Invent. math. 116, 347–392 (1994)

Inventiones mathematicae

Invariant measures for actions of unipotent groups over local fields on homogeneous spaces

G.A. Margulis* **and G.M. Tomanov**

Department of Mathematics, Yale University, P.O. Box 2155 Yale Station, New Haven, CT06520, USA

Oblatum 6-IV-1993

Dedicated to Armand Borel

Study of dynamics of actions of unipotent subgroups on homogeneous spaces has been attracting considerable attention for the last 30 years. One of the main reasons for this was that some problems in number theory and, in particular, in Diophantine approximations can be reformulated in terms of such actions. M.S. Raghunathan made a remarkable observation that a long-standing conjecture due to A. Oppenheim on values of quadratic forms at integral points can be deduced from some results about actions of unipotent subgroups. More precisely, he formulated a conjecture that a closure of an orbit of a unipotent subgroup in the quotient of a Lie group G by a lattice $\Gamma \subset G$ is an orbit of a bigger subgroup and noted the connection of his conjecture with Oppenheim's conjecture.

Oppenheim's conjecture was proved in [Mar2] and [Mar3] (see also [D-Mar3] and [Mar4]) where it was deduced from a theorem about orbits of $SO(2,1)$ in $SL_3(\mathbf{R})/SL_3(\mathbf{Z})$. In later papers [D-Mar2] and [D-Mar3], various strengthenings of these results were obtained. In [D-Mar3], Raghunathan's conjecture was also proved for actions of generic unipotent subgroups on the quotients of $G = SL(3, \mathbf{R})$ by a lattice $\Gamma \subset SL(3, \mathbf{R})$. Borel and Prasad proved in [Bo-Pra] a generalization of Oppenheim's conjecture in a S-arithmetic setting. The reader is referred to [Mar6] for a general survey of the area.

Major progress in the area was made in the last years by Ratner who, in a series of papers [R2–5], proved Raghunathan's conjecture for a general real Lie group G, obtained a classification of all finite invariant measures for actions of unipotent groups U on G/Γ, and proved uniform distribution for actions of one-parameter unipotent groups. The classification of the finite U-invariant measures (measure rigidity) was obtained in [R2–4] and the other results were deduced from the measure rigidity in [R5].

The main purpose of our paper is to give a proof of measure rigidity valid for a product of algebraic groups over local fields of characteristic zero. The impetus for our paper is the path breaking result of M. Ratner for the case of real Lie groups. Our proof is similar in principle to Ratner's, but it

* On leave from the Institute of Information Transmission of Russian Academy of Science

is different in many aspects. In particular, we extensively use algebraic group theory, as well as some facts about entropy of transformations of homogeneous spaces.

We use ideas and techniques from [R2–4] and also from [Bo-Pra, D-Mar1–4, Mar2–4, 6]. Let us note that some of the ideas can be tracked back to [Mar1, D1–5, R1, W].

Although there are many similarities between our proof and M. Ratner's, (in particular, use of dynamical properties of actions of unipotent groups in combination with ergodic theorems for actions of nilpotent groups), we think that it would be superficial and misleading to give any specific references to [R2–4] because of the substantial differences in approach and methods. We would like to add that we were strongly influenced by arguments from [R3] showing how to obtain and to use the information about the local structure of the set of uniform convergence in the proof of measure rigidity. Inspired by these arguments of Ratner, we finally came to our Proposition 8.3. Subsequently we were able to replace in our proof analogs of some of Ratner's decisive but intricate arguments by more transparent arguments using entropy (it seems that a similar replacement can not be done in Ratner's proof itself). On the other hand, some of the most important ingredients in our approach such as: the idea of enlarging the group preserving an ergodic invariant measure (a minimal invariant subset in [Bo-Pra, D-Mar1–4, Mar2–4, 6]) by using rational maps into the normalizers of unipotent subgroups, and the use of properties of multidimensional unipotent actions derived from basic properties of polynomials and Chevalley's theorem, are motivated largely by [Bo-Pra, D-Mar1–4, Mar2–4, 6].

We now introduce some notation and give the formal statements of the main results. Let $\mathscr{T}$ be a finite set and, for every $v \in \mathscr{T}$, let K_v be a local (i.e. nondiscrete locally compact) field of characteristic 0 and $\mathbf{G}_v$ an algebraic group defined over K_v. Denote by G the direct product $\prod_{v\in\mathscr{T}} \mathbf{G}_v(K_v)$ of locally compact groups. Let Γ be a discrete subgroup of G and let μ be a Borel probability measure on G/Γ. The group G acts by left multiplication on G/Γ. Denote by Σ the (closed) subgroup of all elements of G preserving μ. The measure μ is called *algebraic* if there exists a point $x \in G/\Gamma$ such that the orbit Σx is closed in G/Γ and $\mu(\Sigma x) = 1$. For every $v \in \mathscr{T}$, let $\mathbf{U}_v$ be a unipotent K_v-subgroup of $\mathbf{G}_v$. Let us denote the subgroup $\prod_{v\in\mathscr{T}} \mathbf{U}_v(K_v)$ by $\mathscr{U}$. The main result of this paper is the following.

Theorem 1 *If the measure μ is $\mathscr{U}$-invariant and $\mathscr{U}$-ergodic, then μ is algebraic.*

Let $H = \prod_{v\in\mathscr{T}} H_v$ be a subgroup of G such that every H_v is generated by groups of K_v-rational points of unipotent K_v-subgroups of $\mathbf{G}_v$. It is known that if μ is H-unvariant and H-ergodic, then μ is V-ergodic for any maximal unipotent subgroup V of H. (If G is a real group, this result immediately follows from the results in [Mo]. When G contains nonarchimedean factors, the proof is the same as in the real case.) Thus we obtain the following strengthening of Theorem 1.

Theorem 2 *With the above notation if μ is H-invariant and H-ergodic, then μ is algebraic.*

Theorems 1 and 2 are analogous to Ratner's measure rigidity theorems for real Lie groups (see [R4]). Note that in [R4], the measure rigidity for groups generated by their unipotent subgroups is deduced from the measure rigidity for unipotent subgroups in a different way without the use of Mautner's phenomenon (see [Mo]).

The paper is organized as follows: The Sects. 1–4 have auxiliary character. After fixing in Sect. 1 the appropriate terminology and recollecting some known facts from the theory of linear algebraic groups, we define in Sect. 2 the notion of *elements of class* $\mathscr{A}$ and establish some facts related to the horospherical subgroups and the existence of K-rational cross-sections in K-algebraic groups. Section 3 contains the proofs of assertions we need from ergodic theory. Some of our arguments are analogous to the arguments used in the proof of the Borel-Wang density theorem. In Sect. 4, we prove a technical result about the structure of algebraic groups over local fields. In Sect. 5, we introduce a special kind of rational maps (called quasiregular maps) from a unipotent subgroup U of G to its normalizer $\mathscr{N}_G(U)$. In Sect. 6, we investigate the properties of the quasiregular maps and in Sect. 7 we show that under certain conditions there exists a quasiregular map φ such that the elements from $\mathrm{Im}(\varphi)$ preserve a given probability measure μ on G/Γ. (In fact, we prove this result under the weaker assumption that Γ is *any* closed in the Hausdorff topology subgroup of G.) Using the results from Sects. 6–7, we prove in Sect. 8 that there exist elements from the class $\mathscr{A}$ in G preserving μ and also having many other "nice" properties. In Sect. 9, we prove some results about entropy of measure-preserving transformations of G/Γ. The central is Theorem 9.7 which represents interest of its own. The proof of Theorem 9.7 is modeled over the proofs of some results in the paper of Ledrapier and Young [Led-Y]. In Sect. 10 we complete the proof of Theorem 1. Finally, in Sect. 11, we formulate some theorems about closures of orbits of unipotent subgroups, uniform distribution and values of families of quadratic forms. We shortly explain how the proofs in the real case can be adopted to our more general setting.

Theorems 1 and 2 of this paper were announced in [Mar-To] together with a detailed sketch of the proofs. Almost simultaneously with the appearance of [Mar-To], the authors learned about [R7] where Ratner announced the generalization of her results from [R4] and [R5] for the S-arithmetic case. In particular, she announced Theorem 1 and 2 above as well as Theorems 11.1 and 11.2 from Sect. 11 in a slightly more general setting (more precisely, for a class of central extensions of linear groups).

1 Preliminaries

1.1 Notation and Terminology. Let $\mathscr{T}$ be a finite set. For $v\in\mathscr{T}$, let K_v be a local (i.e. nondiscrete locally compact) field of characteristic 0 with the normalized absolute value $|\ \ |_v$. Denote by $K_{\mathscr{T}}$ the direct sum of all K_v, $v\in\mathscr{T}$. By an *extension* $K'_{\mathscr{T}}$ of $K_{\mathscr{T}}$ we mean a direct product of field extensions K'_v of K_v, $v\in\mathscr{T}$. Define a function $|\ \ |\colon K_{\mathscr{T}}\to\mathbf{R}^+\cup\{0\}$ as follows: if $x\in K_{\mathscr{T}}$ then $|x| = \prod_{v\in\mathscr{T}}|x_v|_v$ where x_v denotes the v-component of x. If K'_v is an algebraic extension of K_v then the unique extension of $|\ \ |_v$ (resp. of $|\ \ |$) to an absolute value on K'_v (resp. on $K'_{\mathscr{T}}$) will also be denoted by $|\ \ |_v$ (resp. by $|\ \ |$).

By a *$K_{\mathscr{T}}$-algebraic group* $\mathbf{H}$ (resp. a *$K_{\mathscr{T}}$-algebraic variety* $\mathbf{M}$) we mean a (formal) direct product $\prod_{v\in\mathscr{T}}\mathbf{H}_v$ of K_v-algebraic groups $\mathbf{H}_v$ (resp. a direct product

$\prod_{v \in \mathscr{T}} \mathbf{M}_v$ of K_v-algebraic varieties $\mathbf{M}_v$). A map $f\colon \mathbf{M} \to \mathbf{M}'$, where $\mathbf{M}$ and $\mathbf{M}'$ are $K_{\mathscr{T}}$-algebraic varieties, is called *$K_{\mathscr{T}}$-rational* (resp. *$K_{\mathscr{T}}$-regular*) if f is a product of K_v-rational (resp. K_v-regular) maps $f_v\colon \mathbf{M}_v \to \mathbf{M}'_v$, $v \in \mathscr{T}$. Analogously we define other similar notions such as *$K_{\mathscr{T}}$-rational representation*, *$K_{\mathscr{T}}$-rational character* of a $K_{\mathscr{T}}$-algebraic group, *$K_{\mathscr{T}}$-algebraic subgroup* etc. By $\dim \mathbf{M}$ we mean the dimension of $\mathbf{M}$ that is the sum of the dimensions of $\mathbf{M}_v \in \mathscr{T}$.

As usual $\mathbf{V}(k)$ denotes the set of k-rational points of a k-variety $\mathbf{V}$. If $K'_{\mathscr{T}}$ is an extension of $K_{\mathscr{T}}$ and $\mathbf{M} = \prod_{v \in \mathscr{T}} \mathbf{M}_v$ is a $K_{\mathscr{T}}$-algebraic variety we denote the product $\prod_{v \in \mathscr{T}} \mathbf{M}_v(K'_v)$ by $\mathbf{M}(K'_{\mathscr{T}})$. We will call $\mathbf{M}(K'_{\mathscr{T}})$ the *set of $K'_{\mathscr{T}}$-rational points of* $\mathbf{M}$ or shortly the *set of $K'_{\mathscr{T}}$-points of* $\mathbf{M}$. In case of groups, $\mathbf{H}(K'_{\mathscr{T}})$ will be called the *group of $K'_{\mathscr{T}}$-rational points* or the *group of $K'_{\mathscr{T}}$-points* of a *$K_{\mathscr{T}}$ algebraic group* $\mathbf{H}$.

If $\mathbf{M}_v$ are linear spaces defined over K_v then $\mathbf{M}$ will be called a *linear $K_{\mathscr{T}}$-space*. In this case $\mathbf{M}(K_{\mathscr{T}})$ is a finitely generated $K_{\mathscr{T}}$-module. By the Grassmannian variety $\mathrm{Gr}(\mathbf{M})$ (resp. $\mathrm{Gr}(\mathbf{M}(K_{\mathscr{T}}))$) we mean the direct product of Grassmannian varieties $\mathrm{Gr}(\mathbf{M}_v)$, $v \in \mathscr{T}$ (resp. $\mathrm{Gr}(\mathbf{M}_v(K_v))$, $v \in \mathscr{T}$). There is a natural structure of a projective $K_{\mathscr{T}}$-variety on $\mathrm{Gr}(\mathbf{M})$, and $\mathrm{Gr}(\mathbf{M})(K_{\mathscr{T}})$ is naturally identified with $\mathrm{Gr}(\mathbf{M}(K_{\mathscr{T}}))$.

1.2 If $\mathbf{H} = \prod_{v \in \mathscr{T}} \mathbf{H}_v$ is a $K_{\mathscr{T}}$-algebraic group we denote by $\mathrm{Lie}(\mathbf{H})$ the direct product $\prod_{v \in \mathscr{T}} \mathrm{Lie}(\mathbf{H}_v)$ of the Lie algebras $\mathrm{Lie}(\mathbf{H}_v)$ of $\mathbf{H}_v$. Every Lie algebra $\mathrm{Lie}(\mathbf{H}_v)$ has a K_v-structure. By $\mathrm{Lie}(\mathbf{H}_v(K_v))$ we will denote the Lie algebra of K_v-rational points of $\mathrm{Lie}(\mathbf{H}_v)$. Note that $\mathrm{Lie}(\mathbf{H}_v(K_v))$ is naturally identified with the Lie algebra of the group $\mathbf{H}_v(K_v)$ considered as a Lie group over K_v. We set

$$\mathrm{Lie}(\mathbf{H}(K_{\mathscr{T}})) = \prod_{v \in \mathscr{T}} \mathrm{Lie}(\mathbf{H}_v(K)_v)).$$

We will call $\mathrm{Lie}(\mathbf{H})$ (resp. $\mathrm{Lie}(\mathbf{H}(K_{\mathscr{T}}))$) the *Lie algebra of* $\mathbf{H}$ (resp. $\mathbf{H}(K_{\mathscr{T}})$). One can naturally define the *adjoint representation* Ad of $\mathbf{H}$ (resp. $\mathbf{H}(K_{\mathscr{T}})$) on $\mathrm{Lie}(\mathbf{H})$ (resp. $\mathrm{Lie}(\mathbf{H}(K_{\mathscr{T}}))$).

Let $\mathbf{H}^{(u)}$ (resp. $\mathrm{Lie}(\mathbf{H})^{(n)}$) denote the set of unipotent (resp. nilpotent) elements in $\mathbf{H}$ (resp. in $\mathrm{Lie}(\mathbf{H})$, i.e. $\mathbf{H}^{(u)}$ (resp. $\mathrm{Lie}(\mathbf{H})^{(n)}$) is the direct product of $\mathbf{H}_v^{(u)}$ (resp. $\mathrm{Lie}(\mathbf{H}_v)^{(n)}$), $v \in \mathscr{T}$. Denote by exp: $\mathrm{Lie}(\mathbf{H})^{(n)} \to \mathbf{H}^{(u)}$ (resp. ln: $\mathbf{H}^{(u)} \to \mathrm{Lie}(\mathbf{H})^{(n)}$) the product of exponential maps $\exp_v$: $\mathrm{Lie}(\mathbf{H}_v)^{(n)}$ (resp. the product of logarithmic maps $\ln_v$: $\mathbf{H}_v^{(u)} \to \mathrm{Lie}(\mathbf{H}_v)^{(n)}$), $v \in \mathscr{T}$. Since $\mathbf{H}_v^{(n)}$ (resp. $\mathrm{Lie}(\mathbf{H}_v)^{(u)}$) is a K_v-subvariety in $\mathbf{H}_v$ (resp. in $\mathrm{Lie}(\mathbf{H}_v)$) we have that $\mathbf{H}^{(n)}$ (resp. $\mathrm{Lie}(\mathbf{H})^{(u)}$) is a $K_{\mathscr{T}}$-algebraic subvariety in $\mathbf{H}$(resp. in $\mathrm{Lie}(\mathbf{H})$). Since the maps $\exp_v$ and $\ln_v$ are K_v-regular isomorphisms and $\ln_v = \exp_v^{-1}$, we have that exp and ln are $K_{\mathscr{T}}$-regular isomorphisms and $\ln = \exp^{-1}$. We also have that the maps exp and ln are $\mathbf{H}$-equivariant, i.e. $\exp(\mathrm{Ad}(h)\, y) = h \exp(y)\, h^{-1}$ and $\ln(h x h^{-1}) = \mathrm{Ad}(h) \ln(x)$ for all $h \in \mathbf{H}$, $y \in \mathrm{Lie}(\mathbf{H})^{(n)}$ and $x \in \mathbf{H}^{(u)}$.

1.3 By *Zariski topology* on a $K_{\mathscr{T}}$-algebraic variety $\mathbf{M} = \prod_{v \in \mathscr{T}} \mathbf{M}_v$ we mean the product of the Zariski topologies on $\mathbf{M}_v$, $v \in \mathscr{T}$. The variety $\mathbf{M}$ will be called *connected* if $\mathbf{M}_v$ is connected in the Zariski topology for every $v \in \mathscr{T}$. We say that a subset $X \subset \mathbf{M}$ is *Zariski dense* (resp. *Zariski open, Zariski closed* etc.) if X is dense (resp. open, closed etc.) in the Zariski topology. We will denote by $\bar{X}$ the Zariski closure in $\mathbf{M}$ of a subset $X \subset \mathbf{M}$. Let $X \subset \mathbf{M}(K_{\mathscr{T}})$ and $f\colon \bar{X} \to \mathbf{N}$ be a $K_{\mathscr{T}}$-rational (resp. $K_{\mathscr{T}}$-regular) map to a $K_{\mathscr{T}}$-algebraic variety $\mathbf{N}$. Then the restriction $f|X$ will be also called $K_{\mathscr{T}}$-rational (resp. $K_{\mathscr{T}}$-regular) map.

The topologies on the local fields K_v, $v \in \mathscr{T}$, induce a locally compact Hausdorff topology on $\mathbf{M}(K_v)$. We will refer to this topology as *Hausdorff topology* on $\mathbf{M}(K_v)$. A topology induced on $\mathbf{M}(K_v)$ by the Zariski topology on $\mathbf{M}$ will be called *Zariski topology* on $\mathbf{M}(K_v)$. It is easy to see that the Zariski topology on $\mathbf{M}(K_v)$ is weaker than the Hausdorff topology.

By a *$K_{\mathscr{T}}$-algebraic subvariety* of $\mathbf{M}(K_{\mathscr{T}})$ we mean the Zariski closed subset of $\mathbf{M}(K_{\mathscr{T}})$ or, equivalently, the set of $K_{\mathscr{T}}$-points of a $K_{\mathscr{T}}$-algebraic subvariety of $\mathbf{M}$. Analogously we define the notion of a *$K_{\mathscr{T}}$-algebraic subgroup* of $\mathbf{H}(K_{\mathscr{T}})$ where $\mathbf{H}$ is a $K_{\mathscr{T}}$-algebraic group.

1.4 If k is a local field, ℓ is a finite separable extension of k and $\mathbf{F}$ is a ℓ-group then there is a natural topological isomorphism between group $\mathbf{F}(\ell)$ and $(R_{\ell/k}\mathbf{F})(k)$ where $R_{\ell/k}$ denotes the restriction of scalars functor. Under this isomorphism unipotent elements go to unipotent elements. On the other hand, any local field of characteristic 0 is $\mathbf{R}$, $\mathbf{C}$ or a finite extension of $\mathbf{Q}_p$. Therefore for our purpose (study of actions of unipotent groups or groups generated by unipotent elements) we can assume when it is necessary that $\mathscr{T}$ is a finite set of normalized valuations of the field $\mathbf{Q}$ of rational numbers. Then $K_v, v \in \mathscr{T}$, is either $\mathbf{R}$ or $\mathbf{Q}_p$ and for different v and v' local fields K_v and $K_{v'}$ are not isomorphic.

1.5 If A is a locally compact group, $B \subset A$ is a closed subgroup, and $x \in A$ normalizes B then by $\alpha(x, B)$ we denote the module of the restriction of $\mathrm{Int}(x)$ to B. Thus $\theta(xYx^{-1}) = \alpha(x, B)\,\theta(Y)$ where $Y \subset B$ and θ is a Haar measure on B.

Let $\mathbf{H}$ be a $K_{\mathscr{T}}$-algebraic group, let $\mathbf{L}$ be a $K_{\mathscr{T}}$-algebraic subgroup, and let $x \in \mathbf{H}(K_{\mathscr{T}})$ normalize $\mathbf{L}$. Then $\alpha(x, \mathbf{L}(K_{\mathscr{T}}))$ is equal to the product of the numbers $\alpha(x_v, \mathbf{L}_v(K_v))$, $v \in \mathscr{T}$. For every $v \in \mathscr{T}$, let us denote by $\mathrm{Ad}_{\mathbf{L}}(x_v)$ the restriction of $\mathrm{Ad}(x_v)$ to $\mathrm{Lie}(\mathbf{L}_v)$. Then from the standard description of Haar measures on real and p-adic Lie groups we get that $\alpha(x_v, \mathbf{L}_v(K_v)) = |\det \mathrm{Ad}_{\mathbf{L}}(x_v)|_v$.

Let us formulate some well known results about algebraic groups in terms of $K_{\mathscr{T}}$-algebraic groups.

1.6 Proposition (see [Bo-Ti]) *Suppose that a $K_{\mathscr{T}}$-algebraic group $\mathbf{H}$ acts $K_{\mathscr{T}}$-rationally on a $K_{\mathscr{T}}$-variety $\mathbf{M}$ and x is a point in $\mathbf{M}(K_{\mathscr{T}})$. Then*

(a) *the subset $\mathbf{H}(K_{\mathscr{T}})x$ is closed and open in $(\mathbf{H}x)(K_{\mathscr{T}})$ and hence is locally closed in $\mathbf{M}(K_{\mathscr{T}})$;*

(b) *the natural map $\mathbf{H}(K_{\mathscr{T}})/\mathbf{H}(K_{\mathscr{T}})_x \to \mathbf{H}(K_{\mathscr{T}})x$ is a homeomorphism, where $\mathbf{H}(K_{\mathscr{T}})_x = \{h \in \mathbf{H}(K_{\mathscr{T}}) \mid hx = x\}$.*

1.7 Proposition (see [Bo-Ti]). *Let $f\colon \mathbf{F} \to \mathbf{H}$ be a $K_{\mathscr{T}}$-morphism of $K_{\mathscr{T}}$-*algebraic groups.

(a) *The natural homomorphism $\mathbf{F}(K_{\mathscr{T}})/(\mathrm{Ker} f)(K_{\mathscr{T}}) \to \mathbf{H}(K_{\mathscr{T}})$ is a proper map.*

(b) *If* Ker$f$ *is finite then $f\colon \mathbf{F}(K_{\mathscr{T}}) \to \mathbf{H}(K_{\mathscr{T}})$ is a proper map.*

(c) *If f is an epimorphism then $f\colon \mathbf{F}(K_{\mathscr{T}}) \to \mathbf{H}(K_{\mathscr{T}})$ is an open map.*

1.8 Proposition (see [Bo, 15.7]). *Let $\mathbf{H}$ be a $K_{\mathscr{T}}$-algebraic group and let $\mathbf{F}$ be a solvable $K_{\mathscr{T}}$-split $K_{\mathscr{T}}$-algebraic subgroup of $\mathbf{H}$ (i.e. $\mathbf{F} = \prod_{v \in \mathscr{T}} \mathbf{F}_v$ where $\mathbf{F}_v$ is a* *solvable K_v-algebraic group split over K_v for every $v \in \mathcal{T}$). Let $f\colon \mathbf{H} \to \mathbf{H}/\mathbf{F}$ be the natural $K_{\mathcal{T}}$-morphism from $\mathbf{H}$ to a $K_{\mathcal{T}}$-variety $\mathbf{H}/\mathbf{F}$. Then $f(\mathbf{H}(K_{\mathcal{T}})) = (\mathbf{H}/\mathbf{F})(K_{\mathcal{T}})$.*

1.9 Let $\mathbf{F}$ be a $K_{\mathcal{T}}$-algebraic subgroup of $\mathbf{H}$. We say that a $K_{\mathcal{T}}$-subvariety $\mathbf{L}$ of $\mathbf{H}$ is a *rational cross-section for* $\mathbf{H}/\mathbf{F}$ if $e \in \mathbf{L}$ and the "multiplication map" $\mathbf{L} \times \mathbf{F} \to \mathbf{H}$, $(x, y) \mapsto xy$, is a $K_{\mathcal{T}}$-isomorphism of $\mathbf{L} \times \mathbf{F}$ onto a Zariski open dense subset $\mathbf{A}$ of $\mathbf{H}$. If $\mathbf{A} = \mathbf{H}$ then we say that $\mathbf{L}$ is a *regular cross-section for* $\mathbf{H}/\mathbf{F}$. The set $\mathbf{L}(K_{\mathcal{T}})$ will be called *rational* (resp. *regular*) *cross-section* for $\mathbf{H}(K_{\mathcal{T}})/\mathbf{F}(K_{\mathcal{T}})$.

The following lemma easily follows from the fact that if $\operatorname{char} K = 0$ then any bijective K-morphism of normal K-varieties is a K-isomorphism.

Lemma. *Let a $K_{\mathcal{T}}$-algebraic group $\mathbf{H}$ act $K_{\mathcal{T}}$-rationally on a $K_{\mathcal{T}}$-algebraic variety $\mathbf{M}$. Let $x \in \mathbf{M}(K_{\mathcal{T}})$ and $\mathbf{F} = \{h \in \mathbf{H} \mid hx = x\}$. Assume that $\mathbf{H}x$ is Zariski dense in $\mathbf{M}$. Then for any rational cross-section $\mathbf{L}$ for $\mathbf{H}/\mathbf{F}$, the orbit map $\ell \mapsto \ell x$, $\ell \in \mathbf{L}$, is a $K_{\mathcal{T}}$-biregular isomorphism of $\mathbf{L}$ onto a Zariski open dense subset $\mathbf{L}x$ of $\mathbf{M}$.*

1.10 Let F be a locally compact group and let φ be a continuous automorphism of F. Recall that the automorphism φ is said to be *contracting* if for every compact set $L \subset F$ and for every neighborhood U of the identity, there exists a positive integer $m = m(L, U)$ such that $\varphi^n(L) \subset U$ for all $n > m$.

1.11 If X is a compact metric space and $\mathscr{C}(X)$ is the space of closed non-empty of X subsets then there is a standard Hausdorff metric on $\mathscr{C}(X)$ given by $d(A, B) = \sup_{x \in A,\, y \in B} \{d(x, B),\, d(y, A)\}$. If Y is a locally compact σ-compact metric space then by *Hausdorff topology* on $\mathscr{C}(Y)$ we mean the topology induced on $\mathscr{C}(Y)$ by a Hausdorff metric on $\mathscr{C}(Y')$, where Y' is the one-point compactification of Y.

1.12 The following lemma is a standard fact about differential maps of analytic varieties over local fields and it easily follows from the implicit function theorem.

Lemma. *Let K be a local field, m and r positive integers, Y a neighborhood of 0 in $K^{m+r} = K^m \times K^r$, and $\beta\colon Y \to K^r$ a differentiable map such that $\beta(0) = 0$. For every $x \in K^m$ define $\beta_x\colon Y_x \to K^r$ by $\beta_x(y) = \beta(x, y)$ where $Y_x = \{y \in K^r \mid (x, y) \in Y\}$. Assume that the differential of β_0 at 0 is a surjective map from K^r onto K^r. Then there exists an open neighborhood $\mathcal{O}_1$ of 0 in K^m and open neighborhoods $\mathcal{O}_2$ and $\mathcal{O}_2'$ of 0 in K^r such that for every $x \in \mathcal{O}_1$ the set $\beta_x(\mathcal{O}_2)$ is open in K^r, $\beta_x(\mathcal{O}_2) \supset \mathcal{O}_2'$ and β_x maps $\mathcal{O}_2$ diffeomorphically onto $\beta_2(\mathcal{O}_2)$.*

2 Class $\mathscr{A}$ and horospherical subgroups

2.1 Lemma. *Let K be a local field with an absolute value $|\ |$, let $\mathbf{F}$ be a K-group, and let $g \in \mathbf{F}(K)$ be an element diagonalizable over K. Denote by $\mathbf{T}$ the Zariski closure in $\mathbf{F}$ of the group $\langle g \rangle$ generated by g. Then the following conditions are equivalent:*

(a) *there exists $\pi \in K$ such that $|\pi| > 1$ and all eigenvalues of g are integer powers of π;*

(b) $\mathbf{T}$ *is a* 1*-dimensional* K*-split torus and* $|\chi(g)| \neq 1$ *for any (defined over* K*) nontrivial character* χ *of* $\mathbf{T}$;
(c) $\mathbf{T}$ *is a* 1*-dimensional* K*-split torus and the factor group* $\mathbf{T}(K)/\langle g \rangle$ *is compact.*

The proof of the above lemma easily follows from standard results about algebraic tori (see [Bo, Chap. III]) and from the fact that the quotient of the multiplicative group K^* of K by a cyclic group $\langle \alpha \rangle$ generated by $\alpha \in K^*$ is compact if and only if $|\alpha| \neq 1$.

Definition. Let $g \in \mathbf{H}(K_{\mathscr{T}})$ where $\mathbf{H}$ is a $K_{\mathscr{T}}$-algebraic group. We say that g is an *element from the class* $\mathscr{A}$ if, for every $v \in \mathscr{T}$, the v-component $g_v \in \mathbf{H}_v(K_v)$ of g satisfies one of the conditions (a)–(c) of the above lemma.

2.2 Proposition. *Let a* $K_{\mathscr{T}}$*-algebraic group* $\mathbf{H}$ *act* $K_{\mathscr{T}}$*-rationally on a projective* $K_{\mathscr{T}}$*-algebraic variety* $\mathbf{P}$. *Let* s *be an element from the class* $\mathscr{A}$ *in* $\mathbf{H}(K_{\mathscr{T}})$ *and* $x \in \mathbf{P}(K_{\mathscr{T}})$. *Then a sequence* $\{s^n x\}$ *converges to a point* $y \in \mathbf{P}(K_{\mathscr{T}})$ *in the Hausdorff topology when* $n \to +\infty$.

This proposition easily follows from the fact that every morphism $\alpha\colon \mathbf{V} \to \mathbf{W}$ of an algebraic curve into a projective algebraic variety $\mathbf{W}$ can be extended to a morphism $\bar{\alpha}\colon \mathbf{V}' \to \mathbf{W}$ where $\mathbf{V}'$ is a completion of $\mathbf{V}$. (It is enough to apply this fact to orbit maps $t \mapsto t x_v$, $t \in \mathbf{T}_v$, where $v \in \mathscr{T}$, x_v is the v-component of x and $\mathbf{T}_v$ is the Zariski closure of the group $\langle s_v \rangle$ generated by the v-component s_v of s.)

2.3 Lemma. *Let* F *be a locally compact group,* $D \subset F$ *a close subgroup and* $U \subset F$ *an open subgroup. Assume that* F/D *is compact. Then* $U/U \cap D$ *is compact.*

To prove this well known lemma, it is enough to identify $U/U \cap D$ with the U-orbit of $eD \in F/D$ and notice that, since U is open, all U-orbits in F/D are open and consequently all U-orbits are closed.

2.4 Proposition. *Let* S *be an open subgrop of the group of* $K_{\mathscr{T}}$*-points of a* $K_{\mathscr{T}}$*-algebraic torus. Then there exists a discrete cocompact subgroup* $S_0 \subset S$ *consisting of elements from the class* $\mathscr{A}$.

Proof. In view of Lemma 2.3, we can assume that $S = \mathbf{S}(K_{\mathscr{T}})$ where $\mathbf{S}$ is a $K_{\mathscr{T}}$-algebraic torus. It is enough to consider the case where $\mathscr{T}$ consists of one element v. Let $\mathbf{S}_d$ denote the maximal K_v-split subtorus of the K_v-torus $\mathbf{S}$. Since $\mathbf{S}(K_v)/\mathbf{S}_d(K_v)$ is compact (see 1.7(a) and [Pra]) we can assume that $\mathbf{S}$ is split over K_v. Fix an element $\pi \in K_v$ such that $|\pi|_v > 1$. Put $S_0 = \{x \in \mathbf{S}(K_v) \mid \chi(x)$ is an integer power of π for every K_v-rational character χ of $\mathbf{S}\}$. Since $\mathbf{S}$ is a direct product of 1-dimensional K_v-split tori, we easily get that S_0 is a discrete cocompact subgroup of $\mathbf{S}(K_v)$ and each $x \in S_0$ is an element from the class $\mathscr{A}$.

2.5 Let $\mathbf{H}$ be a $K_{\mathscr{T}}$-algebraic subgroup of a $K_{\mathscr{T}}$-algebraic group $\mathbf{L}$. Set $H = \mathbf{H}(K_{\mathscr{T}})$ and $L = \mathbf{L}(K_{\mathscr{T}})$. For every $g \in L$ normalizing L we set

$$W_H^+(g) = \{x \in H \mid g^n x g^{-n} \to e \text{ when } n \to -\infty\},$$
$$W_H^-(g) = \{x \in H \mid g^n x g^{-n} \to e \text{ when } n \to +\infty\},$$
$$Z_H(g) = \{x \in H \mid g x g^{-1} = x\}.$$

Let us call $W_H^+(g)$ and $W_H^-(g)$ *horospherical subgroups of H corresponding to g*. When this does not lead to confusion we will write $W^+(g)$, $W^-(g)$ and $Z(g)$ instead of $W_H^+(g)$, $W_H^-(g)$ and $Z_H(g)$, respectively.

Proposition. (a) *$W^+(g)$ and $W^-(g)$ are groups of $K_{\mathcal{T}}$-points of unipotent $K_{\mathcal{T}}$-algebraic subgroups $\mathbf{W}^+(g)$ and $\mathbf{W}^-(g)$ of $\mathbf{H}$.*
(b) *The Lie algebra of $\mathbf{W}^+(g)$ (resp. of $\mathbf{W}^-(g)$) coincides with the linear span of the set of eigenvectors x of the transformation* $\mathrm{Ad}(g)$ *with eigenvalues $\lambda(x)$ such that $|\lambda(x)|>1$ (resp. $|\lambda(x)|<1$).*
(c) *The subgroup $Z(g)$ normalizes $W^+(g)$ and $W^-(g)$. Automorphisms* $\mathrm{Int}(g^{-1})|W^+(g)$ *and* $\mathrm{Int}(g)|W^-(g)$ *are contracting.*

Proof. The fact that $W^+(g)$ and $W^-(g)$ are unipotent subgroups normalized by $Z(g)$ easily follows from the definition of the sets $W^+(g)$ and $W^-(g)$. Since the map $\ln\colon \mathbf{H}^{(u)} \to \mathrm{Lie}(\mathbf{H})^{(n)}$ is $\mathbf{H}$-equivariant and $K_{\mathcal{T}}$-biregular (see 1.2), we have that a unipotent element u of $\mathbf{H}(K_{\mathcal{T}})$ belongs to $W^+(g)$ (resp. $W^-(g)$) if and only if $\mathrm{Ad}(g^n)\ln(u)$ converges to 0 when $n\to-\infty$ (resp. $n\to+\infty$). It remains now to notice that if A is a diagonalizable over K linear transformation of a finite-dimensional vector space V over a local field K with an absolute value $|\ \ |$ then the set $\{v\in V | A^n v\to 0$ when $n\to+\infty\}$ coincides with the linear span of the set of eigenvectors x of the transformation A with eigenvalues $\lambda(x)$ such that $|\lambda(x)|<1$.

2.6 Lemma. *Let F be a group and let D and L be subgroups of F. Assume that $D\cap L=\{e\}$. Then the multiplication map*

$$m\colon D\times L\to F,\ m(d,\ell)=d\ell,$$

is injective.

Proof. If $d_1,d_2\in D$, $\ell_1,\ell_2\in L$ and $d_1\ell_1=d_2\ell_2$ then $d_2^{-1}d_1=\ell_2\ell_1^{-1}\in D\cap L=\{e\}$. Hence $d_1=d_2$ and $\ell_1=\ell_2$.

2.7 Proposition. *Let $\mathbf{H}$ be a connected $K_{\mathcal{T}}$-algebraic group and $s\in\mathbf{H}(K_{\mathcal{T}})$ an element from the class $\mathcal{A}$. Then*

(a) *the multiplication map*

$$m\colon W^-(s)\times Z(s)\times W^+(s)\to \mathbf{H}(K_{\mathcal{T}}),\ m(w^-,z,w^+)=w^-zw^+,$$

is a $K_{\mathcal{T}}$-biregular map onto a Zariski open dense subset of $\mathbf{H}(K_{\mathcal{T}})$ containing e;
(b) *the subgroup $N(s)$ generated by $W^+(s)$ and $W^-(s)$ is a normal subgroup of $\mathbf{H}(K_{\mathcal{T}})$ and $H=Z(s)\,N(s)$.*

Proof. (a) It is enough to consider the case where $\mathcal{T}$ consists of one element v. It easily follows from Proposition 2.5 that $W^-(s)\cap Z(s)\,W^+(s)=\{e\}$. On the other hand $W^-(s)$, $Z(s)$, $W^+(s)$ and $Z(s)\,W^+(s)$ are subgroups and $Z(s)\cap W^+(s)=\{e\}$. Therefore, in view of Lemma 2.6, the multiplication map m is injective. But the same is true if we replace K_v by any finite extension $K'_v\supset K_v$. Hence the multiplication map

$$\bar{m}\colon \overline{W^-(s)}\times\overline{Z(s)}\times\overline{W^+(s)}\to\mathbf{H}$$

is injective. Since s is an element from the class $\mathscr{A}$ we have that if λ is an eigenvalue of $\mathrm{Ad}(s)$ and $\lambda \neq 1$ then $|\lambda|_v \neq 1$. From this and Proposition 2.5(b) we get that $\mathrm{Lie}(\mathbf{H})$ is the direct sum of $\mathrm{Lie}(\overline{W^-(s)})$, $\mathrm{Lie}(\overline{Z(s)})$ and $\mathrm{Lie}(\overline{W^+(s)})$. It implies that the image of $\bar{m}$ is Zariski open and dense in the connected group $\mathbf{H}$. This and the injectivity of $\bar{m}$ implies (a).

(b) Since the subgroup $Z(s)$ normalizes both $W^+(s)$ and $W^-(s)$ it also normalizes $N(s)$. Therefore $Z(s)\,N(s)$ is a subgroup. But in view of (a), $Z(s)\,N(s)$ contains a Zariski open dense subset of $\mathbf{H}(K_v)$. Since any Zariski open dense subset of $\mathbf{H}(K_v)$ generates $\mathbf{H}(K_v)$ as a group we get that $\mathbf{H}(K_v)=N(s)\,Z(s)$.

2.8 Let $s \in \mathbf{H}(K_{\mathscr{T}})$ be an element from the class $\mathscr{A}$ and let U be a $K_{\mathscr{T}}$-algebraic subgroup of $\mathbf{H}(K_{\mathscr{T}})$. In view of Proposition 2.2, a sequence $\{\mathrm{Ad}(s^{-n})(\mathrm{Lie}(U)) = \mathrm{Lie}(s^{-n}U s^n)\}$ has a limit in the Grassmannian variety $\mathrm{Gr}(\mathrm{Lie}(W^+(s)))$ when $n \to +\infty$. Denotes this limit by $\mathscr{L}_0$. It is clear that $\mathscr{L}_0$ is a Lie subalgebra of $\mathrm{Lie}(W^+(s))$. Therefore $\mathscr{L}_0 = \mathrm{Lie}(U_0)$ where $U_0 = \exp \mathscr{L}_0$ is a $K_{\mathscr{T}}$-algebraic subgroup of $W^+(s)$. Since the logarithmic map $\ln\colon W^+(s) \to \mathrm{Lie}(W^+(s))$ is $K_{\mathscr{T}}$-biregular we get that U_0 is the limit of $s^{-n}U s^n$ in the Hausdorff topology when $n \to +\infty$. Let us note that $\mathrm{Ad}(s)\,\mathscr{L}_0 = \mathscr{L}_0$ and $s U_0 s^{-1} = U_0$.

Put $\mathbf{U} = \bar{U}$, $\mathbf{U}_0 = \bar{U}_0$, $\mathbf{W}^+(s) = \overline{W^+(s)}$ and $\mathbf{W}^-(s) = \overline{W^-(s)}$. Since $\mathbf{W}^+(s)$ and $\mathbf{U}_0$ are $\mathrm{Int}(s)$-invariant unipotent $K_{\mathscr{T}}$-algebraic groups and $\mathbf{W}^+(s) \supset \mathbf{U}_0$ it follows from [Bo-Spr, 9.13] that there exists an $\mathrm{Int}(s)$-invariant $K_{\mathscr{T}}$-regular cross-section $\mathbf{V}$ for $\mathbf{W}^+(s)/\mathbf{U}_0$.

Proposition. (a) $\mathbf{V}$ *is a* $K_{\mathscr{T}}$-regular cross-section for $\mathbf{W}^+(s)/\mathbf{U}$.
(b) *Denote by* $p\colon \mathbf{U} \to \mathbf{U}_0$ *the projection parallel to* $\mathbf{V}$ *(i.e. for every* $u \in \mathbf{U}$ *we have* $u \in \mathbf{V}\,p(u)$*). Then* p *is* $K_{\mathscr{T}}$-isomorphism.

Proof. (a) We can assume that $\mathscr{T} = \{v\}$. Put $\mathbf{W} = \mathbf{W}^+(s)$. Since $\mathrm{char}(K_v) = 0$ and the multiplication map $\alpha\colon \mathbf{V} \times \mathbf{U} \to \mathbf{W}$, $\alpha(x, y) = xy$, is regular it is enough to show that for any finite extension K'_v of K_v the multiplication map $\mathbf{V}(K'_v) \times \mathbf{U}(K'_v) \to \mathbf{W}(K'_v)$ is bijective. We can assume that $K'_v = K_v$. (The same proof can be applied for arbitrary K'_v because $\mathbf{U}_0(K'_v)$ is the limit of $s^{-n}\mathbf{U}(K'_v)\,s^n$ when $n \to +\infty$.) It follows from the construction of U_0 and the implicit function theorem (see 1.12) that there exists an open neighborhood $\mathscr{O}$ of $e \in W^+(s)$ such that for every positive integer n every point $x \in \mathscr{O}$ can be represented in a unique way as a product yz where $y \in V$ and $z \in s^{-n}U s^n$. Let $\alpha(x_1, y_1) = \alpha(x_2, y_2) = w \in W^+(s)$. There exists n such that the elements $s^{-n}x_i s^n$, $s^{-n}y_i s^n$ (where $i = 1, 2$) and $s^{-n}ws^n$ are in $\mathscr{O}$. Since α is s-equivariant we get that $s^{-n}x_1 s = s^{-n}x_2 s^n$ and $s^{-n}y_1 s^n = s^{-n}y_2 s^n$ and, consequently, $x_1 = x_2$ and $y_1 = y_2$. Thus α is injective. Let w be an arbitrary element from $W^+(s)$. Since $\mathrm{Int}(s^{-1})|_{W^+(s)}$ is contracting (2.5(c)), $s^{-n}ws^n \in \mathscr{O}$ for some n. Then $s^{-n}ws^n = y_1 z_1$ where $y_1 \in V$ and $z_1 \in s^{-n}U s^n$. Hence

$$w = (s^n y_1 s^{-n})(s^n z_1 s^{-n}).$$

Thus α is surjective.

(b) The proof is analogous to the proof of (a).

3 Actions of algebraic groups on measure spaces

3.1 Let $\mathbf{H}$ be a $K_{\mathscr{T}}$-algebraic group acting $K_{\mathscr{T}}$-rationally on a $K_{\mathscr{T}}$-algebraic variety $\mathbf{M}$. Let F be a subgroup of $H = \mathbf{H}(K_{\mathscr{T}})$ generated by unipotent $K_{\mathscr{T}}$-algebraic subgroups of H and elements from the class $\mathscr{A}$.

Lemma. *Let μ be a Borel F-invariant probability measure on $M=\mathbf{M}(K_{\mathcal{T}})$. Then μ is concentrated on the set of F-fixed points in M. In particular, if μ is F-ergodic then μ is concentrated in a point.*

Proof. Let $F_1 \subset F$ denote either a cyclic subgroup generated by an element s from the class $\mathcal{A}$ or a 1-dimensional unipotent $K_{\mathcal{T}}$-algebraic subgroup of H. It is enough to show that the measure μ is concentrated on the set Ω of F_1-fixed points in M. It is known that if K is a local field and the K-group $\mathbf{G}$ acts K-rationally on a K-variety $\mathbf{X}$ then, for any point $x \in \mathbf{X}(K)$, the orbit map $\mathbf{G}(K)/\mathbf{G}(K)_x \to \mathbf{G}(K)x$ is a homeomorphism where $\mathbf{G}(K)_x$ is the stabilizer of x in $\mathbf{G}(K)$ [B-Z]. But every v-component, $v \in \mathcal{T}$, of the $K_{\mathcal{T}}$-algebraic group $\bar{F}_1$, is 1-dimensional. From this and the property (c) of elements of the class $\mathcal{A}$ (see Lemma 2.1) we get that F_1 acts properly on $M-\Omega$. Now one can easily see that μ is concentrated on Ω.

Corollary. *Let (X, μ_0) be a Borel measure space on which F acts ergodically. Let $f\colon (X,\mu_0) \to M$ be a Borel F-equivariant map (i.e. $f(gx)=gf(x)$ for every $g \in F$). Then f is essentially constant, that is there exists a conull subset $X_0 \subset X$ such that the restriction of f on X_0 is constant.*

Proof. Denote by μ the image of μ_0 on M. Then μ is F-invariant ergodic measure and the assertion follows from the lemma.

3.2 Let $\mathbf{H}$ be a $K_{\mathcal{T}}$-algebraic group and let $\mathbf{F}$ be a connected $K_{\mathcal{T}}$-algebraic subgroup of $\mathbf{H}$ such that $F=\mathbf{F}(K_{\mathcal{T}})$ is generated by unipotent elements and elements of the class $\mathcal{A}$. Let Γ be a discrete subgroup of $H=\mathbf{H}(K_{\mathcal{T}})$ and π: $H \to H/\Gamma$ the natural projection.

Proposition. *Let μ be an F-invariant F-ergodic Borel probability measure on H/Γ and let $\mathbf{M}$ be a $K_{\mathcal{T}}$-subvariety of $\mathbf{H}$ such that $\mu(\pi(M))>0$, where $M=\mathbf{M}(K_{\mathcal{T}})$. Then there exists a $K_{\mathcal{T}}$-algebraic subgroup $\mathbf{P}$ of $\mathbf{H}$ and a point $x \in M$ such that $P=\mathbf{P}(K_{\mathcal{T}})$ contains F, $Px \subset M$ and $\mu(\pi(Px))=1$.*

Proof. Since the Zariski topology is Noetherian we may (and will) assume that the $K_{\mathcal{T}}$-variety $\mathbf{M}$ is minimal in the sense that $\mu(\pi(\mathbf{X}(K_{\mathcal{T}})))=0$ for any proper $K_{\mathcal{T}}$-subvariety $\mathbf{X}$ of $\mathbf{M}$. Put $F_0=\{g \in F \mid g\pi(M)=\pi(M)\}$. Cleary F_0 is a subgroup of F. In view of the minimality of $\mathbf{M}$ if $F \neq F_0$ and $g \in F-F_0$ then $\mu(\pi(M) \cap g\pi(M))=0$. Since $\mu(\pi(M))>0$ and the measure μ is finite and F-invariant we obtain that F_0 has finite index in F. On the other hand, for every $g \in F_0$ we have $gM \subset M\Gamma$. Since Γ is countable there exists $\gamma \in \Gamma$ such that $\mu(\pi(gM \cap M\gamma))>0$. Then by the minimality of M we get that $gM \subset M\gamma$. Therefore the quotient F_0/F_1, *where* $F_1=\{h \in F_0 \mid hM=M\}$, is a countable set. (To see this one should use the fact that the inclusion $g\mathbf{M} \subset \mathbf{M}\gamma$ implies $g\mathbf{M}=\mathbf{M}\gamma$ which is equivalent to $gM=M\gamma$.) But $\mathbf{F}$ is connected. Therefore F_1 is Zariski dense in F which implies that $FM=M$.

Put $\Delta=\{d \in \Gamma \mid Md=M\}$ and $Y=M-\bigcup_{\gamma \in \Gamma-\Delta} M\gamma$. One can easily deduce from the minimality of $\mathbf{M}$ that $\mu(\pi(Y))=\mu(\pi(M))$. Note that $FY\Delta=Y$ and $\Gamma\gamma \cap Y=\emptyset$ for every $\gamma \in \Gamma-\Delta$. Therefore the natural map from Y/Δ to H/Γ is injective and we can lift the restriction of μ to $\pi(Y)$ to a non-zero finite F-invariant F-ergodic measure μ_0 on M/Δ. Denote by B the Zariski closure of Δ in H. Then $MB=M$ and the quotient M/B can be embedded into $(\mathbf{H}/\mathbf{B})(K_{\mathcal{T}})$, where $\mathbf{B}$ is the Zariski closure of B in $\mathbf{H}$. (Note that by [B-Z] this embedding is a proper map.) Clearly $\mathbf{F}$ acts $K_{\mathcal{T}}$-rationally on $\mathbf{M}/\mathbf{B}$. Denote by v the image of the measure μ_0 under the natural map $M/\Delta \to M/B$. Then v is an F-invariant ergodic measure on M/B. In view of Lemma 3.1 v is concentrated on a single point. By the construction of v it follows that there exists a point $z \in M$ such that $\mu(\pi(zB)) = \mu(\pi(M))$. Since $zB \subset M$ and $\mathbf{M}$ is minimal we get that $zB = M$. Now to complete the proof it is enough to put $x = z$, $P = zBz^{-1}$ and $\mathbf{P} = \bar{P}$.

3.3 Let H, Γ and π be as in 3.2. We will say that a Borel probability measure μ on H/Γ is *Zariski dense* if there is not a proper $K_{\mathcal{T}}$-algebraic subvariety $\mathbf{M}$ of $\mathbf{H}$ with $\mu(\pi(M)) > 0$ where $M = \mathbf{M}(K_{\mathcal{T}})$. We say that $K_{\mathcal{T}}$-algebraic subvarieties $\mathbf{L}_1$ and $\mathbf{L}_2$ of a $K_{\mathcal{T}}$-algebraic variety $\mathbf{M}$ *are transversal* at $x \in \mathbf{L}_1 \cap \mathbf{L}_2$ if both $\mathbf{L}_1$ and $\mathbf{L}_2$ are smooth at x and $T_x(\mathbf{M}) = T_x(\mathbf{L}_1) \oplus T_x(\mathbf{L}_2)$, where $T_x(\ )$ denote the tangent spaces at x.

Next if $\Omega \subset H/\Gamma$ is a measurable subset we set $\Psi(\Omega) = \{g \in H \mid g\Omega \cap \Omega \neq \emptyset\}$.

Lemma. *Let μ be a Borel probability measure on H/Γ. Assume that μ is Zariski dense and F_0-invariant, where F_0 is an open subgroup of the group of $K_{\mathcal{T}}$-rational points $F = \mathbf{F}(K_{\mathcal{T}})$ of a connected algebraic subgroup $\mathbf{F} \subset \mathbf{H}$. Let $\mathbf{L}$ be a connected $K_{\mathcal{T}}$-algebraic subvariety of $\mathbf{H}$ containing e and transversal to $\mathbf{F}$ and let $\mathbf{M}$ be a proper subvariety of $\mathbf{L}$ containing e. There exists a constant c, $0 < c < 1$, such that if $\Omega \subset H/\Gamma$ is a measurable set with $\mu(\Omega) > 1 - c$, then one can find a converging to e sequence $\{g_n\} \subset \Psi(\Omega) \cap (L - M)$, where $L = \mathbf{L}(K_{\mathcal{T}})$ and $M = \mathbf{M}(K_{\mathcal{T}})$.*

Proof. Let $p \in H/\Gamma$ be a point such that $\mu(W) > 0$ for every neighborhood W of p. Since $\mathbf{F}$ and $\mathbf{L}$ are transversal there exist relatively compact neighborhoods A' and B' of e in F_0 and L, respectively, such that the map $A' \times B' \to A'B'p$, $(x, y) \to xyp$ is a homeomorphism. (Next we will identify $A'B'$ with $A'B'p$ via this homeomorphism.) It follows from the implicit function theorem that there exist neighborhoods A and B of e in A' and B', respectively, such that for every $x, y \in B$ there exist continuous maps $\beta(x, y)\colon Ax \to A'y$ and $\gamma(x, y)\colon Ax \to B'$ uniquely defined by the equation

$$\beta(x, y)(z) = \gamma(x, y)(z)\, z$$

where $z \in Ax$.

Denote by μ_0 the restriction of μ to AB. Since μ is F_0-invariant, $\mu_0 = \int_B v_x \, d\sigma(x)$, where σ is a measure on B and v_x is the measure on Ax induced by the Haar measure on $A \subset F_0$ via the homeomorphism $A \to Ax$, $a \to ax$. Without loss of generality we may (and will) assume that v_x and σ are probability measures. Using the Fubini theorem we can fix a constant e, $0 < c < 1$, such that if $\Omega \subset H/\Gamma$ is a measurable subset and $\mu(\Omega) > \mu(H/\Gamma) - c$ then $\sigma(B_0) > \frac{3}{4}$ where $B_0 = \{x \in B \mid v_x(\Omega \cap Ax) \geqq \frac{4}{5}\}$. Fix a sequence $\{D_i\}$ of measurable subsets of B_0 such that $\sigma(D_i) > 0$ for all i and the diameters of D_i converge to 0 when $i \to \infty$ (recall that L is a measurable space). Passing to a subsequence we can (and will) assume that for every i if $x, y \in D_i$ then $v_y(Ay \cap \beta(x, y)(\Omega \cap Ax)) > \frac{2}{3}$. (We use the fact that if the diameter of D_i is small then the maps $\beta(x, y)$ have Jacobians relatively the Haar measures on Ax and $A'y$ close to 1.) Assume that there exists i such that for all $x, y \in F_i$ if $\beta(x, y)(z) \in Ay \cap \Omega$, where $z \in Ax \cap \Omega$, then $\gamma(x, y)(z) \in M$. In light of the above discussion and the Fubini theorem this implies that there exists a $q \in \Omega$ such that $\mu(MFq) > 0$ which contradicts the assumption that μ is Zariski dense. Therefore for every i there exist $x_i, y_i \in D_i$ and $\gamma(x_i, y_i)(z_i) \in L - M$. Clearly the sequence $\{g_i = \gamma(x_i, y_i)(z_i)\}$ possesses the required properties. The lemma is proved.

4 Groups without unipotent $K_{\mathcal{T}}$-algebraic subgroups

In this section we will assume that $\mathcal{T}$ is a finite set of normalized valuations of the field $\mathbf{Q}$ of rational numbers. Our purpose is to prove the following

4.1 Proposition. *Let* $\mathbf{H}$ *be a connected* $K_{\mathcal{T}}$*-algebraic group and* $H = \mathbf{H}(K_{\mathcal{T}})$. *Let* $F \subset H$ *be a subgroup which is open in the Hausdorff topology and dense in the Zariski topology on* H. *Suppose that* F *does not contain a nontrivial unipotent* $K_{\mathcal{T}}$*-algebraic subgroup of* H. *Then there exists a* $K_{\mathcal{T}}$*-split central torus* $\mathbf{S}$ *in* $\mathbf{H}$ *such that the factor group* $F/F \cap S$, $S = \mathbf{S}(K_{\mathcal{T}})$, *is compact.*

The above proposition and Proposition 2.4 immediately imply

Corollary. *Let* $\mathbf{H}$, H *and* F *be the same as in Proposition* 4.1. *Then there exists a* $K_{\mathcal{T}}$*-split central torus* $\mathbf{S}$ *in* $\mathbf{H}$ *and a discrete subgroup* $S_0 \subset F \cap S$ *such that* F/S_0 *is a compact group and* S_0 *consists of elements from the class* $\mathcal{A}$.

4.2 The proof of Proposition 4.1 uses the following lemma.

Lemma. *Let* $U = \mathbf{U}(K_{\mathcal{T}})$, *where* $\mathbf{U}$ *is a unipotent* $K_{\mathcal{T}}$*-algebraic group. Let* P *be a noncompact open (in the Hausdorff topology on* U*) subgroup of* U. *Then* P *contains a nontrivial* $K_{\mathcal{T}}$*-algebraic subgroup of* U.

Proof. Let us prove first the lemma when $\mathcal{T} = \{v\}$ and $K_v \cong \mathbf{Q}_p$, where p is a prime number. Since the exponential map exp: $\mathrm{Lie}(U) \to U$ maps every 1-dimensional linear subspace of $\mathrm{Lie}(U)$ onto a 1-parameter subgroup of U, it is enough to show that the set $\mathcal{P} = \ln(P)$ contain a 1-dimensional linear subspace of $\mathrm{Lie}(U)$. Fix a coordinate system in $\mathrm{Lie}(U)$ and introduce a norm $\|\ \ \|$ on $\mathrm{Lie}(U)$ by the formula $\|x\| = \sup_i |x_i|_p$, where x_i are the coordinates of $x \in \mathrm{Lie}(U)$.

Since exp is a diffeomorphism and P is a noncompact subgroup of U there exists a sequence $d_i \in \mathcal{P} - \{0\}$ converging to infinity. Denote the line $\mathbf{Q}_p d_i$ by ℓ_i. Passing to a subsequence and considering $\{\ell_i\}$ as a sequence of points in the projectivization of $\mathrm{Lie}(U)$ we may (and will) assume that $\{\ell_i\}$ converges to a line $\ell \subset \mathrm{Lie}(U)$. For every positive integer n we denote $M_n = \{x \in \mathrm{Lie}(U) \mid \|x\| < p^n\}$. For every n and i there exists an integer $m_i(n)$ such that $p^{m_i(n)} d_i \in M_{n+1} - M_n$. For every n, passing to a subsequence we can assume that $\{p^{m_i(n)} d_i\}$ converges to a vector $b_n \in \ell$ with $\|b_n\| = p^n$. Clearly $b_n \in \mathcal{P}$. Hence $\mathbf{Z}_p b_n \subset \mathcal{P}$, where $\mathbf{Z}_p$ is the ring of the p-adic integers in $\mathbf{Q}_p$. Since $\ell = \bigcup_{n \geqq 1} \mathbf{Z}_p b_n$ we conclude that $\ell \subset \mathcal{P}$, which proves the lemma when $\mathcal{T}$ consists of one p-adic valuation.

Let $\mathcal{T}$ contain the archimedean valuation of $\mathbf{Q}$ and $U_\infty \neq \{e\}$. Then $P \cap U_\infty$ is an open subgroup of U_∞. But U_∞ is connected in the Hausdorff topology. Therefore $P \supset U_\infty$.

It remains to consider the case when $\mathcal{T}$ consists of nonarchimedean valuations. It is enough to show that for every $x \in \mathcal{P}$, where $\mathcal{P} = \ln(P)$, all v-components x_v of x, $v \in \mathcal{T}$, are contained in $\mathcal{P}$. Indeed, if this is true then the proof of our assertion is easily reduced to the case when $\mathcal{T}$ consists of one nonarchimedean valuation. Let $x\in\mathcal{P}$. Then $\mathbf{Z}x\subset\mathcal{P}$ and since under the diagonal embedding $\mathbf{Z}$ is dense in $\mathbf{Z}_{\mathcal{T}}=\prod_{v\in\mathcal{T}}\mathbf{Z}_{p(v)}$, where $p(v)$ is a prime number such that $K_v\cong\mathbf{Q}_{p(v)}$, we get that $\mathbf{Z}_{\mathcal{T}}x\subset\mathcal{P}$. In particular, $x_v\in\mathcal{P}$ for each $v\in\mathcal{T}$. This completes the proof of the lemma.

4.3 Proof of Proposition 4.1 Consider the adjoint representation Ad: $H\to\mathrm{GL}(\mathrm{Lie}(H))$. Assume that there exists an element $h\in F$ which has a v-component h_v such that $\mathrm{Ad}(h_v)$ has an eigenvalue α with $|\alpha|_v>1$. Then $W^+(h)\neq\{e\}$ (see 2.5). Since the automorphism $\mathrm{Int}(h^{-1})|_{W^+(h)}$ is contracting (see 2.5(c)) and the subgroup $W^+(h)\cap F$ is $\mathrm{Int}(h)$-invariant and open in $W^+(h)$ we get that $W^+(h)\subset F$. On the other hand, $W^+(h)$ is a unipotent $K_{\mathcal{T}}$-algebraic subgroup of H by 2.5(a) which contradicts the proposition hypothesis. Thus for every $h\in F$ all eigenvalues of $\mathrm{Ad}(h_v)$, for all $v\in\mathcal{T}$, have absolute values equal to 1.

Denote by $R(H)$ the solvable radical of H (i.e. $R(H)$ is the maximal connected in the Zariski topology solvable normal $K_{\mathcal{T}}$-algebraic subgroup of H) and denote by S the maximal split central $K_{\mathcal{T}}$-algebraic torus in H. Note that $S\subset R(H)$. It is enough to prove that $F/F\cap S$ is a compact group. Denote $N=SR_u(H)$ where $R_u(H)$ is the unipotent radical of H. Then N is a $K_{\mathcal{T}}$-algebraic subgroup of H and it follows from 1.7 that H/N is a group of $K_{\mathcal{T}}$-points of a reductive $K_{\mathcal{T}}$-algebraic group. In particular, there exists a reductive $K_{\mathcal{T}}$-algebraic subgroup $L\subset H$ such that $H/N\cong L/L\cap N$ and $L\cap N$ is a finite central subgroup of L. Therefore the restriction of Ad to L induces a representation σ: $H/N\to\mathrm{GL}(\mathrm{Lie}(H))$. By the discussion in the preceding paragraph all elements in $\sigma(FN/N)$ have eigenvalues with absolute values 1. By [Pra, Lemma 1] we obtain that $\sigma(FN/N)$ is a compact group. Note that since $\mathrm{Ker}(\sigma)$ is compact σ is a proper map in view of 1.7(a). Therefore there exists a compact $K\subset F$ such that $F=K(F\cap N)$. This reduces the proof of the proposition to the case when $H=S\times U$ where S is a $K_{\mathcal{T}}$-algebraic split torus in H and U is a $K_{\mathcal{T}}$-algebraic unipotent subgroup of H. Note that $U\cap F$ has finite index in $U\cap SF$ since $U\cap F$ is an open subgroup of U, S is compactly generated and any discrete factor group of any open subgroup of U is a torsion group. Hence $S(U\cap F)$ has finite index in SF. On the other hand, $U\cap F$ is compact in view of Lemma 4.2. Therefore S is cocompact in SF, equivalently, $F/F\cap S$ is a compact group. The proposition is proved.

5 Construction of quasiregular maps

In this section we fix a connected $K_{\mathcal{T}}$-algebraic group $\mathbf{H}$, an element $s\in H$, $H=\mathbf{H}(K_{\mathcal{T}})$, from the class $\mathcal{A}$ and a unipotent $K_{\mathcal{T}}$-algebraic subgroup U in H such that $U\subset W^+(s)$.

5.1 In view of 2.8 the sequence $s^{-n}Us^n$ converges to a $\mathrm{Int}(s)$-invariant unipotent $K_{\mathcal{T}}$-algebraic subgroup U_0 of $W^+(s)$ when $n\to+\infty$. Besides there exists an $\mathrm{Int}(s)$-invariant $K_{\mathcal{T}}$-regular cross-section $V\subset W^+(s)$ when $n\to+\infty$. Besides there exists an $\mathrm{Int}(s)$-invariant $K_{\mathcal{T}}$-regular cross-section $V\subset W^+(s)$ for both W^+/U_0 and $W^+(s)/U$ (see 1.8). On the other hand, by Lemma 2.7(a) $W^-(s)Z(s)W^+(s)$ is a Zariski open subset of H. Therefore $L=W^-(s)Z(s)V$

is a $K_{\mathcal{S}}$-rational cross-section for both H/U and H/U_0. Note that L is Int(s)-invariant. Further we will denote by π: $H \to U_0$ the projection onto U_0 parallel to L and by p the restriction $\pi|_U$. In view of Proposition 2.8(b) p is a $K_{\mathcal{S}}$-isomorphism.

Let us fix relatively compact neighborhoods B^+ and B^- of e in $W^+(s)$ and $W^-(s)$, respectively, such that $sB^+s^{-1} \supset B^+$ and $s^{-1}B^-s \supset B^-$. We put $B_n^+ = s^{-n}B^+s^{-n}$ and $B_n^- = s^{-n}B^-s^n$. Obviously, the sequences $\{B_n^+\}$ and $\{B_n^-\}$ form fundamental systems of neighborhoods of e in $W^+(s)$ and $W^-(s)$, respectively. We define a function ℓ^+ on $W^+(s)$ (resp. ℓ^- on $W^-(s)$) by setting $\ell^+(x)=k$ iff $x \in B_k^+ - B_{k-1}^+$ and $\ell^+(e) = -\infty$ (resp. $\ell^-(x)=k$ iff $x \in B_k^- - B_{k-1}^-$ and $\ell^-(e) = -\infty$). Also, for every integer n we put $C_n = B_n^+ \cap U_0$ and $A_n = p^{-1}(C_n)$.

Let us note that since L and U_0 are Int(s)-invariant the maps π and p commute with Int(s). From this and the definition of U_0 we get

$$\lim_{n \to +\infty} s^{-n} A_n s^n = C_0. \tag{1}$$

5.2 Let us fix a sequence $\{g_n\}$ in H converging to e. We will assume that $\{g_n\} \subset LU - \mathcal{N}_H(U)$, where $\mathcal{N}_H(U)$ denotes the normalizer of U in H. Since L is a $K_{\mathcal{S}}$-rational section for H/U we can define $K_{\mathcal{S}}$-rational maps $\tilde{\varphi}_n$: $U \to L$ and ω_n: $U \to U$ by the following equation

$$u g_n = \tilde{\varphi}_n(u)\, \omega_n(u). \tag{2}$$

By a theorem of Chevalley [Bo, 5.1] there exists a $K_{\mathcal{S}}$-rational representation ρ: $H \to \mathrm{GL}(\Phi)$, where Φ is a sum $\bigoplus_{v \in \mathcal{S}} \Phi_v$ of vector spaces Φ_v over K_v, and a point $q \in \Phi$ such that $U = \{x \in H \mid \rho(x)\, q = q\}$. To simplify the notation we will write xq instead of $\rho(x)\, q$. It is easy to see that

$$\mathcal{N}_H(U)\, q = \{y \in Hq \mid U y = y\}. \tag{3}$$

Fix a relatively compact neighborhood D of q in Φ. Define a sequence of integers $\{r(n)\}$ as follows: $A_{r(n)}\, g_n\, q \not\subset D$ and $A_k\, g_n\, q \subset D$ whenever $k < r(n)$. Next, for every n we define maps α_n and a_n: $U \to U$ by the formulas

$$\alpha_n(u) = p^{-1}(s^n p(u)\, s^{-n}), \tag{4}$$

$$a_n(u) = \alpha_{r(n)}(u), \tag{5}$$

for every $u \in U$.

It follows from (5) and the definition of A_n that for every integer k

$$a_n(A_k) = A_{k+r(n)}. \tag{6}$$

Since p: $U \to U_0$ is a $K_{\mathcal{S}}$-regular isomorphism the maps $\{a_n\}$ are also $K_{\mathcal{S}}$-regular isomorphisms.

We put

$$\varphi_n = \tilde{\varphi}_n \circ a_n\colon\ U \to L. \tag{7}$$

Denote by $\beta: L \to \Phi$ the restriction to L of the orbit map $h \to hq$, $h \in H$, and put

$$\varphi'_n = \beta \circ \varphi_n: \; U \to \Phi. \tag{8}$$

In view of (2), (7) and the equality $Uq = q$ we get

$$\begin{aligned}\varphi'_n(u) &= \beta(\varphi_n(u)) = \varphi_n(u)\, q = \tilde{\varphi}_n(a_n(u))\, q \\ &= \tilde{\varphi}_n(a_n(u))\, \omega_n(a_n(u))\, q = a_n(u)\, g_n\, q.\end{aligned} \tag{9}$$

Hence φ'_n is a $K_{\mathscr{T}}$-regular map from U to Φ. Furthermore, if we identify U with $\mathrm{Lie}(U)$ using the logarithmic map we can (and will) interpret $\{\varphi'_n\}$ as a set of $K_{\mathscr{T}}$-polynomial maps of degrees bounded from above. (According to our terminology a $K_{\mathscr{T}}$-polynomial map f is a set of K_v-polynomial maps f_v, $v \in \mathscr{T}$, and $\deg(f) = \max\{\deg(f_v) \mid v \in \mathscr{T}\}$.)

It follows from (9) and (6) that

$$\varphi'_n(A_{-1}) \subset D \tag{10}$$

and

$$\varphi'_n(A_0) \not\subset D. \tag{11}$$

It is well known that for any vector space Φ_v over a local field K_v a set of polynomials on Φ_v of degrees less than a constant N and uniformly bounded on some nonempty open subset of Φ_v is relatively compact in the topology of uniform convergence on compact subsets. This remark and (10) imply that replacing $\{g_n\}$ by a subsequence we can (as we will) assume that there exists a $K_{\mathscr{T}}$-regular map $\varphi': U \to \Phi$ such that

$$\varphi'(u) = \lim_{n \to \infty} \varphi'_n(u) \tag{12}$$

for every $u \in U$. Since $\varphi'_n(e) = g_n\, q$ and $g_n \to e$ we obtain

$$\varphi'(e) = q. \tag{13}$$

On the other hand, (11) implies that $\varphi'(A_0) \not\subset D$. Therefore φ' is a non-constant $K_{\mathscr{T}}$-polynomial map.

Since L is a rational cross-section for H/U we get from Lemma 1.3 that β is a $K_{\mathscr{T}}$-regular isomorphism of L onto a Zariski open (in the Zariski closure of $\rho(H)q$ in Φ) subset M containing q. But $\varphi'(U) \subset \rho(H)q$. In view of (13) we can define a $K_{\mathscr{T}}$-rational map $\varphi: U \to L$ by the formula

$$\varphi = \beta^{-1} \circ \varphi', \tag{14}$$

where β^{-1} is defined on the Zariski open subset M of $\overline{\rho(H)q}$ containing q. It follows from the definition of φ that $\varphi(e) = e$.

5.3 Definition. Let $\mathbf{F}$ be a $K_{\mathscr{T}}$-algebraic group, I a $K_{\mathscr{T}}$-algebraic subgroup of $\mathbf{F}(K_{\mathscr{T}})$ and $\mathbf{M}$ a $K_{\mathscr{T}}$-algebraic variety. A $K_{\mathscr{T}}$-rational map $f: \mathbf{M}(K_{\mathscr{T}}) \to \mathbf{F}(K_{\mathscr{T}})$ is called *I-quasiregular* if the map from $\mathbf{M}(K_{\mathscr{T}})$ to $\mathbf{V}$ given by $x \to \gamma(f(x))p$

is $K_{\mathscr{T}}$-regular for every $K_{\mathscr{T}}$-rational representation γ: $\mathbf{F} \to \mathrm{GL}(\mathbf{V})$ and every point $p \in \mathbf{V}(K_{\mathscr{T}})$ such that $\gamma(I)\, p = p$.

5.4 Let us prove that the map φ constructed in 5.2 is U-quasiregular.

In view of (14) and (12) we get

$$\varphi(u) = \lim_{n \to \infty} \varphi_n(u), \tag{15}$$

for all $u \in (\varphi')^{-1}(M)$ and the convergence in (15) is uniform on every compact subset of $(\varphi')^{-1}(\mathrm{M})$.

Using (2) and (7) we get

$$\varphi_n(u) = a_n(u)\, g_n\, b_n(u), \tag{16}$$

where $b_n(u) = \omega_n(a_n(u))^{-1}$. Therefore (15) can be written in the form

$$\varphi(u) = \lim_{n \to \infty} a_n(u)\, g_n\, b_n(u), \tag{17}$$

where $u \in (\varphi')^{-1}(M)$ and the convergence is uniform on every compact subset of $(\varphi')^{-1}(M)$.

Now let γ: $H \to \mathrm{GL}(W)$ be a $K_{\mathscr{T}}$-rational representation and $w \in W$ be such that $\gamma(U)\, w = w$. In view of (17) and the inclusion $b_n(U) \subset U$

$$\gamma(\varphi(u))\, w = \lim_{n \to \infty} \gamma(a_n(u)\, g_n)\, w$$

for all $u \in U$ from a nonempty Zariski open subset of U. Note that the maps ψ_n: $U \to W$, $u \to \gamma(a_n(u)\, g_n)\, w$, are $K_{\mathscr{T}}$-regular. Moreover, if we identify U with $\mathrm{Lie}(U)$ we obtain that $\{\psi_n\}$ are $K_{\mathscr{T}}$-polynomial maps of bounded degrees and the restrictions of $\{\psi_n\}$ to some nonempty open subset of U are bounded. Therefore the sequence $\{\psi_n\}$ converges to a $K_{\mathscr{T}}$-polynomial map i.e. the map from U to W given by $x \to \gamma(\varphi(x))\, w$ is $K_{\mathscr{T}}$-polynomial. This proves that φ is a U-quasiregular map.

5.5 Remark. Note that in the above proof the U-quasiregularity of φ was deduced from (17). An arbitrary $K_{\mathscr{T}}$-rational map φ: $U \to H$ will be called *strongly U-quasiregular* if there exist a sequence $\{a_n$: $U \to U\}$ of $K_{\mathscr{T}}$-regular maps, a sequence $\{b_n$: $U \to U\}$ of $K_{\mathscr{T}}$-rational maps and a Zariski open nonempty subset $A \subset U$ such that φ is defined by (17) and the convergence in (17) is uniform on every compact subset of A.

6 Properties of φ

In this section we prove some basic properties of the U-quasiregular map φ constructed in 5.2. We preserve the notations and assumptions from Sect. 5.

6.1 Proposition. *The set* $\mathrm{Im}(\varphi)$ *is contained in* $\mathscr{N}_H(U)$. *Furthermore there is not a compact subset* $K \subset H$ *such that* $\mathrm{Im}(\varphi) \subset KU$.

Proof. The second assertion follows from (14) in Sect. 5 and the fact that φ' is a non-constant $K_{\mathcal{T}}$-polynomial map. In order to prove that $\mathrm{Im}(\varphi)\subset \mathcal{N}_H(U)$ it is enough (in view of (3), Sect. 5) to show that $v\varphi'(u)=\varphi'(u)$ for all $v, u\in U$.

By (12), Sect. 5

(1) $$v\varphi'(u)=\lim_{n\to+\infty} v\varphi'_n(u).$$

Using (9), Sect. 5 we obtain

(2) $$\begin{aligned} v\varphi'_n(u)&=v\,a_n(u)\,g_n\,q\\ &=a_n(a_n^{-1}(v\,a_n(u)))\,g_n)\,q=\varphi'_n(a_n^{-1}(v\,a_n(u))). \end{aligned}$$

It follows easily from the relations (4) and (5) in Sect. 5 that for any $x\in U$ we have

$$a_n^{-1}(x)=\pi'(s^{-r(n)}x\,s^{r(n)}),$$

where π' is the projection parallel to L of H onto U. Therefore

(3) $$a_n^{-1}(v\,a_n(u))=\pi'(s^{-r(n)}v\,s^{r(n)}s^{-r(n)}a_n(u)\,s^{r(n)}).$$

Since $\lim_{n\to\infty} r(n)=+\infty$ and $U_0=\lim_{n\to\infty} s^{-n}U\,s^n$ we have

(4) $$\lim_{n\to\infty} s^{-r(n)}a_n(u)\,s^{r(n)}=p(u).$$

On the other hand, $\pi'(p(u))=u$ and $\lim_{n\to\infty} s^{-r(n)}v\,s^{r(n)}=e$. Therefore, in view of (3) and (4) we obtain

(5) $$a_n^{-1}(v\,a_n(u))=v_n\,u,$$

where $\lim_{n\to\infty} v_n=e$.

Now since $\{\varphi'_n\}$ is a sequence of $K_{\mathcal{T}}$-polynomial maps converging to φ' (1), (2) and (5) imply

$$v\varphi'(u)=\lim_{n\to\infty}\varphi'_n(v_n\,u)=\lim_{n\to\infty}\varphi'_n(u)=\varphi'(u).$$

The proof of the proposition is complete.

6.2 The next properties of φ will be deduced from the formula (17) in 5.2, i.e. from the fact that φ is a strongly quasiregular map (see 5.4). In particular, we can reduce the proofs of these properties to the case when $\mathcal{T}=\{v\}$.

6.3 Denote by F the subgroup of H generated by $\mathrm{Im}(\varphi)$ and U. In view of Proposition 6.1 the subgroup U is normal in F. Let H_1 be the Zariski closure of F in H. It is well known (see, for example, [Bo-Pra, 2.2]) that F is an open in the Hausdorff topology subgroup of H_1.

Proposition. *With the above notation assume that if V is a unipotent $K_{\mathcal{T}}$-algebraic subgroup of H_1 and $V\subset F$ then $V\subset U$. The group H_1 contains a split $K_{\mathcal{T}}$-algebraic torus S such that*

(a) *SU/U is a central subgroup of H_1/U and the group $F/F\cap SU$ is compact;*

(b) *there exists an element* $s \in F \cap S$ *from the class* $\mathscr{A}$ *with the following properties*: (i) s *does not centralize* U, *and* (ii) $\alpha(s, M) \geqq 1$ *for every* $K_{\mathscr{T}}$*-algebraic subgroup* M *of* H *normalized by* SU. (*Recall that* $\alpha(s, M)$ *is defined in* 1.5).

The existence of a split $K_{\mathscr{T}}$-torus S in H_1 which satisfies (a) follows from Proposition 4.1. We are going to prove that S satisfies also (b).

We need the following

6.4 Lemma. *The group* $S \cap Z_F(U)$, *where* $Z_F(U)$ *is the centralizer of* U *in* F, *is finite.*

Proof. Note that $S \cap Z_{H_1}(U)$ is a normal subgroup of H_1. Since H_1 is connected in the Zariski topology and $S \cap Z_{H_1}(U)$ is split $S \cap Z_{H_1}(U)$ is central in H_1. On the other hand, for every $v \in \mathscr{T}$ the torsion subgroup of the multiplicative group K_v^* of K_v is finite. Therefore it is enough to prove that S does not contain elements of infinite order which centralize F.

In view of 6.2 we can reduce the proof of the lemma to the case when $\mathscr{T} = \{v\}$.

Let $H_v \subset \mathrm{GL}(V)$ where V is a vector space over K_v. Let $s \in S$ and s centralize F. Since s is diagonalizable $V = V_{\lambda_1} \oplus V_{\lambda_2} \oplus \ldots \oplus V_{\lambda_2}$, where $\lambda_1, \lambda_2, \ldots, \lambda_r$ are different eigenvalues of s and $V_{\lambda_i}, i = 1, 2, \ldots, r$, are the corresponding eigenspaces. Since s centralizes F we get that $F V_{\lambda_i} = V_{\lambda_i}$ for all i, in particular, the subspaces V_{λ_i} are SU-invariant.

If $g \in \mathrm{GL}(V)$ we denote by $g^{(i)}$, $i = 1, 2, \ldots, r$, the linear transformation of V_{λ_i} given by the formula $g^{(i)} = p_i \circ g|_{V_{\lambda_i}}$ where p_i is the projection of V on V_{λ_i} and $g|_{V_{\lambda_i}}$ is the restriction of g on V_{λ_i}. Since V_{λ_i} are U-invariant it follows from (17), Sect. 5 that for every i we have

$$\varphi(u)^{(i)} = \lim_{n \to \infty} a_n(u)^{(i)} g_n^{(i)} b_n(u)^{(i)},$$

where u is an element from a Zariski open nonempty subset of U. Since $a_n(u)^{(i)}$ and $b_n(u)^{(i)}$ are unipotent elements and $\lim_{n \to \infty} g_n^{(i)} = e$ we obtain

$$\det(\varphi(u)^{(i)}) = 1.$$

On the other hand $\det(u^{(i)}) = 1$ for every $u \in U$ and U and $\varphi(U)$ generate the Zariski dense subgroup F in H_1. Therefore $\det(g^{(i)}) = 1$ for every $g \in H_1$ and every $i = 1, \ldots, n$. In particular, $\det(s^i) = 1$ which implies that every λ_i is a root of unity in K_v^* i.e. s is an element of finite order. The lemma is proved.

6.5 *Proof of Proposition* 6.3. In view of 6.2 it is enough to prove the proposition in the particular case $\mathscr{T} = \{v\}$. So, assume that $\mathscr{T} = \{v\}$ and for every positive integer r denote by V_r the r-th exterior power $\Lambda^r \mathrm{Lie}(H)$ of the Lie algebra of H and by f_r the r-th exterior power of the adjoint representation of H. Let M be a $K_{\mathscr{T}}$-algebraic subgroup of H normalized by the subgroup SU. Let r be the dimension of M. Fix a nonzero vector q_M on the line $\Lambda^r \mathrm{Lie}(M) \subset V_r$. Since SU normalizes M we have

$$f_r(u)\, q_M = q_M \tag{6}$$

for every $u \in U$, and

$$f_r(s)\, q_M = \chi_M(s)\, q_M,$$

for every $s \in S$, where $\chi_M(s)$ is a character of the torus S. In view of 1.5

$$|\chi_M(s)| = \alpha(s, M)$$

for every $s \in S$. Note that although the $K_{\mathscr{T}}$-algebraic subgroups of H normalized by SU form, in general, an infinite set we obtain only a finite number of characters $\chi_i = \chi_{M_i}$, where $M_i, i = 1, 2, \ldots, m$ are $K_{\mathscr{T}}$-algebraic subgroups of H normalized by SU. Let $r_i = \dim M_i$. For every i denote by V_i the r_i-exterior power of Lie(H), by f_i the r_i-exterior power of the adjoint representation of H and by q_i a nonzero vector from $\Lambda^{r_i}\mathrm{Lie}(M_i)$. Now for every i we define a rational map ψ_i: $U \to V_i$ by the following formula

$$\psi_i(u) = f_i(\varphi(u))\, q_i, \qquad u \in U.$$

Since φ is a quasiregular map we obtain that $\psi_1, \psi_2, \ldots, \psi_m$ are polynomial maps. In view of Lemma 6.4 it is enough to find a nontrivial element $s \in S \cap F$ from the class $\mathscr{A}$ such that $|\chi_i(s)|_v \geqq 1$ for all i. Without loss of generality we can (and will) assume that $\chi_i \neq \chi_j$ if $i \neq j$.

By part (a) of the proposition $F \cap SU$ is a cocompact subgroup of F. According to Corollary 4.1 $F \cap S$ contain a closed cocompact subgroup S_0 consisting of elements from the class $\mathscr{A}$. Therefore there exists a compact set K in F such that $F = K S_0 U$. In view of (6) and the centrality of the image of S_0 in F/U for every i there exists a compact neighborhood $\mathcal{O}_i$ of q_i such that if $u \in U$ and $\varphi(u) = ksw$ where $k \in K$, $s \in S_0$ and $w \in U$ we have

$$\psi_i(u) \subset \chi_i(s)\, \mathcal{O}_i \tag{7}$$

for all $u \in U$. Put $\mathcal{O}'_i = \{cx \mid x \in \mathcal{O}_i, c \in K_v, |c|_v \leqq 1\}$. Since $\psi_i, i = 1, 2, \ldots m$, are nonconstant polynomial maps there exists $u_0 \in U$ such that $\psi_i(u_0) \notin \mathcal{O}'_i$ for all i, $1 \leqq i \leqq m$. It follows from (7) that $|\chi_i(s_0)| \geqq 1$ for all i, where $s_0 \in S_0$ is such that $\varphi(u_0) = k_0 s_0 w_0$, $k_0 \in K$, $w_0 \in U$. The proposition is proved.

6.6 Recall that the map φ was constructed starting from a sequence $\{g_n\}$ converging to e and an element $s \in H$ from the class $\mathscr{A}$ (see 5.2). We need some additional definitions and notatons related to $\{g_n\}$ and s.

Definition. We say that the sequence $\{g_n\}$ satisfies the condition $(*)$ with respect to s if there exists a compact subset C in H such that $s^{-r(n)} g_n s^{r(n)} \in C$ for all n.

Next denote $\mathscr{F} = \{x \in H \mid U_0 x \subset \overline{W^-(s) Z(s) U_0}\}$. Since $\overline{W^-(s) Z(s)}$ is a subgroup we obtain that $U_0 x \subset \overline{W^-(s) Z(s) U_0}$ if and only if $\overline{W^-(s) Z(s) U_0}\, x \subset \overline{W^-(s) Z(s) U_0}$. On the other hand, for any $K_{\mathscr{T}}$-algebraic subvariety $X \subset H$ we have

$$\{h \in H \mid X h \subset X\} = \{h \in H \mid X h = X\}.$$

Therefore $\mathscr{F}$ is a $K_{\mathscr{T}}$-algebraic subgroup of H.

Set $U^- = W^-(s) \cap \mathscr{F}$. Since the subgroups $U_0, Z(s)$ and $W^-(s)$ are Int(s)-invariant the subgroups $\mathscr{F}$ and U^- are also Int(s)-invariant. It follows from [Bo-Spr] that there exists a $K_{\mathscr{T}}$-regular Int(s)-invariant cross-section $\mathbf{V}^-$ for $\mathbf{W}^-(s)/\mathbf{U}^-$, where $\mathbf{U}^-$ is the Zariski closure of U^- in $\mathbf{W}^-(s)$. We put $V^- =$

$V^-(K_{\mathcal{T}})$. In view of Proposition 2.7(a) and Proposition 2.8 the set $\Omega = U^- V^- Z(s) VU = W^-(s) Z(s) W^+(s)$ is a Zariski open dense subset of H and for each $g \in \Omega$ we have the unique decomposition

(8) $$g = u^-(g)\, v^-(g)\, z(g)\, v(g)\, u(g) = w^-(g)\, z(g)\, w(g),$$

where $u^-(g) \in U^-$, $v^-(g) \in V^-$, $z(g) \in Z(g)$, $v(g) \in V$, $u(g) \in U$, $w^-(g) = u^-(g)\; v^-(g)$ and $w(g) = u(g)\; v(g)$.

It follows from (8) and the definitions of ℓ^+ and ℓ^- in 5.1 that for every integer k we have

(9) $$\ell^-(s^k w^-(g)\, s^{-k}) = \ell^-(w^-(s^k g s^{-k})) = \ell^-(w^-(g)) - k,$$

(10) $$\ell^+(s^k w^+(g)\, s^{-k}) = \ell^+(w^+(s^k g s^{-k})) = \ell^+(w^+(g)) + k,$$

(11) $$s^k z(g)\, s^{-k} = z(g) = z(s^k g s^{-k}).$$

The next lemma is an easy consequence from the definitions of ℓ^+ and ℓ^-.

Lemma. *A sequence $\{x_n\} \subset W^\pm(s)$ is bounded (resp. tends to e) if and only if the sequence $\ell^\pm(x_n)$ is bounded from above (resp. tends to $-\infty$).*

The equalities (9), (10), (11) and the above lemma imply that the sequence $\{g_n\}$ has the property $(*)$ with respect to s if and only if

(12) $$\sup_n \{r(n) + \ell^-(w^-(g_n))\} < \infty.$$

6.7 Proposition. *Suppose that at least one of the following conditions holds:*
(a) *the sequence $\ell^-(v^-(g_n)) - \ell^-(u^-(g_n))$ is bounded from below;*
(b) $\mathcal{N}_H(U_0) \cap W^-(s) = \{e\}$.

Then the sequence $\{g_n\}$ has the property $(*)$ with respect to s. Furthermore, if (a) *is satisfied then* $\mathrm{Im}(\varphi) \subset W^+(s)$.

Proof. Denote $\ell^-(w^-(g_n))$ by $k(n)$. Set $h_n = s^{k(n)} g_n s^{-k(n)}$. We get from (9) that $w^-(h_n) \in B_0^- - B_{-1}^-$. On the other hand, since $g_n \to e$ we deduce from (10), (11), and Lemma 6.6 that $\lim_{n\to\infty} w(h_n) = \lim_{n\to\infty} z(h_n) = e$. Therefore passing to a subsequence we can assume that

(13) $$\lim_{n\to\infty} h_n = h,$$

where $h \in W^-(s)$ and $h \neq e$.

In view of the relation (1) in 5.1 we have that

(14) $$\lim_{n\to\infty} s^{k(n)} A_{-k(n)}\, s^{k(n)} = C_0.$$

Without loss of generality (choosing B small enough) we can assume that

(15) $$C_0\, h \subset W^-(s)\, Z(s)\, W^+(s).$$

In 5.2 we introduced the $K_{\mathscr{T}}$-polynomial maps φ'_n which converge to a polynomial map φ' such that $\varphi'(e)=q$. According to (9) in 5.2 $\varphi'_n(u)=a_n(u)\, g_n\, q$ for every $u\in U$. Also, by (6) in 5.2 $A_{-k(n)}=a_n(A_{-k(n)-r(n)})$. Therefore,

(16) $$A_{-k(n)}\, g_n\, q=a_n(A_{-k(n)-r(n)})\, g_n\, q=\varphi'_n(A_{-r(n)-k(n)})$$

for all n.

Assume that the property $(*)$ does not hold. Then (12) is not fulfilled and passing to a subsequence we can (and will) assume that

(17) $$\lim_{n\to\infty}(r(n)+k(n))=+\infty.$$

In particular, the sequence $\{A_{-k(n)-r(n)}\}$ converges to $\{e\}$. Therefore $\{\varphi'_n(A_{-k(n)-r(n)})\}$ converges to $\{\varphi'(e)=q\}$. In view of (16)

(18) $$\lim_{n\to\infty}\{A_{-k(n)}\, g_n\, q\}=\{q\}.$$

Taking the compact neighborhood D of q in the definition of the sequence $\{r(n)\}$ (see 5.2) small enough we can assume that

$$A_{r(n)}\, g_n\subset W^-(s)\, Z(s)\, VU$$

for all n. In view of (17) we can also assume without restrictions that $A_{r(n)}\, g_n\supset A_{-k(n)}\, g_n$ for all n. Now it follows from (18) and the fact that $U=\{g\in H\,|\, g\, q=q\}$ that

(19) $$A_{-k(n)}\, g_n\subset W_n^-\, Z_n\, V_n\, U,$$

where $W_n^-\subset W^-(s)$, $Z_n\subset Z(s)$ and $V_n\subset V$ are compact subsets containing $\{e\}$ and such that $\lim_{n\to+\infty} W_n^-=\lim_{n\to+\infty} Z_n=\lim_{n\to\infty} V_n=\{e\}$. Using (10), (11), Lemma 6.6 and the fact that $\lim_{n\to\infty} k(n)=-\infty$ one can deduce that

$$\lim_{n\to\infty}\{s^{k(n)}\, V_n\, s^{-k(n)}\}=\lim_{n\to\infty}\{s^{k(n)}\, Z_n\, s^{-k(n)}\}=\{e\}$$

and $\lim_{n\to\infty} s^{k(n)}\, U\, s^{-k(n)}=U_0$. Since $W^+(s)$, $Z(s)$ and V are Int(s)-invariant subsets of H, the above considerations and (13), (14) and (19) imply that

$$C_0\, h\subset\overline{W^-(s)\, U_0}.$$

Since $C_0\, h$ is Zariski dense in $U_0\, h$ we obtain that

(20) $$U_0\, h\subset\overline{W^-(s)\, U_0},$$

in particular, $h\in U^-$ according to the definition of U^- in 6.6.

Assume that (a) holds i.e. the sequence $\ell^-(v^-(g_n))-\ell^-(u^-(g_n))$ is bounded from below. Then it is easy to see that $v^-(h)\neq e$ which contradicts the fact that $h\in U^-$. Hence the condition (a) implies the property $(*)$ for $\{g_n\}$. Next taking the compact neighborhood D of q in the definition of the sequence $\{r(n)\}$ (see 5.2) small enough we can assume that

$$A_{r(n)}\, g_n \subset W^-(s)\, Z(s)\, L U$$

where L is a compact subset of V. Conjugating this inclusion by $s^{k(n)}$ and going to limits we get that $C_m h \subset W^-(s)\, Z(s)\, U_0$ where $m = \liminf_{n\to\infty}(r(n)+k(n))$. If $m > -\infty$, then C_m is Zariski dense in U_0 and we have that $U_0 h \subset \overline{W^-(s)\, Z(s)\, U_0}$. But $v^-(h) \neq e$ and, hence, $h \notin U_0$ (because (a) holds). Thus $m = -\infty$. Let now $u \in \varphi'^{-1}(M)$ where φ' and M are defined in 5.2. Then $\varphi(u) = \lim_{n\to\infty} \varphi_n(u)$ and $\varphi_n(u) = a_n(u)\, g_n\, b_n(u)$ (see 5.4). In view of (17)

$$\lim_{n\to\infty} s^{-r(n)}\, g_n\, s^{r(n)} = e.$$

Also it is easy to see that

$$\lim_{n\to\infty} s^{-r(n)}\, a_n(u)\, s^{r(n)} = u',\ u' \in U_0,$$

and

$$\lim_{n\to\infty} s^{-r(n)}\, b_n(u)\, s^{r(n)} = u'',\ u'' \in U_0.$$

Hence $\lim_{n\to\infty} s^{-r(n)}\, \varphi_n(u)\, s^{r(n)} \in U_0$. It implies that $\varphi(u) \in W^+(s)$. Thus $\varphi(\varphi'^{-1}(M)) \subset W^+(s)$. But M is Zariski open in $\overline{\rho(H)\, q}$ and hence, $\varphi'^{-1}(M)$ is Zariski open and dense in U. Therefore $\operatorname{Im}(\varphi) \subset W^+(s)$.

To prove that (b) also implies the property $(*)$ first note that

$$\mathbf{E} = \{x \in \mathbf{H} \mid U_0\, x \subset \overline{W^-(s)\, U_0}\}$$

is a $K_{\mathscr{S}}$-algebraic subgroup of $\mathbf{H}$. Indeed, the inclusion $U_0\, x \subset \overline{W^-(s)\, U_0}$ is equivalent to the inclusion $\overline{W^-(s)\, U_0}\, x \subset \overline{W^-(s)\, U_0}$ which is equivalent to the equality $\overline{W^-(s)\, U_0}\, x = \overline{W^-(s)\, U_0}$. Therefore $\mathbf{E}$ is a $K_{\mathscr{S}}$-algebraic subgroup of $\mathbf{H}$. Fix a Borel subgroup $\mathbf{P}$ of $\mathbf{M}$ containing U_0 and denote by $\mathbf{P}_u$ its unipotent radical. Since $\mathbf{P} \subset \overline{W^-(s)\, U_0}$ and $W^-(s)\, U_0$ is a Zariski open subset of $\overline{W^-(s)\, U_0}$ containing e we deduce that the set $(P \cap W^-(s))\, U_0$ is Zariski dense in $\mathbf{P}$. Hence the quotient group $\mathbf{P}/\mathbf{P}_u$ contains a Zariski dense subset of unipotent elements. Therefore $\mathbf{P} = \mathbf{P}_u$. This implies that every Borel subgroup of $\mathbf{E}$ is unipotent. Hence $\mathbf{E}$ is a unipotent $K_{\mathscr{S}}$-algebraic group. We denote $E = \mathbf{E}(K_{\mathscr{S}})$. In view of (20) $h \in E$. Since $h \in W^-(s) - \{e\}$ we have that $E \neq U_0$. This implies that $\mathscr{N}_E(U_0) \neq U_0$ (because in a nilpotent group the normalizer of a proper subgroup F is not equal to F). Using the same argument as above we obtain that $(\mathscr{N}_E(U_0) \cap W^-(s))\, U_0$ is a Zariski dense subgroup of $\mathscr{N}_E(U_0)$. Therefore $W^-(s) \cap \mathscr{N}_E(U_0) \neq \{e\}$ which proves (b) in view of the assumption that the property $(*)$ does not hold.

The proposition is proved.

6.8 The next lemma shows that if the group $\mathbf{H}$ is sufficiently large then given a unipotent $K_{\mathscr{S}}$-algebraic subgroup $\mathbf{U}$ of $\mathbf{H}$ we can always find an element

$s \in \mathbf{H}$ from the class $\mathscr{A}$ such that $U \subset W^+(s)$ and $\mathscr{N}_H(U_0) \cap W^-(s) = \{e\}$ where $U = \mathbf{U}(K_{\mathscr{T}})$.

Lemma. *Let* $\mathbf{U}$ *be a unipotent* $K_{\mathscr{T}}$*-algebraic subgroup of* $\mathbf{L} = \prod_{v \in \mathscr{T}} \mathbf{L}_v$, *where* $\mathbf{L}_v = \mathbf{SL}_{m_v}$. *Then there exists an element* $s \in L$ *(where* $L = \mathbf{L}(K_{\mathscr{T}})$*) from class* $\mathscr{A}$ *such that* $U \subset W_L^+(s)$ *and*

$$\mathscr{N}_L(U_0) \cap W_L^-(s) = \{e\}.$$

Proof. It is enough to prove the lemma in the case when $\mathscr{T}$ contains only one element. Denote by $\mathscr{L}$ the Lie algebra of U and by V the vector space $K_v^{m_v}$. For every $k \geqq 0$ let $\langle \mathscr{L}^k V \rangle$ be the linear subspace of V spanned by $\{g_1 g_2 \dots g_k(v) | g_i \in \mathscr{L}, v \in V\}$. (If $k=0$ we put $V = \langle \mathscr{L}^0 V \rangle$.) Since $\mathscr{L}$ is isomorphic to a subalgebra of the Lie algebra of all strictly upper triangular matrices in $\mathrm{SL}(m_v, K_v)$ we obtain a decreasing sequence of subspaces

$$V \supset \langle \mathscr{L} V \rangle \supset \dots \supset \langle \mathscr{L}^{r-1} V \rangle \supset \langle \mathscr{L}^r V \rangle = \{0\},$$

where $\langle \mathscr{L}^{r-1} V \rangle \neq \{0\}$. For every $i = 1, 2, \dots, r$ fix a subspace V_i such that $\langle \mathscr{L}^{r-i} V \rangle = \langle \mathscr{L}^{r-i+1} V \rangle \bigoplus V_i$. Then $V = V_1 \oplus V_2 \oplus \dots \oplus V_r$. Choose an element s from the class $\mathscr{A}$ such that for every i, s acts as a multiplication by a constant λ_i on V_i and $\lambda_i \lambda_{i-1}^{-1} = c$ where c does not depend on i and $|c|_v > 1$. Fix a basis in V which consists of the bases of $V_1, V_2, \dots, V_r$ taken in the same order. If h is an endomorphism of V we will denote $m(h)$ the matrix corresponding to h in this basis. A trivial computation shows that for every $u \in U$

$$m(u) = \begin{pmatrix} 0 & u_{12} & u_{13} & \dots & u_{1r} \\ 0 & 0 & u_{23} & \dots & u_{2r} \\ \cdot & \cdot & \cdot & \dots & \cdot \\ 0 & 0 & 0 & \dots & u_{r-1r} \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix},$$

where u_{ij} is a matrix corresponding to an endomorphism from V_i to V_j, and

$$\lim_{n \to \infty} c^n m(s^{-n} u s^n) = \begin{pmatrix} 0 & u_{12} & 0 & \dots & 0 \\ 0 & 0 & u_{23} & \dots & 0 \\ \cdot & \cdot & \cdot & \dots & \cdot \\ 0 & 0 & 0 & \dots & u_{r-1r} \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}. \tag{21}$$

The matrix in (21) defines an element from the Lie algebra $\mathscr{L}_0$ of U_0. It follows from (21) that for every k the subspace $\langle \mathscr{L}_0^k V \rangle$ of V spanned by $\{g_1 g_2 \dots g_k(v) | g_i \in \mathscr{L}_0, v \in V\}$ coincides with $\langle \mathscr{L}^k V \rangle$. Let $g \in \mathscr{N}_L(U_0)$. Then $g \mathscr{L}_0 g^{-1} = \mathscr{L}_0$. Therefore

$$g \langle \mathscr{L}_0^k V \rangle = \langle g \mathscr{L}_0^k V \rangle = \langle g \mathscr{L}_0^k g^{-1} g V \rangle = \langle \mathscr{L}_0^k V \rangle.$$

Since $g\langle \mathscr{L}^k V\rangle = \langle \mathscr{L}^k V\rangle$ we obtain

$$m(g)=\begin{pmatrix} g_{11} & g_{12} & \dots & g_{1r} \\ 0 & g_{22} & \dots & g_{2r} \\ \cdot & \cdot & \dots & \cdot \\ 0 & 0 & \dots & g_{rr} \end{pmatrix}.$$

An easy computation shows that for every $x \in \mathrm{Lie}(W^-(s))$,

$$m(x)=\begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ x_{21} & 0 & \dots & 0 & 0 \\ x_{31} & x_{32} & \dots & 0 & 0 \\ \cdot & \cdot & \dots & \cdot & \cdot \\ x_{r1} & x_{r2} & \dots & x_{rr-1} & 0 \end{pmatrix}.$$

This implies that $\mathscr{N}_L(U_0) \cap W_L^-(s) = \{e\}$. The lemma is proved.

7 Basic Lemma

7.1 Let X be a second countable locally compact unimodular group and let θ be its Haar measure. Let V be a separable complete metric space with Borel probability measure μ. Assume that X acts continuously on V and that X preserves μ. Let $\mu = \int_Y \mu_y$ be the decomposition of μ into X-invariant X-ergodic probability measures μ_y, where y is identified with a point from a measure space (Y, σ). For $x \in V$, we denote by $y(x)$ the corresponding point from (Y, σ).

Definition. A sequence of measurable non-null sets $A_n \subset X$ is called *averaging net* if for the action of X on (V, μ) the following analog of the Birkhoff individual ergodic theorem is valid: if f is a continuous function on V with compact support then

$$\lim_{n\to\infty} \frac{1}{\theta(A_n)} \int_{A_n} f(gx)\, d\theta(g) = \int_V f(h)\, d\mu_{y(x)}(h) \tag{1}$$

for almost all $x \in V$.

The following result directly follows from [Tem, Corollary 3.2, Chap. 6].

Proposition. *Let A_n be a sequence of measurable non-null subsets in X. Then $\{A_n\}$ is an averaging net if the following conditions hold:*

(i) $\lim_{n\to\infty} \dfrac{\theta(A_n \Delta g A_n)}{\theta(A_n)} = 0$ *for every $g \in X$, where $A_n \Delta g A_n$ denotes the symmetric difference between A_n and gA_n;*

(ii) *$\{A_n\}$ is increasing;*

(iii) $\sup_{1 \leq n < \infty} \dfrac{\theta(A_n^{-1} A_n)}{\theta(A_n)} < \infty.$

7.2 Next we are going to apply Proposition 7.1 to our original situation. Recall that $\mathbf{H}$ is a $K_{\mathscr{S}}$-algebraic group, U and U_0 are unipotent $K_{\mathscr{S}}$-algebraic sup-groups of $H=\mathbf{H}(K_{\mathscr{T}})$, s is an element from the class $\mathscr{A}$ such that $U\subset W^+(s)$ and $U_0=\lim_{n\to+\infty} s^{-n}Us^n$. Also, recall the following notations from 5.1 and 5.2:

$L=W^-(s)\,Z(s)\,V$ is an $\mathrm{Int}(s)$-invariant $K_{\mathscr{T}}$-rational cross-section both for H/U and H/U_0, $p\colon U\to U_0$ is the projection of U on U_0 parallel to L and $\alpha_n\colon U\to U$ is a $K_{\mathscr{T}}$-regular isomorphism of $K_{\mathscr{T}}$-algebraic varieties given by (4), Sect. 5.

Lemma. *Let Γ be a closed subgroup of H and μ a U-invariant Borel probability measure on H/Γ. Let A be a relatively compact measurable non-null subset of U. Then $\{A_n=\alpha_n(A)\}$ is an averaging net. (Further on, $\{A_n\}$ will be call averaging net corresponding to A.)*

Proof. Assume that A is such that either (a) $A_{i+1}\supset A_i$ for every $i\geqq 1$, or (b) $A_i\cap A_j=\emptyset$ whenever $i\neq j$. We put $\tilde{A}_n=A_n$ in case (a) and $\tilde{A}_n=\bigcup_{i=1}^{n} A_i$ in case (b). Let us show that $\{\tilde{A}_n\}$ satisfies the conditions (i)–(iii) of Proposition 7.1. For every n the Jacobian $J(\alpha_n)$ of the map $\alpha_n\colon U\to U$ is constant. Therefore

$$\frac{\theta(\tilde{A}_n\,\Delta\, g\,\tilde{A}_n)}{\theta(\tilde{A}_n)}=\frac{\theta(\alpha_n^{-1}\,\tilde{A}_n)\,\Delta\,\alpha_n^{-1}(g\tilde{A}_n)}{\theta(\alpha_n^{-1}\,\tilde{A}_n)} \tag{2}$$

for every $g\in U$, and

$$\frac{\theta(\tilde{A}_n^{-1}\,\tilde{A}_n)}{\theta(\tilde{A}_n)}=\frac{(\alpha_n^{-1}(\tilde{A}_n^{-1}\,\tilde{A}_n))}{\theta(\alpha_n^{-1}(\tilde{A}_n))}. \tag{3}$$

It is easy to see that $\theta(\alpha_n^{-1}(\tilde{A}_n))\geqq\theta(A)$ and for every g

$$\lim_{n\to\infty}\theta(\alpha_n^{-1}(\tilde{A}_n)\,\Delta\,\alpha_n^{-1}(g\tilde{A}_n))=0.$$

This, in view of (2), proves (i). On the other hand, taking into account (3) and the fact that $\bigcup_{n=1}^{\infty}\alpha_n^{-1}(\tilde{A}_n^{-1}\,\tilde{A}_n)$ is a relatively compact set we obtain (iii). Since $\{\tilde{A}_n\}$ is increasing it follows from Proposition 7.1 that $\{\tilde{A}_n\}$ is an averaging net.

Let $c=J(\alpha_1)$ and $d=\theta(A)$. Then $c^n=J(\alpha_n)$, $dc^n=\theta(A_n)$ and $\theta(\tilde{A}_n)=d(c+c^2+\ldots+c^n)$. (Recall that $A_i\cap A_j=\emptyset$ if $i\neq j$.) Since $c>1$ we get $\lim_{n\to\infty}\frac{\theta(\tilde{A}_n)}{\theta(A_n)}=\frac{c}{c-1}$ and $\lim_{n\to\infty}\frac{\theta(\tilde{A}_{n-1})}{\theta(A_n)}=\frac{1}{c-1}$. Since $A_n=\tilde{A}_n-\tilde{A}_{n-1}$ the above relations and the fact that $\{\tilde{A}_n\}$ is an averaging net imply that $\{A_n\}$ is also an averaging net.

Let A be an arbitrary relatively compact measurable non-null subset of U. Recall that the automorphism $\mathrm{Int}(s^{-1})|_{W^-(s)}$ is contracting. From this and the definition of α_n (see (4), Sect. 5) one easily gets that if $x\in U$ and $x\neq e$ (resp. $x=e$) then there exists a compact neighborhood A' of x such that $\alpha_i(A')\cap\alpha_j(A')=\emptyset$ when $i\neq j$ (resp. $\{\alpha_i(A')\}$ is increasing). Since A is a relatively compact measurable set this implies that there exist measurable non-null subsets $A^{(1)}$, $A^{(2)}, \dots,$ of A with the following properties:

$$\theta\left(A-\bigcup_{i=1}^{\infty} A^{(i)}\right)=0, \quad A^{(i)}\cap A^{(j)}=\emptyset \quad \text{if} \quad i\neq j,$$

and for every i $\{\alpha_n(A^{(i)})\}$ is either increasing or $\alpha_n(A^{(i)})\cap\alpha_m(A^{(i)})=\emptyset$ when $n\neq m$. It follows from the above discussion that $\{\alpha_n(A^{(i)})\}$ is an averaging net for every i, $1\leq i<\infty$. Now using an elementary argument one can conclude that $\{A_n\}$ is an averaging net. The lemma is proved.

7.3 Let $A\subset U$ be a relatively compact, measurable, non-null set and $\{A_n\}$ be the corresponding to A averaging net.

Definition. We say that $M\subset H/\Gamma$ is a *set of uniform convergence relative to* $\{A_n\}$ if for every $\varepsilon>0$ and every continuous function f on H/Γ with compact support there exists a positive number $N(\varepsilon,f)$ such that for all $x\in M$ and $n>N(\varepsilon,f)$ we have

$$\left|\frac{1}{\theta(A_n)}\int_{A_n} f(gx)\,d\theta(g)-\int_{H/\Gamma} f(h)\,d\mu_{y(x)}(h)\right|<\varepsilon. \tag{4}$$

Lemma. *Let $\varepsilon>0$. There exists a measurable subset $M\subset H/\Gamma$ with $\mu(M)>1-\varepsilon$ which is a set of uniform convergence relative to $\{A_n=\alpha_n(A)\}$ for each relatively compact measurable non-null subset A of U.*

Proof. Since the Hausdorff topology on U is second countable there exists a sequence $\{B_i\}$ of open relatively compact subsets of U such that for every $\eta>0$ and every relatively compact measurable non-null subset A of U there exists a positive integer n with $\theta(B_n\Delta A)<\eta$. Fix a sequence of positive numbers ε_i such that $\sum_{i=1}^{\infty}\varepsilon_i<\varepsilon$. Using Lemma 7.2, the Egoroff theorem and the fact that the space $C_0(H/\Gamma)$ of continuous functions on H/Γ with compact support contains a countable everywhere dense subset, a standard argument shows that for every i there exists a set of uniform convergence M_i relative to $\{B_{i,n}=\alpha_n(B_i)\}$ with $\mu(M_i)>1-\varepsilon_i$. Put $M=\bigcap_{i=1}^{\infty} M_i$. Let us prove that M is a set of uniform convergence relative to $\{A_n=\alpha_n(A)\}$, where A is an arbitrary relatively compact non-null subset of U. Assume the contrary, i.e. there exist a function $f\in C_0(U/\Gamma)$, an increasing sequence of positive integers n_i, a sequence $x_i\in M$ and a positive constant d such that

$$\left|\frac{1}{\theta(A_{n_i})}\int_{A_{n_i}} f(gx_i)\,d\theta(g)-\int_{H/\Gamma} f(h)\,d\mu_{y(x_i)}(h)\right|>d, \tag{5}$$

for all i. Choosing B_m such that $\theta(A\Delta B_m)$ is sufficiently small we deduce from the fact that f has compact support that for all i

$$\left|\frac{1}{\theta(A)}\int_{A} f(\alpha_{n_i}(g)\,x_i)\,d\theta(g)-\frac{1}{\theta(B_m)}\int_{B_m} f(\alpha_{n_i}(g)\,x_i)\,d\theta(g)\right|<\frac{d}{3}. \tag{6}$$

Note that

$$(7)\qquad \frac{1}{\theta(A)}\int_A f(\alpha_{n_i}(g)\,x_i)\,d\theta(g)=\frac{1}{\theta(A_{n_i})}\int_{A_{n_i}} f(g\,x_i)\,d\theta(g)$$

and

$$(8)\qquad \frac{1}{\theta(B_m)}\int_{B_m} f(\alpha_{n_i}(g)\,x_i)\,d\theta(g)=\frac{1}{\theta(B_{m,n_i})}\int_{B_{m,n_i}} f(g\,x_i)\,d\theta(g).$$

On the other hand, in view of the choice of M taking i large enough we obtain

$$\left|\frac{1}{\theta(B_{m,n_i})}\int_{B_{m,n_i}} f(g\,x_i)\,d\theta(g)-\int_{H/\Gamma} f(h)\,d\mu_{y(x_i)}(h)\right|<\frac{d}{3},$$

which, after taking into account (6), (7) and (8), contradicts (5). The lemma is proved.

7.4 let f: $U\to U$ be a $K_{\mathcal{T}}$-rational map. Using the logarithmic map and fixing a basis in the Lie algebra Lie(U) we get a coordinate system on U. By *degree* of f we mean the maximum of the degrees of nominators and the denominators of the $K_{\mathcal{T}}$-rational functions which determine f in this coordinate system.

Lemma. *Let $\{f_n$: $U\to U\}$ be a sequence of $K_{\mathcal{T}}$-rational maps, $\mathcal{M}$ a Zariski open and dense subset of U and f: $U\to U$ a $K_{\mathcal{T}}$-rational isomorphism such that $f|_{\mathcal{M}}$ is a biregular map from $\mathcal{M}$ to $f(\mathcal{M})$. Assume that the degrees of f_n are bounded and that the sequence $\{f_n\}$ converges to f uniformly on compact subsets of $\mathcal{M}$. Then for any $x\in\mathcal{M}$ there exist a neighborhood $\mathcal{O}_x$ of x and a neighborhood $\mathcal{O}'_x$ of $f(x)$ such that for all sufficiently large n, $f_n(\mathcal{O}_x)\supset\mathcal{O}'_x$ and the restriction of f_n to $\mathcal{O}_x$ is a diffeomorphism of $\mathcal{O}_x$ onto $f_n(\mathcal{O}_x)$.*

To prove the above lemma one should apply Lemma 1.12 and the following observation. Let $\Phi_d(U)$ be the set of all $K_{\mathcal{T}}$-rational maps from U to U with degrees less than d. Then there exists a positive integer m, a $K_{\mathcal{T}}$-rational map F: $K_{\mathcal{T}}^m\times U\to U$ and a $K_{\mathcal{T}}$-regular map α: $\Phi_d(U)\to K_{\mathcal{T}}^m$ such that for every $f\in\Phi_d(U)$, $F(a(f),x)=f(x)$ on a Zariski open dense subset of U.

7.5 **Basic Lemma.** *Let M be a set of uniform convergence relative to every averaging net $\{A_n\}$ corresponding to a relatively compact non-null subset $A\subset U$. Let $\{x_n\}$ be a sequence in M converging to $x\in M$. Let $\{g_n\}$ be a sequence of elements in $H-\mathcal{N}_H(U)$ which satisfies the condition $(*)$ with respect to s (see 6.6). Suppose that $g_n x_n\in M$ for all n. Let φ be a U-quasiregular map corresponding to $\{g_n\}$ and constructed in 5.2. Then the ergodic component $\mu_{y(x)}$ is* Im(φ)*-invariant.*

Proof. We will use the notation of Sect. 5. Recall that φ was constructed as a limit of $K_{\mathcal{T}}$-rational maps φ_n: $U\to U$. To prove the lemma we need to establish some additional facts about φ_n and φ. Set $w_n=s^{-r(n)}g_n s^{r(n)}$. Since $\{g_n\}$ satisfies the condition $(*)$ with respect to s, passing to a subsequence we can (as we will) assume that w_n converges to an element $w\in W^-(s)$. Define a $K_{\mathcal{T}}$-rational map δ: $U\to U$ by the formula

$$(9)\qquad p(u)\,w\in Lp(\delta(u)),$$

where $L=W^-(s)\,Z(s)\,V$ and p: $U\to U_0$ is a projection parallel to L (see 5.1).

In view of (2) and (7) in 5.2, for every n there exists a $K_{\mathcal{T}}$-rational map $\delta_n: U \to U$ such that

$$a_n(u)\, g_n = \varphi_n(u)\, a_n(\delta_n(u)). \tag{10}$$

Note that (a) $\lim_{n\to\infty} s^{-r(n)} a_n(x)\, s^{r(n)} = p(x)$ for every $x \in U$, (b) φ_n are $K_{\mathcal{T}}$-rational maps from U to L, and (c) if $x \in L$ and the sequence $s^{-n} x s^n$ tends to an element $y \in H$ when $n \to \infty$ then $y \in W^-(s)\, Z(s)$ (because $W^-(s)\, Z(s)$ is a closed $\operatorname{Int}(s)$-invariant subgroup of H and $\operatorname{Int}(s^{-1})|_V$ acts as a contracting automorphism of V). This and (10) imply that the element

$$p(u)\, w = \lim_{n\to+\infty} s^{-r(n)} a_n(u)\, g_n\, s^{r(n)}$$

is contained in $W^-(s)\, Z(s)\, V$ for every u from the Zariski open subset $\mathcal{M} \overset{\text{def}}{=} (\varphi')^{-1}(M)$ (for the definition of $(\varphi')^{-1}(M)$ see 5.2). Therefore

$$U_0\, w \subset \overline{W^-(s)\, Z(s)\, U_0} \tag{11}$$

and the sequence $\{\delta_n\}$ of $K_{\mathcal{T}}$-rational maps converges to δ uniformly on compact subsets of $\mathcal{M}$. (Note that since the degrees of the $K_{\mathcal{T}}$-rational maps $\{\varphi_n\}$ and $\{a_n\}$ are bounded (see Sect. 5) we get from (10) that the degrees of $\{\delta_n\}$ are also bounded.)

It follows from (11) that $\overline{W^-(s)\, Z(s)\, U_0}$ is $\operatorname{Int}(w)$-invariant. Since the multiplication map $\mathbf{W}^-(s) \times \mathbf{Z}(s) \times \mathbf{U}_0 \to \overline{W^-(s)\, Z(s)\, U_0}$, $(w^-, z, u) \to w^- z u$, is a $K_{\mathcal{T}}$-isomorphism onto a Zariski open dense subset of $\overline{W^-(s)\, Z(s)\, U_0}$ and the subgroup $W^-(s)\, Z(s)$ is $\operatorname{Int}(w)$-invariant we obtain that the projection of $w^{-1} \mathbf{U} w$ onto $\mathbf{U}_0$ parallel to $\overline{W^{-1}(s)\, Z(s)}$ is a $K_{\mathcal{T}}$-rational isomorphism. This, in view of (9), implies that δ is a $K_{\mathcal{T}}$-rational isomorphism of $K_{\mathcal{T}}$-algebraic varieties.

Now let $u_0 \in \mathcal{M}$. Put $q = \varphi(u_0)$. We need to prove that the ergodic component $\mu_{y(x)}$ is q-invariant. This is equivalent to the fact that for all continuous functions f on H/Γ with compact support we have

$$\int_{H/\Gamma} f(h)\, d\mu_{y(x)}(h) = \int_{H/\Gamma} f^q(h)\, d\mu_{y(x)}(h), \tag{12}$$

where $f^q(h) = f(qh)$.

Let $A \subset \mathcal{M}$ be a compact neighborhood of u_0 in U such that $\{\varphi_n\}$ and $\{\delta_n\}$ converge to φ and δ, respectively, uniformly on A. Put $B = \delta(A)$ and $B(n) = \delta_n(A)$. It follows from Lemma 7.4 that

$$\lim_{n\to\infty} \theta(B \Delta B(n)) = 0. \tag{13}$$

Lemma 7.4 also implies that without loss of generality we can (and will) assume that there exists a compact subset $\tilde{B}$ such that $\tilde{B} \supset B \cup \left(\bigcup_{n=1}^{\infty} B(n)\right)$ and the sequence $\{\delta_n^{-1}\}$ converges uniformly to δ^{-1} on $\tilde{B}$.

Let f be a continuous function on H/Γ with compact support. For every n we put $A_n = \alpha_n(A)$ and $B_n = \alpha_n(B)$. In view of Lemma 7.2 $\{A_n\}$ and $\{B_n\}$ are averaging nets corresponding to A and B, respectively. Since $a_n(A) = A_{r(n)}$ (see

(5), Sect. 5) and the Jacobian of the K_τ-biregular map $a_n\colon U \to U$ is constant, in view of (10) we obtain

$$\text{(14)}\qquad \frac{1}{\theta(A_{r(n)})} \int_{A_{r(n)}} f(u g_n x_n)\, d\theta(u) = \frac{1}{\theta(A)} \int_A f(a_n(u)\, g_n x_n)\, d\theta(u)$$
$$= \frac{1}{\theta(A)} \int_A f(\varphi_n(u)\, a_n(\delta_n(u))\, x_n)\, d\theta(u).$$

Let $\varepsilon > 0$. Choosing A small enough we can find $n_0 > 0$ such that for all $n > n_0$

$$\text{(15)}\qquad \left| \frac{1}{\theta(A)} \int_A f(\varphi_n(u)\, a_n(\delta_n(u))\, x_n)\, d\theta(u) - \frac{1}{\theta(A)} \int_A f^q(a_n(\delta_n(u))\, x_n)\, d\theta(u) \right| < \varepsilon.$$

Since f^q is bounded and the Jacobian of δ_n converges uniformly to the Jacobian of δ on A, substituting $v = \delta_n(u)$, using (13) and replacing (if necessary) A by a smaller neighborhood of u_0, one can easily see that there exists a constant $n_1 \geqq n_0$ such that for all $n \geqq n_1$ we have

$$\text{(16)}\qquad \left| \frac{1}{\theta(A)} \int_A f^q(a_n(\delta_n(u))\, x_n)\, d\theta(u) - \frac{1}{\theta(B)} \int_B f^q(a_n(v)\, x_n)\, d\theta(v) \right| < \varepsilon.$$

Therefore, in view of (14), (15) and (16)

$$\text{(17)}\qquad \left| \frac{1}{\theta(A_{r(n)})} \int_{A_{r(n)}} f(u g_n x_n)\, d\theta(u) - \frac{1}{\theta(B_{r(n)})} \int_{B_{r(n)}} f^q(u x_n)\, d\theta(u) \right| < 2\varepsilon$$

for all $n > n_1$ (we use again that the Jacobian of a_n is constant).

On the other hand, M is a set of uniform convergence for both $\{A_n\}$ and $\{B_n\}$. Therefore there exists a constant $N(\varepsilon, f)$ such that if $r(n) \geqq N(\varepsilon, f)$ then

$$\left| \frac{1}{\theta(A_{r(n)})} \int_{A_{r(n)}} f(u g_n x_n)\, d\theta(u) - \int_{H/\Gamma} f(h)\, d\mu_{y(g_n x_n)}(h) \right| < \varepsilon$$

and

$$\left| \frac{1}{\theta(B_{r(n)})} \int_{B_{r(n)}} f^q(u x_n)\, d\theta(u) - \int_{H/\Gamma} f^q(h)\, d\mu_{y(x_n)}(h) \right| < \varepsilon.$$

Hence for all n such that $n > n_1$ and $r(n) > N(\varepsilon, f)$, in view of (17), we obtain

$$\left| \int_{H/\Gamma} f(h)\, d\mu_{y(g_n x_n)}(h) - \int_{H/\Gamma} f^q(h)\, d\mu_{y(x_n)}(h) \right| < 4\varepsilon.$$

So, to complete the proof, it is enough to show that if f is a continuous function on H/Γ with compact support and $\{z_n\}$ is a sequence from M converging to $z \in M$ then

$$\lim_{n \to +\infty} \int_{H/\Gamma} f(h)\, d\mu_{y(z_n)}(h) = \int_{H/\Gamma} f(h)\, d\mu_{y(z)}(h).$$

Let $\varepsilon_1 > 0$ and $N(\varepsilon_1, f)$ be such that if $n > N(\varepsilon_1, f)$ then

$$\left|\frac{1}{\theta(A_n)} \int_{A_n} f(u\bar{z})\, d\theta(u) - \int_{H/\Gamma} f(h)\, d\mu_{y(\bar{z})}(h)\right| < \varepsilon_1 \tag{18}$$

for every $\bar{z} \in M$.

Choosing n large enough, since $z_n \to z$ and f has compact support, we get

$$\left|\frac{1}{\theta(A_n)} \int_{A_n} f(u z_n)\, d\theta(u) - \frac{1}{\theta(A_n)} \int_{A_n} f(u z)\, d\theta(u)\right| < \varepsilon_1 .$$

Now, in view of (18)

$$\left| \int_{H/\Gamma} f(h)\, d\mu_{y(z_n)}(h) - \int_{H/\Gamma} f(h)\, d\mu_{y(z)}(h)\right| < 2\varepsilon_1 ,$$

which completes the proof of the assertion and with this the proof of the Basic Lemma.

8 Applications of the Basic Lemma and of the properties of φ

8.1 Let $G = \mathbf{G}(K_{\mathcal{T}})$, where $\mathbf{G}$ is a connected $K_{\mathcal{T}}$-algebraic group, $\mathcal{U}$ a unipotent $K_{\mathcal{T}}$-algebraic subgroup of G, Γ a discrete subgroup of G, and μ a Borel probability $\mathcal{U}$-invariant and $\mathcal{U}$-ergodic measure on G/Γ.

Up to the end of Sect. 8 *we will assume that the measure* μ is Zariski dense (*see* 3.3), $\mathcal{U}$ *is a maximal subgroup in the class of all unipotent* $K_{\mathcal{T}}$*-algebraic subgroups of* G *preserving* μ *and* $\mathcal{U}$ *is not a normal subgroup of* G.

Let $s \in \mathcal{N}_G(\mathcal{U})$ be an element from the class $\mathcal{A}$ preserving μ. Denote by $U^+(s)$ the maximal $K_{\mathcal{T}}$-algebraic subgroup of $W_G^+(s)$ preserving μ. Since $s\mu = \mu$ the element s normalizes $U^+(s)$. We set $\mathcal{F}(s) = \{g \in G \mid U^+(s)\, g$ is contained in the Zariski closure of $W_G^-(s)\, Z_G(s)\, U^+(s)\}$ and $U^-(s) = \mathcal{F}(s) \cap W_G^-(s)$. It follows from the discussion in 6.6 that $\mathcal{F}(s)$ and $U^-(s)$ are $K_{\mathcal{T}}$-algebraic subgroups of G. (Note that $\mathcal{F}(s)$ coincides with the group $\mathcal{F}$ introduced in 6.6 if we substitute $U^+(s)$ by U_0 from 6.6.)

We claim that $\mathcal{F}(s)$ contains $\mathcal{U}$. Indeed, denote by R the subgroup of G generated by $\mathcal{U}$ and $U^+(s)$. Let $\tilde{R}$ be the Zariski closure of R in G. Then R is open in the Hausdorff topology of $\tilde{R}$ [Bo-Pra, 2.2] and R is $\mathrm{Int}(s)$-invariant. Therefore, $R \cap W_G^+(s)$ is open $\mathrm{Int}(s)$-invariant subgroup of $\tilde{R} \cap W_G^+(s)$. Since $\mathrm{Int}(s^{-1})$ acts as a contraction on $W_G^+(s)$ we obtain that $R \cap W_G^+(s) = \tilde{R} \cap W_G^+(s)$. But $R \cap W_G^+(s)$ preserves μ and contains $U^+(s)$. In view of the definition of $U^+(s)$, this implies that $\tilde{R} \cap W_G^+(s) = U^+(s)$. By Proposition 2.7

$$\tilde{R} \subset \overline{(W_G^-(s) \cap \tilde{R})(Z_G(s) \cap \tilde{R})(W_G^+(s) \cap \tilde{R})}.$$

Thus $\tilde{R} \subset \overline{(W_G^-(s)\, Z_G(s)\, U^+(s)}$. Hence $\mathcal{U} \subset \mathcal{F}(s)$.

As in 6.6 one can write $W_G^+(s) = V^+(s)\, U^+(s)$ and $W_G^-(s) = U^-(s)\, V^-(s)$, where $V^+(s)$ and $V^-(s)$ are $K_{\mathcal{T}}$-rational sections for $W_G^+(s)/U^+(s)$ and $W_G^-(s)/U^-(s)$, respectively. In view of Proposition 2.7, there exists a Zariski open subset of

G containing e such that every element g from this subset has a unique representation $g=u^-(g)\,v^-(g)\,z(g)\,v^+(g)\,u^+(g)$, where $u^-(g)\in U^-(s)$, $v^-(g)\in V^-(s)$, $z(g)\in Z_G(s)$, $v^+(g)\in V^+(s)$ and $u^+(g)\in U^+(s)$.

8.2 Proposition. *With the above notation and assumption, let $N\subset G$ be a subgroup which is maximal in the class of normal subgroups of G preserving μ and generated by unipotent $K_{\mathscr{T}}$-algebraic subgroups of G. Assume that $\mathscr{U}\not\subset N$. Then there exists a $\mathscr{U}$-quasiregular map $\varphi\colon \mathscr{U}\to \mathscr{N}_G(\mathscr{U})$ such that*

(i) $\mathrm{Im}(\varphi)$ *consists of elements preserving* μ;
(ii) *if F is the subgroup of G generated by $\mathscr{U}$ and* $\mathrm{Im}(\varphi)$ *then F contains an element s from the class $\mathscr{A}$ with the following properties:*

(a) $U^+(s)\neq\{e\}$;
(b) $\alpha(s,\mathscr{F}(s))\geqq 1$;
(c) *if $N(s)$ denotes the subgroup of G generated by $W_G^+(s)$ and $W_G^-(s)$ then $N(s)/N(s)\cap N$ is an infinite group.*

Proof. Let us embed $\mathbf{G}$ in a $K_{\mathscr{T}}$-algebraic group $\mathbf{H}=\prod_{v\in\mathscr{T}}\mathbf{H}_v$, where $\mathbf{H}_v=\mathbf{SL}_m$. According to Lemma 6.8, there exists an element $t\in H, H=\mathbf{H}(K_{\mathscr{T}})$ from the class $\mathscr{A}$ such that $\mathscr{U}\subset W_H^+(t)$ and $\mathscr{N}_H(U_0)\cap W_H^-(t)=\{e\}$, where $U_0=\lim_{n\to+\infty} t^{-n}\mathscr{U}t^n$. Given a relatively compact non-null subset $A\subset\mathscr{U}$ we will denote by $\{A_n\}$ the averaging net corresponding to A as defined in 7.2 (i.e. $A_n=\alpha_n(A)$). In view of Lemma 7.3, for every $\varepsilon>0$ there exists a measurable subset $M_\varepsilon\subset H/\Gamma$ with $\mu(M_\varepsilon)>1-\varepsilon$ which is a set of uniform convergence for all averaging nets $\{A_n\}$ corresponding to relatively compact non-null subsets $A\subset\mathscr{U}$. (Note that G/Γ is contained in H/Γ, so the measure μ on G/Γ can be also considered as a measure on H/Γ.)

Denote by $\mathbf{N}$ the Zariski closure of N in $\mathbf{G}$. It follows from the Levi decomposition of $\mathbf{G}$ that there is a connected $K_{\mathscr{T}}$-algebraic subvariety $\mathbf{L}$ of $\mathbf{G}$ which contains e and is transversal to $\mathbf{N}$ at e and has the following property: $r(\mathbf{L})$ is a normal $K_{\mathscr{T}}$-algebraic subgroup of $\mathbf{G}/R_u(\mathbf{G})$ and $\mathbf{G}/R_u(\mathbf{G})$ is an almost direct product of $r(\mathbf{N})$ and $r(\mathbf{L})$ where $R_u(\mathbf{G})$ is the unipotent radical of $\mathbf{G}$ and $r\colon \mathbf{G}\to\mathbf{G}/R_u(\mathbf{G})$ is the natural epimorphism.

Put $P=\mathscr{N}_G(\mathscr{U})$. Let us show that $P\not\supset L$, where $L=\mathbf{L}(K_{\mathscr{T}})$. Assume the contrary. Then the set LN normalizes the group $\mathscr{U}N$. Since LN is Zariski dense in G this implies that G normalizes the Zariski closure E of $\mathscr{U}N$ in G. Therefore G normalizes the subgroup E^+ of E generated by all unipotent elements of E. But $\mathscr{U}N$ has finite index in E. Therefore $\mathscr{U}N$ contains all unipotent $K_{\mathscr{T}}$-algebraic subgroups of E i.e. $E^+=\mathscr{U}N$. In view of the maximality of N we obtain that $N\supset\mathscr{U}$ which contradicts our hypothesis. Therefore $P\not\supset L$.

It follows from Lemma 3.3 that for all sufficiently small ε there exists a converging to e sequence $\{g_n\}\subset\Psi(M_\varepsilon)\cap(L-P)$, where $\Psi(M_\varepsilon)=\{x\in G\,|\,xM_\varepsilon\cap M_\varepsilon\neq\phi\}$. Denote by $\varphi\colon\mathscr{U}\to H$ the quasiregular map corresponding to $\{g_n\}$ (see 5.2). Since $\{g_n\}\subset G$, the formula (17) in 5.2 implies that $\mathrm{Im}(\varphi)\subset G$. On the other hand, it follows from Proposition 6.7 and the choice of t that the sequence $\{g_n\}$ has the property $(*)$ with respect to t. Using the Basic Lemma, we deduce that $\mathrm{Im}(\varphi)$ preserves μ. This proves (i). Denote by F the subgroup generated by $\mathrm{Im}(\varphi)$ and $\mathscr{U}$. Then F is contained in $\mathscr{N}_G(\mathscr{U})$ (Proposition 6.1) and it is open in its Zariski closure in G. By virtue of our assumptions about

$\mathscr{U}$ (see 8.1), if V is a unipotent $K_{\mathscr{T}}$-algebraic subgroup of G and $V \subset F$ then $V \subset \mathscr{U}$. Now Proposition 6.3 implies that there exists a split $K_{\mathscr{T}}$-torus S in the Zariski closure $\bar{F}$ of F in G such that (a) $F/F \cap S\mathscr{U}$ is a compact group and (b) there exists an element $s \in S \cap F$ from the class $\mathscr{A}$ such that $U^+(s) \neq \{e\}$ and $\alpha(s, D) \geq 1$ for every $K_{\mathscr{T}}$-algebraic subgroup D of G normalized by SU. According to 8.1, $\mathscr{U} \subset \mathscr{F}(s)$. On the other hand, since $S \cap F$ commutes with s we obtain that $S \cap F$ normalizes $W_G^+(s)$, $Z_G(s)$ and $W_G^-(s)$. In view of the definition of $U^+(s)$ in 8.1, it follows that $S \cap F$ normalizes $U^+(s)$ and, therefore, $S \cap F$ normalizes $\mathscr{F}(s)$. Since $S \cap F$ is Zariski dense in S, we get that S normalizes $\mathscr{F}(s)$. Hence $\alpha(s, \mathscr{F}(s)) \geq 1$. So, we have proved that s has the properties (a) and (b) in the formulation of the proposition.

Since $\{r(g_i)\} \subset r(\mathbf{L})$ and $r(\mathbf{L})$ is a normal subgroup of the reductive group $\mathbf{G}/R_u(\mathbf{G})$ it follows from (17) in 5.2, that $r(S \cap F) \subset r(\mathbf{L})$. Note that $r \circ \varphi$ is a strongly quasiregular map. Therefore, in view of 6.2 and Lemma 6.4, $r(s)$ does not centralize $r(\mathscr{U})$. This implies that the subgroup $N(s)$ generated by $W_G^+(s)$ and $W_G^-(s)$ has nontrivial projection into $r(\mathbf{L})$ which proves that s has the property (c). The proposition is proved.

8.3 Proposition. *Let $s \in \mathscr{N}_G(\mathscr{U})$, $s \neq e$, be an element from the class $\mathscr{A}$ preserving μ. For every $\varepsilon > 0$, there exists a compact subset $M_\varepsilon \subset G/\Gamma$ with $\mu(M_\varepsilon) > 1 - \varepsilon$ such that if $\{g_i\}$ is a sequence of elements from $G - \mathscr{N}_G(U^+(s))$ converging to e and $g_i M_\varepsilon \cap M_\varepsilon \neq \phi$ for all i then the sequence $\ell^-(v^-(g_i)) - \ell^-(u^-(g_i))$ tends to $-\infty$ when i tends to $+\infty$. (Recall that the function ℓ^-: $W^-(s) \to \mathbf{Z}$ has been defined in 5.1.)*

Proof. Put $U = U^+(s)$ and $\mathscr{U}^0 = \mathscr{U} \cap Z_G(s)$. Denote by R the closure in the Hausdorff topology of G of the subgroup generated by $\mathscr{U}^0$ and s. It follows from the generalized Mautner Lemma [Mar6, Lemma 3] that R acts ergodically on $(G/\Gamma, \mu)$. Let $\mu = \int_{(Y,\nu)} \mu_y \, d\nu(y)$ be the decomposition of μ into U-invariant U-ergodic probability measures μ_y, where $y \in Y$ and (Y, ν) is a finite mesure space. If $x \in G/\Gamma$, we will denote by $y(x)$ the corresponding point from (Y, ν).

For every Borel probability measure σ on G/Γ we denote by W_σ the maximal $K_{\mathscr{T}}$-algebraic subgroup of $W_G^+(s)$ preserving σ. It is easy to see that if $\sigma = \lim_{i \to \infty} \sigma_i$ and the sequence $\ln(W_\sigma)$ converges to a $K_{\mathscr{T}}$-subspace L of $\mathrm{Lie}(W_G^+(s))$ then $\exp L \subset W_\sigma$. From this and the compactness of the Grassmannian variety $\mathrm{Gr}(\mathrm{Lie}(W_G^+(s)))$ one can easily get that (1) if $\sigma = \lim_{i \to \infty} \sigma_i$ then $\dim W_\sigma \geq \overline{\lim} \dim W_{\sigma_i}$; (2) the map $\sigma \mapsto \ln(W_\sigma)$ is continuous on the set $\{\sigma \mid \dim W_\sigma = \ell\}$ for every ℓ. Therefore, the following assertion is true

(A) The map $\sigma \mapsto \ln(W_\sigma)$ from the space of Borel probability measures on G/Γ into $\mathrm{Gr}(\mathrm{Lie}(W_G^+(s)))$ is Borel.

Set $W_x = W_{\mu_{y(x)}}$. Since R normalizes $\mathscr{U}$ we have that for every $g \in R$ the equality $\mu_{y(gx)} = \mu_{gy(x)}$ is true for almost all $x \in G/\Gamma$. Therefore, for every $g \in R$ we have that $W_{gx} = g W_x g^{-1}$ for almost all $x \in G/\Gamma$.

Denote by Ω the space of all $K_{\mathscr{T}}$-algebraic subgroups of $W_G^+(s)$. Then the above remark implies that the map f: $(G/\Gamma, \mu) \to \Omega$, $x \to W_x$, is R-equivariant. Since the logarithmic map defines an imbedding of Ω into $\mathrm{Gr}(\mathrm{Lie}(W_G^+(s)))$, it follows from the assertion (A) that f is a Borel map. Now, in view of Corollary 3.1 and the ergodicity of the action of R on $(G/\Gamma, \mu)$ we get that f is essentially constant. Therefore, there exists a conull subset $M_0 \subset G/\Gamma$ such that $W_x = U$ for all $x \in M_0$.

For every $\varepsilon > 0$ fix a compact subset $M_\varepsilon \subset M_0$ such that $\mu(M_\varepsilon) > 1 - \varepsilon$ and M_ε is a set of uniform convergence for all averaging nets $\{A_n\}$ corresponding to non-null relatively compact subsets A of U. Let $g_i \in G - \mathcal{N}_G(U)$ be a sequence converging to e and $g_i M_\varepsilon \cap M_\varepsilon \neq \phi$ for all i. Assume that the sequence $\{\ell^-(v^-(g_i)) - \ell^-(u^-(g_i))\}$ does not tend to $-\infty$ when $i \to \infty$. Passing to a subsequence, we will assume (without loss of generality) that the sequence is bounded from below and that for every i there exists an $x_i \in M_\varepsilon$ such that $g_i x_i \in M_\varepsilon$ and $\lim_{i \to \infty} x_i = x$ where $x \in M_\varepsilon$. By Proposition 6.7, the sequence $\{g_i\}$ satisfies the property $(*)$ with respect to s. Let φ be a U-quasiregular map corresponding to $\{g_i\}$ and constructed in 5.2. It follows from Basic Lemma, Proposition 6.1 and Proposition 6.7 that $\operatorname{Im}(\varphi) \subset W_G^+(s) \cap \mathcal{N}_G(U)$ and that $\operatorname{Im}(\varphi)$ preserves the ergodic component $\mu_{y(x)}$. Let F be the subgroup generated by U and $\operatorname{Im}(\varphi)$ and $\tilde{F}$ be the Zariski closure of F in $W_G^+(s)$. Note that $\tilde{F}/U$ is a group of $K_{\mathscr{T}}$-rational ponts points of a $K_{\mathscr{T}}$-algebraic group (Proposition 1.8) and that F/U is a noncompact open subgroup of $\tilde{F}/U$ (see 6.3 and Proposition 6.1). In view of Proposition 4.1, this implies that F/U contains a nontrivial unipotent $K_{\mathscr{T}}$-algebraic subgroup of $\tilde{F}/U$. Since F preserves $\mu_{y(x)}$, we obtain that $W_x \neq U$ which contradicts the fact that $x \in M_0$. The proposition is proved.

8.4 Corollary. *Let $s \neq e$ be an element from the class $\mathscr{A}$* preserving *μ and $s \in \mathcal{N}_G(\mathscr{U})$. Then there exists a conull subset $M \subset G/\Gamma$ such that $M \cap W_G^-(s)\, x \subset U^-(s)\, x$ for every $x \in M$.*

Proof. For every $\varepsilon > 0$, let M_ε be a subset of G/Γ as given by Proposition 8.3. Let $\mu = \int_{(Z,\rho)} \mu_z \, d\rho(z)$ be the decomposition of μ into $\langle s \rangle$-ergodic components, where $\langle s \rangle$ denotes the cyclic subgroup generated by s. As usual, if $x \in M_\varepsilon$, we will denote by $z(x)$ the corresponding point from (Z, ρ).

For every $z \in (Z, \rho)$ denote by C_z the intersection $\operatorname{Supp}(\mu_z) \cap M_\varepsilon$ where $\operatorname{Supp}(\mu_z)$ denotes the support of v_z. Let $\delta = \rho\{z \in (Z, \rho) \mid \mu_z(C_a) \geqq \frac{2}{3}\}$. Then using Fubini's theorem, $\delta + \frac{2}{3}(i - \delta) \geqq 1 - \varepsilon$. Whence $1 - \delta \geqq 3\varepsilon$. Let $M_1 = \{x \in M_\varepsilon \mid \mu_{z(x)}(C_{z(x)}) \geqq \frac{2}{3}\}$ and $M_1' = \{x \in M_\varepsilon \mid v_{z(x)}(C_{z(x)}) \leqq \frac{2}{3}\}$. It is easy to see that $\mu(M_n') \geqq 2\varepsilon$. Hence $\mu(M_1) \geqq 1 - 3\varepsilon$. It follows from the Birkhoff ergodicity theorem, that there exists a measurable subset $M_2 \subset M_1$ with $\mu(M_1 - M_2) = 0$ which has the following property: if χ denotes the characteristic function of M_2 then for every $x \in M_2$ the sequence $\frac{1}{n} \sum_{i=1}^{n} \chi(s^i x)$ tends to a number greater than or equal to $\frac{2}{3}$.

Let $x_1, x_2 \in M_2$ and $x_2 = w x_1$ where $w \in W_G^-(s)$. Assume that $v^-(w) \neq e$. In view of the above property of M_2, there exists an increasing sequence of positive integers $\{n_i\}$ such that $s^{n_i} x_1, s^{n_i} x_2 \in M_2$ for all i. Put $g_i = s^{n_i} w s^{-n_i}$. Clearly, $s^{n_i} x_2 = g_i s^{n_i} x_1$ for all i and $\lim_{i \to \infty} g_i = e$. Note that $g_i \notin \mathcal{N}_G(U^+(s))$ for every i. Indeed, if $g_i \in \mathcal{N}_G(U^+(s))$ then $w \in \mathcal{N}_G(U^+(s))$, because $\mathcal{N}_G(U^+(s))$ is $\operatorname{Int}(s)$-invariant. Hence, $w \in U^-(s)$, contradicting the assumption that $v^-(w) \neq e$. By Proposition 8.3, the sequence $\{\ell^-(v^-(g_i)) - \ell^-(u^-(g_i))\}$ tends to $-\infty$ when $i \to \infty$. On the other hand, if $v^-(w) \neq e$, then one can easily deduce from (9) in 6.6 that the sequence is bounded from below. So, the assumption that $v^-(w) \neq e$ leads to contradiction and, therefore, $W_G^-(s)\, x \cap M_\varepsilon \subset U^-(s)\, x$ for every $x \in M_2$. Recall that $\mu(M_2)\geqq 1-3\varepsilon$. Now, passing to a limit when $\varepsilon\to 0$, it is easy to obtain a conull subset M such that $W_G^-(s)x\cap M\subset U^-(s)x$ for all $x\in M$. The corollary is proved.

9 Entropy of translations of homogeneous spaces

In this section, G and Γ denote the same as in Sect. 8. We fix an element $s\in G$ from the class $\mathscr{A}$ and write $W^-=W_G^-(s)$, $Z=Z_G(s)$ and $W^+=W_G^+(s)$. Let μ be a Borel s-invariant probability measure on G/Γ.

We can consider G as a $K_{\mathscr{T}}$-algebraic subgroup of $\mathrm{GL}_n(K_{\mathscr{T}})$. The absolute values $|\ \ |_v$ on K_v induce a norm $\|\ \ \|$ on the ring of $K_{\mathscr{T}}$-endomorphisms $\mathrm{End}(K_{\mathscr{T}}^n)$. Define a metric ρ' on $\mathrm{End}(K_{\mathscr{T}}^n)$ by the formula $\rho'(A,B)=\|A-B\|$. Since $\mathrm{GL}_n(K_{\mathscr{T}})\subset\mathrm{End}(K_{\mathscr{T}})$ the metric ρ' induces a metric on G which we denote also by ρ'. Let us fix a right invariant metric ρ on G such that on every compact subset $L\subset G$ the metrics $\rho|_L$ and $\rho'|_L$ are equivalent in a sense that their ratio is bounded. This metric induces a metric on G/Γ which will also be denoted by ρ.

9.1 Fix a point $p\in G/\Gamma$ such that every neighborhood of p in G/Γ has positive measure μ. Fix relatively compact neighborhoods B' and C' of e in W^- and ZW^+ respectively, such that the map

$$x\mapsto xp,\ x\in D'\overset{\mathrm{def}}{=}B'C',$$

is a homeomorphism onto an open subset $D\overset{\mathrm{def}}{=}D'p$ of G/Γ. We write $C=C'p$.

Lemma. *Assume that* $\mathrm{diam}(sXs^{-1})\leqq\frac{1}{10}\mathrm{diam}(X)$ *for every* $X\subset B'$. *For every* $c\in C$, *there exists a containing* c *subset* E_c *of* W^-c *such that:*

(1) $E_c\subset B'c$;

(2) E_c *is open in* W^-c *(in the orbit topology) and the subset* $E\overset{\mathrm{def}}{=}\bigcup_{c\in C}E_c$ *is open in* G/Γ;

(3) *whenever* $s^nE_c\cap E\neq\phi$, $c\in C$, $n>0$, *we have* $s^nE_c\subset E$.

Proof. We can assume that B' is a sphere of radius $a/2$ centered at e, i.e. $B'=\left\{x\in W^-\,|\,\rho(e,x)<\frac{a}{2}\right\}$. Let B_0 denote the sphere in W^- or radius $\frac{1}{10}$ centered at e.

For every $c\in C$ we define the set E_c as follows: $x\in E_c$ if and only if there exists a nonnegative integer p, a sequence $\{c_0=c,c_1,\dots,c_p\}$ of elements in C and sequence $\{n_0=0,n_1,\dots,n_p\}$ of nonnegative integers such that $x\in s^{n_i}B_0c_p$ and $s^{n_{i-1}}B_0c_{i-1}\cap s^{n_i}B_0c_i\neq\phi$ for every $1\leqq i\leqq p$. The minimal p for which such sequences exist will be denoted by $p(x)$.

It easily follows from the above definition that E_c has the properties (2) and (3). Let us prove (1) by induction on $p(x)$. The assertion is trivial if $p(x)=0$. Assume that (1) is proved for every $y\in E_d$, $d\in C$, with $p(y)\leqq k-1$. Let $x\in E_c$, $p(x)=k>0$ and let $\{c_0=c,c_1,\dots,c_k\}$ and $\{n_0=0,\ n_1,\dots,n_k\}$ be corresponding sequences. Let $n=n_j=\min\{n_1, \ldots, n_k\}$. Recall that $B'd_1 \cap B'd_2 = \phi$ if $d_1, d_2 \in C$, $d_1 \neq d_2$. From this and the induction assumption, we get that $n>0$. The induction assumption also implies that

$$\bigcup_{i=1}^{j} s^{n_i-n} B_0 c_i \subset B' c_j \quad \text{and} \quad \bigcup_{i=j}^{k} s^{n_i-n} B_0 c_i \subset B' c_j.$$

But $n>0$, $\operatorname{diam}(B') \leqq a$ and $\operatorname{diam}(sXs^{-1}) \leqq \frac{1}{10} \operatorname{diam}(X)$ for every $X \subset B'$. Therefore

$$\operatorname{diam}\left(\bigcup_{i=1}^{k} s^{n_i} B_0 c_i\right) \leqq \frac{a}{10}.$$

This implies that

$$\operatorname{diam}\left(\bigcup_{i=1}^{k} s^{n_i} B_0 c_i\right) \leqq \operatorname{diam}(B_0 c_0) + \operatorname{diam}\left(\bigcup_{i=1}^{k} s^{n_i} B_0 c_i\right) \leqq \frac{a}{10} + \frac{a}{10} < \frac{a}{2}.$$

But the union $\bigcup_{i=0}^{k} s^{n_i} B_0 c_i$ contains both c *and* x. Hence $x \in B'c$.

9.2 Lemma. *Let M be a relatively compact open subset in a $K_{\mathscr{S}}$-analytic variety V. If μ is a probability measure on M* and q: $M \to (0, 1)$ *is such that* $\log q$ is *μ-integrable, then there exists a countable partition $\mathscr{P}$ of M with entropy $H(\mathscr{P}) < \infty$ such that, if $\mathscr{P}(x)$ denotes the atom of $\mathscr{P}$ containing x, then* $\operatorname{diam} \mathscr{P}(x) \leqq q(x)$.

The above lemma is an analog for $K_{\mathscr{S}}$-analytic varieties of Lemma 2 in [Ma] and its proof is virtualy the same.

9.3 We will use the standard terminology and results from ergodic theory (see [Roh]).

Definition. We say that a measurable partition ξ of the measure space $(G/\Gamma, \mu)$ is *subordinate* to a closed subgroup V of G if for almost all (with respect to μ) $x \in G/\Gamma$, we have

(a) $\xi(x) \subset Vx$ where $\xi(x)$ denotes, as usual, the element of ξ containing x;
(b) $\xi(x)$ is relatively compact in Vx in the orbit topology.
(c) $\xi(x)$ contains a neighborhood of x in Vx.

Let η and η' be measurable partitions of $(G/\Gamma, \mu)$. We write $\eta \leqq \eta'$ if $\eta(x) \supset \eta'(x)$ for almost all (with respect to μ) $x \in G/\Gamma$. We define a partition $g\eta, g \in G$, by $(g\eta)(x) = g(\eta(g^{-1}x))$.

Proposition. *Assume that μ is s-ergodic. Then there exists a measurable partition η of the measure space $(G/\Gamma, \mu)$ with the following properties*:

(i) *η is subordinate to W^-*;
(ii) *η is s-invariant, i.e. $\eta \leqq s\eta$*;
(iii) *the mean conditional entropy $H(s\eta \mid \eta)$ is equal to the entropy $h(s, \mu)$ of the automorpism $x \mapsto sx$, $x \in G/\Gamma$, of the measure space $(G/\Gamma, \mu)$.*

Proof. Let E_c and E denote the same as in Lemma 9.1. Denote by π: $E \to C$ the natural projection ($\pi(x) = c$ if $x \in E_c$). We set $\eta(x) = E_{\pi(x)}$ for every $x \in E$. It is enough to find a countable measurable partition ξ of $(G/\Gamma, \mu)$ such that

$H(\xi)<\infty$ and $\eta(x)=\xi^-(x)$ for almost all $x\in E$ where $\xi^-=\bigvee_{n=0}^{\infty} s^{-n}\xi$ is the product of the partitions $s^{-n}\xi, 0\leqq n<\infty$. Indeed, let us set $\eta=\xi^-$. It is clear that η is s-invariant. The set of $x\in G/\Gamma$ for which properties (a) and (b) (resp. (c)) in the definition of a subordinate partition are satisfied is s^{-1}-invariant (resp. s-invariant) and contains E. But $\mu(E)>0$ and μ is s-ergodic. Therefore, η^- is subordinate to W^-. To check the property (iii) it is enough to show that the partition $\xi_s=\bigvee_{-\infty}^{\infty} s^k\xi$ is the partition into points (see [Roh, Sect. 9]). We have that $\xi^-(x)=\eta(x)\subset B'\cdot B'^{-1}x$ if $x\in E$. On the other hand, the automorphism $\mathrm{Int}(s)|_{W^-}$ is contracting. Therefore, $\xi_s(x)=\{x\}$ if $s^{-n}x\in E$ for infinitely many positive n. But $\mu(E)>0$ and μ is s-ergodic. Hence $\xi_s(x)=x$ for almost all x.

Let us construct the desired partition ξ. For $x\in E$, let $n(x)$ be the smallest positive integer n such that $s^n x\in E$. Since $\mu(E)>0$ and μ is s-invariant and s-ergodic, we get (using standard arguments from ergodic theory) that

(1) $$\int_E n(x)\,d\mu(x)=1.$$

Define a probability measure μ' on C by

(2) $$\mu'(X)=\frac{\mu(\pi^{-1}(X))}{\mu(E)},\ X\subset C.$$

Property (3) of the family $\{E_c|c\in C\}$ implies that $n(x)$ is constant on every E_c, $c\in C$. Therefore, in view of (1) and (2)

(3) $$\int_C n(c)\,d\mu'(c)<\infty.$$

There exists $\lambda>1$ such that $\rho(sg_1, sg_2)\leqq\lambda\rho(g_1,g_2)$ for all $g_1,g_2\in G$. Since the function $n(c)$ is μ'-integrable, one can find a positive function $q(c)<\lambda^{-n(c)}$, $c\in C$, such that the function $\log q(c)$ is μ'-integrable and the μ'-essential infinum $\operatorname{ess\,inf}_{c\in C} q(c)$ is 0.

The multiplication map $W^-\times ZW^+\to G$, $(x,y)\to xy$, is a diffeomorphism onto an open subset of G. Therefore replacing, if necessary, B' and C' by smaller subsets we can find $\varepsilon>0$ such that (a) $\|g\|\leqq 2\rho(\pi(x),\pi(y))$ whenever $x,y\in E$, $y=gx$, $g\in ZW'$ and $\|g\|<\varepsilon$; (b) if $x,y\in C$ there exists $g\in ZW^+$ such that $y=gx$ and $\|g\|<\varepsilon$.

Since the function $\log q(c)$ is μ'-integrable, there exists a countable measurable partition $\mathscr{P}$ of C such that $H(\mathscr{P})<\infty$ and $\operatorname{diam}\mathscr{P}(x)<\frac{\varepsilon}{2}q(x)$ for almost all $x\in C$ (see 9.2). Now we define a countable measurable partition ξ of G/Γ by

$$\xi(x)=\begin{cases}\pi^{-1}(\mathscr{P}(\pi(x))) & \text{if } x\in E\\ (G/\Gamma)-E & \text{if } x\notin E.\end{cases}$$

Since $H(\mathscr{P})<\infty$, we get using (2) that $H(\xi)<\infty$. It remains to show that $\eta(x)=\xi(s)$ for almost all $x\in E$. It follows from the property (3) of the family

$\{E_c\}$ that $\eta(z) \subset \xi(z)$. Let x and y be elements in E with $\xi^-(x) = \xi^-(y)$. Since $\eta(z) \subset \xi(z)$ we can assume that $x, y \in C$. Then $y = gx$ where $g \in ZW^+$ and $\|g\| < \varepsilon$. Set $x_1 = x$, $y_1 = y$, $g_1 = g$ and define by induction

$$x_{k+1} = s^{n(x_k)} x_k,\ y_{k+1} = s^{n(x_k)} y_k,\ g_{k+1} = s^{n(x_k)} g_k\, s^{-n(x_k)}.$$

A trivial induction argument shows that

$$y_k = g_k\, x_k. \tag{4}$$

Let us prove that

$$\|g_k\| < \varepsilon q(\pi(x_k)) \quad \text{for all } k \geqq 0. \tag{5}$$

If $k=1$, the inequality (5) is true because $\operatorname{diam} \mathscr{P}(x) < \frac{\varepsilon}{2} q(\pi(x))$ and $\mathscr{P}(x) = \mathscr{P}(y)$. Assume that (5) is proved for k. Then

$$\|g_{k+1}\| = \|s^{n(x_k)} g_k\, s^{-n(x_k)}\| \leqq \lambda^{n(x_k)} \|g_k\| \leqq \varepsilon \lambda^{n(x_k)} q(\pi(x_k)) \leqq \varepsilon.$$

Then since x_{k+1} and $y_{k+1} = g_{k+1}\, x_{k+1}$ belong to the same element of the partition ξ (because $\xi^-(x) = \xi^-(y)$) and $\operatorname{diam} \mathscr{P}(\pi(x_k)) \leqq \frac{\varepsilon}{2} q \pi(x_k))$ we get from the definition of $\varepsilon > 0$ that (5) is true for $k+1$.

Since the measure μ is s-ergodic and $\operatorname{ess\,inf}_{c \in C} q(c) = 0$ we have that $\liminf_{k \to \infty} q(\pi(x_k)) = 0$ for almost all $x \in E$. On the other hand, if $h \in ZW^+$ and $h \neq e$ then e is not a limit point of the sequence $\{s^n h s^{-n} | n \geqq 0\}$. Therefore (5) implies that $g = e$ and $x = y$.

Remark. It follows from the construction of η that for almost all $x \in G/\Gamma$ the map $W^- \to W^- x$, $w \to wx$, is bijective. Indeed, let $x \in G/\Gamma$ be such that the set of positive integers $I = \{n | s^n \in E\}$ is infinite. Let W_0 be a relatively compact subset of W^-. Since the automorphism $\operatorname{Int}(s)|_{W^-}$ is contracting, we get $s^n W_0 x = s^n W_0 s^{-n} s^n x \subset E$ for large enough $n \in I$. Therefore, the map $w \mapsto wx$, $w \in W_0$, is bijective for every relatively compact W_0. This proves our assertion.

9.4 Lemma (see [Led-Str, Proposition 2.2]). *Let T be an automorphism of a measure space (X, σ), $\sigma(X) < \infty$, and let f be a positive finite measurable function defined on X such that*

$$\log_2^- \frac{f \circ T}{f} \in L^1(X, \sigma), \qquad \textit{where} \quad \log_2^-(a) = \min(\log_2 a, 0).$$

Then

$$\int_X \log_2 \frac{f \circ T}{f}\, d\mu = 0.$$

9.5 Lemma. *Let V be a closed subgroup of W^- normalized by s and η be a measurable partition of $(G/\Gamma, \mu)$ subordinate to V. Assume that $\eta \leqq s\eta$, and that for almost all $x \in G/\Gamma$, the conditional measure $\mu_{x,\eta}$ of μ on $\eta(x)$ is proportional to the restriction to $\eta(x)$ of a V-invariant measure on Vx. Then the measure μ is V-invariant.*

Proof. The measure μ induces in a standard way conditional measures $\mu_{x,V}$ on the orbits Vx. These measures are defined and unique up to a proportionality for almost all $x \in X$. From the assumption about $\mu_{x,\eta}$ we get that for almost all $x \in G/\Gamma$, the restriction of $\mu_{x,V}$ to $\eta(x)$ is proportional to the restriction of the V-invariant measure. Thus the uniqueness of $\mu_{x,V}$ and the s-invariance of μ imply that the restriction of $\mu_{x,V}$ to $(s^{-n}\eta)(x)$ is proportional to the restriction of the V-invariant measure. On the other hand, since the automorphism $\mathrm{Int}(s)|_{V^-}$ is contracting, we have that $\bigcup_{0 \leqq n \leqq \infty} (s^{-n}\eta)(x) = Vx$ for almost all $x \in X$. Therefore the measures $\mu_{x,V}$ are V-invariant and, hence, the measure μ is V-invariant.

9.6 Proposition. *Let V be a closed subgroup of W^- normalized by s and let η be a s-invariant measurable partition of $(G/\Gamma, \mu)$ subordinate to V.*
(i) *If μ is V-invariant, then $H(s\eta|\eta) = \log_2 \alpha(s^{-1}, V)$ where $H(s\eta|\eta)$ is the mean conditional entropy and $\alpha(s, V)$ is defined in* 1.5.
(ii) $H(s\eta|\eta) \leqq \log_2 \alpha(s^{-1}, V)$. *The equality $H(s\eta|\eta) = \log_2 \alpha(s^{-1}, V)$ implies that $V\mu = \mu$.*

Proof. Since $\eta \leqq s\eta$ for almost all $x \in G/\Gamma$ we have a partition η_x of $\eta(x)$ such that $\eta_x(y) = (s\eta)(y)$ for almost all $y \in \eta(x)$. Denote by τ the Haar measure on V. Since $\eta(x) \subset Vx$, τ induces a measure on $\eta(x)$ which we will denote also by τ. Put $L(x) = \tau(\eta(x))$ and $\tau_x = \tau/L(x)$, $x \in G/\Gamma$. Note that on $\eta(x)$ we have a conditional probability measure μ_x induced by μ. Put $p(x) = \tau_x(\eta_x(s))$ and $r(x) = \mu_x(\eta_x(x))$. Then since $\eta_x(x) = s(\eta(s^{-1}x))$ one easily sees that $p(x) = \dfrac{L(s^{-1}x)\,\alpha^{-1}}{L(x)}$, where $\alpha = \alpha(S^{-1}, V)$ (see 1.5). Since η is a measurable partition subordinate to V, $L(x)$ is a positive finite measurable function. Note that $p(x) \leqq 1$. Therefore $\log_2^- \dfrac{L(s^{-1}x)}{L(x)} \in L^1(G/\Gamma, \mu)$. In view of Lemma 9.4, we obtain

$$-\int_{G/\Gamma} \log_2 p(x)\, d\mu(x) = \log_2 \alpha. \tag{6}$$

Assume that μ is V-invariant. Then $\mu_x = \tau_x$ for almost all $x \in G/\Gamma$, in particular, $p(x) = r(x)$ for almost all $x \in G/\Gamma$. But

$$-\int_{G/\Gamma} \log_2 r(x)\, d\mu(x) = H(s\eta|\eta). \tag{7}$$

This in view of (6) proves (i).

Let $Y_i(x)$, $1 \leqq i < \infty$, denote the elements of the countable partition η_x of $\eta(x)$. Then we have

$$\int_{\eta(x)} \log_2 p(y)\, d\mu_x(y) - \int_{\eta(x)} \log_2 r(y)\, d\mu_x(y) \tag{8}$$

$$= \sum_{i=1}^{\infty} \log_2 \frac{\tau_x(Y_i(x))}{\mu_x(Y_i(x))}\, \mu_x(Y_i(x)).$$

We have that

(9) $$\sum_{i=1}^{\infty} \tau_x(Y_i(x)) \leq 1$$

and

(10) $$\sum_{i=1}^{\infty} \mu_x(Y_i(x)) = 1.$$

(In (9), we can have inequality because apriori it is possible that the measure τ_x of $\eta(x) - \bigcup_{1 \leq i \leq \infty} Y_i(x)$ is positive). From (8), (9) and (10), using the convexity of log we get that

$$\int_{\eta(x)} \log_2 p(y)\, d\mu_x(y) \leq \int_{\eta(x)} \log_2 r(y)\, d\mu_x(y)$$

and the equality holds if and only if $p(y) = r(y)$ i.e. $\tau_x(\eta_x(y)) = \mu_x(\eta_x(y))$ for all $y \in \eta(x)$. Now using integration over the quotient space $(G/\Gamma, \mu)/\eta$ of the measure space $(G/\Gamma, \mu)$ by η we get from (6) and (7) that $H(s\eta|\eta) \leq \log_2 \alpha$ and the equality holds if and only if $\tau_x((s\eta)(x)) = \mu_x((s\eta)(x))$ for almost all $x \in G/\Gamma$.

Assume that $H(s\eta|\eta) = \log_2 \alpha(s^{-1}, V)$. Then $H(s^k\eta|\eta) = \log_2 \alpha(s^{-k}, V)$ for every $k > 0$. Using the same argument as above and replacing s by s^k, we get that $\tau_x((s^k\eta)(x)) = \mu_x((s^k\eta)(x))$ for any $k > 0$ and almost all $x \in G/\Gamma$. On the other hand since η is subordinate to V and the automorphism $\operatorname{Int}(s)$ is contracting on V we have that $\bigvee_{k=1}^{\infty} s^k\eta$ is the partition into points. Hence $\mu_x = \tau_x$ for almost all $x \in G/\Gamma$. In view of Lemma 9.5, it implies that μ is V-invariant.

9.7 Theorem. *Assume that the element s acts ergodically on the measure space $(G/\Gamma, \mu)$. Let V be a closed subgroup of W^- normalized by s. Put $\alpha = \alpha(s^{-1}, V)$.*

(i) *If μ is V-invariant, then $h(s, \mu) \geq \log_2 \alpha$.*

(ii) *Assume that there exists a subset $\Psi \subset G/\Gamma$ with μ-measure 1 such that $\Psi \cap W^- x \subset Vx$ for every $x \in \Psi$. Then $h(s, \mu) \leq \log_2(\alpha)$ and the equality implies that μ is V-invariant.*

Proof. According to Propostion 9.3, there exists a measurable s-invariant subordinate to W^- partition η of $(G/\Gamma, \mu)$ such that $H(s\eta|\eta) = h(s, \mu)$. Let $x \in G/\Gamma$ be such that the map $w \mapsto wx$, $w \in W^-$, is bijective. (In view of Remark 9.3, the set of all $x \in G/\Gamma$ with this property is conull.) Set $\eta'(x) = Vx \cap \eta(x)$. Then η' is a measurable s-invariant partition of $(G/\Gamma, \mu)$ subordinate to V. Since $h(s, \mu) \geq H(s\eta'|\eta')$, the part (i) of the theorem follows from the equality $H(s\eta'|\eta') = \log_2(\alpha)$ (Proposition 9.6 (i)).

Now assume that $\Psi \cap W^- x \subset Vx$ for every x from a conull subset $\Psi \subset G/\Gamma$. Then η and η' coincide on Ψ (i.e. $\eta(x) \cap \Psi = \eta'(x) \cap \Psi$). Hence $H(s\eta|\eta) = H(s\eta'|\eta')$. By Proposition 9.3(iii), $h(s, \mu) = H(s\eta|\eta)$. Using Proposition 9.6(ii) we obtain that $h(s, \mu) \leq \log_2 \alpha$ and the equality implies that μ is V-invariant. The theorem is proved.

10 Proof of Theorem 1

Let $G = \mathbf{G}(K_{\mathscr{T}})$, where $\mathbf{G}$ is a $K_{\mathscr{T}}$-algebraic, $\mathscr{U}$ a unipotent $K_{\mathscr{T}}$-algebraic subgroup of G, Γ a discrete subgroup of G and μ a Borel probability $\mathscr{U}$-ergodic $\mathscr{U}$-invariant measure on G/Γ.

We need the following simple

10.1 Lemma. *If there exists a closed (in the Hausdorff topology) normal unimodular subgroup N of G such that μ is N-invariant and N-ergodic then μ is algebraic.*

Proof. Let π: $G \to G/\Gamma$ be the natural projection. Denote by μ' the lifting of μ to G i.e.

$$\mu'(X) = \int_{G/\Gamma} a_X(y)\, d\mu(y)$$

where $a_X(y)$ is the number of elements in $\pi^{-1}(y) \cap X$. Then $\mu' \Gamma = \mu'$. On the other hand $N\mu' = \mu'$, and since the subgroup N is unimodular and normal in G, $\mu' N = \mu'$. Thus $\mu' N\Gamma = \mu'$ and hence $\mu' F = \mu'$ where $F \subset G$ is the closure of $N\Gamma$ in the Hausdorff topology. Since μ is N-ergodic, we have that μ' is F-ergodic. From this, we get that μ' is a F-invariant measure on a coset gF. Hence μ is algebraic (here we use that $F \supset \Gamma$).

10.2 Proposition 2.7(a) easily implies the following.

Lemma. *Let $s \in G$ be an element from the class $\mathscr{A}$ and let H be a $K_{\mathscr{T}}$-algebraic subgroup of G normalized by s. Then*

$$\alpha(s, H) = \alpha(s, W_H^-(s))\, \alpha(s, W_H^+(s)).$$

10.3 In proving Theorem 1, we may (and will) assume the following: (i) $\mathscr{U}$ is a maximal subgroup in the class of all unipotent $K_{\mathscr{T}}$-algebraic subgroups of G preserving μ; (ii) the measure μ is Zariski dense, i.e. G does not contain a proper $K_{\mathscr{T}}$-algebraic subvariety M of G such that $\mu(\pi(M)) > 0$ (in view of Proposition 3.2); (iii) the $K_{\mathscr{T}}$-algebraic group $\mathbf{G}$ is connected (in view of (ii)); (iv) G does not contain a normal unimodular subgroup N of G such that μ is N-invariant and N-ergodic (in view of Lemma 10.1).

10.4 Let N be the maximal subgroup in the class of all normal subgroups of G preserving μ and generated by unipotent $K_{\mathscr{T}}$-algebraic subgroups of G. (A standard argument from the theory of linear algebraic groups shows that N is closed in the Hausdorff topology on G.) In view of assumption (iv) in 10.3, we have that $\mathscr{U} \not\subset N$. According to Proposition 8.2, there exists a $\mathscr{U}$-quasiregular map φ: $\mathscr{U} \to \mathscr{N}_G(\mathscr{U})$ such that $\mathrm{Im}(\varphi)$ consists of elements preserving μ and the subgroup F generated by $\mathscr{U}$ and $\mathrm{Im}(\varphi)$ contains an element from the class $\mathscr{A}$ such that: (1) $U^+(s) \neq \{e\}$, (2) $\alpha(s, \mathscr{F}(s)) \geqq 1$, (3) $N(s)/N(s) \cap N$ is an infinite group, where $N(s)$ is the (normal) subgroup generated by $W_G^+(s)$ and $W_G^-(s)$. (We use the notation from Sect. 8).

Denote $|\det \mathrm{Ad}\, h|$, $h \in G$, by $d(h)$. Since $\varphi(u) = \lim_{n \to \infty} a_n(u)\, g_n\, b_n(u)$, the elements $a_n(u)$ and $\mathrm{b}_n(\mathrm{u})$ are unipotent, $d(h) = 1$ if h is unipotent, and $\lim_{n \to \infty} g_n = 1$, we have that

$$d(\varphi(u)) = \lim_{n \to \infty} d(a_n(u))\, d(g_n)\, d(b_n(u)) = 1.$$

Thus $\alpha(g, G)=1$ for every $g \in \operatorname{Im}(\varphi)$ and, consequently, for all $g \in F$. In particular, $\alpha(s, G)=1$.

10.5 Now the proof of Theorem 1 can be completed in three steps.

Step 1 In view of 8.4, there exists a conull subset of M such that $M \cap W_G^-(s)\, x \subset U^-(s)\, x$ for every $x \in M$. Let $\mu = \int_{(Z,\rho)} \mu_z \, d\rho(z)$ be the decomposition of μ into $\langle s \rangle$-ergodic components. It follows from Mautner's lemma [Mar6, Lemma 3, p. 31] that every $\langle s \rangle$-ergodic component is $U^+(s)$-invariant. By Fubini's theorem, $\mu_z(M)=1$ for almost all (with respect to ρ) $\langle s \rangle$-ergodic components μ_z. Fix an $\langle s \rangle$-ergodic component μ_z of the measure μ with the property $\mu_z(M)=1$. Since $h(s, \mu_z)=h(s^{-1}, \mu_z)$, Theorem 9.7 implies

$$\log_2 \alpha(s, U^+(s)) \leqq h(s, \mu_z) \leqq \log_2 \alpha(s^{-1}, U^-(s)).$$

But

$$\alpha(s^{-1}, U^-(s)) = \alpha(s, U^-(s))^{-1}$$

and in view of Lemma 9.2

$$(1) \qquad \alpha(s, \mathscr{F}(s)) = \alpha(s, U^+(s))\, \alpha(s, U^-(s)) \geqq 1.$$

Therefore

$$h(s, \mu_z) = \log_2 \alpha(s^{-1}, U^-(s)).$$

It follows from Theorem 9.7(ii), that μ_z is a $U^-(s)$-invariant measure. Therefore the measure μ is $U^-(s)$-invariant.

Step 2 Assume that $U^-(s) \neq W_G^-(s)$. This, in view of the definition of $U^-(s)$ in 8.1, implies that $U^+(s)$ is not a normal subgroup of G. It follows from Lemma 3.3 that there exist a constant c, $0<c<1$, such that if $\Omega \subset G/\Gamma$ is a measurable set with $\mu(\Omega)>1-c$ then there exists a converging to e sequence $\{g_n\} \subset \Psi(\Omega)$ such that

$$\{g_n\} \subset (V^-(s)\, Z_G(s)\, W_G^+(s) - (Z_G(s)\, W_G^+(s) \cup \mathcal{N}_G(\mathrm{U}^+(s)))) \cap \Psi(\Omega).$$

Then $\ell^-(v^-(g_n)) > -\infty$ and $\ell^-(u^-(g_n)) = -\infty$. This, in view of Proposition 8.3, leads to contradiction. Thus $U^-(s)=W^-(s)$, and hence, μ is $W_G^-(s)$-invariant.
Step 3 In view of 10.2 we have that

$$(2) \qquad \alpha(s, G) = \alpha(s, W_G^+(s))\, \alpha(s, W_G^-(s)) = 1.$$

The restriction of the automorphism $\operatorname{Int}(s^{-1})$ to $W^+(s)$ is contracting. But $U^+(s) \subset W^+(s)$. Therefore $\alpha(s, U^+(s)) \leqq \alpha(s, W^+(s))$ and the equality holds if and only if $U^+(s)=W^+(s)$. From this, (1) and (2) and the equality $U^-(s)=W^-(s)$ we get that $U^+(s)=W^+(s)$. Therefore, μ is $N(s)$-invariant which contradicts the maximality of N and the choice of s. The theorem is proved.

11 Some applications

We formulate in this section some theorems about closures of orbits of unipotent subgroups, uniform distribution and values of families of quadratic forms. These results, are analogs of corresponding results for real Lie groups (see [D-Mar 5, 6; R 5, 6]). We will give only indications what should be changed in the proofs for the real case to get the proofs for the case of $K_{\mathcal{S}}$-algebraic groups. As for real Lie groups the description of finite measures invariant and ergodic relative to unipotent subgroups is used in a major way. Another important ingredient is an analog of Dani's theorem about the finiteness of ergodic measures invariant under actions of unipotent subgroups.

As in Sect. 10, let $G=\mathbf{G}(K_{\mathcal{S}})$ where $\mathbf{G}$ is a $K_{\mathcal{S}}$-algebraic group, $\mathcal{U}$ a unipotent $K_{\mathcal{S}}$-algebraic subgroup of G, and Γ a discrete subgroup of G.

11.1 Theorem. *Assume that Γ is a lattice in G, i.e. the volume of G/Γ with respect to the Haar measure is finite. Then, for any $x\in G/\Gamma$, there exists a closed subgroup $L=L(x)\subset G$ containing $\mathcal{U}$* such that the closure of the orbit *$\mathcal{U}x$ coincides with Lx.*

This theorem which is an analog of Theorem A in [R 5], is easily deduced from Theorem 11.2 and Proposition 11.3. Note that Theorem 11.2 is an analog of Theorem B in [R 5] and Proposition 11.3 is an analog of Proposition 2.1 in [D-Mar6] and Theorem 1.1 in [R 5].

11.2 Theorem (Uniform distribution) *Let $v\in\mathcal{S}$ and let $\mathcal{U}=\{u(t)\,|\,t\in K_v\}$ be a one-parameter unipotent $K_{\mathcal{S}}$-algebraic subgroup of $\mathbf{G}_v(K_v)$. Denote by σ_v the Haar measure on K_v. Let A be a Borel relatively compact subset of K_v with $\sigma_v(A)>0$. Assume that Γ is a lattice in G. Then for any $x\in G/\Gamma$, there exists a closed subgroup $L=L(x)\subset G$ containing $\mathcal{U}$ such that closure of the orbit $\mathcal{U}x$ coincides with Lx, Lx admits L-invariant Borel probability measure θ and*

$$\lim_{|T|_v\to\infty}\frac{1}{|T|_v}\int_{TA} f(u(t)\,x)\,d\sigma_v(t)=\int_{Lx} f(y)\,d\theta(y)$$

for any bounded continuous function f on G/Γ.

Note that for one-parameter $\mathcal{U}$, Theorem 11.2 is a stronger version of Theorem 11.1. Theorem 11.2 is an easy consequence of Theorem 1 and Theorems 11.4 and 11.6 formulated below.

11.3 Proposition. *Denote by C the set of all closed subgroups H of G such that $H\cap\Gamma$ is a lattice in H and the Zariski closures of $H\cap\Gamma$ and H coincide. Then C is countable.*

11.4 Theorem. *Let $\mathcal{U}=\{u(t)\,|\,t\in K_v\}$ and σ_v be the same as in Theorem 11.2. Assume that Γ is a lattice in G. Let F be a compact subset of G/Γ and let $\varepsilon>0$ be given. Then there exists a compact subset M of G/Γ such that for any $x\in F$ and $B>0$*

$$\sigma_v(\{t\in K_v\,|\,|t|_v<B \quad \textit{and} \quad u(t)\,x\in M\})\geqq(1-\varepsilon)\,B.$$

This theorem is an analog of Theorem 6.1 in [D-Mar6] and Proposition 1.3 in [R 5]. Let us make some remarks about the proof.

It is easy to make a reduction to the case where the groups $\mathbf{G}_v$ are semisimple and have no $K_{\mathcal{S}}$-anisotropic factors. Then, in view of the arithmeticity theorem, either rank $G \overset{\text{def}}{=} \sum_{v \in \mathscr{T}} \text{rank}_{K_v} \mathbf{G}_v = 1$ or Γ is an arithmetic subgroup of G. In the former case, we can assume that G is a real group (because as it is well known, any lattice in a p-adic Lie group is cocompact) and we can use results from [D4] and [D5] (see also Theorem 6.1 in [D-Mar6]). If Γ is arithmetic, one can assume that $\Gamma = \text{SL}_n(\mathbf{Q}(S))$ and $G = \prod_{p \in S} \text{SL}_n(\mathbf{Q}_p)$ where S is a containing ∞ finite set of valuations of $\mathbf{Q}$ and $\mathbf{Q}(S)$ denote the ring of S-integers in $\mathbf{Q}$. Then if $\mathbf{Q}(S) = \mathbf{Z}$, Theorem 11.4 is essentially Theorem 3.2 in [D5]. In the general case, we can use the same type of arguments as in [D2] and [D5] and also as in the proof of Theorem 1 in [Mar1] (which can be considered as a weak version of Theorem 2.1 in [D2]). These arguments are based on some properties of polynomials and on the study of maps of some partially ordered sets into the space of polynomials.

11.5 Theorem. *Let H be a subgroup of G generated by unipotent $K_{\mathscr{T}}$-algebraic subgroups of G contained in H. Let ν be a locally finite H-invariant measure on G/Γ. Assume that Γ is a lattice in G. Then there exist Borel H-invariant subsets X_i, $1 \leqq i < \infty$, such that $\nu(X_i) < \infty$ for all i and $G/\Gamma = \bigcup_{i=1}^{\infty} X_i$. In particular, every locally finite H-ergodic H-invariant measure on G/Γ is finite.*

For unipotent H, Theorem 11.5 is easily deduced from Theorem 11.4. One can reduce the general case to the case of unipotent H using analogs for $K_{\mathscr{T}}$-algebraic groups of Moore's results on Mautner phenomenon (see [Mo]).

11.6 As in [D-Mar6] for any closed subgroup W of G we denote by $S(W)$ the set of all $x \in G/\Gamma$ for which there exists a proper closed subgroup H of G containing W such that Hx admits a finite H-invariant measure; under this condition Hx is automatically a proper closed subset of G/Γ. We put $\mathscr{G}(W) = G/\Gamma - S(W)$.

Theorem. *Let W be a subgroup of G generated by unipotent $K_{\mathscr{T}}$-algebraic sub-groups of G contained in W. Let F be a compact subset of $\mathscr{G}(W)$. Assume that Γ is a lattice in G. Then for any $\varepsilon > 0$, there exists a neighborhood Ω of $S(W)$ such that for any one-parameter $\{u(t)\}$ of G, where $t \in K_v$, $v \in \mathscr{T}$, any $x \in F$ and any $B \geqq 0$*

$$\sigma_v\{t \in K_v \mid |t|_v < B, u(t)\,x \in \Omega\} \leqq \varepsilon B.$$

The proof of the above theorem is analogous to the proof of Theorem 1 in [D-Mar6] and is independent of the results on invariant measures.

11.7 The following theorem is an analog of Theorem 2 in [D-Mar6] and can be considered as a generalization of Theorem 11.2.

Theorem. *Assume that Γ is a lattice in G. Let θ be the G-invariant probability measure on G/Γ. Let $v \in \mathscr{T}$ and let $\{u_i(t)\}$, $t \in K_v$, be a sequence of one-parameter unipotent $K_{\mathscr{T}}$-algebraic subgroups of G converging to a unipotent one-parameter $K_{\mathscr{T}}$-algebraic subgroup $\{u(t)\}$, $t \in K_v$; that is, $u_i(t) \to u(t)$ for all t. Let $\{x_i\}$ be* *a sequence in G/Γ converging to a point in $\mathscr{G}(\{u(t)\})$, let A and σ_v denote the same as in Theorem 11.2, and let $\{T_i\}$ be a sequence in K_v such that $|T_i|_v$ tends to infinity. Then for any bounded continuous function f on G/Γ*

$$\lim_{i\to\infty} \frac{1}{|T_i|_v} \int_{T_iA} f(u_i(t)x_i)\, d\sigma_v(t) = \int_{G/\Gamma} f(y)\, d\theta(y).$$

11.8 We will use some notation and terminology from [Bo-Pra]. Let k be a number field. For every place v of k, let k_v denote the completion of k at v. Let S be a finite set of places of k containing the set S_∞ of archimedean ones, k_S the direct sum of the field $k_s (s\in S)$ and $\mathcal{O}_S$ the ring of S-integers of k.

Let F be a quadratic form on k_S^n. Equivalently, F can be viewed as a collection $F_s (s\in S)$, where F_s is a quadratic form on k_s^n. We say that F is *non-degenerate* (resp. *isotropics*) if each F_s is non-degenerate (resp. isotropic). The form F will be said to be *rational* (over k) if it is a multiple of a form on k^n, i.e. if there exists a form F_0 on k^n and λ invertible in k_S such that $F=\lambda F_0$, and *irrational* otherwise.

We have that $\mathcal{O}_s^n$ is a cocompact lattice in k_S^n. Let θ be the Haar measure on k_S^n such that the volume of $k_S^n/\mathcal{O}_S^n$ with respect to θ is 1.

Let $Q_S(n)$ denote the space of non-degenerate indefinite quadratic forms on k_S^n. The space $Q_S(n)$ has a natural locally compact topology given by pointwise convergence as functions on k_S^n.

The following theorem is a generalization of Corollary 5 in [D-Mar6]. The proof is based on some modifications of Theorem 11.7 and is analogous to the proof of Corollary 5 in [D-Mar6].

Theorem. *Let M be a compact subset of $Q_S(n)$ and let Ω be a relatively compact neighborhood of 0 in k_S^n. Then we have the following:*

(i) *for any relatively compact open subset I in k_S and $\alpha>0$ there exists a finite subset L of M such that each quadratic form $F\in L$ is rational and for any compact subset C of $M-L$ there exists $r_0>0$ such that for all F in C and all $t=\{t_s\}\in k_S$ with $|t_s|_s>r_0$ (as usual $|x|_s$ denotes the value of $s\in S$ at $x\in K_S$),*

$$|\{z\in t\Omega\cap \mathcal{O}_S^n \mid F(z)\in I\}| \geqq (1-\alpha)\,\theta(\{v\in t\Omega \mid F(v)\in I\});$$

(ii) *if $n\geqq 5$, for every $\varepsilon>0$ there exist $c>0$ and $r_0>0$ such that for all $F=\{F_s\}\in M$ and $t=\{t_s\}\in k_S$ with $|t_s|_s>r$,*

$$\begin{aligned} &|\{z=\{z_s\}\in t\Omega\cap \mathcal{O}_S^n) \mid |F_s(z_s)|_s<\varepsilon\}| \\ &\quad \geqq c\,\theta(\{v=\{v_s\}\in t\Omega \mid |F_s(v_s)|<\varepsilon)\}. \end{aligned}$$

11.9 It is possible to prove analogs for algebraic groups over local fields of other results about actions of unipotent groups on homogeneous spaces of real Lie groups. In particular, it is possible to prove analogs of recent results of Mozes and Shah about limits of invariant measures.

Acknowledgements. We would like to thank A. Borel who initiated the interest of the second author to this area and stimulated our work. We would also like to thank A. Katok for the many useful discussions about entropy, and also for pointing us to the paper [Led-Y].

G. Margulis was partially supported by N.S.F. grant (DMS-9204270) and G. Tomanov was partially supported by N.S.F. grants (DMS-9100383) and (INT-8819854).

References

[B-Z] Bernstein, I.N., Zelevinski, A.V.: Representation of the group GL(n, F) where F is a non-archimedean local field. Russ. Math. Surv. **31.3**, 1–68 (1976)

[Bo1] Borel, A.: Linear Algebraic Groups, second enlarged edition. Berlin Heidelberg New York: Springer 1991

[Bo2] Borel, A.: Density properties of certain subgroups of semisimple groups. Ann. Math. **72**, 179–188 (1960)

[Bo-Pra] Borel, A., Prasad, G.: Values of isotropic quadratic gorms at S-integral points. Compos. Math. **83**, 347–372 (1992)

[Bo-Spr] Borel, A., Springer, T.A.: Rationality properties of linear algebraic groups. Tôhoku Math. J. **20**, 443–497 (1968)

[D1] Dani, S.G.: Invariant measures of horospherical flows on noncompact homogeneous spaces. Invent. Math. **47**, 101–138 (1978)

[D2] Dani, S.G.: On invariant measures, minimal sets, and a lemma of Margulis. Invent. Math. **51**, 239–260 (1979)

[D3] Dani, S.G.: Invariant measures and minimal sets of horospherical flows. Invent. Math. **64**, 357–385 (1981)

[D4] Dani, S.G.: On orbits of unipotent flows on homogeneous spaces. Ergod. Th. Dyn. Syst. 4, 25–34 (1984)

[D5] Dani, S.G.: On orbits of unipotent flows on homogeneous spaces – II. Ergod. Th. Dyn. Syst. 6, 167–182 (1986)

[D-Mar1] Dani, S.G., Margulis, G.A.: Values of quadratic forms at primitive integral points. Invent. Math. **98**, 405–424 (1989)

[D-Mar2] Dani, S.G., Margulis, G.A.: Orbit closures of generic unipotent flows on homogeneous spaces of SL(3, **R**). Math. Ann. **286**, 101–128 (1990)

[D-Mar3] Dani, S.G., Margulis, G.A.: Values of quadratic forms at integral points; an elementary approach. Enseign. Math. **36**, 143–174 (1990)

[D-Mar4] Dani, S.G., Margulis, G.A.: Asymptotic behavior of trajectories of unipotent flows on homogeneous spaces. Proc. Indian Acad. Sci. Math. Sci. **101**, 1–17 (1991)

[D-Mar5] Dani, S.G., Margulis, G.A.: On the limit distributions of orbits of unipotent flows and integral solutions of quadratic inequalities. C.R. Acad. Sci., Paris, Ser. I **314**, 698–704 (1992)

[D-Mar6] Dani, S.G., Margulis, G.A.: Limit distributions of orbits of unipotent flows and values of quadratic forms (to appear)

[D-Smi] Dani, S.G., Smillie, J.: Uniform distribution of horocycle flows for Fuchsian groups. Duke Math. J. **51**, 185–194 (1984)

[Led-Str] Ledrapier, F., Strelcyn, J.-M.: A proof of the estimation from below in Pesin's entropy formula. Ergodic Theory Dyn. Syst. **2**, 203–219 (1982)

[Led-Y] Ledrapier, F., Young, L.-S.: The metric entropy of diffeomorphisms. I. Ann. Math. **122**, 503–539 (1985)

[Ma] Mañé, R.: A proof of Pesin's formula. Ergodic Theory Dyn. Syst. **1**, 95–102 (1981)

[Mar1] Margulis, G.A.: On the action of unipotent groups in the space of lattices. In: Gelfand, I.M. (ed.) Proc. of the summer school on group representations. Bolyai Janos Math. Soc., Budapest, 1971, pp. 365–370. Budapest: Akadémiai Kiado 1975

[Mar2] Margulis, G.A.: Formes quadratiques indefinies et flots unipotents sur les espaces homogénes. C.R. Acad. Sci., Paris, Ser. I **304**, 249–253 (1987)

[Mar3] Margulis, G.A.: Discrete subgroups and ergodic theory. In: Aubert, K.E. et al. (eds.) Proc. of the conference "Number theory, trace formula and discrete grops" in honor of A. Selberg. Oslo 1987, pp. 377–388. London New York: Academic Press 1988

[Mar4] Margulis, G.A.: Orbits of group actions and values of quadrtic forms at integral points. In: Festschrift in honour of I.I. Piatetski-Shapiro. (Isr. Math. Conf. Proc., vol. 3, pp. 127–151) Jerusalem: The Weizmann Science Press of Israel 1990
[Mar5] Margulis, G.A.: Discrete Subgroups of Semisimple Groups. Berin Heidelberg New York: Springer 1990
[Mar6] Margulis, G.A.: Dynamical and ergodic properties of subgroup actions on homogeneous spaces with applications in number theory. In: Sutake, I. (ed.) Proceedings of the International Congress of Mathematicians, Kyoto 1990, pp. 193–215. Tokyo: The Mathematical Society of Japan and Berlin Heidelberg New York: Springer 1991
[Mar-To] Margulis, G.A., Tomanov, G.M.: Measure rigidity for algebraic groups over local fields. C.R. Acad. Sci., Paris **315**, 1221–1226 (1992)
[Mo] Moore, C.C.: The Mautner phenomenon for general unitary representations. Pac. J. Math. **86**, 155–169 (1980)
[Pra1] Prasad, G.: Elementary proof of a theorem of Bruhat-Tits-Rousseau and a theorem of Tits. Bull. Sci. Math. Fr. **110**, 197–202 (1982)
[Pra2] Prasad, G.: Ratner's Theorem in S-arithmetic setting. In: Workshop on Lie Groups, Ergodic Theory and Geometry. (Publ., Math. Sci. Res. Inst., p. 53) Berlin Heidelberg New York: Springer 1992
[R1] Ratner, M.: Horocycle flows: joining and rigidity of products. Ann. Math. **118**, 277–313 (1983)
[R2] Ratner, M.: Strict measure rigidity for unipotent subgroups of solvable groups. Invent. Math. **101**, 449–482 (1990)
[R3] Ratner, M.: On measure rigidity of unipotent subgroups of semi-simple groups. Acta. Math. **165**, 229–309 (1990)
[R4] Ratner, M.: On Raghunathan's measure conjecture. Ann. Math. **134**, 545–607 (1992)
[R5] Ratner, M.: Raghunathan topological conjecture and distributions of unipotent flows. Duke Math. J. **63**, 235–280 (1991)
[R6] Ratner, M.: Invariant measures and orbit closures for unipotent actions on homogeneous spaces. (Publ., Math. Sci. Res. Inst.) Berlin Heidelberg New York: Springer (to appear)
[R7] Ratner, M.: Raghunathan's conjectures for p-adic Lie Groups. Int. Math. Research Notices. Number 5, 141–146 (1993) (in Duke Math. J. 70:2)
[Roh] Rohlin, V.A.: Lectures on the theory of entropy of transformations with invariant measures. Russ. Math. Surv. **22:5**, 1–54 (1967)
[Sh] Shah, N.: Uniformly distributed orbits of certain flows on homogeneous spaces. Math. Ann. **289**, 315–334 (1991)
[Tem] Tempelman, A.: Ergodic Theorems for Group Actions. Dordrecht Boston London: Kluwer 1992
[W] Witte, D.: Rigidity of some translations on homogeneous spaces. Invent Math. **81**, 1–27 (1985)
[Zi] Zimmer, R.: Ergodic Theory and Semisimple Groups. Boston Basel Stuttgart: Birkhäuser 1984

Note added in proofs

Recently the authors obtained some generalizations and corollaries from Theorem 2. In these results, G is a group from a class of central extensions of Kg-algebraic groups, Γ is a closed subgroup of G and H is a subgroup from a class of closed subgroups of G. In particular, we reduce the question about algebraicity of an H-invariant, H-ergodic, probability measure μ on G/Γ to the case where H is a central extension of a split algebraic torus. Using known results about Mautner phenomenon, we also obtaine simple argument deducing the measure rigidity for general real Lie groups from the measure rigidity for real algebraic groups.