n(d, \Sigma)^s$) to denote the subset of stable points. We put the relevant results concerning the quotient morphism $\pi: \mathcal{Q}_n(d, \Sigma)^{ss} \to \mathfrak{M}_H(d, \Sigma)$ into the following proposition. The proof of which can be found in [7, 16].

Proposition 2.1 *There is a $\kappa: \mathbb{Z}^+ \to \mathbb{Z}^+$ depending on (X, H, Σ) such that for any d and $n \geq \kappa(d)$, there is a good quotient of $\mathcal{Q}_n(d, \Sigma)^{ss}$ by G that is isomorphic to $\mathfrak{M}_H(d, \Sigma)$. $\mathfrak{M}_H(d, \Sigma)$ is projective. Further, any closed point of $\mathfrak{M}_H(d, \Sigma)$* corresponds to a unique S-*equivalent class of semistable sheaves in $\mathscr{E}(d, \Sigma)$.* Finally, when restricted to subset of stable points,

$$\pi: \mathcal{Q}_n(d, \Sigma)^s \to \mathfrak{M}_H(d, \Sigma)^s$$

is a principle G-bundle.

We now prove the results parallel to the Theorem 0.1 regarding the moduli $\mathfrak{M}_H(d, \Sigma)^s$.

Theorem 2.2 *There is a universal constant C depending on (X, H, Σ) such that whenever $d \geq C$, then the moduli scheme $\mathfrak{M}_H(d, \Sigma)$* is normal and *$\mathfrak{M}_H(d, \Sigma)^s$ is a local complete intersection.*

Proof. By Theorem 1.2, when $d \geq A$ and $n \geq \kappa(d)$, $\mathcal{Q}_n(d, \Sigma)^{ss}$ is normal and a l.c.i. Thus by universal mapping property, $\mathfrak{M}_H(d, \Sigma)$ is normal [19, p. 5]. Further, since $\pi: \mathcal{Q}_n(d, \Sigma)^s \to \mathfrak{M}_H(d, \Sigma)^s$ is a principle G-bundle and since $\mathcal{Q}_n(d, \Sigma)^s$ is a l.c.i., $\mathfrak{M}_H(d, \Sigma)^s$ is a l.c.i. also. This establishes Theorem 2.2. □

So far, we have dealt solely with the moduli space of sheaves whose determinants lie in Σ. As to the moduli space $\mathfrak{M}_H(d, I)$ of sheaves with det $= I$, it can be realized either as a closed subscheme of $\mathfrak{M}_H(d, \Sigma)$, where $I \in \Sigma$, or as a quotient of $\mathfrak{M}_H(d, \Sigma)$ by $\mathrm{Pic}(X)^0$. To view $\mathfrak{M}_H(d, I)$ as a subscheme of $\mathfrak{M}_H(d, \Sigma)$, we proceed as follows: Let

$$\det_Q: \mathcal{Q}_n(d, \Sigma)^{ss} \to \Sigma \subseteq \mathrm{Pic}(X)$$

be the morphism induced by the invertible sheaf det $\mathscr{F}$ on $X \times \mathscr{Q}_n(d, \Sigma)^{ss}$, where $\mathscr{F}$ is the universal family of $\mathscr{Q}_n(d, \Sigma)^{ss}$. Clearly, $\det_Q$ is G-equivalent, where G acts on Σ trivially. Thus $\det_Q$ descends to a morphism $\det: \mathfrak{M}_H(d, \Sigma) = \mathscr{Q}_n(d, \Sigma)^{ss}//G \to \Sigma$. We define $\mathfrak{M}_H(d, I) = \det^{-1}(I)$. We leave it to the readers to check that $\mathfrak{M}_H(d, I) = \det^{-1}(I)$ is the coarse moduli space of rank two H-semistable sheaves E with $\det E = I$ and $c_2(E) = d$. To compare the local structure of $\mathfrak{M}_H(d, I)$ with that of $\mathfrak{M}_H(d, \Sigma)$, it is easier to use the quotient morphism

$$\phi: \mathfrak{M}_H(d, \Sigma) \to \mathfrak{M}_H(d, I)$$

constructed as follows: Let P be the identity component of $\operatorname{Pic}(X)$. P is a smooth group scheme. There is a canonical action

$$\Phi: P \times \mathfrak{M}_H(d, \Sigma) \to \mathfrak{M}_H(d, \Sigma)$$

that sends any line bundle L_S on $X \times S$ (in P) and any family of semistable sheaves E_S on $X \times S$ (in $\mathfrak{M}_H(d, \Sigma)$) to the family $L_S \otimes E_S$. Let $\Lambda \subseteq P$ be the discrete subgroup of all order two elements and let $\hat{P} = P/\Lambda$ be the quotient group scheme. It is easy to see that the action Φ descends to a $\hat{P}$-action

$$\hat{\Phi}: \hat{P} \times \mathfrak{M}_H(d, \Sigma) \to \mathfrak{M}_H(d, \Sigma).$$

We claim that $\hat{\Phi}$ is free and the quotient $\mathfrak{M}_H(d, \Sigma)/\hat{P}$ is isomorphic to $\mathfrak{M}_H(d, I)$.

Indeed, let $\Phi_2: \hat{P} \times \Sigma \to \Sigma$ be the $\hat{P}$-action that sends $[L] \in \hat{P}$ and $L' \in \Sigma$ to the line bundle $L^{\otimes 2} \otimes L'$. Φ_2 is well-defined. Clearly, there is a commutative diagram

$$\begin{array}{ccc} \hat{P} \times \mathfrak{M}_H(d, \Sigma) & \xrightarrow{(\mathrm{id}, \det)} & \hat{P} \times \Sigma \\ \downarrow \Phi & & \downarrow \Phi_2 \\ \mathfrak{M}_H(d, \Sigma) & \xrightarrow{\det} & \Sigma. \end{array}$$

Since Σ is a principal bundle over $\Sigma/\hat{P} = \text{point}$, $\mathfrak{M}_H(d, \Sigma)$ is a principal bundle over $\mathfrak{M}_H(d, \Sigma)/\hat{P}$. Therefore, $\mathfrak{M}_H(d, \Sigma)/\hat{P}$ is isomorphic to the section $\det^{-1}(I)$. This completes the proof of claim.

Finally, for large d, because $\mathfrak{M}_H(d, \Sigma)$ is normal and $\hat{P}$ is smooth, $\mathfrak{M}_H(d, I)$ is normal also. Similarly, $\mathfrak{M}_H(d, I)^s$ is a l.c.i. because $\mathfrak{M}_H(d, \Sigma)^s$ is a l.c.i.. Thus Theorem 2.2 and (2) of the Theorem 0.2 have been established. The (1) of the Theorem 0.2 follows from (1) of Theorem 1.2 and that $\dim \hat{P} = h^1(\mathcal{O}_X)$. □

In the following, we shall discuss the existence of universal families (sheaves) on $X \times \mathfrak{M}_H(d, I)$. A sheaf E on $X \times \mathfrak{M}_H(d, I)$ is said to be a universal family if for any closed point $s \in \mathfrak{M}_H(d, I)$, the restriction of E to the fiber of $X \times \mathfrak{M}_H(d, I)$ over $s \in \mathfrak{M}_H(d, I)$, say E_s, belongs to the S-equivalent class represented by s. It is known that in some cases, especially when there are strictly semistable sheaves present, the universal family does not exist even locally. For our purpose, we will introduce the following concept: For any scheme $\mathfrak{W}$ and $f: \mathfrak{W} \to \mathfrak{M}_H(d, I)$, if there is a classical (or étale) open covering $\{U_i\}$ of $\mathfrak{W}$, a collection of sheaves E_i on $X \times U_i$ such that for any closed $s \in U_i$, $E_{i,s}$ belongs to the S-equivalent class of semistable sheaves represented by $f(s) \in \mathfrak{M}_H(d, I)$ and further, over each $U_i \cap U_j$ there is an isomorphism

$$E_{i|X \times (U_i \cap U_j)} \xrightarrow{\cong} E_{j|X \times (U_i \cap U_j)},$$

then we say $\{E_i\}$ is a local universal family of $\mathfrak{W}$. In the rest of this section, we will construct a desingularization $\tilde{\mathfrak{M}}_H(d, I)$ of $\mathfrak{M}_H(d, I)$ and study the existence of local universal families on $\tilde{\mathfrak{M}}_H(d, I)$. As we will see, there basically are two types of singularities of $\mathfrak{M}_H(d, I)$. One type comes from the singularities of the Quot-scheme $\mathcal{Q}_n(d, I)^{ss}$. Such singularities can be taken care of easily by using Hironaka's desingularization result. Another type comes from the presence of strictly semistable sheaves. It turns out that in general, we can not find local universal families even on a resolution of these singularities. It is to circumvent this technical difficulty that we will use the partial desingularization of $\mathfrak{M}_H(d, I)$ introduced by Kirwan [12].

Following Gieseker, any flat family of rank two torsion free quotient sheaves of $\mathcal{O}_X^{\oplus N}$ over $X \times S$, say $\varphi: \mathcal{O}_{X\times S}^{\oplus N} \to E_S$ with $\det E_S = p_X^* I(2n) \otimes p_S^* L$, where L is a line bundle on S, associates to a canonical section

$$\wedge^2\varphi \in H^0(S, \mathrm{Hom}_S(\wedge^2 \mathcal{O}_S^{\oplus N}, H^0(X, I(2n)) \otimes p_S^* L))$$

which induces a morphism

$$[\wedge^2\varphi]: S \to \mathbf{P}(\mathrm{Hom}(\wedge^2\mathbb{C}^N, H^0(I(2n)))).$$

Here we adopt the notation that $\mathbf{P}(\mathbb{C}^l)$ is the space of lines in $\mathbb{C}^l$. We denote $M = H^0(I(2n))$. Now, if we apply the previous construction to the Quot-scheme $\mathcal{Q}_n(d, I)^{ss}$ and its universal quotient family $\mathcal{F}$, we obtain a morphism

$$\mu: \mathcal{Q}_n(d, I)^{ss} \to \mathbf{P}((\wedge^2\mathbb{C}^N)^\vee \otimes M). \tag{2.1}$$

Proposition 2.4 (Maruyama [16]) *There is a $\kappa: \mathbb{Z}^+ \to \mathbb{Z}^+$ such that when $n \geqq \kappa(d)$, the morphism μ of* (2.1) *is a locally closed immersion.*

Certainly, under the dual action of G on $\mathbf{P}((\wedge^2\mathbb{C}^N)^\vee \otimes M)$, μ is G-equivariant. Now if we denote by $\mathbf{P}((\wedge^2\mathbb{C}^N)^\vee \otimes M)^s$ (resp. $\mathbf{P}((\wedge^2\mathbb{C}^N)^\vee \otimes M)^{ss}$) the set of Mumford stable (resp. semistable) points, then there is a $\kappa(\cdot): \mathbb{Z}^+ \to \mathbb{Z}^+$ so that when $n \geqq \kappa(d)$, we have

$$\mu^{-1}(\mathbf{P}((\wedge^2\mathbb{C}^N)^\vee \otimes M)^s) = \mathcal{Q}_n(d, I)^s$$

and

$$\mu^{-1}(\mathbf{P}((\wedge^2\mathbb{C}^N)^\vee \otimes M)^{ss}) = \mathcal{Q}_n(d, I)^{ss}.$$

(For definition of Mumford stability, see [19]). Let $Z = \mathbf{P}((\wedge^2\mathbb{C}^N)^\vee \otimes M)$. By *Mumford* [19], the geometric invariant theory quotient $Z^{ss}//G$ does exist. $Z^{ss}//G$ is projective and further, μ induces a closed immersion

$$\tilde{\mu}: \mathfrak{M}_H(d, I) \to Z^{ss}//G.$$

In [12], Kirwan described how to blow up Z along a sequence of non-singular G-invariant subvarieties to obtain new projective variety $\tilde{Z}$ acted on by G with G-equivariant morphism $\varphi: \tilde{Z} \to Z$ such that all semistable points of $\tilde{Z}$ are automatically stable. Let $\tilde{Z}^{ss}$ be the set of semistable points of $\tilde{Z}$. Then $\tilde{\varphi}: \tilde{Z}^{ss}//G \to Z^{ss}//G$ is a partial desingularization of $Z^{ss}//G$ in the sense that all singularities of $\tilde{Z}^{ss}//G$ are finite quotient singularities.

The scheme $\tilde{Z}^{ss}$ is obtained as follows: (The proof of the statements below can be found in [12]). Since $Z^s \neq \emptyset$, $Z^{ss}\backslash Z^s \neq \emptyset$ if and only if there are semistable points $z \in Z^{ss}$ such that the stabilizers $\mathrm{stab}(z) \subset G$ (of z) contain non-trivial connected reductive subgroups. Let $r = \max\{\dim \mathrm{stab}(z) | z \in Z^{ss}\}$ and let $\mathcal{R}(r)$ be a set of representatives of conjugacy classes of all connected reductive subgroups $R \subseteq G$ with $\dim R = r$ such that

$$\mathfrak{W}_R^{ss} = \{z \in Z^{ss} \mid R \text{ fixes } z\} \tag{2.2}$$

is non-empty. Then

$$\mathfrak{B}_r = \bigcup_{R \in \mathscr{R}(r)} G\mathfrak{W}_R^{ss} \tag{2.3}$$

is a disjoint union of non-singular closed subvarieties of Z^{ss}. Since it is G-invariant, the action G on Z^{ss} lifts to an action on the blowing-up Z_r of Z^{ss} along $\mathfrak{B}_r$ and the G-linearization of $\mathcal{O}_Z(1)$ can canonically be lifted to a G-linearization of $\pi_r^* \mathcal{O}_Z(k)(-W)$, where W is the exceptional divisor of $Z_r \to Z^{ss}$, k is any integer and $\mathcal{O}_Z(1)$ is the ample line bundle on Z of which the stability of points in Z was defined. When k is sufficiently large, the set Z_r^{ss} of G-semistable points of Z_r, semistable with respect to the G-linearization of $\pi_r^* \mathcal{O}_Z(k)(-W)$, is independent of k. In [12, §1], Kirwan proved:

Lemma 2.5 (Kirwan) (1) *The complement of* Z_r^{ss} *is the proper transform of the subset* $p^{-1}(p(\mathfrak{B}_r)) \subseteq Z^{ss}$ *where* $p: Z^{ss} \to Z^{ss}//G$ *is the quotient morphism.*
(2) *No points of* Z_r^{ss} *are fixed by a reductive subgroup of* G *of dimension at least* r, *and a point in* $W^{ss} = Z_r^{ss} \cap W$ *is fixed by a reductive subgroup* $R \subseteq G$ *of dimension less than* r *if and only if it belongs to the proper transform of the subvariety* $\mathfrak{W}_R^{ss} \subseteq Z^{ss}$.

To obtain $\tilde{Z}^{ss}$, we first blow up Z^{ss} along $\mathfrak{B}_r$ to get Z_r. Let Z_r^{ss} be the set of semistable points of Z_r (semistable with respect to the G-linearization of $\pi_r^* \mathcal{O}_Z(k)(-W)$, $k \gg 0$). We then blow up Z_r^{ss} along $\mathfrak{B}_{r-1} \subseteq Z_r^{ss}$ to get Z_{r-1}, where $\mathfrak{B}_{r-1}$ is the set of points in Z_r^{ss} that are fixed by some reductive subgroups $R \subseteq G$ of dimension $r-1$. Then we blow up Z_{r-1}^{ss} again along $\mathfrak{B}_{r-2} \subseteq Z_{r-1}^{ss}$, and so on until we obtain Z_1 that has the property that no connected (non-trivial) reductive subgroup of G fix any semistable closed point in Z_1. Then by [12], all closed points of Z_1^{ss} are stable. Now, we assume d is large so that $\mathfrak{M}_H(d, I)$ is normal. Consider the normal subscheme $\mathcal{Q}_n(d, I)^{ss} \subseteq Z^{ss}$. Let $\mathcal{Q}_n(d, I)_1^{ss} \subseteq Z_1^{ss}$ be the proper transform of $\mathcal{Q}_n(d, I)^{ss} \subseteq Z^{ss}$. Note that because $Z_1^s = Z_1^{ss}$, all G-orbits of $\mathcal{Q}_n(d, I)_1^{ss}$ are closed.

Next, we seek to desingularize $\mathcal{Q}_n(d, I)_1^{ss}$. According to Hironaka, a resolution of $\mathcal{Q}_n(d, I)_1^{ss}$ can be derived by performing successive blowing ups along smooth centers contained in the singular locus of $\mathcal{Q}_n(d, I)_1^{ss} \subseteq Z_1^{ss}$ and its blowing ups. Because our goal is to find local universal families on the resolution, we need to keep track the inclusion $\mathcal{Q}_n(d, I)_1^{ss} \subseteq Z_1^{ss}$ as we do blowing ups. Thus, each time we blow-up $\mathcal{Q}_n(d, I)_1^{ss}$, we will take a smooth G invariant subvariety Y contained in the singular locus of $\mathcal{Q}_n(d, I)_1^{ss}$ and blow up $\mathcal{Q}_n(d, I)_1^{ss}$ and Z_1^{ss} along Y simultaneously to get $'\mathcal{Q}_n(d, I)_1^{ss} \subseteq {}'Z_1^{ss}$. Note that since Z_1^{ss} is smooth and has stable points only, the closed points of the blowing-up $'\mathcal{Q}_n(d, I)_1^{ss} \subseteq {}'Z_1^{ss}$ are all stable under G. Let $\mathcal{Q}_n(d, I)_0^{ss} \subseteq Z_0^{ss}$ be the pair of smooth schemes that is the result of this series of blowing-ups. By our previous argument, $\operatorname{stab}(z) \subseteq G$ is discrete for any $z \in \mathcal{Q}_n(d, I)_0^{ss}$. Thus $\mathcal{Q}_n(d, I)_0^{ss}/G$ is a partial resolution of $\mathfrak{M}_H(d, I)$ in the sense that all singularities of $\mathcal{Q}_n(d, I)_0^{ss}/G$ are finite quotient singularities.

In order to get a smooth resolution of $\mathfrak{M}_H(d, I)$, we need to blow up $\mathcal{Q}_n(d, I)_0^{ss}$ further. First, we need a list of all possible $\operatorname{stab}(z)$ for $z \in \mathcal{Q}_n(d, I)_0^{ss}$. We have the following lemma the proof of which will be postponed until the end of this section.

Lemma 2.6 *For any closed* $z \in \mathcal{Q}_n(d, I)_0^{ss}$, stab$(z)$ *can possibly be* $\{e\}$, $\mathbb{Z}_2$ *and* $\mathbb{Z}_2 \times \mathbb{Z}_2$.

We first blow up $\mathcal{Q}_n(d, I)_0^{ss}$ along the subset $\mathfrak{W}_2 \subseteq \mathcal{Q}_n(d, I)_0^{ss}$ of points whose stabilizers are $\mathbb{Z}_2 \times \mathbb{Z}_2$($\mathfrak{W}_2$ is smooth and has codimension bigger than one). Let $\mathfrak{W}_1$ be the set of points in the blowing-up whose stabilizer is $\mathbb{Z}_2$ (the only remaining possible case). We then perform one more blowing up along components of $\mathfrak{W}_1$ whose codimension are bigger than one. Let the resulting scheme be $\tilde{\mathcal{Q}}_n(d, I)^{ss}$. Note that the set $\mathfrak{W} = \{ y \in \tilde{\mathcal{Q}}_n(d, I)^{ss} | \text{stab}(\tilde{y}) \neq \{e\}\}$ is a smooth divisor and all points in $\mathfrak{W}$ have stabilizer $\mathbb{Z}_2$. Therefore, $\tilde{\mathfrak{M}}_H(d, I) = \tilde{\mathcal{Q}}_n(d, I)^{ss}/G$ is smooth. In particular, we obtain a desingularization $\tilde{\mathfrak{M}}_H(d, I)$ of $\mathfrak{M}_H(d, I) \subseteq Z^{ss}//G$. Let $\Phi_M : \tilde{\mathfrak{M}}_H(d, I) \to \mathfrak{M}_H(d, I)$, let $\Phi_Q : \tilde{\mathcal{Q}}_n(d, I)^{ss} \to \mathcal{Q}_n(d, I)^{ss}$ and let $\tilde{\pi} : \tilde{\mathcal{Q}}_n(d, I)^{ss} \to \tilde{\mathfrak{M}}_H(d, I)$.

Proposition 2.7 *Let* $\xi \in \tilde{\mathfrak{M}}_H(d, I)$ *be any closed point, let* $\tilde{\xi} \in \tilde{\pi}^{-1}(\xi) \subset \tilde{\mathcal{Q}}_n(d, I)^{ss}$ *and let* $\Gamma = \text{stab}(\tilde{\xi})$. *Then*
(1) $\Gamma = \{e\}$ *of* $\mathbb{Z}_2$,
(2) *there is a* Γ*-invariant smooth locally closed set* $V \subseteq \tilde{\mathcal{Q}}_n(d, I)^{ss}$ *with* $\tilde{\xi} \in V$ *such that* $\tilde{\pi} : V \to \tilde{\mathfrak{M}}_H(d, I)$ *is* Γ*-invariant and further,* V/Γ *is an étale neighborhood of* $\xi \in \tilde{\mathfrak{M}}_H(d, I)$.

Proof. (1) has already been established in the previous argument. As of (2), since $\tilde{\mathcal{Q}}_n(d, I)^{ss}$ is smooth and since $\tilde{\mathcal{Q}}_n(d, I)^{ss} \to \tilde{\mathfrak{M}}_H(d, I)$ is a good quotient, we can apply the étale slice theorem of Luna [19, p. 152] directly to our situation to obtain the desired Γ-invariant subset $V \subseteq \tilde{\mathcal{Q}}_n(d, I)^{ss}$. □

The remainder of this section is devoted to the proof of Lemma 2.6. To accomplish this, we need to find a more manageable account of each blowing-up $\phi_i : Z_i^{ss} \to Z_{i+1}^{ss}$. We first state the following fact:

Lemma 2.8 *Let* $\xi \in \mathcal{Q}_n(d, I)^{ss}$ *be any closed point with closed orbit* $G \cdot \xi$. *Then* stab(ξ) *can only be* $\{e\}$, $\mathbb{C}^*$ *or* PGL$(2, \mathbb{C})$. *More precisely, if we denote by* E_ξ *the quotient sheaf associated to* ξ, *then*: (1) stab$(\xi) = \{e\}$ *if* E_ξ *is stable*; (2) stab$(\xi) = \mathbb{C}^*$ *if* $E_\xi = F_1 \oplus F_2$ *with* $F_1 \neq F_2$ *and* (3) *stab*$(\xi) =$ PGL$(2, \mathbb{C})$ *if* $E_\xi = F \oplus F$.

Proof. [5, 16]. □

For any $\xi \in \mathcal{Q}_n(d, I)^{ss}$, we know that dim stab$(\xi) \leqq 3$. Thus no blowing-up Z_i^{ss} will affect $\mathcal{Q}_n(d, I)^{ss} \subseteq Z^{ss}$ unless possibly $i \leqq 3$. Let $\phi_3 : Z_3^{ss} \to Z_4^{ss}$ be the blowing-up of Z_4^{ss} along

$$\mathfrak{B} = \bigcup_{R \in \mathscr{R}(3)} G\mathfrak{W}_R^{ss}$$

and let W be the exceptional divisor. We first suppose that there are semistable quotient sheaves $E \in \mathcal{Q}_n(d, I)^{ss}$ so that $E = F \oplus F$. Let $\xi \in \mathcal{Q}_n(d, I)^{ss}$ be such a closed point. Since $Z_4^{ss} \to Z^{ss}$ is an isomorphism along $\mathcal{Q}_n(d, I)^{ss}$, by abuse of notation we will view $\xi \in \mathcal{Q}_n(d, I)^{ss}$ as a closed point of Z_4^{ss}.

Let $m = N/2$ and let $\varphi_1 : \mathbb{C}^m \to H^0(F)$ be an isomorphism. φ_1 induces an isomorphism $\varphi = (\varphi_1, \varphi_1) : \mathbb{C}^N \to H^0(E)$. Let $\{e_1, \ldots, e_{2m}\}$ be the obvious basis of $\mathbb{C}^{2m}$ and let $\{v_i\}_{i=1}^l$ be a basis of $M = H^0(I(2n))$. Then $\xi \in \mathbf{P}((\wedge^2 \mathbb{C}^N)^\vee \otimes M)$ can be described up to scalar by

$$\tilde{\xi} = \sum_{i=1}^{l} \begin{pmatrix} 0 & T_i \\ -T_i & 0 \end{pmatrix} \otimes v_i \in (\wedge^2 \mathbb{C}^N)^\vee \otimes M, \tag{2.4}$$

where T_i are $m \times m$ symmetric matrices defined by

$$T_{i,kh} = (\varphi(e_k) \wedge \varphi(e_{m+h}), v_i^\vee).$$

We have

Lemma 2.9 *Let $R_0 = \mathrm{stab}(\xi) \subseteq G$. Then under the given choice of the basis of $H^0(E)$ and M, $\mathfrak{W}_{R_0}^{ss} \subseteq Z^{ss}$ has the expression $\mathfrak{W}_{R_0}^{ss} = \mathbf{P}(\mathbf{V}_0) \cap Z^{ss}$ where*

$$\mathbf{V}_0 = \left\{ \sum_{i=1}^{l} \begin{pmatrix} 0 & T_i \\ -T_i & 0 \end{pmatrix} \otimes v_i \mid T_i \textit{ are symmetric } m \times m \textit{ matrixes} \right\}.$$

Proof. Clearly $R_0 = \mathrm{PGL}(2, \mathbb{C})$ acts on $\mathbf{P}((\wedge^2 \mathbb{C}^N)^\vee \otimes M)$ via

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \left[\sum \begin{pmatrix} A_i & B_i \\ C_i & D_i \end{pmatrix} \otimes v_i \right] = \left[\sum \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} A_i & B_i \\ C_i & D_i \end{pmatrix} \begin{pmatrix} a & c \\ b & d \end{pmatrix} \otimes v_i \right], \tag{2.5}$$

where A_i, B_i, C_i and D_i are $m \times m$ matrixes with A_i, D_i antisymmetric, $B_i^t = -C_i$ and $a, \ldots, d$ are $m \times m$ scalar matrixes. Here B^t is the transpose of B. Suppose $w \in \mathfrak{W}_{R_0}^{ss}$ has the expression $\sum \left(\begin{smallmatrix} A_i & B_i \\ C_i & D_i \end{smallmatrix}\right) \otimes v_i$ and that for some i_0, $A_{i_0} \neq 0$. Then one checks directly that w is invariant under $\mathrm{PGL}(2, \mathbb{C})$ only if $B_i, C_i, D_i = 0$ for all i. But then w can not be semistable. So A_i has to be zero for all i. Similarly, $D_i = 0$ for all i. One further checks that w is invariant under $\mathrm{PGL}(2, \mathbb{C})$ implies that $B_i = -C_i$ for all i. Thus the lemma is true. □

Let $N_\xi \mathfrak{B} = T_\xi Z^{ss} / T_\xi \mathfrak{B}$ be the normal vector space of $\mathfrak{B} \subseteq Z^{ss}$ at ξ and let $N_\xi \mathfrak{W}_{R_0}^{ss}$ be the normal vector space of $\mathfrak{W}_{R_0}^{ss} \subseteq Z^{ss}$ at ξ. Note that both $\mathfrak{B}$ and $\mathfrak{W}_{R_0}^{ss} \subseteq Z^{ss}$ are smooth at ξ [12, Corollary 5.10]. Then $\phi_3^{-1}(\xi) \subseteq W$ is isomorphic to $\mathbf{P}(N_\xi \mathfrak{B})$. Since $N_\xi \mathfrak{B} \subseteq N_\xi \mathfrak{W}_{R_0}^{ss}$,

$$\phi_3^{-1}(\xi) = \mathbf{P}(N_\xi \mathfrak{B}) \subseteq \mathbf{P}(N_\xi \mathfrak{W}_{R_0}^{ss}). \tag{2.6}$$

Further, since $\mathfrak{W}_{R_0}^{ss}$ is fixed by R_0, R_0 acts on $\mathbf{P}(N_\xi \mathfrak{W}_{R_0}^{ss})$ and the inclusion (2.6) is R_0-equivalent. Because for any $\zeta \in \phi_3^{-1}(\xi)$, $\mathrm{stab}(\zeta) \subseteq R_0$. Thus to classify those $\zeta \in \phi_3^{-1}(\xi)$ with $\mathrm{stab}(\zeta) \neq \{e\}$, it suffices to classify $\zeta \in \mathbf{P}(N_\xi \mathfrak{W}_{R_0}^{ss})$ whose stabilizers $\mathrm{stab}_{R_0}(\zeta) \subseteq R_0$ are non-trivial.

Set $\mathbf{V}$ be the space of $N \times N$ antisymmetric matrixes and set

$$\mathbf{V}_1 = \left\{ \begin{pmatrix} T_0 & T_2 \\ T_2 & T_1 \end{pmatrix} \mid T_0, T_1 \text{ and } T_2 \text{ are antisymmetric } m \times m \text{ matrixes} \right\}.$$

Then $\mathbf{V} = \mathbf{V}_0 \oplus \mathbf{V}_1$ and further, $\mathbf{V}_0$ is fixed by $\mathrm{SL}(2, \mathbb{C})$ and V_1 is invariant under $\mathrm{SL}(2, \mathbb{C})$. Note that $\mathfrak{W}_{R_0}^{ss} = \mathbf{P}(\mathbf{V}_0 \otimes M)$ (Lemma 2.9) and that the normal vector space $N_\xi \mathfrak{W}_{R_0}^{ss}$ of $\mathfrak{W}_{R_0}^{ss} \subseteq Z^{ss}$ is isomorphic to $\mathbf{V}_1 \otimes M$. Thus we only need to classify points of $\mathbf{P}(V_1 \otimes M)$ with non-trivial stabilizers in R_0.

Lemma 2.10 *Let $\mathbf{P}(\mathbf{V}_1 \otimes M)$ be the projective space acted on by $\mathrm{PGL}(2, \mathbb{C})$ as described. Assume $\zeta \in \mathbf{P}(\mathbf{V}_1 \otimes M)^{ss}$ is a semistable point (under $\mathrm{PGL}(2, \mathbb{C})$) such that $\mathrm{stab}(\zeta)$ is non-trivial, then either $\mathrm{stab}(\zeta) = \mathbb{Z}_2$ or $\mathrm{stab}(\zeta) = \mathbb{C}^* \rtimes \mathbb{Z}_2$. In the* *later case, the fixed point set is* $\mathrm{PGL}(2, \mathbb{C})\cdot \mathbf{P}(\mathbf{V}_3 \otimes M) \subseteq \mathbf{P}(\mathbf{V}_1 \otimes M)$ *where*

$$\mathbf{V}_3 = \left\{ \begin{pmatrix} 0 & T_2 \\ T_2 & 0 \end{pmatrix} \middle|\, T_2 \text{ antisymmetric} \right\}.$$

Proof. Assume that

$$\zeta = \left[\sum_{i=1}^{l} \begin{pmatrix} T_0^i & T_2^i \\ T_2^i & T_1^i \end{pmatrix} \otimes v_i \right] \in \mathbf{P}(\mathbf{V}_1 \otimes M)^{ss}$$

is a semistable point with $\mathrm{stab}(\zeta) \subseteq \mathrm{SL}(2, \mathbb{C})/\{1, -1\}$ non-trivial. Since ζ is semistable under $\mathrm{SL}(2, \mathbb{C})$, then possibly after a change of the basis $\{v_i\}$ of M, we can assume $\mathrm{rank}\begin{pmatrix} T_0^1 & T_2^1 \\ T_2^1 & T_1^1 \end{pmatrix} \geqq 2$. Further, we can find a $g_0 \in \mathrm{SL}(2, \mathbb{C})$ and integers $1 \leqq i < j \leqq m$ such that if we write $g_0 \cdot \begin{pmatrix} T_0^1 & T_2^1 \\ T_2^1 & T_1^1 \end{pmatrix} \cdot g_0^t = (t_{ij})$, then

$$\begin{pmatrix} t_{i,j} & t_{i,m+j} \\ t_{m+i,j} & t_{m+i,m+j} \end{pmatrix} = \begin{pmatrix} 0 & \alpha \\ \alpha & 0 \end{pmatrix}, \quad \alpha \neq 0. \tag{2.7}$$

Now let $g \in \mathrm{stab}(\zeta)$. Then $\tilde{g} = g_0 g g_0^{-1}$ satisfies

$$\tilde{g} \cdot g_0 \cdot \begin{pmatrix} T_0^1 & T_2^1 \\ T_2^1 & T_1^1 \end{pmatrix} \cdot g_0^t \cdot \tilde{g}^t = \lambda g_0 \cdot \begin{pmatrix} T_0^1 & T_2^1 \\ T_2^1 & T_2^1 \end{pmatrix} \cdot g_0^t, \quad \lambda \in \mathbb{C}^*. \tag{2.8}$$

One checks directly by using (2.7) that $\tilde{g}$ must belongs to the subgroup

$$R_1 = \left\{ \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \middle|\, t \in \mathbb{C}^* \right\} \cup \left\{ \begin{pmatrix} 0 & t^{-1} \\ -t & 0 \end{pmatrix} \middle|\, t \in \mathbb{C}^* \right\} \cong \mathbb{C}^* \rtimes \mathbb{Z}_2. \tag{2.9}$$

One further checks that the fixed point set of R_1 in $\mathbf{P}(\mathbf{V}_1 \otimes M)$ is $\mathbf{P}(\mathbf{V}_3 \otimes M)$. Thus if $\zeta \in g_0^{-1}\, \mathbf{P}(\mathbf{V}_3 \otimes M)$, $\mathrm{stab}(\zeta) = g_0^{-1} \cdot R_1 \cdot g_0 = \mathbb{C}^* \rtimes \mathbb{Z}_2$.

Now assume $\zeta \notin g_0^{-1} \cdot \mathbf{P}(\mathbf{V}_3 \otimes M)$. Let $g_0(\zeta) = \sum \begin{pmatrix} \tilde{T}_0^i & \tilde{T}_2^i \\ \tilde{T}_2^i & \tilde{T}_1^i \end{pmatrix} \otimes v_i$. Then we can find k and $1 \leqq i' < j' \leqq m$ so that if we write $\begin{pmatrix} \tilde{T}_0^k & \tilde{T}_2^k \\ \tilde{T}_2^k & \tilde{T}_1^k \end{pmatrix} = (s_{ij})$, then

$$\begin{pmatrix} s_{i',j'} & s_{i',j'+m} \\ s_{i'+m,j'} & s_{i'+m,j'+m} \end{pmatrix} = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \tag{2.10}$$

with $a \neq 0$ or $c \neq 0$. We first assume that $\tilde{g} = \begin{pmatrix} t_0 & \\ & t_0^{-1} \end{pmatrix}$, then

$$\tilde{g}\left(\sum \begin{pmatrix} \tilde{T}_0^i & \tilde{T}_2^i \\ \tilde{T}_2^i & \tilde{T}_1^i \end{pmatrix} \otimes v_i \right) \tilde{g}^t = \lambda \left(\sum \begin{pmatrix} \tilde{T}_0^i & \tilde{T}_2^i \\ \tilde{T}_2^i & \tilde{T}_1^i \end{pmatrix} \otimes v_i \right), \quad \lambda \in \mathbb{C}^* \tag{2.11}$$

and (2.10) implies that $\lambda = 1$ and $t_0^2 = 1$. Namely, $\tilde{g} = \mathrm{id}$ and then $g = \mathrm{id}$ (in $\mathrm{PGL}(2, \mathbb{C})$). Now suppose $\tilde{g} = \begin{pmatrix} 0 & t_0 \\ -t_0^{-1} & 0 \end{pmatrix}$, using (2.10) again, we get $\lambda = -1$ and $a = -t_0^2 c$. Thus when $\zeta \notin \mathrm{PGL}(2, \mathbb{C}) \cdot \mathbf{P}(\mathbf{V}_3 \otimes M)$, $\mathrm{stab}(\zeta) = \mathbb{Z}_2$. □

Corollary 2.11 *With the notation as before, suppose* $\zeta \in Z_3^{ss}$ *over* $\xi \in \mathcal{Q}_n(d, I)^{ss}$ *with* $\mathrm{stab}(\xi) = \mathrm{PGL}(2, \mathbb{C})$ *and* $\mathrm{stab}(\zeta) \neq \{e\}$, *then* $\mathrm{stab}(\zeta) = \mathbb{Z}_2$ *or* $\mathbb{C}^* \rtimes \mathbb{Z}_2$.

Proof. This is clear because $\eta: \mathbf{P}(N_\xi \mathfrak{B}) \subsetneq \mathbf{P}(N_\xi \mathfrak{W}^{ss}_{R_0})$ is R_0 equivariant, where $R_0 = \mathrm{stab}(\xi)$. Since $\mathrm{stab}(\zeta) \subseteq \mathrm{stab}(\xi) = R_0$, $\zeta \in \mathbf{P}(N_\xi \mathfrak{B})$ is fixed by $g \in R_0$ if and only if $\eta(\zeta)$ is fixed by g. Therefore, the corollary follows from Lemma 2.10. □

Note that the stabilizer of points of the proper transform $\mathcal{Q}_n(d, I)^{ss}_3 \subseteq Z^{ss}_3$ has dimension at most one. Thus $\mathcal{Q}_n(d, I)^{ss}_2 \subseteq Z^{ss}_2$ is isomorphic to $\mathcal{Q}_n(d, I)^{ss}_3$. Let $\mathfrak{W}_1 \subseteq Z^{ss}_2$ be the set of points whose stabilizers contain $\mathbb{C}^*$ and let Z_1 be the blowing-up of Z^{ss}_3 along $\mathfrak{W}_1$.

Lemma 2.12 *Let $\xi \in \mathcal{Q}_n(d, I)^{ss} \subseteq Z^{ss}$ be any closed point with* $\mathrm{stab}(\xi) = \mathbb{C}^*$, *then for any $\zeta \in W_1$ lying over ξ, where W_1 is the exceptional divisor of $\mathcal{Q}_n(d, I)^{ss}_1 \to \mathcal{Q}_n(d, I)^{ss}_2$,* $\mathrm{stab}(\zeta) = \mathbb{Z}_2$.

Proof. The proposition can be proved similar to that of Corollary 2.10. We leave the proof to the readers. □

Lemma 2.13 *With the notation as above and suppose $\zeta \in W_1$ is any closed point lying over $\xi \in W_2 \cap \mathcal{Q}_n(d, I)^{ss}_2$, where W_2 is the exceptional divisor of $Z^{ss}_2 \to Z^{ss}_4$. Then* $\mathrm{stab}(\zeta)$ *can possibly be* $\{e\}$, $\mathbb{Z}_2$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Proof. Since $\zeta \in W_1$, $\dim \mathrm{stab}(\xi) \geq 1$. By Corollary 2.11, $\mathrm{stab}(\xi) = \mathbb{C}^* \rtimes \mathbb{Z}_2$. Let $R = \mathbb{C}^* \subseteq R_1 = \mathrm{stab}(\xi)$, let $\mathfrak{W}^{ss}_R$ be the fixed point set of R in Z^{ss}_2 and let $\mathfrak{B}_1 = \{z \in Z^{ss}_2 \mid \dim \mathrm{stab}(z) = 1\}$. Since every $F \oplus F \in \mathcal{Q}_n(d, I)^{ss}$ can be deformed to sheaves of the form $F_t \oplus F'_t$ (in $\mathcal{Q}_n(d, I)^{ss}$) with $F_t \neq F'_t$ for general t(at least when d is large), by (2) of Lemma 2.4, the projection

$$\phi: \mathfrak{B}_1 \cap W_2 \subseteq Z^{ss}_2 \to \mathfrak{B}_3 \subseteq Z^{ss}$$

has the property that $\phi(\mathfrak{B}_1 \cap W_2) = \mathfrak{B}_3$ and that $T_\xi \mathfrak{B}$ and $T_\xi W_2$ span $T_\xi Z^{ss}_2$. Thus $\phi_*: T_\xi(\mathfrak{B}_1 \cap W_2) \to T_{\phi(\xi)} \mathfrak{B}_3$ is surjective. (Here, all sets involved are smooth thanks to [12].) Therefore the normal vector space $N_\xi \mathfrak{B}_1$ of $\mathfrak{B}_1 \subseteq Z^{ss}_2 \cap \mathcal{Q}_n(d, I)^{ss}_2$ at ξ is contained in the tangent space $T_\xi(W_2 \cap \phi^{-1}\phi(\xi))$ and therefore $N_\xi \mathfrak{B}_1$ is contained in the normal vector space of $\phi^{-1}\phi(\xi) \cap \mathfrak{B}_1 \subset \phi^{-1}\phi(\xi)$. By the proof of Lemma 2.10, the normal vector space of $\phi^{-1}\phi(\xi) \cap \mathfrak{B}_1$ in $\phi^{-1}\phi(\xi)$ is contained in the normal vector space of $\mathbf{P}(\mathbf{V}_3 \otimes M)$ in $\mathbf{P}(\mathbf{V}_1 \otimes M)$ at ξ. Clearly, the inclusion is R equivariant. Similar to the argument of Corollary 2.11, to prove the lemma, we only need to study the stabilizer $\mathrm{stab}_R(\zeta)$ of all $\zeta \in \mathbf{P}(\mathbf{V}')$ where $\mathbf{V}'$ is the normal vector space of $\mathbf{P}(\mathbf{V}_3 \otimes M)$ in $\mathbf{P}(\mathbf{V}_1 \otimes M)$. Here R acts on $\mathbf{P}(\mathbf{V}')$ via conjugation.

The normal vector space $\mathbf{V}'$ at ξ is isomorphic to $(\mathbf{V}_1/\mathbf{V}_3) \otimes M)$ where

$$\mathbf{V}_1/\mathbf{V}_3 = \left\{ \begin{pmatrix} T_0 & 0 \\ 0 & T_1 \end{pmatrix} \middle| \, T_0, T_1 \text{ are antisymmetric} \right\}.$$

For any $w \in (\mathbf{V}_1/\mathbf{V}_3) \otimes M$, $w = \sum \begin{pmatrix} T_0 & 0 \\ 0 & T_1 \end{pmatrix} \otimes v_1$, suppose

$$\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} w \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} = \lambda w, \quad \lambda \in \mathbb{C}^*.$$

Then $t^4 = 1$. Thus $\mathrm{stab}_R(\zeta) = \{e\}$ or $\mathbb{Z}_2$. However $\mathrm{stab}(\zeta) \subseteq \mathbb{C}^* \rtimes \mathbb{Z}_2$. Therefore since $\mathrm{stab}_R(\zeta) \cap \mathbb{C}^* = \mathbb{Z}_2$ or $\{e\}$, the order of $\mathrm{stab}(\zeta)$ can only be 1, 2 or 4. We claim that when the order of $\mathrm{stab}(\zeta)$ is four, $\mathrm{stab}(\zeta) = \mathbb{Z}_2 \times \mathbb{Z}_2$. Indeed, all

$g \in \mathbb{C}^* \rtimes \mathbb{Z}_2 \backslash \mathbb{C}^*$ satisfy $g^2 = 1$. Thus there are at least two order two elements in stab(ζ). So stab(ζ) $= \mathbb{Z}_2 \times \mathbb{Z}_2$. This completes the proof of Lemma 2.6. □

3 Symplectic forms on $\tilde{\mathfrak{M}}_H(d, I)$

Let $\tilde{\mathfrak{M}}_H(d, I)$ be the desingularization of $\mathfrak{M}_H(d, I)$ introduced in §2 and let $\tilde{\mathfrak{M}}_H(d, I)_0 \subseteq \tilde{\mathfrak{M}}_H(d, I)$ be the open subset of closed points lying over stable sheaves in $\mathfrak{M}_H(d, I)$. For d large, $\tilde{\mathfrak{M}}_H(d, I)_0$ is dense in $\tilde{\mathfrak{M}}_H(d, I)$ thanks to the estimate (1.5). In this section, we assume X is a smooth minimal surface of general type with $h^0(K_X) \geqq 1$. We will explain how to construct (possibly degenerate) holomorphic symplectic form

$$\Theta_\theta \colon T\tilde{\mathfrak{M}}_H(d, I)_0 \times T\tilde{\mathfrak{M}}_H(d, I)_0 \to \mathbb{C} \tag{3.1}$$

associated to an $\theta \in H^0(K_X)$ originated in [22]. Then we will study when such a section is non-degenerate at generic points of $\tilde{\mathfrak{M}}_H(d, I)_0$. If this indeed is the case, then

$$\det \Theta_\theta \in \left(\bigwedge^{\text{top}} T\,\tilde{\mathfrak{M}}_H(d, I)_0\right)^{\otimes(-2)} \tag{3.2}$$

will provide us a two-canonical section of $\tilde{\mathfrak{M}}_H(d, I)_0$. One remarkable feature of this section is that when d is large, it extends over $\tilde{\mathfrak{M}}_H(d, I)$ and further, it vanishes along the exceptional divisor of $\tilde{\mathfrak{M}}_H(d, I) \to \mathfrak{M}_H(d, I)$.

First, let us give a brief account of Kodaira-Spencer map of a family of sheaves on X which is needed in constructing the symplectic form φ_θ. For any smooth quasi-projective variety S and any family E_S of coherent sheaves on $X \times S$ flat over S, we have the Kodaira-Spencer map associated to this family:

$$\kappa_{E_S} \colon TS \to \mathscr{E}xt^1_{X \times S/S}(E_S, E_S). \tag{3.3}$$

Here $\mathscr{E}xt^i_{X \times S/S}(E_S, E_S)$ is the relative extension sheaf over S such that for any open set $U \subseteq S$, $\mathscr{E}xt^i_{X \times S/S}(E_S, E_S)(U)$ is the $\mathcal{O}_U$-module $\operatorname{Ext}^i_{X \times U}(E_{S|X \times U}, E_{S|X \times U})$. κ_{E_S} is defined as follows: We first consider a sheaf of graded algebra $\mathscr{G}$ on S, where $\mathscr{G}_0 = \mathcal{O}_S$, $\mathscr{G}_1 = \Omega_S$ is the sheaf of differentials and $\mathscr{G}_k = \{0\}$ for $k \geqq 2$ and then form the associated scheme Proj $\mathscr{G}$. We let η: Proj $\mathscr{G} \to S$ be the morphism defined by the homomorphism

$$\bar{\eta} \colon \mathcal{O}_S \to \mathcal{O}_S \oplus \Omega_S, \quad \bar{\eta}(f) = (f, d_S f).$$

The differential d_S is defined as follows: Following [EGAIV, 0.20], $\mathcal{O}_S \oplus \Omega_S = \mathcal{O}_S \otimes_{\mathbb{C}} \mathcal{O}_S/\mathscr{T}$, where $\mathscr{T} \subseteq \mathcal{O}_S \otimes \mathcal{O}_S$ is the ideal generated by $1 \otimes f - f \otimes 1$, and $d_S f = 1 \otimes f - f \otimes 1$. Clearly, pointwise, $d_S f \otimes k(s) = f - f(s)$ (modulo $\mathfrak{m}_s^2$) $\in \Omega_S \otimes k(s)$ for closed point $s \in S$. It is easy to see that the set of closed points of Proj $\mathscr{G}$ is isomorphic to S. Indeed, S embeds into Proj $\mathscr{G}$ via the projection $\mathcal{O}_S \oplus \mathscr{G} \to \mathcal{O}_S$. Since E_S is flat over S, $(1_X \times \eta)^* E_S$ is a sheaf on $X \times$ Proj $\mathscr{G}$ flat over Proj $\mathscr{G}$. Thus

$$0 \to (1_X \times \eta)^* E_S \otimes \mathscr{J} \to (1_X \times \eta)^* E_S \to (1_X \times \eta)^* E_{|X \times S} \to 0 \tag{3.4}$$

is exact, where $\mathscr{J}$ is the ideal sheaf of $X \times S$ in $X \times$ Proj $\mathscr{G}$. (This is the sheaf of first principal part of E_S given in [EGA IV].) Clearly, $\mathscr{J} = p_S^* \Omega_S$ as $\mathcal{O}_{X \times S}$-modules.

Now let

$$\rho_{E_S} \in \mathrm{Ext}^1_{X\times S}(E_S, E_S \otimes p_S^* \Omega_S)$$

be the extension class defined by the exact sequence of $\mathcal{O}_{X\times S}$-modules (3.4). Combined with the canonical map $\mathrm{Ext}^1_{X\times S}(E_S, E_S \otimes p_S^*\Omega_S) \otimes_{\mathbb{C}} \mathcal{O}_S \to \mathcal{E}xt^1_{X\times S/S}(E_S, E_S \otimes p_S^*\Omega_S)$, ρ_{E_S} then defines a section

$$\mathcal{O}_S \to \mathcal{E}xt^1_{X\times S/S}(E_S, E_S \otimes p_S^* \Omega_S),$$

or equivalently, the Kodaira–Spencer map

$$\kappa_{E_S}: \mathcal{T}_S \to \mathcal{E}xt^1_{X\times S/S}(E_S, E_S). \tag{3.5}$$

Here, $\mathcal{T}_S = \Omega_S^\vee$ is the tangent bundle of S. Finally, we remark that the Kodaira–Spencer map κ_{E_S} is canonical in the sence that if F_S is another family of sheaves on $X \times S$ that is isomorphic to E_S via $f: F_S \to E_S$, then

$$\kappa_F = L(f)\cdot \kappa_E \cdot R(f)^{-1}, \tag{3.6}$$

where $L(f): \mathrm{Ext}(E,\cdot) \to \mathrm{Ext}(F,\cdot)$ and $R(f): \mathrm{Ext}(\cdot, F) \to \mathrm{Ext}(\cdot, E)$ are the induced homomorphisms. Now we assume $h^0(K_X) \geqq 1$. Pick an $\theta \in H^0(K_X)$. We can define a bilinear form $\Theta_\theta: \mathcal{T}_S \times \mathcal{T}_S \to \mathcal{O}_S$ as the compositions of κ_{E_S} with the Yoneda product

$$\mathcal{E}xt^1_{X\times S/S}(E_S, E_S) \times \mathcal{E}xt^1_{X\times S/S}(E_S, E_S) \xrightarrow{\times} \mathcal{E}xt^2_{X\times S/S}(E_S, E_S) \tag{3.7}$$

followed by tensoring θ

$$\mathcal{E}xt^2_{X\times S/S}(E_S, E_S) \xrightarrow{\otimes\theta} \mathcal{E}xt^2_{X\times S/S}(E_S, E_S \otimes p_X^* K_X) \tag{3.8}$$

and then by taking the trace

$$\mathcal{E}xt^2_{X\times S/S}(E_S, E_S \otimes p_X^* K_X) \xrightarrow{\mathrm{tr}} R^2 ps_*(p_X^* K_X) \cong \mathcal{O}_S.$$

The bilinear form Θ_θ was first introduced by Mukai [18] and Tyurin [25].

Lemma 3.1 *The Mukai–Tyurin bilinear form Θ_θ is skew-symmetric.*

Proof. See [22]. □

There is a point-wise construction of the bilinear form Θ_θ. For any rank two torsion free sheaf E on X, there is an antisymmetric bilinear map

$$\varphi_\theta(E): \mathrm{Ext}^1(E, E) \times \mathrm{Ext}^1(E, E) \to \mathbb{C}$$

associated to $\theta \in H^0(K_X)$ defined as the compositions of the following maps:

$$\mathrm{Ext}^1(E, E) \times \mathrm{Ext}^1(E, E) \xrightarrow{\times} \mathrm{Ext}^2(E, E) \xrightarrow{\otimes\theta} \mathrm{Ext}^2(E, E \otimes K_X) \xrightarrow{\mathrm{tr}} H^2(K_X).$$

Now let $s \in S$ be any closed point. The Kodaira-Spencer map $\kappa_s: T_s S \to \mathrm{Ext}^1(E_s, E_s)$ of the family E_S at $s \in S$ then induces a bilinear form on $T_s S \times T_s S$:

$$\phi_\theta(E_s): T_s S \times T_s S \xrightarrow{(\kappa_s, \kappa_s)} \mathrm{Ext}^1(E_s, E_s) \times \mathrm{Ext}^1(E_s, E_s) \xrightarrow{\varphi_\theta(E_s)} \mathbb{C}.$$

Lemma 3.2 *Under the canonical restriction homomorphisms*

$$r_s\colon \mathscr{E}xt^1_{X\times S/S}(E_S, E_S) \to \operatorname{Ext}^1(E_s, E_s)$$

and $r_s'\colon \mathscr{T}_S \to T_sS$, *we have* $r_s \circ \kappa_{E_S} = \kappa_s \circ r_s'$ *and further, the following diagram is commutative*:

$$\begin{array}{ccc} \mathscr{T}_S \times \mathscr{T}_S & \xrightarrow{\Theta_\theta} & \mathscr{O}_S \\ \downarrow \text{res} & & \downarrow \text{res} \\ T_sS \times T_sS & \xrightarrow{\phi_\theta(E_s)} & \mathbb{C}. \end{array} \tag{3.9}$$

Proof. It is a local problem, so we can assume S is affine. The existence of r_s follows from [10, III.9.3.1]. Here we have used the fact that $\mathscr{E}xt^i_{X\times S/S}(E_S, E_S)$ is defined as hypercohomology of a complex of (finite locally free) sheaves $\mathscr{H}om(F_S, F_S)$, where $F_S^1 \to F_S^0 \to E_S$ is a locally free resolution of E_S [8, p. 705]. Then the identity $r_s \circ \kappa_{E_S} = \kappa_s \circ r_s'$ follows directly. For the proof of the second part, we use the following commutative diagram

$$\begin{array}{ccccc} \mathscr{E}xt^1_{X\times S/S}(E_S, E_S) \times \mathscr{E}xt^1_{X\times S/S}(E_S, E_S) & \to & \mathscr{E}xt^2_{X\times S/S}(E_S, E_S \otimes p_X^* K_X) & \to & R^2 p_{S*}(p_X^* K_X) \\ \downarrow (r_s, r_s) & & \downarrow r_s & & \downarrow \text{res} \\ \operatorname{Ext}^1(E_s, E_s) \times \operatorname{Ext}^1(E_s, E_s) & \to & \operatorname{Ext}^2(E, E \otimes K_X) & \to & H^2(K_X). \end{array}$$

Here the two left row arrows are defined as the composition of (3.7) and (3.8). The lemma will be established if we can show that $R^2 p_{S*}(p_X^* K_X) \xrightarrow{\text{res}} H^2(K_X)$ is surjective. But this is clear by using the base change theorem since $R^2 p_{S*}(p_X^* K_X)$ is locally free. □

Once we have the bilinear form Θ_θ, we can take the determinant of Θ_θ to obtain a homomorphism $(\det \mathscr{T}_S)^{\otimes 2} \to \mathscr{O}_S$. By abuse of notation, we denote the corresponding section in $H^0(S, \omega_S^{\otimes 2})$ by $\det \Theta_\theta$. In the following, we show that this construction yields a two-canonical section of $\widetilde{\mathfrak{M}}_H(d, I)_0$.

Proposition 3.3 *Associated to every* $\theta \in H^2(K_X)$, *there is a* $\Lambda_\theta \in H^0(\widetilde{\mathfrak{M}}_H(d, I)_0, \omega^{\otimes 2})$, *where* ω *is the canonical bundle of* $\widetilde{\mathfrak{M}}_H(d, I)$.

Proof. Let $\tilde{\pi}\colon \widetilde{\mathfrak{Q}}_n(d, I)^{ss} \to \widetilde{\mathfrak{M}}_H(d, I)$ be the projection. By assumption, all closed points of $\tilde{\pi}^{-1}(\widetilde{\mathfrak{M}}_H(d, I)_0)$ have stabilizer $\{e\}$. Thus by applying (2) of Proposition 2.7, there is an open covering $\{U\}$ of $\widetilde{\mathfrak{M}}_H(d, I)_0$ by classical open sets such that over each U, there is a lifting $\rho_U\colon U \to \widetilde{\mathfrak{Q}}_n(d, I)^{ss}$ of $U \subsetneq \widetilde{\mathfrak{M}}_H(d, I)_0$. Let E_U (on $X \times U$) be the pull-back of the universal family of $\widetilde{\mathfrak{Q}}_n(d, I)^{ss}$. Clearly, E_U is a local universal family. Then by the previous construction, for any $\theta \in H^2(K_X)$ and any U, there is a two-canonical form $\det \Theta_{\theta, U} \in H^0(U, \omega^{\otimes 2})$. (Since U's can be taken as analytic open subset of an étale neighborhood in $\widetilde{\mathfrak{M}}_H(d, I)_0$, the previous construction is still valid). Now let U, $V \subseteq \widetilde{\mathfrak{M}}_H(d, I)_0$ be two open subsets with $U \cap V \neq \emptyset$. Proposition 3.3 will be proved if we can show that

$$(\det \Theta_{\theta, U})_{|U \cap V} = (\det \Theta_{\theta, V})_{|U \cap V}. \tag{3.10}$$

Since this is a local problem, we only need to check (3.10) near each points of $U \cup V$. Since $\mathfrak{Q}_n(d, I)^s \to \mathfrak{M}_H(d, I)^s$ is a principal bundle, so is $\widetilde{\mathfrak{Q}}(d, I)_0 \to \widetilde{\mathfrak{M}}_H(d, I)_0$. Thus there is an $f_{UV}\colon U \cap V \to G$ such that when restricted to $U \cap V$, $f_{UV} \cdot \rho_V = \rho_U$. Now let $z \in U \cap V$ be any closed point. At a neighborhood O of $z \in U \cap V$, we can lift f_{UV} to $\tilde{f}_{UV}: O \to \mathrm{GL}(N, \mathbb{C})$. Thus on $X \times O$, we have an isomorphism

$$\tilde{f}_{UV}: E_{V|X\times O} \to E_{U|X\times O}.$$

Now let κ_U and κ_V be the corresponding Kodaira–Spencer maps of E_U and E_V respectively, then by (3.6), $\kappa_V = L(\tilde{f}_{UV})\cdot\kappa_U\cdot R(\tilde{f}_{UV})^{-1}$ when restricted to O. Therefore near z,

$$\begin{aligned}\Theta_{\theta,V} &= \mathrm{tr}(\kappa_V \times \kappa_V \otimes \theta)\\ &= \mathrm{tr}(L(\tilde{f}_{UV})\cdot\kappa_U\cdot R(\tilde{f}_{UV})^{-1} \times L(\tilde{f}_{UV})\cdot\kappa_U\cdot R(\tilde{f}_{UV})^{-1}\otimes\theta) \qquad (3.11)\\ &= \mathrm{tr}(\kappa_U \wedge \kappa_U \otimes \theta) = \Theta_{\theta,U}.\end{aligned}$$

Thus $\Theta_{\theta,U|U\cap V} = \Theta_{\theta,V|U\cap V}$ and therefore (3.10) holds. □

To ensure that the two-canonical form $\det\Theta_\theta$ so constructed is non-trivial, we need to check that the bilinear form Θ_θ is non-degenerate (or $\det\Theta_\theta$ is non-vanishing) at the general points of $\tilde{\mathfrak{M}}_H(d, I)_0$. We quote the following observation made by O'Grady:

Lemma 3.4 *Let $z\in\tilde{\mathfrak{M}}_H(d, I)_0$ be a closed point corresponding to the sheaf E, then the symplectic form Θ_θ is non-degenerate at z if and only if the map*

$$\mathrm{Ext}^1(E, E)^0 \xrightarrow{\otimes\theta} \mathrm{Ext}^1(E, E\otimes K_X)^0 \qquad (3.12)$$

is an isomorphism.

Now assume E is locally free at $D\in|K_X|$, where $D = \theta^{-1}(0)$. Then the map (3.12) fits into the following long exact sequence

$$\longrightarrow \mathrm{Ext}^0(E, E\otimes K_X)^0 \to H^0_D(\mathscr{E}nd^0(E_{|D})\otimes K_X) \to \mathrm{Ext}^1(E, E)^0 \to$$

$$\xrightarrow{\otimes\theta} \mathrm{Ext}^1(E, E\otimes K_X)^0 \to H^1_D(\mathscr{E}nd^0(E_{|D})\otimes K_X) \to \mathrm{Ext}^2(E, E)^0 \to\ .$$

By Donaldson's generic smoothness result, if we assume d large and E generic, then $\mathrm{Ext}^0(E, E\otimes K_X)^0 = \mathrm{Ext}^2(E, E)^0 = \{0\}$. Further, since D is a canonical curve, $h^0_D(\mathscr{E}nd^0(E_{|D})\otimes K_X) = h^1_D(\mathscr{E}nd^0(E_{|D})\otimes K_X)$. Thus (3.12) is an isomorphism if and only if $h^0_D(\mathscr{E}nd^0(E_{|D})\otimes K_X) = 0$. Thus, the question whether Θ_θ is non-degenerate at general $z\in\tilde{\mathfrak{M}}_H(d, I)_0$ has been reduced to the question whether $h^0_D(\mathscr{E}nd^0(E_{|D})\otimes K_X) = 0$ for general $E\in\mathfrak{M}_H(d, I)$. To this end, we observe

Lemma 3.5 *Suppose we can find a rank two locally free sheaf V on D with $\det V = I_{|D}$ such that $h^0_D(\mathscr{E}nd^0(V)\otimes K_X) \leqq l$. Then there is a constant C depending on (X, H, I) such that for $d \geqq C$, we have $h^0_D(\mathscr{E}nd^0(E_{|D})\otimes K_X) \leqq l$ for general $E\in\mathfrak{M}_H(d, I)$.*

Proof. By Donaldson's generic smoothness result [4, 26], we can assume that there is a constant C such that for $d \geqq C$, $\mathrm{Ext}^2(E, E(-D))^0 = \{0\}$ for general $E\in\mathfrak{M}_H(d, I)$. We can also assume that the general sheaves E in $\mathfrak{M}_H(d, I)$ are locally free (which follows from the estimate (1.7) and the Theorem 0.1). Thus, we will have

$$\mathrm{Ext}^1_X(E, E)^0 \to \mathrm{Ext}^1_D(E_{|D}, E_{|D})^0$$

is surjective for general E. We fix such a general E.

Let V be the given vector bundle on D with $\det V = I_{|D}$ and $h^0_D(\mathscr{E}nd^0(V) \otimes K_X) \leqq l$. We claim that there is a smooth affine curve S, $s_0, s_1 \in S$ and a locally free sheaf F_S on $D \times S$ of determinant $p_D^* I_{|D}$ such that $F_{s_0} = V$ and $F_{s_1} = E_{|D}$. Indeed, let L be a very ample line bundle on D so that both $E_{|D} \otimes L$ and $V \otimes L$ are generated by global sections. Then V (as well as $E_{|D}$) belongs to the exact sequence

$$0 \to L^{-1} \to V \to L \otimes I_{|D} \to 0 .$$

In particular, there are $s_0, s_1 \in \mathrm{Ext}^1_D\ (L \otimes I_{|D}, L^{-1})$ whose corresponding extension sheaves are isomorphic to V and $E_{|D}$ respectively. Since $\mathrm{Ext}^1(L \otimes I_{|D}, L^{-1})$ is affine, we can choose S to be a line in $\mathrm{Ext}^1(L \otimes I_{|D}, L^{-1})$ containing s_0 and s_1. Thus the claim has been established.

Because $h^1_D(\mathscr{E}nd^0(V) \otimes K_X) \leqq l$, by upper-semicontinuity of cohomology groups, for general $s \in S$, $h^1_D(\mathscr{E}nd^0(F_s) \otimes K_X) \leqq l$. Therefore, the lemma will be established if we can show that there is a deformation E_t, $t \in T$ is a curve, of E such that for general $t \in T$, E_t is isomorphic to general F_s.

Let $s \in \mathfrak{m}_{s_1} - \mathfrak{m}^2_{s_1}$ be uniformizing parameter of S. Let $R_k = \mathrm{Spec}\, \mathbb{C}[s]/(s^k) \subseteq S$. We claim that for any $k \geqq 2$, there is a sheaf E_k on $X \times R_k$ flat over R_k that induces E when restricted to the closed $X \subseteq X \times R_k$ such that $E_{k|D \times R_k} \cong F_{S|D \times R_k}$. Indeed, if we have already found E_{k-1}, then the obstruction to the existence of E_k lies in $\mathrm{Ext}^2(E, E(-D))^0$ which is zero by our assumption on E. Thus E_k exists for all k. Therefore, we can find an irreducible curve S' in $\mathfrak{M}_H(d, I)$ containing E such that for general $s' \in S'$, $E_{s'|D}$ is isomorphic to a general F_s, $s \in S$. In other words, we have $h^0(D, \mathscr{E}nd^0(E_{s'|D}) \otimes K_X) \leqq l$. This completes the proof of Lemma 3.5. □

The existence problem of the desired vector bundles on D is largely solved by the following proposition:

Proposition 3.6 *Let X be a minimal surface of general type and let $D \in |K_X|$ be a reduced canonical divisor. Then for any line bundle I on X, there is at least one rank two vector bundle V over D of* $\det V = I_{|D}$ *such that $h^0(D, \mathscr{E}nd^0(V) \otimes K_X) \leqq 1$.*

The case when D is smooth was first established by Beauville [3] and by [22]. Because the argument for the general case is quite independent from the the rest of this paper, we will include the proof of this proposition in the Appendix.

Now we are ready to prove

Proposition 3.7 *Assume X is a surface of general type admitting a reduced canonical divisor $D \in |K_X|$ and assume $\chi(\mathscr{O}_X) + I \cdot I$ is even, then there is a constant C depending on (X, H, I) such that for any $d \geqq C$ and for general $E \in \mathfrak{M}_H(d, I)$, we have*

$$h^0(D, \mathscr{E}nd^0(E_{|D}) \otimes K_X) = 0 .$$

Proof. Let C be the constant given by the Lemma 3.5 and Theorem 0.1. Then by Lemma 3.5 and Proposition 3.6, for $d \geqq C$ and $E \in \mathfrak{M}_H(d, I)$ general, $\mathrm{Ext}^2(E, E)^0 = \{0\}$ and $h^0(D, \mathscr{E}nd^0(E_{|D}) \otimes K_X) \leqq 1$. On the other hand, the assumption $\chi(\mathscr{O}_X) + I \cdot I = \text{even}$ implies that $\mathfrak{M}_H(d, I)$ is of even dimension. Thus Lemma 3.1 implies that the kernel of $\mathrm{Ext}^1(E, E)^0 \to \mathrm{Ext}^1(E, E \otimes K_X)^0$ is of even dimension. Namely, $h^0(D, \mathscr{E}nd^0(E_{|D}) \otimes K_X)$ is even. Thus it must be zero because it is no more than 1. This completes the proof of the proposition. □

To this end, we know that under the assumption of Proposition 3.7, the symplectic form Θ_θ (on $\tilde{\mathfrak{M}}_H(d, I)_0$) is non-degenerate at general points. Thus the section

$$\Lambda_\theta = \det \Theta_\theta \in H^0(\tilde{\mathfrak{M}}_H(d, I)_0, \omega^{\otimes 2})$$

is non-trivial at each irreducible component of $\tilde{\mathfrak{M}}_H(d, I)_0$. For our study, we will show that when d is large, it extends over $\tilde{\mathfrak{M}}_H(d, I)$ to section $\bar{\Lambda}_\theta \in H^0(\tilde{\mathfrak{M}}_H(d, I), \omega^{\otimes 2})$. We will also show that the extended section $\bar{\Lambda}_\theta$ vanishes along the exceptional divisor W of $\Psi: \tilde{\mathfrak{M}}_H(d, I) \to \mathfrak{M}_H(d, I)$. We state it as a proposition.

Proposition 3.8 *Assume $d \gg 0$, then there is an extension $\bar{\Lambda}_\theta \in H^0(\tilde{\mathfrak{M}}_H(d, I), \omega^{\otimes 2})$ of Λ_θ having the property that $\bar{\Lambda}_\theta | W = 0$, where W is the exceptional divisor of $\Psi: \tilde{\mathfrak{M}}_H(d, I) \to \mathfrak{M}_H(d, I)$.*

Proof. Let $\xi \in \tilde{\mathfrak{M}}_H(d, I)$ be any closed point over a singular point $\Psi(\xi)$ of $\mathfrak{M}_H(d, I)$. First we assume that $\Psi(\xi)$ corresponds to a stable sheaf. Let $U \subseteq \tilde{\mathfrak{M}}_H(d, I)$ be a classical neighborhood of $\xi \in \tilde{\mathfrak{M}}_H(d, I)$ so that a local universal family E_U exists on $X \times U$. By Lemma 3.2, we have following commutative diagram,

$$\begin{array}{ccc} T_\xi U \times T_\xi U & \xrightarrow{\Theta_\theta(\xi)} & \mathbb{C} \\ \downarrow (\kappa_\xi, \kappa_\xi) & & \| \\ \operatorname{Ext}^1(E_\xi, E_\xi) \times \operatorname{Ext}^1(E_\xi, E_\xi) & \xrightarrow{\varphi_\theta} & \mathbb{C}\,. \end{array}$$

Clearly, $\Lambda_\theta(\xi) = 0$ when $\det \Theta_\theta(\xi) = 0$ and when $\kappa_\xi: T_\xi U \to \operatorname{Ext}^1(E_\xi, E_\xi)$ is not injective. Since $\mathfrak{M}_H(d, I)$ is normal, we have $\dim \Psi^{-1}\Psi(\xi) \geqq 1$. Further, since E_ξ is stable, E_U restricted to $X \times \Psi^{-1}\Psi(\xi)$ is a constant family. Thus $\kappa_\xi(T_\xi(\Psi^{-1}\Psi(\xi))) = 0$. Therefore,

$$\Lambda_\theta(\xi) = \det \Theta_\theta(\xi) = 0\,. \tag{3.13}$$

Next we consider the case when $\Psi(\xi) \in \mathfrak{M}_H(d, I)$ corresponds to a strictly semistable sheaf. By Proposition 2.7, there is a group Γ, $\Gamma = \{e\}$ or $\mathbb{Z}_2$, a Γ-invariant smooth (analytic) variety $\tilde{U}$ and a (classical) neighborhood U of $\xi \in \tilde{\mathfrak{M}}_H(d, I)$ such that $\tilde{U}/\Gamma = U$ and such that there is a local universal family $E_{\tilde{U}}$ on $X \times \tilde{U}$ with a Γ-linearization. Now consider the Mukai-Tyurin bilinear form $\Theta_\theta(\tilde{U})$ on $\tilde{U}$ associated to the family $E_{\tilde{U}}$. By (3.11), $\det \Theta_\theta(\tilde{U})$ is invariant under Γ and thus descends to a meromorphic two-canonical form $\bar{\Lambda}_\theta$ on U. We need to show that $\bar{\Lambda}_\theta$ is indeed regular.

We consider the case where $\Gamma = \mathbb{Z}_2$. $\Gamma = \{e\}$ can be settled similarly. Since $\tilde{U} \to U$ is a Galois quotient branched over a smooth divisor $S \subseteq \tilde{U}$, $\bar{\Lambda}_\theta$ is regular if $\det \Theta_\theta(\tilde{U})$ has vanishing order at least two along S and $\bar{\Lambda}_\theta$ vanishes at ξ if $\det \Theta_\theta(\tilde{U})$ has vanishing order at least four along S. Let $\zeta \in S$ be a general closed point. We again consider the following diagram

$$\begin{array}{ccc} T_\zeta \tilde{U} \times T_\zeta \tilde{U} & \xrightarrow{\Theta_\theta(\tilde{U})(\zeta)} & \mathbb{C} \\ \downarrow (\kappa_\zeta, \kappa_\zeta) & & \downarrow \cong \\ \operatorname{Ext}^1(E_\zeta, E_\zeta) \times \operatorname{Ext}^1(E_\zeta, E_\zeta) & \xrightarrow{\varphi_\theta(E_\zeta)} & \mathbb{C}\,. \end{array}$$

We claim that if

$$\dim \ker \{\kappa_\zeta: T_\zeta \tilde{U} \to \operatorname{Ext}^1(E_\zeta, E_\zeta)\} \geqq 4, \tag{3.14}$$

then $\det \Theta_\theta(\tilde{U}) \in \omega_{\tilde{U}}^{\otimes 2} \otimes \mathfrak{m}_\zeta^{\otimes 4}$, where m_ζ is the ideal sheaf of $\zeta \in \tilde{U}$. In fact, (3.14) implies that the value of the antisymmetric bilinear form $\Theta_\theta(\tilde{U})$ at ζ, considered as a $k \times k$ scalar matrix, has at least 4 zero eigenvalues. Thus by choosing an appropriate local frame of $T\tilde{U}$, we can assume

$$\Theta_\theta(\tilde{U})(\zeta) = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix},$$

where A is a $(k-4)\times(k-4)$ antisymmetric matrix. In other words, we have

$$\Theta_\theta(\tilde{U}) = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} + B,$$

where $B \in \mathfrak{m}_\zeta^{\oplus k \times k}$. Therefore, we must have $\det \Theta_\theta(\tilde{U}) \in \omega_{\tilde{U}}^{\otimes 2} \otimes \mathfrak{m}_\zeta^{\otimes 4}$.

To this end, we only need to establish (3.14) by assuming d large. Let $\psi: \tilde{U} \subseteq \mathfrak{Q}_n(d, I)^{ss} \to \mathfrak{M}_H(d, I)$ be the induced map. Then by (1) of Lemma 2.3, for any $\xi \in \psi^{-1}(\psi(\zeta))$, $E_\xi = \mathrm{gr}(E_\xi)$ is a direct sum of stable sheaves. Thus the subfamily of $E_{\tilde{U}}$ parameterized by the set $\psi^{-1}(\psi(\zeta))$ is also a constant family. Hence

$$T_\zeta(\psi^{-1}(\psi(\zeta))) \subseteq \ker\{\kappa: T_\zeta\tilde{U} \to \mathrm{Ext}^1(E_\zeta, E_\zeta)\}. \tag{3.15}$$

If we can show that for large d, we always have $\dim \psi^{-1}(\psi(\zeta)) \geqq 4$, then (3.14) holds and consequently, $\bar{\Lambda}_\theta$ is regular and $\bar{\Lambda}_\theta|_W = 0$. Indeed, the set of closed points in $\mathfrak{M}_H(d, I)$ correspond to strictly semistable sheaves are parameterized by sheaves $F_1 \oplus F_2$, where $c_1(F_1)\cdot H = c_1(F_2)\cdot H = \frac{1}{2}H\cdot I$ and $\chi(F_1) = \chi(F_2)$. Let $l_i = \ell(F_i^{\vee\vee}/F_i)$. Then because $d = l_1 + l_2 + c_1(F_1)\cdot c_1(F_2)$ and $c_1(F_1) + c_1(F_2) = I$, by Hodge index theorem,

$$l_1 + l_2 \leqq d - \frac{I^2}{4}.$$

Thus the number of moduli of the set of split semistable sheaves is at most

$$(h^1(\mathcal{O}_X) + 2l_1) + (h^1(\mathcal{O}_X) + 2l_2) \leqq 2d + 2h^1(\mathcal{O}_X) - \tfrac{1}{2}I^2 \ll 4d - I^2 - 3\chi(\mathcal{O}_X), \quad \text{for } d \gg 0.$$

Because $S \subseteq \tilde{U}$ is of codimension one, for general $\zeta \in S$, $\dim \psi^{-1}(\psi(\zeta)) \geqq 4$ when d is large. Therefore, the proposition has been established. □

4 Canonical sheaf of the moduli scheme

In the last section, we will first relate the dualizing sheaf of the resolution $\tilde{\mathfrak{M}}_H(d, I)$ to a determinant line bundle on $\tilde{\mathfrak{M}}_H(d, I)$. We will then show that the space of sections of the k-th tensor product of this determinant line bundle has maximal growth rate. Finally, by using the special sections constructed in §3, we will prove that the moduli schemes $\mathfrak{M}_H(d, I)$ are of general type under the constraint in Theorem 0.2 provided that d is sufficiently large.

In the following, we assume d is sufficiently large so that $\mathfrak{M}_H(d, I)$ is normal. Let $\tilde{\mathfrak{M}}_H(d, I)$ and $\tilde{\mathfrak{Q}}_n(d, I)^{ss}$ be the resolution of $\mathfrak{M}_H(d, I)$ and $\mathfrak{Q}_n(d, I)^{ss}$ introduced in §2, where $\tilde{\mathfrak{M}}_H(d, I) = \tilde{\mathfrak{Q}}_n(d, I)^{ss}/G$ and $G = \mathrm{PGL}(N, \mathbb{C})$. Let $\mathcal{E}$ be the family of sheaves on $X \times \tilde{\mathfrak{Q}}_n(d, I)^{ss}$ that is the pull-back of the universal quotient family on

$X \times \mathcal{Q}_n(d, I)^{ss}$. By using a finite length locally free resolution of $\mathscr{E}$ (which exists because $\mathscr{E}$ is flat over $\tilde{\mathcal{Q}}_n(d, I)^{ss}$), one checks that the complex of sheaves

$$\mathscr{E}xt^{\cdot}_{X \times Q/Q}(\mathscr{E}, \mathscr{E})^0 \tag{4.1}$$

is a perfect complex on $\tilde{\mathcal{Q}}_n(d, I)^{ss}$. Here the subscript $X \times Q/Q$ is an abbreviation of $X \times \tilde{\mathcal{Q}}_n(d, I)^{ss}/\tilde{\mathcal{Q}}_n(d, I)^{ss}$. Thus according to [13] (see [15] also), there is a determinant line bundle

$$\det(\mathscr{E}xt^{\cdot}_{X \times Q/Q}(\mathscr{E}, \mathscr{E})^0) \tag{4.2}$$

of the complex (4.1) on $\tilde{\mathcal{Q}}_n(d, I)^{ss}$.

Lemma 4.1 $\det(\mathscr{E}xt^{\cdot}_{X \times Q/Q}(\mathscr{E}, \mathscr{E})^0)$ *is a G-line bundle on* $\tilde{\mathcal{Q}}_n(d, I)^{ss}$.

Proof. In general $\mathscr{E}$ does not admit a G-linearization. However, if we think of $\tilde{\mathcal{Q}}_n(d, I)^{ss}$ as a $\mathrm{GL}(N, \mathbb{C})$ scheme, where $\mathrm{GL}(N, \mathbb{C})$ acts on $\tilde{\mathcal{Q}}_n(d, I)^{ss}$ via $\mathrm{GL}(N, \mathbb{C}) \to \mathrm{PGL}(N, \mathbb{C})$, then there is a canonical $\mathrm{GL}(N, \mathbb{C})$-linearization of $\mathscr{E}$. Thus $\det(\mathscr{E}xt^{\cdot}_{X \times Q/Q}(\mathscr{E}, \mathscr{E})^0)$ is a $\mathrm{GL}(N, \mathbb{C})$ line bundle. To show that the $\mathrm{GL}(N, \mathbb{C})$-action descends to a G-action it suffices to show that the group $\mathbb{C}^* \subseteq \mathrm{GL}(N, \mathbb{C})$ acts trivially on $\det(\mathscr{E}xt^{\cdot}_{X \times Q/Q}(\mathscr{E}, \mathscr{E})^0)$.

Indeed, for any sheaf E on X with $g = c \cdot \mathrm{id}: E \to E, c \in \mathbb{C}^*$, the induced homomorphism on the similar complex (4.1), say

$$g^i_*: \mathrm{Ext}^i_X(E, E)^0 \to \mathrm{Ext}^i_X(E, E)^0$$

is identity. Thus

$$\det(g_*): \bigotimes_{i=0}^{2} \left(\bigwedge^{\mathrm{top}} \mathrm{Ext}^i_X(E, E)^0 \right)^{(-1)^i} \to \bigotimes_{i=0}^{2} \left(\bigwedge^{\mathrm{top}} \mathrm{Ext}^i_X(E, E)^0 \right)^{(-1)^i}$$

is also an identity homomorphism. Now by combining the base change property of the determinant [13] and the smoothness of $\tilde{\mathcal{Q}}_n(d, I)^{ss}$, we conclude that the $\mathbb{C}^*$ action on the line bundle (4.2) is trivial. Therefore the $\mathrm{GL}(N, \mathbb{C})$ descends to G-action on the line bundle (4.2). □

In the following, we denote the G-bundle $\det(\mathscr{E}xt^{\cdot}_{X \times Q/Q}(\mathscr{E}, \mathscr{E})^0)$ by $\mathbf{Det}_{/Q}(\mathscr{E}, \mathscr{E})$. Our next task is to study when the line bundle $\mathbf{Det}_{/Q}(\mathscr{E}, \mathscr{E})$ descends to $\mathfrak{M}_H(d, I)$. We need the following descent lemma of Kempf:

Lemma 4.2 (Descent lemma) *For any G-vector bundle V on* $\tilde{\mathcal{Q}}_n(d, I)^{ss}$, *V descends to* $\mathfrak{M}_H(d, I)$ *if and only if for every closed point* $\xi \in \tilde{\mathcal{Q}}_n(d, I)^{ss}$ *with closed orbit* $G \cdot \xi$, *the stabilizer* $\mathrm{stab}(\xi) \subset G$ *of* ξ *acts trivially on* V_ξ.

Proof. See [5]. □

Recall that for any closed $\xi \in \tilde{\mathcal{Q}}_n(d, I)^{ss}$, $G \cdot \xi$ is always closed and further, if $\mathrm{stab}(\xi)$ is non-trivial, then $\mathrm{stab}(\xi)$ is $\mathbb{Z}_2$.

Lemma 4.3 *For any closed* $\xi \in \tilde{\mathcal{Q}}_n(d, I)^{ss}$, $\mathrm{stab}(\xi)$ *acts trivially on* $\mathbf{Det}_{/Q}(\mathscr{E}, \mathscr{E}) \otimes k(\xi)$.

Proof. According to §2, if $\mathrm{stab}(\xi)$ is nontrivial, then the induced action on E by $\mathrm{stab}(\xi) = \mathbb{Z}_2$ is generated either by $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right): F_1 \oplus F_2 \to F_1 \oplus F_2$ or by $\left(\begin{smallmatrix} 0 & \mathrm{id} \\ \pm \mathrm{id}^{-1} & 0 \end{smallmatrix}\right): F \oplus F \to F \oplus F$. Here, $\mathscr{E}_\xi = F_1 \oplus F_2$ or $F \oplus F$. We will check the case where $g \in \mathrm{stab}(\xi)$ has the form $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. The other case can be proved similarly. First note that

$$\mathrm{Ext}^i(\mathscr{E}_\xi,\mathscr{E}_\xi)^0=\mathrm{Ext}^i(F_1,F_2)\oplus\mathrm{Ext}^i(F_2,F_1)\oplus (\mathrm{Ext}^i(F_1,F_1)\oplus\mathrm{Ext}^i(F_2,F_2))/H^i(\mathcal{O}_X))$$

and that $\mathrm{Ext}^i(F_1,F_2)\oplus\mathrm{Ext}^i(F_2,F_1)$ (resp. $(\mathrm{Ext}^i(F_1,F_1)\oplus\mathrm{Ext}^i(F_2,F_2))/H^i(\mathcal{O}_X))$ is the eigenspace of the homomorphism

$$g^i_*:\mathrm{Ext}^i(\mathscr{E}_\xi,\mathscr{E}_\xi)^0\to\mathrm{Ext}^i(\mathscr{E}_\xi,\mathscr{E}_\xi)^0$$

with eigenvalue -1 (resp. 1). Since $\chi(\mathrm{Ext}^\cdot(F_1,F_2))+\chi(\mathrm{Ext}^\cdot(F_2,F_1))$ is even, $\det(g_*)$ acts as identity on

$$\bigotimes_{i=0}^{2}\left(\bigwedge^{\mathrm{top}}\mathrm{Ext}^i(\mathscr{E}_\xi,\mathscr{E}_\xi)^0\right)^{(-1)^i}.$$

Therefore by base change,

$$\det(g_*):\det(\mathscr{E}xt^\cdot_{X\times Q/Q}(\mathscr{E},\mathscr{E})^0)\otimes k(\xi)\to\det(\mathscr{E}xt^\cdot_{X\times Q/Q}(\mathscr{E},\mathscr{E})^0)\otimes k(\xi)$$

is the identity homomorphism. □

Now we can apply the descent lemma to the G-bundle $\mathbf{Det}_{/Q}(\mathscr{E},\mathscr{E})$ to obtain a descent line bundle on $\widetilde{\mathfrak{M}}_H(d,I)$. We denote the descended line bundle by $\mathbf{Det}_{/M}(\mathscr{E},\mathscr{E})$. Our next proposition shows that $\mathbf{Det}_{/M}(\mathscr{E},\mathscr{E})$ is very close to the canonical line bundle of $\widetilde{\mathfrak{M}}_H(d,I)$.

Proposition 4.4 *Let ω be the canonical line bundle of $\widetilde{\mathfrak{M}}_H(d,I)$ and let $W\subseteq\widetilde{\mathfrak{M}}_H(d,I)$ be the exceptional divisor of $\widetilde{\mathfrak{M}}_H(d,I)\to\mathfrak{M}_H(d,I)$. Then the restriction of $\mathbf{Det}_{/M}(\mathscr{E},\mathscr{E})$ to the open set $\widetilde{\mathfrak{M}}_H(d,I)\backslash W$ is isomorphic to the canonical line bundle ω of $\widetilde{\mathfrak{M}}_H(d,I)$.*

Proof. Consider the complex (4.2) over $\widetilde{\mathfrak{Q}}_n(d,I)^{ss}$. When restricted to $\tilde\pi^{-1}(\widetilde{\mathfrak{M}}_H(d,I)\backslash W)\subseteq\widetilde{\mathfrak{Q}}_n(d,I)^{ss}$, $\mathscr{E}xt^i_{X\times Q/Q}(\mathscr{E},\mathscr{E})^0=\{0\}$ for $i=0,2$ and

$$\mathscr{E}xt^1_{X\times Q/Q}(\mathscr{E},\mathscr{E})^0 \tag{4.3}$$

is locally free. (We remark that since $\mathscr{E}xt^\cdot(\cdot\,,\cdot)$ is defined as hypercohomology of complex of locally free sheaves, the base change theorem still holds in this setting [10, III.12.11].) One checks that (4.3) is a G-bundle and thus descends to a vector bundle over $\widetilde{\mathfrak{M}}_H(d,I)\backslash W$ by descent lemma. We claim that the descent of $\mathscr{E}xt^1_{X\times Q/Q}(\mathscr{E},\mathscr{E})^0$ is isomorphic to the tangent bundle $T(\widetilde{\mathfrak{M}}_H(d,I)\backslash W)$. Indeed, let $\mathscr{F}$ be the kernel of $\mathcal{O}_{X\times Q}^{\oplus N}\to\mathscr{E}$, then by [2, 9, 18], the tangent bundle of $\tilde\pi^{-1}(\widetilde{\mathfrak{M}}_H(d,I)\backslash W)\subseteq\widetilde{\mathfrak{Q}}_n(d,I)^{ss}$ is the kernel of the composition

$$\mu:\mathscr{H}om_{X\times Q/Q}(\mathscr{F},\mathscr{E})\xrightarrow{\delta}\mathscr{E}xt^1_{X\times Q/Q}(\mathscr{E},\mathscr{E})\xrightarrow{\mathrm{tr}}R^1p_{Q*}(\mathcal{O}_{X\times Q}).$$

Clearly, the image sheaf

$$\mathscr{H}om_{X\times Q/Q}(\mathcal{O}_{X\times Q}^{\oplus N},\mathscr{E})\subsetneq\mathrm{Ker}(\mu) \tag{4.4}$$

is the relative tangent bundle of the fibration $\tilde\pi:\tilde\pi^{-1}(\widetilde{\mathfrak{M}}_H(d,I)\backslash W)\to(\widetilde{\mathfrak{M}}_H(d,I)\backslash W$. Thus the descent of the cokernel of (4.4) is isomorphic to the tangent bundle $T(\widetilde{\mathfrak{M}}_H(d,I)\backslash W)$. By the vanishing of $\mathscr{E}xt^i_{X\times Q/Q}(\mathscr{E},\mathscr{E})^0$, $i=0,2$, the cokernel of (4.4) is isomorphic to (4.3). So the descent of $\mathscr{E}xt^1_{X\times Q/Q}(\mathscr{E},\mathscr{E})^0$ is isomorphic to $T(\widetilde{\mathfrak{M}}_H(d,I)\backslash W)$. This implies that the descent of the determinant line bundle (4.2), which is canonically isomorphic to the dual of the determinant of the descent of (4.3), is isomorphic to the restriction of ω to $\widetilde{\mathfrak{M}}_H(d,I)\backslash W$. Thus the proposition is established. □

Corollary 4.5 *Let $W_i = 1, \ldots, l$ be irreducible components of W. Then there are integers $a_1, \ldots, a_l$ such that $\omega = \mathbf{Det}_{/M}(\mathscr{E},\mathscr{E})(\sum^l a_i W_i)$.*

Proof. This follows from the fact that both ω and $\mathbf{Det}_{/M}(\mathscr{E},\mathscr{E})$ are locally free and that $\widetilde{\mathfrak{M}}_H(d,I)$ is smooth. □

To show that $\widetilde{\mathfrak{M}}_H(d,I)$ is of general type, we need to show that the space $H^0(\widetilde{\mathfrak{M}}_H(d,I),\omega^{\otimes m})$ has maximal growth rate. As explained in the introduction, our first step is to show that the space $H^0(\widetilde{\mathfrak{M}}_H(d,I),\mathbf{Det}_{/M}(\mathscr{E},\mathscr{E})^{\otimes m})$ has maximal growth rate. Because our argument is based on the assumption that X is a general type surface, at some point, we need to relate the line bundle $\mathbf{Det}_{/M}(\mathscr{E},\mathscr{E})$ directly to canonical divisors of X. We have the following relation:

Proposition 4.6 *For any divisor $D\in|rK_X|$, let $\mathbf{Det}_{/Q}(\mathscr{E},\mathscr{E}_{|D})$ be the determinant line bundle of the perfect complex $\mathscr{E}xt_{X\times Q/Q}(\mathscr{E},\mathscr{E}_{|D})^0$, where $\mathscr{E}_{|D}$ is the restriction of $\mathscr{E}$ to $D\times\widetilde{\mathfrak{Q}}_n(d,I)^{ss}$. Then*

$$\mathbf{Det}_{/Q}(\mathscr{E},\mathscr{E}_{|D}) \cong \mathbf{Det}_{/Q}(\mathscr{E},\mathscr{E})^{\otimes(-2r)}$$

as G-bundles.

We will prove Proposition 4.6 by first establishing the following lemmas:

Lemma 4.7 *Let $p_X : X\times\widetilde{\mathfrak{Q}}_n(d,I)^{ss}\to X$ be the projection. Then the complex*

$$\mathscr{E}xt^{\cdot}_{X\times Q/Q}(\mathscr{E},\mathscr{E}\otimes p_X^*K_X)^0 \tag{4.5}$$

is a perfect complex whose determinant line bundle $\mathbf{Det}_{/Q}(\mathscr{E},\mathscr{E}\otimes p_X^*K_X)$ descends to a line bundle over $\widetilde{\mathfrak{M}}_H(d,I)$. If we denoted the descent by $\mathbf{Det}_{/M}(\mathscr{E},\mathscr{E}\otimes p_X^*K_X)$, then

$$\mathbf{Det}_{/M}(\mathscr{E},\mathscr{E}\otimes p_X^*K_X) \cong (\mathbf{Det}_{/M}(\mathscr{E},\mathscr{E}))^{-1}\,.$$

Proof. The proof of the first part is a direct consequence of Lemma 4.2 and 4.3. To establish the isomorphism, we apply the duality theorem to the (smooth) projective morphism p_2 [11, p.210] to conclude that there is a complex of finite locally free sheaves of finite length, say $\mathscr{R}^{\cdot}$, such that the complex (4.5) is quasi-isomorphic to $\mathscr{R}^{\cdot}$ while the complex (4.1) is quasi-isomorphic to the comples $\mathscr{H}om(\mathscr{R}^{\cdot},\mathcal{O})$. Thus by [13],

$$\mathbf{Det}_{/Q}(\mathscr{E},\mathscr{E}\otimes p_X^*K_X) \cong \mathbf{Det}_{/Q}(\mathscr{E},\mathscr{E})^{-1}\,. \tag{4.6}$$

Therefore their descents satisfy the same identity. □

Lemma 4.8 *Assume A and B are two effective divisors such that $K_X = \mathcal{O}(A-B)$, then*

$$\mathbf{Det}_{/Q}(\mathscr{E},\mathscr{E}_{|A})\otimes\mathbf{Det}_{/Q}(\mathscr{E},\mathscr{E}_{|B})^{-1} = \mathbf{Det}_{/Q}(\mathscr{E},\mathscr{E})^{\otimes 2} \tag{4.7}$$

as G-bundles.

Proof. Since $\mathscr{E}$ is a family of torsion free sheaves on $X \times \tilde{\mathscr{Q}}_n(d, I)^{ss}$ flat over $\tilde{\mathscr{Q}}_n(d, I)^{ss}$, by the local criteria of flatness, the sequence

$$0 \to \mathscr{E} \otimes p_X^* \mathcal{O}_X(-B) \to \mathscr{E} \otimes p_X^* K_X \to (\mathscr{E} \otimes p_X^* K_X)_{|A} \to 0 \tag{4.8}$$

is exact. (4.8) induces a long exact sequence

$$\to \mathscr{E}xt^i_{X \times Q/Q}(\mathscr{E}, \mathscr{E} \otimes p_X^* \mathcal{O}_X(-B))^0 \to \mathscr{E}xt^i_{X \times Q/Q}(\mathscr{E}, \mathscr{E} \otimes p_X^* K_X)^0 \to$$
$$\to \mathscr{E}xt^i_{X \times Q/Q}(\mathscr{E}, (\mathscr{E} \otimes p_X^* K_X)_{|A})^0 \to \mathscr{E}xt^{i+1}_{X \times Q/Q}(\mathscr{E}, \mathscr{E} \otimes p_X^* \mathcal{O}_X(-B))^0$$

which gives rise to a triangle of complexes of sheaves

$$\mathscr{E}xt^{\cdot}_{X \times Q/Q}(\mathscr{E}, \mathscr{E} \otimes p_X^* \mathcal{O}_X(-B))^0 \to \mathscr{E}xt^{\cdot}_{X \times Q/Q}(\mathscr{E}, \mathscr{E} \otimes p_X^* K_X)^0$$
$$\to \mathscr{E}xt^{\cdot}_{X \times Q/Q}(\mathscr{E}, (\mathscr{E} \otimes p_X^* K_X)_{|A})^0 .$$

Then by [13], the determinant line bundle of the respective complexes satisfy

$$\mathbf{Det}_{/Q}(\mathscr{E}, \mathscr{E} \otimes p_X^* K_X) = \mathbf{Det}_{/Q}(\mathscr{E}, (\mathscr{E} \otimes p_X^* K_X)_{|A}) \otimes \mathbf{Det}_{/Q}(\mathscr{E}, \mathscr{E} \otimes p_X^* \mathcal{O}_X(-B)) . \tag{4.9}$$

Similarly, by considering the long exact sequence induced by the short exact sequence

$$0 \to \mathscr{E} \otimes p_X^* Q_X(-B) \to \mathscr{E} \to \mathscr{E}_{|B} \to 0 ,$$

we obtain another triangle of complexes.

$$\mathscr{E}xt^{\cdot}_{X \times Q/Q}(\mathscr{E}, \mathscr{E} \otimes p_X^* \mathcal{O}_X(-B))^0 \to \mathscr{E}xt^{\cdot}_{X \times Q/Q}(\mathscr{E}, \mathscr{E})^0 \to \mathscr{E}xt^{\cdot}_{X \times Q/Q}(\mathscr{E}, \mathscr{E}_{|B})^0$$

and an identity of their determinant line bundles

$$\mathbf{Det}_{/Q}(\mathscr{E}, \mathscr{E} \otimes p_X^* \mathcal{O}_X(-B)) = \mathbf{Det}_{/Q}(\mathscr{E}, \mathscr{E}) \otimes \mathbf{Det}_{/Q}(\mathscr{E}, \mathscr{E}_{|B})^{-1} .$$

Thus combined with (4.9) and Lemma 4.7, we have

$$\mathbf{Det}_{/Q}(\mathscr{E}, \mathscr{E})^{\otimes(-2)} = \mathbf{Det}_{/Q}(\mathscr{E}, \mathscr{E})^{-1} \otimes \mathbf{Det}_{/Q}(\mathscr{E}, \mathscr{E} \otimes p_X^* K_X)$$
$$= \mathbf{Det}_{/Q}(\mathscr{E}, \mathscr{E}_{|B})^{-1} \otimes \mathbf{Det}_{/Q}(\mathscr{E}, (\mathscr{E} \otimes p_X^* K_X)_{|A}) .$$

Thus the lemma follows from

Lemma 4.9 *For any curve $A \subseteq X$, the G-bundle $\mathbf{Det}_{/Q}(\mathscr{E}, \mathscr{E}_{|A} \otimes p_A^* L)$ is independent of the choice of $L \in \mathrm{Pic}(A)$.*

Proof. We only need to show that for any effective Cartier divisor $C \subseteq A$,

$$\mathbf{Det}_{/Q}(\mathscr{E}, \mathscr{E}_{|A}) \cong \mathbf{Det}_{/Q}(\mathscr{E}, \mathscr{E}_{|A} \otimes p_A^* \mathcal{O}_A(C))$$

as G-bundles. Let $\mathscr{R}_1 \to \mathscr{R}_0 \to \mathscr{E}$ be a length two locally free resolution of $\mathscr{E}$. Then the exact sequence

$$0 \to \mathscr{R}_{i|A} \to \mathscr{R}_{i|A} \otimes p_A^* \mathcal{O}_A(C) \to \mathscr{R}_{i|C} \to 0$$

induces a triangle of complexes

$$\mathscr{E}xt^{\cdot}_{A \times Q/Q}(\mathscr{R}_\cdot, \mathscr{R}_{\cdot|A}) \to \mathscr{E}xt^{\cdot}_{A \times Q/Q}(\mathscr{R}_\cdot, \mathscr{R}_{\cdot|A} \otimes p_A^* \mathcal{O}_A(C)) \to \mathscr{E}xt^{\cdot}_{A \times Q/Q}(\mathscr{R}_\cdot, \mathscr{R}_{\cdot|C})$$

and the identity

$$\mathbf{Det}_{/Q}(\mathscr{R}_., \mathscr{R}_{.|A} \otimes p_A^* \mathscr{O}_A(C)) = \mathbf{Det}_{/Q}(\mathscr{R}_., \mathscr{R}_{.|A}) \otimes \mathbf{Det}_{/Q}(\mathscr{R}_., \mathscr{R}_{.|C}) . \quad (4.10)$$

Since the first and the second terms in (4.10) are the line bundles $\mathbf{Det}_{/Q}(\mathscr{E}, \mathscr{E}_{|A} \otimes \mathscr{O}_A(C))$ and $\mathbf{Det}_{/Q}(\mathscr{E}, \mathscr{E}_{|A})$ respectively, the lemma will be established if we can show that $\mathbf{Det}_{/Q}(\mathscr{R}_., \mathscr{R}_{.|C}) = \mathscr{O}_Q$. But this is obvious because

$$\mathscr{E}xt^{\cdot}_{A \times Q/Q}(\mathscr{R}_., \mathscr{R}_{.|C}) = \mathscr{H}om(\mathscr{R}_., \mathscr{R}_{.|C})$$

which of course has trivial determinant. Thus the lemma has been established. □

We quote one more lemma whose proof appear in [15].

Lemma 4.10 *Let C_0, C_1 and C_2 be smooth divisors of X so that C_0 is linearly equivalent to $C_1 + C_2$. Then as G-bundles,*

$$\mathbf{Det}_{/Q}(\mathscr{E}, \mathscr{E}_{|C_0}) \cong \mathbf{Det}_{/Q}(\mathscr{E}, \mathscr{E}_{|C_1}) \otimes \mathbf{Det}_{/Q}(\mathscr{E}, \mathscr{E}_{|C_2}) .$$

Remark. Though the lemma is stated and proved for smooth divisors of X in [15], it is indeed true for any divisors $C_0 \sim C_1 + C_2$.

Proof of Proposition 4.6 First of all, by choosing smooth divisor A and B so that $K_X = \mathscr{O}_X(A - B)$, we obtain the isomorphism (4.7). Then we can apply (4.9) and (4.10) to deduce Proposition 4.6. □

Now we demonstrate how to construct global sections of $\omega^{\otimes k}$ by using the Corollary 4.5 after Donaldson [4]. In the rest of this paper, we assume X is a minimal surface of general type. For any smooth $D \in |rK_X|$, we consider the set

$$\tilde{\mathscr{Q}}_n(d, I)^{ss}[D] = \{ s \in \tilde{\mathscr{Q}}_n(d, I)^{ss} | \mathscr{E}_s \text{ is stable and } \mathscr{E}_{s|D} \text{ is semistable} \} .$$

Clearly, if we let $\mathfrak{M}_D(2, I)$ be the moduli space or rank two semistable vector bundles on D with determinant $I_{|D}$, then the restriction to $D \times \tilde{\mathscr{Q}}_n(d, I)^{ss}[D]$ of the universal family $\mathscr{E}$ induces a morphism

$$\varphi_D : \tilde{\mathscr{Q}}_n(d, I)^{ss}[D] \to \mathfrak{M}_D(2, I) . \quad (4.11)$$

We remark that φ_D is G-equivalent, where G acts on $\mathfrak{M}_D(2, I)$ trivially. We have the following result of Donaldson,

Lemma 4.11 *There is an ample line bundle $\mathscr{L}_D$ on $\mathfrak{M}_D(2, I)$ such that*

$$\varphi_D^*(\mathscr{L}_D) \cong \mathbf{Det}_{/Q}(\mathscr{E}, \mathscr{E}_{|D})^{-1} | \tilde{\mathscr{Q}}_n(d, I)^{ss}[D] . \quad (4.12)$$

Further, this isomorphism is G-equivariant, where G acts on $\mathscr{L}_D$ trivially.

Proof. We first remark that since over $\tilde{\mathscr{Q}}_n(d, I)^{ss}[D]$, $\mathscr{E}_{|D}$ is locally free, the restriction to $\tilde{\mathscr{Q}}_n(d, I)^{ss}[D]$ of the complex of sheaves

$$\mathscr{E}xt^{\cdot}_{X \times Q/Q}(\mathscr{E}, \mathscr{E}_{|D})^0 \quad (4.13)$$

is isomorphic to the complex of sheaves

$$R^{\cdot} p_{Q*}(\mathscr{E}nd^0(\mathscr{E}_{|D})) .$$

Now let $\mathscr{U}_D^{2,I}$ be the functor that sends any separable finite type scheme S over $\mathbb{C}$ to the set of all families of rank two semistable vector bundels F_S on $D \times S$ over S with $\det F_S = p_D^*(I_{|D}) \otimes p_S^* L$, $L \in \mathrm{Pic}(S)$, where two F_S and F'_S are considered identical if $F_S \cong F'_S \otimes p_S^* L'$ for some $L' \in \operatorname{Pic}(S)$. $\mathscr{U}_D^{2,I}$ is coarsely represented by the projective scheme $\mathfrak{M}_D(2, I)$. Next, for any scheme S and any family of semistable vector bundles $F_S \in \mathscr{U}_D^{2,I}(S)$, we assign to it the line bundle

$$\det(R^\cdot p_{S*} \mathscr{E}nd^0(F_S)) \in \operatorname{Pic}(S) . \tag{4.14}$$

Clearly, if $F_S \cong F'_S \otimes p_S^* L$ for some $L \in \operatorname{Pic}(S)$, then $\det(R^\cdot p_{S*} \mathscr{E}nd^0(F_S))$ is canonically isomorphic to $\det(R^\cdot p_{S*} \mathscr{E}nd^0(F'_S))$. Thus combined with the base change property, (4.14) defines an element in $\mathbf{Pic}(\mathscr{U}_D^{2,I})$ – the Picard group of the functor $\mathscr{U}_D^{2,I}$ [5]. It is shown, for instance using Lemma 4.2, that the line bundle (4.14) is indeed the pull-back of a line bundel on $\mathfrak{M}_D(2, I)$. We denote the inverse of this line bundle by $\mathscr{L}_D$. Then (4.12) follows from the universality of the line bundle $\mathscr{L}_D$. Finally, the ampleness of the line bundle $\mathscr{L}_D$ follows from [3]. □

Up to this point, we are able to construct a lot of G-invariant sections of $\mathbf{Det}_{/Q}(\mathscr{E}, \mathscr{E}_{|D})^{\otimes(-m)}$ on $\tilde{\mathscr{Q}}_n(d, I)^{ss}[D]$ by appealing this restriction technique. Namely, for any positive integer m and any $v \in H^0(\mathfrak{M}_D(2, I), \mathscr{L}_D^{\otimes m})$, $\varphi_D^*(v)$ is a G-invariant section of $\mathbf{Det}_{/Q}(\mathscr{E}, \mathscr{E}_{|D})^{\otimes(-m)}$ over $\tilde{\mathscr{Q}}_n(d, I)^{ss}[D]$. Our next task is to show that all of them extend canonically to $\tilde{\mathscr{Q}}_n(d, I)^{ss}$.

Lemma 4.12 *Let $D \in |rK_X|$ be a general smooth divisor. Then any section $\varphi_D^*(v)$, $v \in H^0(\mathfrak{M}_D(2, I), \mathscr{L}_D^{\otimes m})$, extends uniquely to a section in $H^0(\tilde{\mathscr{Q}}_n(d, I)^{ss}, \mathbf{Det}_{/Q}(\mathscr{E}, \mathscr{E}_{|D})^{\otimes(-m)})$.*

Proof. We first consider a special case. Suppose S is a smooth affine curve, $p \in S$, F is a rank two locally free sheaf on $D \times S$ with $\det F = p_D^* I_{|D}$ such that F_s are semistable for all closed $s \in S \backslash p$. We let $\det(R^\cdot p_{S*} \mathscr{E}nd^0(F))$ be the determinant line bundle on S. By the proof of Lemma 4.9, the morphism $\varphi: S\backslash p \to \mathfrak{M}_D(2, I)$ (induced by the family F) has the property

$$\varphi^*(\mathscr{L}_D) \cong \det(R^\cdot p_{S*} \mathscr{E}nd^0(F))^{-1}_{|S\backslash p} .$$

Now let $v \in H^0(\mathfrak{M}_D(2, I), \mathscr{L}_D^{\otimes m})$ be any section and let $\varphi^*(v) \in H^0(S\backslash p, \det(R^\cdot p_{S*} \mathscr{E}nd^0(F))^{\otimes(-m)})$ be the pull-back section. We claim that $\varphi^*(v)$ extends to a regular section $\varphi^*(v)^{ex}$ on S.

Indeed, since $\mathfrak{M}_D(2, I)$ is complete, after possibly taking a base change $\pi: \tilde{S} \to S$, there is a locally free sheaf $\tilde{F}$ on $D \times \tilde{S}$ such that the restriction of $\tilde{F}$ to $D \times (\tilde{S}\backslash \pi^{-1}(p))$ is isomorphic to $\tilde{\pi}^*(F_{|D \times (S\backslash p)})$, where $\tilde{\pi}: D \times \tilde{S} \to D \times S$, and such that $\tilde{F}_s$ is semistable for all $s \in \tilde{S}$. Let $\tilde{\varphi} = \varphi \circ \pi: \tilde{S}\backslash \pi^{-1}(p) \to \mathfrak{M}_D(2, I)$ and let $\tilde{\varphi}^*(v)$ be the pull-back section (over $\tilde{S}\backslash \pi^{-1}(p)$) based on the isomorphism

$$\tilde{\varphi}^*(\mathscr{L}_D) \cong \det(R^\cdot p_{S*} \mathscr{E}nd^0(\tilde{\pi}^* F))^{-1}_{|\tilde{S}\backslash \pi^{-1}(p)} .$$

By base change property of the determinant line bundles,

$$\pi^* \det(R^\cdot p_{S*} \mathscr{E}nd^0(F))^{\otimes(-m)} \cong \det(R^\cdot p_{\tilde{S}*} \mathscr{E}nd^0(\tilde{\pi}^* F))^{\otimes(-m)}$$

and $\tilde{\varphi}^*(v) = \pi^* \varphi^*(v)$. Thus $\varphi^*(v)$ extends if and only if $\tilde{\varphi}_{S*}(v)$ extends. Therefore, without loss of generality, we can assume $S = \tilde{S}$. Clearly, $\tilde{F}$ belongs to the exact sequence

$$0 \to \tilde{F} \to F \to A \to 0 , \tag{4.15}$$

where A is a torsion sheaf supported on $D \times p$. Dualizing the sequence (4.15), we get

$$0 \to F^\vee \to \tilde{F}^\vee \to A' \to 0 . \tag{4.16}$$

By tensoring (4.16) by $\tilde{F}$ and tensoring (4.15) by $F^\vee$, we get

$$0 \to F^\vee \otimes \tilde{F} \to \tilde{F}^\vee \otimes \tilde{F} \to A' \otimes \tilde{F} \to 0\,; \tag{4.17}$$

$$0 \to F^\vee \otimes \tilde{F} \to F^\vee \otimes F \to F^\vee \otimes A \to 0\,. \tag{4.18}$$

Now we apply the functor $\det(R^\cdot p_{S*}(\cdot))$ to the exact sequence (4.17) and (4.18) to get

$$\begin{aligned}\det(R^\cdot p_{S*}(F^\vee \otimes F)) &= \det(R^\cdot p_{S*}(F^\vee \otimes \tilde{F})) \otimes \det(R^\cdot p_{S*}(F^\vee \otimes A)) \\ &= \det(R^\cdot p_{S*}(\tilde{F}^\vee \otimes \tilde{F})) \otimes \det(R^\cdot p_{S*}(A' \otimes \tilde{F}))^{-1} \\ &\quad \otimes \det(R^\cdot p_{S*}(F^\vee \otimes A))\,.\end{aligned}$$

Since $\det(R^\cdot p_{S*}(F^\vee \otimes F)) = \det(R^\cdot p_{S*}\mathscr{E}nd^0(F))$, we have

$$\det(R^\cdot p_{S*}\mathscr{E}nd^0(F)) = \det(R^\cdot p_{S*}\mathscr{E}nd^0(\tilde{F}))(lp),$$

where $l = \chi(F^\vee \otimes A) - \chi(A' \otimes \tilde{F})$. Thus

$$\det(R^\cdot p_{S*}\mathscr{E}nd^0(F))^{\otimes(-m)} = \det(R^\cdot p_{S*}\mathscr{E}nd^0(\tilde{F}))^{\otimes(-m)}(-mlp)\,. \tag{4.19}$$

Clearly, the morphism $\phi: S \to \mathfrak{M}_D(2, I)$ induced by the family $\tilde{F}$ coincides with φ when restricted to $S\backslash p$ and further, the pull-back section $\phi^*(v) \in H^0(S, \det(R^\cdot p_{S*}\mathscr{E}nd^0(\tilde{F}))^{\otimes(-m)})$ coincides with $\varphi^*(v)$ over $S\backslash p$ via the isomorphism (4.19). Thus, $\varphi^*(v)$ extends if

$$l = \chi(F^\vee \otimes A) - \chi(A' \otimes \tilde{F}) \leqq 0\,. \tag{4.20}$$

We prove (4.20) by induction on the length $\ell(A_{|x_0 \times S})$ where $x_0 \in D$ is a general closed point. Since F_S is locally free, without loss of generality, we can assume $\operatorname{rank}_D A \otimes k(p) = 1$. Thus $\operatorname{rank}_D A' \otimes k(p) = 1$ also, Since $\tilde{F} \otimes k(p)$ is semistable, we must have $\deg_D A \otimes k(p) \leqq \frac{1}{2} I \cdot D$. Therefore, $\chi(F^\vee \otimes A \otimes k(p)) \leqq 2\chi(\mathcal{O}_D)$. Similarly, $\chi(A' \otimes \tilde{F} \otimes k(p)) \geqq 2\chi(\mathcal{O}_D)$. Hence

$$\chi(F^\vee \otimes A \otimes k(p)) - \chi(A' \otimes \tilde{F} \otimes k(p)) \leqq 0\,.$$

Next, let F' be the kernel of $F \to A \otimes k(p)$. F' belongs to the exact sequence

$$\begin{array}{ccccccc} 0 \to & \tilde{F} & \to & F' & \to & B & \to 0 \\ & \| & & \cap & & \cap & \\ 0 \to & \tilde{F} & \to & F & \to & A & \to 0\,. \end{array}$$

Clearly, $A/B \cong A \otimes k(p)$, so $\ell(B_{|x_0 \times S}) < \ell(A_{|x_0 \times S})$. By induction hypothesis, $\chi(F^\vee \otimes B) - \chi(B' \otimes \tilde{F}) \leqq 0$. So

$$\begin{aligned}\chi(F^\vee \otimes A) - \chi(A' \otimes \tilde{F}) &= \chi(F^\vee \otimes A \otimes k(p)) - \chi(A' \otimes \tilde{F} \otimes k(p)) \\ &\quad + \chi(F^\vee \otimes B) - \chi(B' \otimes \tilde{F}) \leqq 0\,.\end{aligned}$$

Therefore, $\varphi^*(v)$ extends to a regular section over S.

Now we prove the extension lemma. We first let $\tilde{\mathfrak{Q}}_n(d, I)^{ss}[K_X]$ be the open subset of $\tilde{\mathfrak{Q}}_n(d, I)^{ss}$ consisting of K_X-semistable quotient sheaves and let $\mathfrak{W} = \tilde{\mathfrak{Q}}_n(d, I)^{ss} \backslash \tilde{\mathfrak{Q}}_n(d, I)^{ss}[K_X]$. By [23], $\mathfrak{W}$ is a closed subset of $\tilde{\mathfrak{Q}}_n(d, I)^{ss}$ of codimension at least one when $d \gg 0$. Let $\mathfrak{W}_1, \ldots, \mathfrak{W}_l$ be irreducible components of $\mathfrak{W}$ that have codimension one. By choosing $D \in |rK_X|$ general, we can assume $\mathscr{E}_{s|D}$ is locally free for general $s \in \mathfrak{W}_1, \ldots, \mathfrak{W}_l$. Now we apply the Bogomolov's result which says that if E is μ-semistable with respect to K_X and $r \geqq 2d+1$, then for smooth $D \in |rK_X|$, $E_{|D}$ is semistable provided that $E_{|D}$ is locally free. Therefore, if we assume d large, $r \geqq 2d+1$ and D general, the compliment of $\tilde{\mathfrak{Q}}_n(d,I)^{ss}[D]$ in $\tilde{\mathfrak{Q}}_n(d,I)^{ss}[K_X]$ has codimension at least two. (We know that when d is large, the general points of $\tilde{\mathfrak{Q}}_n(d,I)^{ss}$ corresponds to locally free sheaves.) Take a smooth curve S, $p \in S$, and a morphism $\mu: S \to \tilde{\mathfrak{Q}}_n(d,I)^{ss}$ so that $\mu(S \backslash p) \subseteq \tilde{\mathfrak{Q}}_n(d,I)^{ss}[D]$ and $\mu(p) \in \mathfrak{W}_i$ is general. Let F_S be the pull-back of $\mathscr{E}$ via μ. By shrinking $p \in S$ if necessary, we can assume $F_{S|D}$ is locally free. Then

$$\mu^* \mathbf{Det}_{/Q}(\mathscr{E}, \mathscr{E}_{|D}) = \det(R^{\cdot} p_{S*} \mathscr{E}nd^0(F_{S|D})) .$$

Thus by the previous argument, the pull-back section $\mu^* \varphi_D^*(v)$ extends to a regular section over S. Since $\tilde{\mathfrak{Q}}_n(d,I)^{ss}$ is smooth, $\varphi_D^*(v)$ is regular along $\mathfrak{W}_i$. Therefore, $\varphi_D^*(v)$ extends over $\tilde{\mathfrak{Q}}_n(d,I)^{ss}$. □

Clearly, the extensions of $\varphi_D^*(v)^{ex}$ are G-invariant. Thus they descend to sections in

$$H^0(\tilde{\mathfrak{M}}_H(d,I), \mathbf{Det}_{/M}(\mathscr{E}, \mathscr{E}_{|D})^{\otimes(-m)}) .$$

Therefore, by Lemma 4.7 and 4.8, we proved

Proposition 4.13 *For any $m \geqq 0$ and general $D \in |rK_X|$, there is a homomorphism*

$$\rho_m : H^0(\mathfrak{M}_D(2,I), \mathscr{L}_D^{\otimes m}) \to H^0(\tilde{\mathfrak{M}}_H(d,I), \mathbf{Det}_{/M}(\mathscr{E}, \mathscr{E})^{\otimes 2rm})$$

that is induced by isomorphisms in Lemma 4.11 *and* 4.12.

Proposition 4.14 *Suppose (X, H) is a minimal surface of general type and $I \in \mathrm{Pic}(X)$ is a line bundle has the property that $c_1(I) \cdot Z$ is even for any (-2)-exceptional curve Z of X, then there is a constant C depending on (X, H, I) such that for any $d \geqq C$ and any irreducible component $\mathbf{M} \subseteq \tilde{\mathfrak{M}}_H(d,I)$,*

$$h^0(\mathbf{M}, \mathbf{Det}_{/M}(\mathscr{E}, \mathscr{E})^{\otimes m}) = c \cdot m^{c(d)} + O(m^{c(d)-1})$$

with $c > 0$ and $c(d) = \dim \mathfrak{M}_H(d,I)$.

Proof. Let $\mathbf{Q} \subseteq \tilde{\mathfrak{Q}}_n(d,I)^{ss}$ be the irreducible components corresponding to $\mathbf{M}$ and let $\varphi_D : \mathbf{Q} \to \mathfrak{M}_D(2,I)$, $D \in |rK_X|$ general, be the rational map (4.11). Then by Proposition 4.13.

$$h^0(\mathbf{M}, \mathbf{Det}_{/M}(\mathscr{E}, \mathscr{E})^{\otimes 2rm}) \geqq c \cdot m^{\beta} + O(m^{\beta-1}), \quad c > 0 ,$$

where $\beta = \dim \varphi_D(\mathbf{Q})$.

We claim that $\dim \varphi_D(\mathbf{Q}) = \dim \mathfrak{M}_H(d,I)$ when r is large. Indeed, let Y be the canonical model of X and let $Z \subseteq X$ be the exceptional divisor of $f: X \to Y$. Then for large d and general $E \in \mathbf{M}$, $\mathrm{Ext}^2(E, E(-Z))^0 = \{0\}$. Thus by the assumption on the degrees of E along components of Z, $E_{|Z} \cong \mathcal{O}_Z^{\oplus 2}$. Now let E, F be two general members of $\mathbf{M}$, then $(E^\vee \otimes F)_{|Z} \cong \mathcal{O}_Z^{\oplus 4}$. We claim that then $f_*(E^\vee \otimes F)$ is locally free. Indeed, for any integer k, assume $(E^\vee \otimes F)_{|kZ} \cong \mathcal{O}_{kZ}^{\otimes 4}$, then because

$$\begin{aligned} h^1(Z, (E^\vee \otimes F)_{|Z}(-kZ)) &= h^0(Z, (F^\vee \otimes E)_{|Z}((k+1)Z)) \\ &= h^0(Z, \mathcal{O}_Z((k+1)Z)^{\oplus 4}) = 0 , \end{aligned}$$

$$H^0((k+1)Z, (E^\vee \otimes F)_{|(k+1)Z}) \to H^0(kZ, (E^\vee \otimes F)_{|kZ})$$

is surjective. Therefore, $(E^\vee \otimes F)_{|kZ} = \mathcal{O}_{kZ}^{\oplus 4}$ for any k by induction on k. In particular, $f_*(E^\vee \otimes F)$ is locally free over Y by [10, 3.11]. Next, since K_Y is ample on Y, we can choose r large so that for any locally free $E, F \in \mathfrak{M}_H(d, I)$ with $f_*(E^\vee \otimes F)$ locally free, $h^1(Y, f_*(E^\vee \otimes F)(-rK_Y)) = 0$. Now let $E, F \in \mathfrak{M}_H(d, I)$ be any two stable locally free sheaves. Assume $E_{|D} \cong F_{|D}$, $D \in |rK_X|$ general. Since

$$H^0(Y, f_*(E^\vee \otimes F)) \to H^0(f(D), f_*(E^\vee \otimes F)) = H^0(D, E^\vee \otimes F) \neq \{0\}$$

is surjective, there is a non-trivial homomorphism $g: E \to F$. Since E and F are stable, $E \cong F$. Thus $\varphi: \tilde{\mathscr{Q}}_n(d, I)^{ss}/G \to \mathfrak{M}^{2,I}(D)$ is generically one-to-one. So $\dim \varphi_D(\tilde{\mathscr{Q}}_n(d, I)^{ss}) = \dim \mathfrak{M}_H(d, I)$. □

Now we prove the main theorem:

Theorem 4.15 *Let (X, H) be any smooth minimal surface of general type and I a line bundle on X such that $c_1(I) \cdot Z$ is even for any (-2)-exceptional curve Z of X and that $\chi(\mathcal{O}_X) + I^2$ is even. Suppose there is a reduced $D \in |K_X|$, then there is a constant C depending on (X, H, I) such that for any $d \geqq C$, $\mathfrak{M}_H(d, I)$ is of general type.*

Proof. According to Corollary 4.5, the canonical line bundle $\omega = \mathbf{Det}_{/M}(\mathscr{E}, \mathscr{E})(\sum a_i W_i)$. On the other hand, under the assumption on X and I, the two-canonical section $\bar{\Lambda}_D$ constructed in §3 is non-trivial at each irreducible component $\mathbf{M} \subseteq \tilde{\mathfrak{M}}_H(d, I)$ and is indeed a section of $\omega^{\otimes 2}(-W)$ when d is sufficiently large, where W is the exceptional divisor of $\tilde{\mathfrak{M}}_H(d, I) \to \mathfrak{M}_H(d, I)$. Now let $\alpha = \max\{-a_i\}$. Then there is an injective homomorphism

$$\begin{array}{ccc} H^0(\mathbf{M}, \mathbf{Det}_{/M}(\mathscr{E}, \mathscr{E})^{\otimes m}) & \to & H^0(\mathbf{M}, \omega^{\otimes(1+2\alpha)m}) \\ \cup & & \cup \\ v & \mapsto & v \otimes \bar{\Lambda}_D^{\otimes \alpha m}\ . \end{array}$$

It is well-defined because $v \in H^0(\mathbf{M}, \omega^{\otimes m}(-m\sum a_i W_i))$ while $\bar{\Lambda}_D^{\otimes \alpha m} \in H^0(\mathbf{M}, \omega^{\otimes 2\alpha m}(-\alpha m W))$. Thus for $m \gg 0$,

$$\begin{aligned} \dim H^0(\mathbf{M}, \omega^{\otimes(1+2\alpha)m}) &\geq \dim H^0(\mathbf{M}, \mathbf{Det}_{/M}(\mathscr{E}, \mathscr{E})^{\otimes m}) \\ &= c \cdot m^{c(d)} + O(m^{c(d)-1}), \quad c > 0\ . \end{aligned}$$

Thus, the theorem and the Theorem 0.2 has been established. □

The requirement that there is a reduced $D \in |K_X|$ is a technical condition. All we need is a result similar to Proposition 3.6 for arbitrary canonical curves. The condition that $\chi(\mathcal{O}_X) + I^2$ is even should not be necessary. One indication along this line is that if we assume $h^0(K_X) \geqq 3$, then we can choose three general θ_1, θ_2 and $\theta_3 \in H^0(K_X)$ to form three symplectic forms Θ_{ω_i}. Then it is likely that

$$\begin{pmatrix} \Theta_{\theta_1} & \Theta_{\theta_2} \\ \Theta_{\theta_3} & \Theta_{\theta_1} \end{pmatrix} : T\tilde{\mathfrak{M}}_H(d, I) \times T\tilde{\mathfrak{M}}_H(d, I) \to T\tilde{\mathfrak{M}}_H(d, I) \times T\tilde{\mathfrak{M}}_H(d, I)$$

will be non-degenerate at general points of $\tilde{\mathfrak{M}}_H(d, I)$. Thus its determinant will provide us the desired pluri-canonical section in proving Theorem 0.2 in the general case. The author conjectures

Conjecture. *For any smooth minimal algebraic surface (X, H) of general type and any fixed line bundle I such that $c_1(I) \cdot Z$ are even for any (-2)-exceptional* *curve Z, then there is a constant C depending on (X, H, I) such that whenever $d \geqq C$, then $\mathfrak{M}_H(d, I)$ is of general type.*

The techniques developed in this paper can be employed to study the Kodaira dimension of the moduli $\mathfrak{M}_H(d, I)$ for other surfaces. For example, when $\kappa(X) = -\infty$ and $-rK_X, r > 0$, is effective, then one easily sees that $\mathfrak{M}_H(d, I)$ will have Kodaira dimension $-\infty$. For X with $K_X = \mathcal{O}_X$, $\kappa(\mathfrak{M}_H(d, I)) = 0$ because Proposition 3.8 provides us a two-canonical form on a desingularization of $\mathfrak{M}_H(d, I)$. When $\kappa(X) = 0$ while $K_X \neq \mathcal{O}_X$, one checks that $\kappa(\mathfrak{M}_H(d, I)) \leqq 0$ and the equality holds if $\mathfrak{M}_H(d, I)$ is smooth.

5 Appendix

In this appendix, we are going to prove the following proposition:

Proposition 5.1 *Let X be a minimal surface of general type and let $C \in |K_X|$ be a reduced canonical divisor. Then for any line bundle I on C, there is at least one rank two locally free sheaf E on C with* $\det E = I$ *such that*

$$h^0(C, \mathcal{E}nd^0(E) \otimes K_X) \leqq 1 . \tag{5.1}$$

The tactic used in attacking this problem follows closely with that of Mumford in his study of Prym variety [21]. We outline the proof briefly here. For simplicity, we assume C is irreducible. First of all, let E be a sheaf as in Proposition 5.1 and let $x \in C$ be a general closed point. We consider an elementary transformation F of E by the exact sequence

$$0 \to F \to E \to \mathbb{C}_x \to 0 . \tag{5.2}$$

By a careful study of $H^0(\mathcal{E}nd^0(F) \otimes K_X)$, we can show that for general homomorphism $E \to \mathbb{C}_x$,

$$h^0(\mathcal{E}nd^0(F) \otimes K_X) = \begin{cases} h^0(\mathcal{E}nd^0(E) \otimes K_X) - 1, & \text{if } h^0(\mathcal{E}nd^0(E) \otimes K_X) \geqq 1 \ ; \\ 1, & \text{if } h^0(\mathcal{E}nd^0(E) \otimes K_X) = 0 \ . \end{cases}$$

Thus if $h^0(\mathcal{E}nd^0(E) \otimes K_X) \neq 0$, we can actually decrease it by replacing E with F. One draw back of this maneuver is that since $\det F = I(-x)$, F is not what we want. The solution to this is to perform elementary transformation one more time. More precisely, we construct E' by the following exact sequence

$$0 \to E' \to F \to \mathbb{C}_y \to 0 .$$

The same argument shows that in any case, we will either have $h^0(\mathcal{E}nd^0(E') \otimes K_X) \leqq h^0(\mathcal{E}nd^0(E) \otimes K_X)$ or it is already $\leqq 1$. Finally, let A be an invertible sheaf on C such that $A^{\otimes 2} = \mathcal{O}_C(x + y)$. Then the sheaf $E' \otimes A$ has determinant I. Therefore, if we begin with a general sheaf E, E must satisfy (5.1).

From now on, we always assume $C \in |K_X|$ is reduced, I is an invertible sheaf on C and except when mentioned is made to the contrary, all sheaves considered are rank two locally free sheaves on C. Before we prove Proposition 5.1, we first state the following results:

Lemma 5.2 *Let E_t be a family of locally free sheaves on C with* $\det E_t = I$, $t \in T$. *Suppose the parameter space T is connected, then $h^0(C, \mathcal{E}nd^0(E_t) \otimes K_X)$ is constant modulo 2.*

Proof. The case when C is smooth was established in [3, 22]. By a careful study of their proofs, one sees that the only thing needs to be checked in order to generalize their proofs to the reduced curve case is that one can define a residue map $\omega_C(D)/\omega_C \to \mathbb{C}$, where D is a reduced effective Cartier divisor of C, such that for any $g \in H^0(\omega_C(D))$, $\sum_{x\in D} \mathrm{Res}_x(g) = 0$ and when $\mathrm{Res}_x(g) = 0$ for any $x \in D$, then $g \in H^0(\omega_C)$. But this certainly can be achieved by the exact sequence

$$H^0(\omega_C) \to H^0(\omega_C(D)) \to H^0(\omega_C(D)/\omega_C) \xrightarrow{\text{res}} H^1(\omega_C) = \mathbb{C} .$$

We leave the detail of the proof to the readers. □

Lemma 5.3 *Let ω_C be the dualizing sheaf of C, let D be any Cartier divisor on C and let $S \subseteq C$ be any irreducible component. Then if the restriction homomorphism $H^0(C, \omega_C(-D)) \to H^0(S, \omega_C(-D))$ is trivial, then for general closed point $x \in S$,*

$$H^0(C, \mathcal{O}(D + x)) \to \mathcal{O}(D + x)_{|x}$$

is surjective.

Proof. Assume $H^0(C, \mathcal{O}(D + x)) \to \mathcal{O}(D + x)_{|x}$ is trivial, then $H^0(C, \mathcal{O}(D + x)) = H^0(C, \mathcal{O}(D))$. By R.R. and duality, we then have $h^0(C, \omega_C(-D)) = h^0(C, \omega_C(-D - x)) + 1$. But since we have assumed that $H^0(C, \omega_C(-D)) \to H^0(S, \omega_C(-D))$ is trivial, we must have $h^0(C, \omega_C(-D)) = h^0(C, \omega_C(-D - x))$, a contradiction. □

We also need the following technical lemma:

Lemma 5.4 *Let $C = D_1 + D_2$ be a splitting (D_2 may be empty) and let $S \subseteq D_1$ be an irreducible component. Suppose E is a rank two locally free sheaf on C and suppose F is defined by the exact sequence*

$$0 \to F \to E \xrightarrow{\varphi_x} \mathbb{C}_x \to 0 , \tag{5.3}$$

where $x \in S$ and φ_x are general closed point and homomorphism respectively. Then

$$h^0(D_1, \mathcal{E}nd^0(F)(D_1)) \leqq h^0(D_1, \mathcal{E}nd^0(E)(D_1))$$

unless the restriction homomorphism

$$R_{E,S}: H^0(D_1, \mathcal{E}nd^0(E)(D_1 + D_2)) \to H^0(S, \mathcal{E}nd^0(E)(D_1 + D_2)) \tag{5.4}$$

is trivial. In this case, we have $h^0(D_1, \mathcal{E}nd^0(F)(D_1)) \leqq h^0(D_1, \mathcal{E}nd^0(E)(D_1)) + 1$.

Proof. Let $r = h^0(D_1, \mathcal{E}nd^0(E)(D_1))$ and let $r_0 = h^0(D_1, \mathcal{O}_{D_1}(D_1))$. Because the dualizing sheaf $\omega_{D_1} = K_{X|D_1}(D_1) = \mathcal{O}_{D_1}(2D_1 + D_2)$, by R.R. and Serre duality,

$$\begin{aligned}\mathrm{Hom}_{D_1}(E, E(D_1 + D_2)) &= \chi(D_1, E^\vee \otimes E(D_1 + D_2)) + \mathrm{Ext}^1_{D_1}(E, E(D_1 + D_2)) \\ &= 2D_1 \cdot D_2 + r + r_0 .\end{aligned}$$

We first assume that for general $x \in S$ and φ_x,

$$\mathrm{Hom}_{D_1}(E, E(D_1 + D_2)) \to \mathrm{Hom}(E, \mathbb{C}_x) = \mathbb{C}^{\oplus 2} \tag{5.5}$$

is surjective. Then $\dim \operatorname{Hom}_{D_1}(E, F(D_1 + D_2)) = 2D_1 \cdot D_2 + r + r_0 - 2$. By R.R. and Serre duality, we have

$$\dim \operatorname{Hom}_{D_1}(F, E(D_1)) = \chi(D_1, F^\vee \otimes E(D_1)) + \operatorname{Ext}^1_{D_1}(F, E(D_1)) = r + r_0 .$$

Since $\operatorname{Hom}_{D_1}(F, F(D_1)) \to \operatorname{Hom}_{D_1}(F, E(D_1))$ is injective, we must have $h^0(D_1, \mathscr{E}nd(F)(D_1)) \leqq r + r_0$. Or equivalently, $h^0(D_1, \mathscr{E}nd^0(F)(D_1)) \leqq h^0(D_1, \mathscr{E}nd^0(E)(D_1))$.

Next we consider the case where $R_{E,S}$ (cf. (5.4)) is non-trivial while the homomorphism in (5.5) has image $\mathbb{C}$ for generic x and φ_x. Note that then

$$\operatorname{Im}\{H^0(D_1, \mathcal{O}_{D_1}(D_1 + D_2)) \to H^0(S, \mathcal{O}_{D_1}(D_1 + D_2))\} = \{0\} . \tag{5.6}$$

For similar reason as before, we have $\dim \operatorname{Hom}_{D_1}(E, F(D_1 + D_2)) = 2D_1 \cdot D_2 + r + r_0 - 1$ and by R.R. and duality, we have $\dim \operatorname{Hom}_{D_1}(F, E(D_1)) = r + r_0 + 1$. Thus

$$\dim H^0(D_1, \mathscr{E}nd^0(F)(D_1)) \leqq r + 1 .$$

It remains to show that the equality does not occur when $x \in S$ and φ_x are general. Namely, we need to show that the homomorphism

$$\alpha : \operatorname{Hom}_{D_1}(F, F(D_1)) \to \operatorname{Hom}_{D_1}(F, E(D_1)) \tag{5.7}$$

is not surjective. Here is our argument: We first consider the filtration $E(-x) \subset F \subset E$ and the induced filtration

$$\operatorname{Hom}_{D_1}(F, F(D_1)) \xrightarrow{\alpha} \operatorname{Hom}_{D_1}(F, E(D_1)) \xrightarrow{\beta} \operatorname{Hom}_{D_1}(E(-x), E(D_1)) . \tag{5.8}$$

Under a fixed trivialization $E_x = \mathcal{O}_{S,x}^{\oplus 2}$ and the compatible trivialization $E(D_1)_x = \mathcal{O}_{S,x}^{\oplus 2}$, we assume φ_x has the form $\varphi_x = \binom{1}{t} : E_x = \mathcal{O}_{S,x}^{\oplus 2} \to \mathbb{C}_x$, where $t \in \mathbb{C}$. Then we can find a local trivialization $F_x = \mathcal{O}_{S,x}^{\oplus 2}$ so that inclusion $F_x \to E_x$ is given by

$$\begin{pmatrix} -t & 1 \\ \xi & 0 \end{pmatrix} : \mathcal{O}_{S,x}^{\oplus 2} \to \mathcal{O}_{S,x}^{\oplus 2} ,$$

where ξ is the uniformizing parameter of S at x. Thus the inclusion $E(-x) \to F$ is of the form

$$\begin{pmatrix} 0 & 1 \\ \xi & t \end{pmatrix} : \mathcal{O}_{S,x}^{\oplus 2} \to \mathcal{O}_{S,x}^{\oplus 2} ,$$

where the trivialization $E(-x)_x = \mathcal{O}_{S,x}^{\oplus 2}$ is compatible to $E_x = \mathcal{O}_{S,x}^{\oplus 2}$. Let $f \in \operatorname{Hom}_{D_1}(E(-x), E(D_1))$. Clearly, $f \in \operatorname{Im}\{\beta\}$ (cf. (5.8)) if

$$\begin{pmatrix} 0 & 1 \\ \xi & t \end{pmatrix}^{-1} \circ \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} = \begin{pmatrix} \frac{1}{\xi}(f_{21} - t f_{11}) & \frac{1}{\xi}(f_{22} - t f_{12}) \\ f_{11} & f_{12} \end{pmatrix}$$

is regular. Namely,

$$f_{21}(x) - t f_{11}(x) = 0, \quad f_{22}(x) - t f_{12}(x) = 0 . \tag{5.9}$$

Because for general t, (5.9) imposes two conditions on f. Thus, $\dim \operatorname{coker}\{\beta\} = 2$. On the other hand,

$$\dim \operatorname{Hom}_{D_1}(F, E(D_1)) - \dim \operatorname{Hom}_{D_1}(E, E(D_1)) = (r + r_0 + 1) - (r + r_0) = 1 .$$

Therefore,

$$\dim \operatorname{Hom}_{D_1}(E(-x), E(D_1))/\operatorname{Hom}_{D_1}(E, E(D_1)) = 3 .$$

In other words, there are non-trivial $(a_{11}, a_{12}, a_{21}, a_{22}) \in \mathbb{C}^4$ such that for any $f \in \operatorname{Hom}_{D_1}(E(-x), E(D_1))$,

$$a_{11}f_{11}(x) + a_{12}f_{12}(x) + a_{21}f_{21}(x) + a_{22}f_{22}(x) = 0 .$$

Now assume $h^0(D_1, \mathscr{E}nd^0(F)(D_1)) = r + 1$, then the homomorphism α (cf. (5.7)) must be surjective. That is, for $f \in \operatorname{Hom}_{D_1}(E(-x), E(D_1))$ with $f \in \operatorname{Im}\{\beta\}$, we must have

$$\begin{pmatrix} 0 & 1 \\ \xi & t \end{pmatrix}^{-1} \circ \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} \circ \begin{pmatrix} -t & 1 \\ \xi & 0 \end{pmatrix}^{-1}$$

regular. Thus f automatically satisfies one more constrain:

$$f_{11}(x) + t f_{12}(x) = 0 .$$

Therefore, both

$$\begin{pmatrix} a_{11} & a_{12} & a_{21} & a_{22} \\ -t & 0 & 1 & 0 \\ 0 & -t & 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} a_{11} & a_{12} & a_{21} & a_{22} \\ -t & 0 & 1 & 0 \\ 0 & -t & 0 & 1 \\ 1 & t & 0 & 0 \end{pmatrix}$$

are of rank 3 for general t. An easy argument shows that this is possible only if

$$a_{12} = a_{21} = 0, \quad a_{11} = a_{22} .$$

Namely, for any $f \in \operatorname{Hom}_{D_1}(E(-x), E(D_1))$, $\operatorname{tr}(f(x)) = 0$. But this is impossible because by Lemma 5.3, $H^0(D_1, \mathcal{O}(D_1 + x)) \to \mathcal{O}(D + x)_{|x}$ is surjective thanks to (5.6). Therefore we have established the first part of the Lemma 5.4. The conclusion that when $R_{E,S}$ is trivial, $h^0(D_1, \mathscr{E}nd^0(F)(D_1)) \leqq h^0(D_1, \mathscr{E}nd^0(E)(D_1)) + 1$ can be treated similarly. We leave it to the readers. □

We state and prove the following known fact.

Lemma 5.5 *For any smooth points x, y of C in the same irreducible component $S \subseteq C$, there is an invertiable sheaf A on C so that $A^{\otimes 2} \cong \mathcal{O}_C(x + y)$.*

Proof. It suffices to show that there is an invertible sheaf A on C so that $A^{\otimes 2} \cong \mathcal{O}_C(x - y)$. First note that the set of line bundles on C is isomorphic to $H^1(C, \mathcal{O}_C^*)$ which fits into the exact sequence

$$H^1(C, \mathcal{O}_C) \to H^1(C, \mathcal{O}_C^*) \xrightarrow{c_1} H^2(C, \mathbb{Z}) .$$

Clearly, $c_1(\mathcal{O}_C(x - y)) = 0$. Thus $\mathcal{O}_C(x - y)$ belongs to the image of $H^1(C, \mathcal{O}_C) \to H^1(C, \mathcal{O}_C)$. On the other hand, $H^1(C, \mathcal{O}_C)$ is a linear space. Thus there is an $A \in H^1(C, \mathcal{O}_C)$ such that $2A$ corresponds to the line bundle $\mathcal{O}_C(x - y)$. □

Proof of Proposition 5.1 Let E be a rank two vector bundle with $\det E = I$ that attains the minimum value $h^0(C, \mathscr{E}nd^0(E) \otimes K_X)$ among all vector bundles of the same type. We first show that for any irreducible component $S \subseteq C$, the image of the restriction homomorphism

$$R_{E,S}: H^0(C, \mathscr{E}nd^0(E) \otimes K_X) \to H^0(S, \mathscr{E}nd^0(E) \otimes K_X) \tag{5.10}$$

has dimension at most one. Indeed, suppose $\dim \operatorname{Im}\{R_{E,S}\} \geqq 2$. We let $x_1, x_2 \in S$ be two general closed points and let $\varphi_i: E \to \mathbb{C}_{x_i}$ be general homomorphism. Then by applying Lemma 5.4, with $F = \ker\{E \xrightarrow{\varphi_1} \mathbb{C}_{x_1}\}$, we have $h^0(C, \mathscr{E}nd^0(F) \otimes K_X) \leqq h^0(C, \mathscr{E}nd^0(E) \otimes K_X)$. On the other hand, thanks to Lemma 5.2, one checks easily that

$$h^0(C, \mathscr{E}nd^0(F) \otimes K_X) - h^0(C, \mathscr{E}nd^0(E) \otimes K_X) \equiv 1 \bmod (2) . \tag{5.11}$$

Thus we must have

$$h^0(C, \mathscr{E}nd^0(F) \otimes K_X) \leqq h^0(C, \mathscr{E}nd^0(E) \otimes K_X) - 1 . \tag{5.12}$$

There are two possible situations: The first is when the strict inequality holds in (5.12). In this case, we apply Lemma 5.4 to $E' = \ker\{F \xrightarrow{\varphi_2} \mathbb{C}_{x_2}\} \subset F$ to get

$$\begin{aligned} h^0(C, \mathscr{E}nd^0(E') \otimes K_X) &\leqq h^0(C, \mathscr{E}nd^0(F) \otimes K_X) + 1 \\ &\leqq h^0(C, \mathscr{E}nd^0(E) \otimes K_X) - 1 . \end{aligned}$$

Now if we let A be the line bundle with $A^{\otimes 2} = \mathcal{O}_C(x + y)$, $E' \otimes A$ has determinant I. This certainly violates our assumption that $h^0(C, \mathscr{E}nd^0(E) \otimes K_X)$ is minimal. The second case is when $h^0(C, \mathscr{E}nd^0(F) \otimes K_X) = h^0(C, \mathscr{E}nd^0(E) \otimes K_X) - 1$. By our construction, the kernel of

$$R_{F,S}: H^0(C, \mathscr{E}nd^0(F) \otimes K_X) \to H^0(S, \mathscr{E}nd^0(F) \otimes K_X)$$

is isomorphic to the kernel of

$$R_{E,S}: H^0(C, \mathscr{E}nd^0(E) \otimes K_X) \to H^0(S, \mathscr{E}nd^0(E) \otimes K_X) .$$

Hence, $\dim \operatorname{Im}\{R_{F,S}\} = \dim \operatorname{Im}\{R_{E,S}\} - 1 \geqq 1$. Thus if we apply Lemma 5.4 to the sheaf $E' = \ker\{\xrightarrow{\varphi_2} \mathbb{C}_{x_2}\}$, we get

$$h^0(C, \mathscr{E}nd^0(E') \otimes K_X) \leqq h^0(C, \mathscr{E}nd^0(E) \otimes K_X) - 2 .$$

This again violates our assumption on E. Therefore, $\dim \operatorname{Im}\{R_{E,S}\} \leqq 1$ for any irreducible component S of C.

Now assume $h^0(C, \mathscr{E}nd^0(E) \otimes K_X) \geqq 2$. By our previous argument, this is possible only when all sections $f \in H^0(C, \mathscr{E}nd^0(E) \otimes K_X)$ have the property that $f^{-1}(0)$ contains at least one irreducible component of C. Let $f \in H^0(C, \mathscr{E}nd^0(E) \otimes K_X)$ and let $C = D_1 + D_2$ be the splitting so that $f_{|D_2} \equiv 0$ while f is non trivial at general points of D_1. Possibly by replacing f with other sections and shrinking D_1 accordingly, we can assume that f can not be expressed as the sum of $f_1, f_2 \in H^0(C, \mathscr{E}nd^0(E) \otimes K_X)$ such that $\operatorname{supp}(f_1)$ and $\operatorname{supp}(f_2)$ have no common components. We fix such an f and the corresponding splitting $C = D_1 + D_2$. Note that then $h^0(D_1, \mathscr{E}nd^0(E) \otimes K_X(-D_2)) = 1$.

We now claim that there is at least one irreducible component $S \subseteq D_1$ such that

$$\dim \operatorname{Im}\{R'_{E,S}: H^0(D_1, \mathscr{E}nd^0(E) \otimes K_X) \to H^0(S, \mathscr{E}nd^0(E) \otimes K_X)\} \geqq 2 . \tag{5.13}$$

Indeed, because $f \in H^0(D_1, \mathscr{E}nd^0(E) \otimes K_X(-D_2))$ and that $K_X(-D_2)_{|D_1} \subset K_{X|D_1}$, f lifts to an $\tilde{f} \in H^0(D_1, \mathscr{E}nd^0(E) \otimes K_X)$ which is non-trivial along any irreducible component of D_1. In general, it is possible that $\tilde{f} = g_1 + g_2$, where $g_1, g_2 \in H^0(D_1, \mathscr{E}nd^0(E) \otimes K_X)$ while $\operatorname{supp}(g_1)$ and $\operatorname{supp}(g_2)$ have no common components. Let $D_1 = S_1 + \ldots + S_k$ be the maximal splitting so that there are $g_i \in H^0(D_1, \mathscr{E}nd^0(E) \otimes K_X)$ such that $\tilde{f} = g_1 + \ldots + g_k$ and that $\operatorname{supp}(g_i) = S_i$. By our assumption of f, this is possible only if $D_2 \cdot S_i > 0$ for all i. (Otherwise, f itself can be written as $g_1 + g_2$ with $g_1, g_2 \in H^0(D_1, \mathscr{E}nd^0(E) \otimes K_X(-D_2))$ and $\operatorname{supp}(g_1)$ has no common components with $\operatorname{supp}(g_2)$.) Note that when $k = 1$, $D_1 \cdot D_2 > 0$ since X is of general type. Therefore

$$\begin{aligned} h^0(D_1, \mathscr{E}nd^0(E) \otimes K_X) &= \chi(D_1, \mathscr{E}nd^0(E) \otimes K_X) + h^0(D_1, \mathscr{E}nd^0(E) \otimes K_X(-D_2)) \\ &= \tfrac{3}{2} D_1 \cdot D_2 + 1 > k + 1 . \end{aligned}$$

In particular, there is at least one $\xi \in H^0(D_1, \mathscr{E}nd^0(E) \otimes K_X)$ that cannot be expressed as linear combination of $g_1, \ldots g_k$. Let $S \subseteq D_1$ be the component such that $\xi_{|S}$ is non-trivial. Then S is the component satisfying (5.13).

Now let $x_1, x_2 \in S$ be two general closed points and let $\varphi_i: E \to \mathbb{C}_{x_i}$ be general homomorphism. Let $F = \ker\{E \xrightarrow{\varphi_1} \mathbb{C}_{x_1}\}$. Then by Lemma 5.4, (5.11) and the fact that $\dim \operatorname{Im}\{(R_{E,S}\} = 1$,

$$h^0(C, \mathscr{E}nd^0(F) \otimes K_X) = h^0(C, \mathscr{E}nd^0(E) \otimes K_X) - 1 .$$

Clearly, then we must have $h^0(D_1, \mathscr{E}nd^0(F) \otimes K_X(-D_2)) = 0$ because $\operatorname{Im}\{R_{F,S}\} = \{0\}$. Next, by R.R. and the Serre duality, $h^0(D_1, \mathscr{E}nd^0(E) \otimes K_X) - h^0(D_1, \mathscr{E}nd^0(F) \otimes K_X) = 1$. Finally, because the kernels of

$$R'_{E,S}: H^0(D_1, \mathscr{E}nd^0(E) \otimes K_X) \to H^0(S, \mathscr{E}nd^0(E) \otimes K_X)$$

$$R'_{F,S}: H^0(D_1, \mathscr{E}nd^0(F) \otimes K_X) \to H^0(S, \mathscr{E}nd^0(F) \otimes K_X)$$

are isomorphic,

$$\dim \operatorname{Im}\{R'_{F,S}\} = \dim \operatorname{Im}\{R'_{E,S}\} - 1 \geqq 1 .$$

Therefore, we can apply Lemma 5.4 to the sheaf $E' = \ker\{F \xrightarrow{\varphi_2} \mathbb{C}_{x_2}\}$ to obtain

$$h^0(D_1, \mathscr{E}nd^0(E') \otimes K_X(-D_2)) \leqq h^0(D_1, \mathscr{E}nd^0(F) \otimes K_X(-D_2)) = 0 .$$

Therefore,

$$h^0(C, \mathscr{E}nd^0(E') \otimes K_X) < h^0(C, \mathscr{E}nd^0(E) \otimes K_X) .$$

Combined with Lemma 5.5, this contradicts to our assumption on E and thus completes the proof of Proposition 5.1. □

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