Invent. math. 116, 309–328 (1994)

Inventiones mathematicae

The asymptotic behavior of invariant collective motion

Friedrich Knop *

Department of Mathematics, Rutgers University, New Brunswick NJ 08903, USA
Mathematisches Institut, Rheinsprung 21, CH-4051 Basel, Switzerland

Oblatum 20-XI-1992 & 10-II-1993

For Armand Borel

1 Introduction

Let G be a connected complex reductive group acting on a smooth algebraic variety X. Then the cotangent bundle T_X^* on X carries a canonical symplectic structure, and the G-action induces a moment map $\Phi: T_X^* \to \mathfrak{g}^*$. Consider the Hamiltonian vector fields attached to functions of the form $f \circ \Phi$ with $f \in \mathbb{C}[\mathfrak{g}^*]^G$. In this paper we study the asymptotic behavior of the associated flow (a so-called invariant collective motion) and show that it possesses a symmetry with respect to a finite reflection group W_X. This is applied to the theory of equivariant embeddings of X. The approach is purely algebraic.

More specifically: Choose any generic point $\alpha \in T_X^*$. Because the functions $f \circ \Phi$ with $f \in \mathbb{C}[\mathfrak{g}^*]^G$ are in involution (i.e., their Poisson product vanishes), the flow through α is in the orbit of an abelian group A_α. It is known (see [GS]) that this orbit is also the orbit for the connected isotropy group $G^0_{\Phi(\alpha)}$. This implies that A_α is a linear algebraic group and it turns out that it is a torus. The projection of this orbit to X is called a *flat* of X and just equals $G^0_{\Phi(\alpha)}\pi(\alpha)$. In case, X is the complexification of a symmetric space, a flat in our sense is the complexification of a usual flat (= maximal totally geodesic, flat submanifold).

Let $X \subseteq \bar{X}$ be a normal equivariant embedding. The main point of this paper is to study the closure of a generic flat in $\bar{X}$. This will be done in two different steps.

The first one is to show that a certain finite group W_X acts on them. Consider the family of tori $\alpha \mapsto A_\alpha$. Although every two of these groups are isomorphic to each other, the family cannot in general be trivialized globally. But we show that it can be trivialized on a finite cover $\hat{T}_X$ of an open subset of T_X^*. Hence, there is an action of a torus A_X on $\hat{T}_X$ such that flats are just the projections of the orbits to X. The map of $\hat{T}_X$ onto its image in T_X^* is a Galois covering

* Partially supported by: Schweizerischer Nationalfonds zur Förderung der wissenschaftlichen Forschung

such that its group W_X is a subquotient of the Weyl group W of G. This group W_X also acts on A_X (Theorem 4.1, Theorem 4.2).

Let $\bar{F}^{\hat{\alpha}}$ be the closure of $\pi(A_X\hat{\alpha})$ in $\bar{X}$ where $\hat{\alpha}\in\hat{T}_X$. Observe that $\bar{F}\hat{\alpha}=\bar{F}_{w\hat{\alpha}}$ for every $w\in W_X$. There is a rigidity lemma (Lemma 6.1) which says that for any torus action the closures of any two generic orbits are canonically isomorphic. Applied to our situation, this gives an isomorphism $\bar{F}_{\hat{\alpha}}\xrightarrow{\sim}\bar{F}_{w\hat{\alpha}}=\bar{F}_{\hat{\alpha}}$, which 060implies:

1.1. Theorem *For $\hat{\alpha}\in\hat{T}_X$ generic, the action of W_X on $A_X\cong F_\alpha$ can be extended to $\bar{F}_{\hat{\alpha}}$.* [Corollary 6.3]

The second part is as follows: For each A_X-orbit $E\subseteq\bar{F}_{\hat{\alpha}}$ consider the set of one-parameter subgroups λ: $\mathbf{G}_m\to A_X$ such that $\lim_{t\to 0}\pi(\lambda(t)\hat{\alpha})$ exists and is contained in E. Then let $\mathscr{C}(E)$ be the closure of the convex cone generated by this set inside $\mathscr{H}(X):=\mathrm{Hom}(\mathbf{G}_m, A_X)\otimes\mathbb{Q}$. Because $\bar{F}_{\hat{\alpha}}$ contains only finitely many orbits, we get a finite set of cones, a so-called fan. It is known that the normalization of $\bar{F}_{\hat{\alpha}}$ is completely determined by its fan.

The main point is that we are able to control the limit behavior of one-parameter subgroups which are in a certain cone $\mathscr{Z}(X)\subseteq\mathscr{H}(X)$. To describe this cone, let $\mathscr{V}(X)^G$ be the set of all G-invariant discrete valuations (with values in $\mathbb{Q}$) of the field of rational functions $K:=\mathbb{C}(X)$. For a Borel subgroup $B\subseteq G$ let $K^{(B)}$ be the multiplicative group of B-semiinvariant rational functions, i.e., for any $f\in K^{(B)}$ there is a character $\chi_f\in\mathscr{X}(B)$ with $f^b=\chi_f(b)f$ for all $b\in B$. One can prove that each $v\in\mathscr{V}(X)^G$ is completely determined by its restriction to $K^{(B)}$. Hence, $\mathscr{V}(X)^G$ may be identified with a subset of Hom $(K^{(B)}, \mathbb{Q})$.

Now we define $\mathscr{Z}(X)$ to be the set of those invariant valuations, which restrict to the trivial valuation on K^B. These valuations are called *central*. Then $\mathscr{Z}(X)$ can be viewed as a subset of $\mathscr{H}(X)$ as follows: Let $\Gamma_X:=\{\chi_f\mid f\in K^{(B)}\}=K^{(B)}/(K^*)^B$. This is a subgroup of $\mathscr{X}(B)$, hence finitely generated free abelian. It follows from the definition, that $\mathscr{Z}(X)$ can be regarded as a subset of $\mathrm{Hom}(\Gamma_X, \mathbb{Q})$. Next, one shows that Γ_X is canonically isomorphic to the character group $\mathscr{X}(A_X)$, so one gets

$$\mathscr{Z}(X)\hookrightarrow \mathrm{Hom}\,(\Gamma_X, \mathbb{Q})=\mathrm{Hom}\,(\mathscr{X}(A_X), \mathbb{Q})\cong\mathrm{Hom}\,(\mathbf{G}_m, A_X)\otimes\mathbb{Q}=\mathscr{H}(X).$$

Having this, we prove:

1.2. Theorem *Let λ: $\mathbf{G}_m\to A_X$ be a one-parameter subgroup of A_X. Assume it corresponds to a valuation $v\in\mathscr{Z}(X)$. Let Y be the center of v in $\bar{X}$. Then for $\hat{\alpha}\in\hat{T}_X$ generic, the limit $\lim_{t\to 0}\pi(\lambda(t)\hat{\alpha})$ exists and is contained in Y.* [Theorem 7.3]

The main idea of proof is the use of a slice theorem, essentially due to Brion et al. [BLV], which describes the local structure of $\bar{X}$ around Y as a B-variety.

These two theorems above combine to give a full picture of the limit behavior of one-parameter subgroup because we can prove:

1.3. Theorem *The cone $\mathscr{Z}(X)$ is a fundamental domain for the group W_X, i.e., every orbit intersects $\mathscr{Z}(X)$ in exactly one point.* [Theorem 7.4]

This is the main theorem of this paper. Its proof is roughly as follows: It is known that $\mathscr{Z}(X)\subseteq\mathscr{H}(X)$ is a finitely generated convex cone. Let W_X' be the group generated by the reflections at the codimension-one-faces of $\mathscr{Z}(X)$. Then general functorial properties of W_X and W'_X imply $W'_X \subseteq W_X$, and in particular, that $\mathscr{Z}(X)$ meets every W_X-orbit at least once. On the other hand, the first two theorems imply that every W_X-orbit intersects $\mathscr{Z}(X)$ in at most one point. As a corollary we get that $W_X = W'_X$ is generated by reflections and that $\mathscr{Z}(X)$ is a simplicial cone.

If X is spherical, i.e., X contains an open B-orbit, then $K^B = \mathbb{C}$ which implies that every invariant valuation is central. That means that W_X completely describes $\mathscr{V}(X)^G$. But also in the general case, W_X determines at least the qualitative structure of $\mathscr{V}(X)^G$: It is the union of simplicial cones which intersect along their common face $\mathscr{Z}(X)$ (see [Kn4, § 9] for details).

In this paper only the third theorem is proved in full generality, while the first two are only proved for varieties which I have named *non-degenerate.* There are three reasons to justify this: "Most" (e.g. all quasiaffine) varieties are non-degenerate. There are techniques (affine cones, Kostant's shifted cotangent bundles) to reduce the general case to the non-degenerate one. Finally, the proof in the general case is so much more involved that it would have almost doubled the size of this paper. But I want to mention that I have a proof for the general case which I will publish elsewhere.

The surprising fact, that $\mathscr{Z}(X)$ is the fundamental domain of a finite group was, for spherical X, first proved by Brion [Br], but his construction of W_X was not very enlightening. The present paper is the result of my effort for a more geometric construction of W_X. It has been announced in [Kn1], a paper which contains the basic results on the moment map, while [Kn4] does the same for invariant valuations. Unfortunately, the preparation of [Kn4] took quite a time, during which I was able to simplify some proofs in [Kn1]. Therefore, the present paper is quite independent of [Kn1] since it reproves most results in the non-degenerate case. It replaces the never published preprint [Kn2].

Notation. All varieties are defined over an algebraically closed field k of characteristic zero. The group G is always reductive and connected. We choose a Borel subgroup $B \subseteq G$ with unipotent radical U and maximal torus T. The Lie algebra of any group is usually denoted by the corresponding fraktur letter. The character group of a group H (or Lie algebra $\mathfrak{h}$) is denoted by $\mathscr{X}(H)$ (or $\mathscr{X}(\mathfrak{h}^*) = (\mathfrak{h}^*)^{\mathfrak{h}})$). If $H \to \mathrm{GL}(V)$ is a representation then $V^{(H)}$ is the set of H-eigenvectors. The character for $v \in V^{(H)}$ is χ_v. If H acts on an affine variety X then $X//H := \operatorname{Spec} k[X]^H$.

2 The local structure theorem

In this section we present a refined version of the local structure theorem of Brion et al. [BLV]. Let X be a normal, but not necessarily smooth G-variety. A *B-divisor* is a formal linear combination $D = \sum a_i D_i$ where the D_i are B-stable prime Cartier divisors of X and the coefficients are elements of k. If they are in $\mathbb{Z}$ or $\mathbb{N}$ then we call D integral or effective, respectively. The support of D is the union of those D_i with $a_i \neq 0$. We also will denote it with D, when no confusion is possible. The stabilizer of the support of D is denoted by $P[D]$. It is a parabolic subgroup of G.

Being Cartier, each D_i defines a line bundle $\mathcal{O}(D_i)$ on X with a canonical section σ_i. There is a finite covering $\tilde{G}$ of G such that every $\mathcal{O}(D_i)$ can be $\tilde{G}$-linearized. We will, for convenience of notation, replace G by $\tilde{G}$. The sections σ_i are eigenvectors for $P[D]$ and hence determine infinitesimal characters $\chi_{D_i} \in \mathcal{X}(\mathfrak{p}[D])$. We define $\chi_D := \sum_i a_i \chi_{D_i}$. Observe, that the linearizations are unique up to a character of G. Therefore, also χ_D is only well defined up to a character of $\mathfrak{g}$.

For any parabolic subgroup P let Δ_P^+ be the set of roots in the unipotent radical of P. Consider a character $\chi \in \mathcal{X}(\mathfrak{p})$. Then we call it *P-regular*, if $\langle \chi, \alpha^\vee \rangle \neq 0$ for all $\alpha \in \Delta_P^+$. A *B-divisor D is regular*, if χ_D is $P[D]$-regular. There are plenty of them:

2.1. Lemma *Every effective B-divisor is regular.*

Proof. Let D be an effective B-divisor. It induces a line bundle $\mathcal{L} = \mathcal{O}(D)$ with a section σ. We may assume that $\mathcal{L}$ is G-linearized. Then, $P[D]$ is clearly just the normalizer of the line $k\sigma$ and χ_D is the character with which $\mathfrak{p}[D]$ acts on it. Because the action of G on $H^0(X, \mathcal{L})$ is locally finite, σ is a highest weight vector and χ_D is an integral dominant weight. Hence the roots of the unipotent radical of $P[D]$ are just those α such that $\langle \chi_D, \alpha^\vee \rangle > 0$.

If τ is another section of $\mathcal{O}(D_i)$ then $\frac{\tau}{\sigma_i}$ is a regular function on $X \setminus D$. Hence, we can define a morphism

$$\psi_D : X \setminus D \to \mathfrak{g}^* : x \mapsto l_x \qquad \text{where } l_x(\xi) := \sum_i a_i \frac{\xi \sigma_i}{\sigma_i}(x),$$

which is well-defined up to a translation by a character of $\mathfrak{g}$.

If D is an effective B-divisor, then it induces directly a line bundle $\mathcal{O}(D)$ with section σ_D, character χ_D and morphism ψ_D. Note, that our previous definitions are compatible with these notions.

Later on, there may occur divisors which are not Cartier. The next lemma tells us that the morphism ψ_D is defined anyway because we may replace X by $X \setminus \bigcap_{g \in G} gD$.

2.2. Lemma *Let X be a normal G-variety and $D \subseteq G$ a prime divisor. Then, D is a Cartier divisor outside $Z = \bigcap_{g \in G} gD$.*

Proof. We may assume that Z is empty. Let $\iota: X_r \hookrightarrow X$ be the inclusion of the subset of smooth points of X. Because $D \cap X_r$ is a Cartier divisor, it defines a line bundle $\mathcal{L}_r$ on X_r. There is a finite cover $\tilde{G} \twoheadrightarrow G$ such that the action of $\tilde{G}$ on X_r can be lifted to a linearization of $\mathcal{L}_r$. Therefore, $\tilde{G}$ acts also on $\mathcal{L} := \iota_* \mathcal{L}_r$. Because X is normal, L is a (trivial) line bundle on $X \setminus D$. Hence it is a line bundle everywhere, because Z is empty by assumption. But that means precisely that D is a Cartier divisor. □

Now we are in the position to state the local structure theorem:

2.3. Theorem *Let X be a normal G-variety with a B-divisor $D = \sum_i a_i D_i$. Assume that χ_D is $P[D]$-regular. Then*:
(a) *The image of ψ_D is a single $P[D]$-orbit with isotropy subgroup a Levi complement of $P[D]$.*

(b) *For some* $x_0 \in X \setminus D$ *let*

$$\eta_0 := \psi_D(x_0), \quad L := G_{\eta_0}, \quad \Sigma := \psi_D^{-1}(\eta_0).$$

Then L is a Levi subgroup of $P[D]$ *and there is an isomorphism*

$$P[D] \overset{L}{\times} \Sigma \to X \setminus D.$$

Proof. (a) Let $\mathfrak{l}_0$ be a Levi complement of $\mathfrak{p} = \mathfrak{p}[D]$ with center $\mathfrak{z}$ and semisimple part $\mathfrak{l}_0'$. We will identify $\mathfrak{g}^*$ with $\mathfrak{g}$ by means of an invariant scalar product.

For $x \in X \setminus D$ let $\eta := \psi_D(x)$. Because all σ_i are $P[D]$-eigenvectors, we have $\eta \in (\mathfrak{l}_0' \oplus \mathfrak{p}_u)^\perp = \mathfrak{z} \oplus \mathfrak{p}_u$. Because the eigencharacter is χ_D, we actually have

$$\operatorname{Im} \psi_D \subseteq \xi_D + \mathfrak{p}_u.$$

Here, ξ_D is the element in $\mathfrak{z}$ which is dual to $\chi_D \in \mathfrak{z}^*$, i.e., $(\xi_D, \xi) = \chi_D(\xi)$ holds for all $\xi \in \mathfrak{l}_0$. Therefore, the $P[D]$-regularity of χ_D implies, that $\mathfrak{l}_0$ is the centralizer of ξ_D in $\mathfrak{g}$. It further follows that $\xi_D + \mathfrak{p}_u$ is a single $P[D]$-orbit.
(b) By (a) also $L = G_{\eta_0}$ is a Levi subgroup of $P[D]$ and ψ_D induces a $P[D]$-morphism $X \setminus D \to P[D]/L$. But that is equivalent to $P[D] \overset{L}{\times} \Sigma \to X \setminus D$ being an isomorphism. □

Because of $P[D \cup D'] = P[D] \cap P[D']$ there exists a B-divisor D such that $P[D]$ is absolutely minimal. Let $P(X) := P[D]$ and $P_u(X)$ its unipotent radical. Hence, $P(X)$ is the largest subgroup $H \subseteq G$ such that every B-stable divisor is H-stable. It is easy to see [Kn4, 2.5] that the definition is compatible with that in [Kn1]:

$$P(X) = \{g \in G \mid gBx = Bx \text{ for } x \in X \text{ generic}\}.$$

2.4 Proposition *Let D be regular B-divisor with* $P[D] = P(X)$ *and let* L, Σ *as in Theorem* 2.3. *Then the commutator subgroup* L' *acts trivially on* Σ.

Proof. Let $B_L := B \cap L$. Then B_L-stable divisors in Σ correspond to B-stable divisors in X. Hence, they are L-stable. By [Su] there is a dense open L-stable subset $\Sigma_0 \subseteq \Sigma$ which can be equivariantly embedded into a projective space $\mathbf{P}(V)$. Consider the closure of the affine cone over Σ_0 in V. Let Σ' be its normalization. It has the property that every $B_L \times k^*$-stable divisor is $L \times k^*$-stable. Because Σ' is normal, this implies that every highest weight vector in $k[\Sigma']$ is an eigenvector for L. Therefore, L' acts trivially on $k[\Sigma']$ and hence on Σ' (being affine), Σ_0 and Σ. □

Consider a situation as in the local structure theorem. Let $L_0 \subseteq L$ be the kernel of the action on Σ, and $S(X) := L_0 P_u(X)$. Then, the orbits of U, $S(X)$ and $P_u(X)$ on $X \setminus D$ coincide, the latter acting freely. Furthermore, $A_X := L/L_0 = P(X)/S(X)$ is a torus acting effectively on Σ. Because $k(X)^{(B)} = k(\Sigma)^{(L)}$ we get for the character group

$$\mathscr{X}(A_X) = \{\chi_f \mid f \in k(X)^{(B)}\}.$$

This shows that A_X does not depend on the particular choice of D.

3 Polarized cotangent vectors

Assume that X is smooth. Then the cotangent bundle $\pi: T_X^* \to X$ is a vector bundle. The G-action induces the *moment map*

$$\Phi: T_X^* \to \mathfrak{g}^*: \alpha \mapsto l_\alpha \qquad \text{where } l_\alpha(\xi) = \alpha(\xi_{\pi(\alpha)}).$$

Assume that D is a principal B-divisor, i.e., $D = \sum_j b_j [f_j]$ where $b_j \in k$ and $[f_j]$ is the principal divisor attached to a rational function $f_j \in k(X)^{(B)}$. Then the map ψ_D factors through the moment map: In fact, define

$$\psi_D^*: X \setminus D \to T_X^*: x \mapsto \sum_j b_j f_j(x)^{-1} (df_j)_x.$$

Then $\xi f(x) = (df)_x(\xi)$ implies $\psi_D = \Phi \circ \psi_D^*$. It is easily checked that ψ_D^* is $P[D]$-equivariant. Note that it can be expressed formally as $\psi_D^* = d \log \prod_j f_j^{b_j}$.

This map will be our link between the geometries of X and T_X^*. Unfortunately, there are cases when there exists no regular principal divisor with $P[D] = P(X)$, e.g., for $X = G/B$ every principal B-divisor is trivial but $P(X) = B$. This leads to the following

Definition. A G-variety X is called *non-degenerate* if there is a principal B-divisor D such that χ_D is $P(X)$-regular.

Note, that $P(X)$-regularity of χ_D implies $P[D] = P(X)$. In fact, $P(X)$ is contained in $P[D]$ by minimality and contains it by regularity. There are many non-degenerate varieties, e.g.:

3.1. Lemma *Every quasiaffine G-variety is non-degenerate.*

Proof. Let D be any B-divisor of a quasiaffine G-variety X with $P[D] = P(X)$. Then there is $f \in k[X]^{(B)}$ vanishing on the support of D. Hence, $P[[f]] = P(X)$ and $[f]$ is regular because it is effective. □

Let $f_1, \ldots, f_s \in k(X)^{(B)}$ be a transcendence basis of $k(X)^U$ and let $D_0 := \bigcup_j [f_j]$. For any s-tuple $\underline{b} \in k^s$ let $D(\underline{b}) := \sum_j b_j [f_j]$. Then we get the map

$$\psi^*: (X \setminus D_0) \times \mathbf{A}^s \to T_X^*: (x, \underline{b}) \mapsto \psi_{D(\underline{b})}^*(x).$$

Suppose X is non-degenerate. Then all U-orbits in $X \setminus D_0$ will have the same dimension by Proposition 2.4. Consider

$$X_0 := \{x \in X \setminus D_0 \mid (df_1)_x, \ldots, (df_s)_x \text{ are linearly independent}\}.$$

Because the f_i form a transcendence basis of $k(X)^U$ this set is not empty and for any $x \in X_0$ the common kernel of the $(df_j)_x$ is the tangent space to the orbit Ux. Hence, ψ^* induces an isomorphism of $X_0 \times \mathbf{A}^s$ onto

$$C := \Phi^{-1}(\mathfrak{u}^\perp) \cap \pi^{-1}(X_0) = \{\alpha \in T_X^* \mid x = \pi(\alpha) \in X_0 \text{ and } \alpha(\mathfrak{u}x) = 0\}.$$

In particular, we get

$$\operatorname{codim} C=\dim T_X^*-(\dim X+\operatorname{tr.deg} k(X)^U)=\dim X-\dim \Sigma=\dim P_u(X).$$

Before I proceed let me set up some notation. Sometimes we will identify $\mathfrak{g}$ and $\mathfrak{g}^*$ by means of an invariant scalar product. Let L be a Levi complement of $P=P(X)$ and let L_0 be the intersection of L with $S=S(X)$. We identify the Lie algebra of A_X with the orthogonal complement $\mathfrak{a}$ of $\mathfrak{l}_0$ in $\mathfrak{l}$. Let $P^-=P(X)^-$ be the parabolic opposite to P with $P\cap P^-=L$. Let $\mathfrak{a}^r\subseteq\mathfrak{a}^*$ be the set of P-regular elements. It is the complement of finitely many hyperplanes.

3.2. Theorem *Let X be non-degenerate. Then $G\cdot C$ is dense in T_X^*.*

Proof. By the local structure theorem, the generic U- and S-orbits in X coincide. Hence, $\Phi(C)\subseteq\mathfrak{s}^\perp=\mathfrak{a}\oplus\mathfrak{p}_u$. Because $\mathfrak{a}^*$ is spanned by the characters χ_{f_i}, the composition $C\to\mathfrak{a}\oplus\mathfrak{p}_u\to\mathfrak{a}$ is surjective. Via P_u, every element of $\mathfrak{a}^r\times\mathfrak{p}_u$ can be moved into $\mathfrak{a}^r$. This shows that the generic P_u-orbit of C contains an element α with $\xi:=\Phi(\alpha)\in\mathfrak{a}^r$.

Consider the map between tangent spaces $d\Phi\colon T_\alpha(T_X^*)\to\mathfrak{g}$. Then $d\Phi(T_\alpha(C))\subseteq\subseteq\mathfrak{a}\oplus\mathfrak{p}_u$ while $d\Phi(\mathfrak{p}_u^-\alpha)=[\mathfrak{p}_u^-,\xi]=\mathfrak{p}_u^-$. The last equality again holds because ξ is regular. Hence, C and $P_u^-\alpha$ intersect transversally in α. This shows $\dim P_u^-\cdot C=\dim P_u^-+\dim C=\dim T_X^*$, which implies the assertion. □

3.3. Corollary *Let X be non-degenerate. Then $\overline{\Phi(T_X^*)}=\overline{G\cdot\mathfrak{a}_X^*}$.*

This recovers [Kn1, 5.4] for non-degenerate varieties. Note, that by the same theorem, the last two statements are definitely wrong for degenerate varieties.

The Weyl group W of $\mathfrak{g}$ acts also on $\mathfrak{t}^*$. Let $N(\mathfrak{a}^*):=\{w\in W\mid w\mathfrak{a}^*=\mathfrak{a}^*\}$ and let $W(\mathfrak{a}^*)$ be the image of $N(\mathfrak{a}^*)$ in the automorphism group of $\mathfrak{a}^*$. Then the map $\mathfrak{a}^r\to\mathfrak{t}^*/W$ is unramified and its image can be identified with $\mathfrak{a}^r/W(\mathfrak{a}^*)$. Consider the following diagram:

$$\begin{array}{ccccccc} & & T^*_X\times_{\mathfrak{t}^*/W}\mathfrak{a}^r & \xrightarrow{\Phi} & G/L\times\mathfrak{a}^r & \longrightarrow & \mathfrak{a}^r \\ & \overset{\hat\psi}{\nearrow} & \downarrow & & \downarrow{\scriptstyle 1} & & \downarrow \\ X_0\times\mathbf{A}^s_r & \xrightarrow{\psi^*} & T^*_X & \xrightarrow{\Phi} & \overline{G\mathfrak{a}^*} & \xrightarrow{2} & \mathfrak{t}^*/W. \end{array}$$

Here, the arrow 1 is the map $(gL,\xi)\mapsto g\xi$ and arrow 2 uses the Chevalley isomorphism $\mathfrak{g}^*G\to\mathfrak{t}^*/W$. It easy to see that the right and hence also the left hand square is cartesian. Finally, the main point is that ψ^* lifts to $\hat\psi$ by setting $\hat\psi(x,\underline{b})=(\psi^*(x),\chi_{D(\underline{b})})$. Here $\mathbf{A}_r^s\subseteq\mathbf{A}^s$ is the preimage of $\mathfrak{a}^r$ by the surjective linear map $\mathbf{A}^s\to\mathfrak{a}^*\colon\underline{b}\mapsto\chi_{D(\underline{b})}=\sum_j b_j\chi_{f_j}$.

The set $Z:=T_X^*\times_{\mathfrak{t}^*/W}\mathfrak{a}^r$ is smooth, but in general not connected. Thus, $\hat\psi$ singles out a specific component which we denote by $\hat T_X$. Its elements are called *polarized cotangent vectors*. The group $W(\mathfrak{a}^*)$ permutes the components of Z transitively. The subgroup $W_X\subseteq W(\mathfrak{a}^*)$ of those elements which map $\hat T_X$ into itself is called the *little Weyl group* of X. It acts freely on $\hat T_X$ and the map $\hat T_X/W_X\to T_X^*$ is an open embedding.

A little Weyl group has been already defined in [Kn1]. These definitions are compatible:

3.4. Lemma *Let X be non-degenerate. Then W_X is the same as defined in* [Kn1].

Proof. To see this, choose a linear section τ of $\mathbf{A}^s \to \mathfrak{a}^*$ and $x \in X_0$ and let

$$\hat{\sigma}: \mathfrak{a}^r \xrightarrow{x \times \tau} X_0 \times \mathbf{A}^s_r \xrightarrow{\hat{\psi}} \hat{T}_X.$$

This is a section of $\hat{\Psi}: \hat{T}_X \to \mathfrak{a}^r$ which composed with $\hat{T}_X \to \hat{T}^*_X$ gives the map σ in [Kn1, 6.2]. The existence of $\hat{\sigma}$ implies that the generic fibers of $\hat{\Psi}$ and therefore also of $\hat{T}_X/W_X \to \mathfrak{a}^r/W_X$ are connected. This implies that $k[\mathfrak{a}^r/W_X]$ is integrally closed in $k[T^*_X]$, hence equals the variety L_X of [Kn1, 6]. This shows the claim. □

4 Flats

Consider the composed map $\hat{T}_X \to G/L \times \mathfrak{a}^r \to G/L$ and let $\hat{\Sigma}$ and $\hat{C}$ be the preimage of eL and $P_u eL$ respectively. Then we have

$$\hat{C} = P \overset{L}{\times} \hat{\Sigma} \quad \text{and} \quad \hat{T}_X = G \overset{P}{\times} \hat{C} = G \overset{L}{\times} \hat{\Sigma}.$$

The set $P_u eL \times \mathfrak{a}^r \subseteq G/L \times \mathfrak{a}^*$ is mapped isomorphically onto $\mathfrak{a}^r \times \mathfrak{p}_u \subseteq \mathfrak{g}^*$. Hence, the map $\hat{C} \cap \hat{\pi}^{-1}(X_0) \to C \subseteq T^*_X$ is an open embedding (where $\hat{\pi}: \hat{T}_X \to X$ is the canonical projection). This implies that

$$\hat{\psi}: X_0 \times \mathbf{A}^s_r \hookrightarrow \hat{C}$$

is an open embedding. Because $\hat{\psi}^{-1}(\hat{\Sigma}) \twoheadrightarrow \mathbf{A}^s_r$ is a parameterized family of slices as in Theorem 2.3, the Levi group L acts also on $\hat{\Sigma}$ only via its quotient A_X. In particular, the generic isotropy group of $\hat{T}_X$ and that of T^*_X is conjugated to L_0. This is a new proof of [Kn1, 8.2] for non-degenerate varieties.

Because A_X is abelian *this induces an A_X-action on* $\hat{T}_X = G \overset{L}{\times} \hat{\Sigma}$ by $a \cdot [g, \hat{\alpha}] := [g, a\hat{\alpha}]$. It commutes with the action of G, but not with that of W_X as we will see later. There is another description of the A_X-action which does not refer to a particular choice of L: By means of $\hat{\Phi}$ each element $\hat{\alpha} \in \hat{T}_X$ determines a coset $gL \in G/L$, hence a homomorphism

$$\varphi_{\hat{\alpha}}: G_{\hat{\Phi}(\hat{\alpha})} = gLg^{-1} \xrightarrow{\sim} L \twoheadrightarrow A_X.$$

Note, that $\varphi_{\hat{\alpha}}$ does not depend on the particular choice of g in its coset. Then we have the formula

$$a \cdot \hat{\alpha} = \bar{a}\hat{\alpha} \quad \text{with any lift } \bar{a} \text{ of } a \text{ in } G_{\hat{\Phi}(\hat{\alpha})}.$$

Of course, it suffices to check this for $\hat{\alpha} \in \hat{\Sigma}$ where it is just the definition. It also shows $\ker \varphi_{\hat{\alpha}} \subseteq G_{\hat{\alpha}}$ with equality for generic $\hat{\alpha}$. Thus we get a homomorphism

$$\Theta_{\hat{\alpha}}: A_X \to G_{\hat{\Phi}(\hat{\alpha})}/G_{\hat{\alpha}}$$

which is an isomorphism for generic $\hat{\alpha}$.

Let A be any connected group acting on a symplectic variety Z and let $\Phi: Z \to \mathfrak{a}^*$ be a morphism. Then every $\xi \in \mathfrak{a}$ induces two vector fields on Z: The first one, ξ_*, is induced by the A-action. The second is the Hamiltonian vector field H_{l_ξ} attached to the function $l_\xi(x) = \Phi(x)(\xi)$. Then Φ is called a moment map if these two vector fields coincide: $\xi_* = H_{l_\xi}$. The existence of a moment map has (among others) two important consequences:

1. Every Hamiltonian vector field is an infinitesimal symplectomorphism. Hence, the A-action preserves the symplectic structure of Z.
2. The A-action is uniquely determined by Φ.

We apply this to $\hat{T}_X$ which carries as an étale cover of T_X^* a canonical symplectic structure.

4.1. Theorem *The morphism $\hat{\Psi}: \hat{T}_X \to \mathfrak{a}^*$ is a moment map for the A_X-action on $\hat{T}_X$. In particular, this action is symplectic.*

Proof. Without loss of generality let $\hat{\alpha} \in \hat{\Sigma}$. Choose $\xi \in \mathfrak{a}$ which, considered as an element of $\mathfrak{l} \subseteq \mathfrak{g}$, we denote by $\bar{\xi}$. Consider the commutative diagram

$$\begin{array}{ccccc} \hat{T}_X & \xrightarrow{\hat{\Phi}} & G/L \times \mathfrak{a}^r & \longrightarrow & \mathfrak{a}^r \\ \downarrow & & \downarrow & & \\ T_X^* & \xrightarrow{\Phi} & \mathfrak{g}^*. & & \end{array}$$

By definition of the A_X-action we have $\xi_{\hat{\alpha}} = \bar{\xi}_{\hat{\alpha}}$. The well known fact that Φ is a moment map implies $\bar{\xi}_{\hat{\alpha}} = (H_{l_{\bar{\xi}}})_{\hat{\alpha}}$. Hence we have to show $(H_{l_{\bar{\xi}}})_{\hat{\alpha}} = (H_{l_\xi})_{\hat{\alpha}}$ or equivalently $(d l_{\bar{\xi}})_{\hat{\alpha}} = (d l_\xi)_{\hat{\alpha}}$. This follows from the fact that ξ and $\bar{\xi}$ (considered as linear functions on $\mathfrak{a}^*$ and $\mathfrak{g}^*$, respectively) induce the same covector in the point $(eL, \hat{\Psi}(\hat{\alpha}))$ of $G/L \times \mathfrak{a}^r$. □

4.2. Theorem *There is a W_X-action on A_X which is compatible with that on $\mathfrak{a}_X = \mathrm{Lie}\, A_X$. With this action we have*

$$w(a \cdot \hat{\alpha}) = wa \cdot w\hat{\alpha} \quad \textit{for all } w \in W_X,\ a \in A_X,\ \hat{\alpha} \in \hat{T}_X,$$

i.e., there is an action of $(W_X \ltimes A_X) \times G$ on $\hat{T}_X$.

Proof. Let $w \in W_X$ and $\hat{\alpha} \in \hat{T}_X$ generic. Because the actions of W_X and G commute we have $G_{w\hat{\alpha}} = G_{\hat{\alpha}}$ and $G_{\hat{\Phi}(w\hat{\alpha})} = G_{\hat{\Phi}(\hat{\alpha})}$. Now we define the action of w on A_X by

$$A_X \xrightarrow{\Theta_{\hat{\alpha}}} G_{\hat{\Phi}(\hat{\alpha})}/G_{\hat{\alpha}} = G_{\hat{\Phi}(w\hat{\alpha})}/G_{w\hat{\alpha}} \xrightarrow{\Theta_{w\hat{\alpha}}^{-1}} A_X.$$

This automorphism is independent of $\hat{\alpha}$ by the rigidity of automorphisms of a torus [Sp, 2.5.10] and the irreducibility of $\hat{T}_X$. Let $\bar{a} \in G_{\hat{\Phi}(\hat{\alpha})} \subseteq G$ be a lift of $\Theta_{\hat{\alpha}}(a)$. Then, by definition, it is also a lift of $\Theta_{w\hat{\alpha}}(wa)$. Hence we get

$$wa \cdot w\hat{\alpha} = \bar{a}\, w\hat{\alpha} = w\bar{a}\hat{\alpha} = w(a \cdot \hat{\alpha}).$$

That the two actions on $\mathfrak{a}_X$ are the same follows from Theorem 4.1: $w(\xi_*) = wH_{l_\xi} = H_{l_{w\xi}}$. □

Definition. Let $\hat\pi\colon \hat T_X \to X$ be the projection and $\hat\alpha \in \hat T_X$. Then $F_{\hat\alpha} := \hat\pi(A_X \cdot \hat\alpha)$ is called a *flat* of X and the map $\xi_{\hat\alpha}\colon A_X \to F_{\hat\alpha} : a \mapsto \hat\pi(a\cdot\hat\alpha)$ a *polarization* of it.

Observe that $F_{\hat w\alpha} = F_{\hat\alpha}$, i.e., as a set $F_{\hat\alpha}$ depends only on the image $\alpha \in T_X^*$. In fact, it can also be described entirely in terms of T_X^* as $F_\alpha = \pi(G_{\Phi(\alpha)}\alpha)$.

5 Twisted flats

Next we will describe how to deal with degenerate varieties. For this and for another purpose (proof of Theorem 7.4) we need the following

5.1. Theorem *Let $p\colon \tilde X \to X$ be an G-equivariant principal H-bundle, where H is a torus. Assume that $k(\tilde X)^B = k(X)^B$, i.e., H acts trivially on $k(\tilde X)^B$. Then*
1. $P(X) = P(\tilde X)$.
2. There is a canonical exact sequence

$$1 \to H \to A_{\tilde X} \to A_X \to 1.$$

3. Assume $\tilde X$ is non-degenerate. Then, the little Weyl group $W_{\tilde X}$ fixes H pointwise. In particular it acts on A_X.

Proof. Let $X_0 = P(X) \overset{L}{\times} \Sigma$ as in the local structure Theorem 2.3 and put $\tilde\Sigma := p^{-1}(\Sigma)$ and $\tilde X_0 := p^{-1}(X_0) = G \overset{L}{\times} \tilde\Sigma$. Choose $\tilde x \in \tilde\Sigma$ generic and let $x = p(\tilde x)$. Then the orbit map $g \mapsto g\tilde x$ induces a morphism $\omega\colon L_x \to p^{-1}(x) = H\tilde x = H$. Because H is a torus and $\omega(1) = 1$, this map is a homomorphism on L_x^0 (see [KKV, 1.2]). In particular, $L' \subseteq L_x^0$ acts trivially on $\tilde\Sigma$. This implies $P(\tilde X) = P(X)$.

Because X and $\tilde X$ have the same field of B-invariants, the image of ω is dense, hence equals H. This implies $H = H\tilde x \subseteq L\tilde x = A_{\tilde X}$ with quotient $Lx = A_X$. This shows the exact sequence. It is easy to see that the H-action on $\tilde\Sigma$ coincides with that induced by $H \subseteq A_{\tilde X}$. This implies the commutativity of the diagram

$$\begin{array}{ccc} \hat T_{\tilde X} & \longrightarrow & \mathfrak{a}_{\tilde X}^* \\ \downarrow & & \downarrow \\ T_{\tilde X}^* & \twoheadrightarrow & \mathfrak{h}^*. \end{array}$$

Because all maps are $W_{\tilde X}$-equivariant, the little Weyl group acts trivially on $\mathfrak{h} = \operatorname{Lie} H$. □

Let now X be any smooth but possibly degenerate variety. Choose an effective B-divisor D with $P[D] = P(X)$. It is regular by Lemma 2.1. Replace it by a multiple such that $\mathcal{O}(D)$ carries a G-linearization. Then

$$p\colon\ L := \operatorname{Spec}_X \bigoplus_{n\in\mathbb{Z}} \mathcal{O}(nD) \to X$$

is the geometric realization of the line bundle $\mathcal{O}(D)$ where the zero-section is removed. In particular, it is a principal k^*-bundle. Let σ_D be the canonical section of $\mathcal{O}(D)$. Then $\sigma_D \in k[L]$, hence $p^{-1}(D)$ is a principal B-divisor. This shows that L is non-degenerate.

A little bit more generally, we choose once and for all a principal H-bundle $p\colon \tilde X \to X$, where H is a torus and such that $\tilde X$ is non-degenerate. To force

$k(\tilde{X})^B = k(X)^B$, we replace G by the product $G \times H$, the second factor acting trivially on X. We define $W_X := W_{\tilde{X}}$. We will see later (Corollary 7.5) that this definition is independent of the choice of $\tilde{X}$ and is compatible with that in [Kn1].

Any $\hat{\alpha} \in \hat{T}_{\tilde{X}}$ gives rise to a map $A_{\tilde{X}} \to X$: $a \mapsto p \circ \hat{\pi}(a \cdot \hat{\alpha})$. Because it is H-invariant, it factors through $A_X = A_{\tilde{X}}/H$.

Definition. This map $\xi_{\hat{\alpha}}: A_X \to X$ is called a *twisted* (*polarized*) *flat* of X. The map $\xi_*: \hat{T}_{\tilde{X}} \times A_X \to X$ is the *generic flat* (with respect to $\tilde{X}$).

On a degenerate variety there are no untwisted flats. But even for non-degenerate varieties there may be more twisted flats than ordinary ones: Let $G = \mathrm{GL}_2(k)$, $T_0 := \mathrm{diag}(*, 1)$ and $T := \mathrm{diag}(*, *)$. Put $\varphi: \tilde{X} = G/T_0 \to X = G/T$. Then a flat of X through the base point is the orbit of a generic isotropy group of $\mathfrak{t}^\perp = \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix}$ which is just $\begin{pmatrix} a & b \\ b & a \end{pmatrix}$. Hence, there is only one flat. But $\mathfrak{t}_0^\perp = \begin{pmatrix} 0 & * \\ * & * \end{pmatrix}$. So the isotropy subgroup may be every maximal torus of G, i.e., any orbit of a one parameter subgroup of G is a twisted flat.

6 The closure of a flat

We are interested in the closure of flats. For this we need the following rigidity lemma. Recall, that an action of a connected algebraic group on a variety X is called *locally linear* if every point of X has a stable open neighborhood which can be equivariantly embedded into a projective space. Normality of X implies local linearity by a result of Sumihiro [Su]. Therefore, any stable subvariety of a normal variety is locally linear.

6.1. Lemma *Let A be a torus, Z a locally linear A-variety, $\varphi: Z \to S$ an A-invariant morphism and $\sigma: S \to Z$ a section of φ. Assume, that S is irreducible and that for every $s \in S$ the orbit $A\sigma(s)$ is open and dense in the fiber $Z_s := \varphi^{-1}(s)$. Then there is a unique non-empty open subset $S' \subseteq S$ which is maximal with the following property:*

Choose any $s_0 \in S'$. Then there is a unique A-equivariant S-isomorphism

$$Z \times_S S' \xrightarrow{\sim} Z_{s_0} \times S'$$

such that $\sigma(s)$ is mapped for all $s \in S'$ to $(\sigma(s_0), s)$.

Proof. Uniqueness is clear because $A\sigma(s)$ is dense in Z_s, for every $s \in S$. For that reason, the union of all S' with the property above has this property. Hence there exists a unique maximal such subset if there exists any.

It remains to show existence. Because A is a torus and the action is locally linear, Z can be covered by A-stable affine open subsets [Su]. Hence, again by uniqueness, we may assume that Z is affine. Consider the quotient $Z/\!/A = \mathrm{Spec}\, k[Z]^A$. Then φ factors through $Z/\!/A$ and $Z/\!/A \to S$ is birational. Because of the section σ it is an isomorphism, i.e., $k[S] = k[Z]^A$.

For a character $\chi \in \mathcal{X}(A)$, let $k[Z]_\chi$ be the corresponding space of eigenfunctions. Let $\mathcal{M} := \{\chi \in \mathcal{X}(A) | k[Z]_\chi \neq 0\}$. This is a finitely generated monoid. For $\chi \in \mathcal{M}$ choose any non-zero $g \in k[Z]_\chi$. Let $g_0(z) := g(\sigma\varphi(z))$. Then $g_0 \in k[Z]^A$ and $f_\chi := g/g_0 \in k(Z)_\chi$ is a rational function with the property $f_\chi|_{\sigma(S)} \equiv 1$. Because $A\sigma(s)$

is dense in Z_s, the function f_χ is uniquely determined by this property, hence independent of the choice of g. For the same reason, we have $f_\chi f_\eta = f_{\chi+\eta}$ for all $\chi, \eta \in \mathcal{M}$. Because $\mathcal{M}$ is finitely generated, we can replace S and Z by affine open subsets, such that all f_χ are regular functions.

Let $g \in k[Z]_\chi$. Because every fiber of φ contains a dense orbit, we have $k(Z)^A = k(S)$. Hence $g/f_\chi \in k(S)$. But then it is actually regular, because $(g/f_\chi)|_{\sigma(S)} = g|_{\sigma(S)}$ is. This shows that $k[S] \to k[Z]_\chi : g \mapsto g f_\chi$ is an isomorphism for every $\chi \in \mathcal{M}$. Hence we get an isomorphism $k[S] \bigotimes_k k[\mathcal{M}] \to k[Z]$ or equivalently a trivialization $Z \to S \times F$ where F is the torus embedding $\operatorname{Spec} k[\mathcal{M}]$. □

6.2. Corollary *Let X be a smooth G-variety and $X \hookrightarrow \bar{X}$ a normal equivariant embedding. Then there exists a unique equivariant embedding $A_X \hookrightarrow \bar{A}_{\bar{X}}$ (the closure of a generic twisted flat) and a non-empty open subset $S' \subseteq \hat{T}_{\bar{X}}$ such that the following holds:*

There is a unique closed embedding $S' \times \bar{A}_{\bar{X}} \to S' \times X$ such that every $(\hat{\alpha}, a) \in S' \times A_X$ is mapped to $(\hat{\alpha}, \zeta_{\hat{\alpha}}(a))$.

Furthermore, S' can be chosen to be stable under G, $A_{\bar{X}}$ and W_X.

Proof. Let Z_0 be the closure of the image of

$$\hat{T}_{\bar{X}} \times A_X \to \hat{T}_{\bar{X}} \times X : (\hat{\alpha}, a) \mapsto (\hat{\alpha}, \zeta_{\hat{\alpha}}(a)).$$

Then there is a non-empty open subset $S \subseteq \hat{T}_{\bar{X}}$ such that the fibers of $Z_0 \to \hat{T}_{\bar{X}}$ are, over S, exactly the closures of flats. The section is given by $\sigma(\hat{\alpha}) = (\hat{\alpha}, p \circ \hat{\pi}(\hat{\alpha}))$. Then the existence of S' follows from the preceding lemma. It is easily checked that Z_0 is stable under G, $A_{\bar{X}}$, and W_X, and that φ is equivariant. Hence, if we choose S' to be maximal, then it is unique and stable under all three groups. □

6.3. Corollary *The action of W_X on A_X extends to $\bar{A}_{\bar{X}}$.*

Finally, let me explain the case of a symmetric variety $X = G/H$. Here G is semisimple and H is the set of fixed points of an involution $\vartheta \in \operatorname{Aut} G$. Let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ be the decomposition into eigenspaces of ϑ. Then $\mathfrak{h}$ is the Lie algebra of H and $\mathfrak{p}$ can be identified with $\mathfrak{h}^\perp$, hence $T_X^* = G \overset{H}{\times} \mathfrak{p}$. Choose a generic $\xi \in \mathfrak{p}$. Then $G_\xi H/H \subseteq X$ is a flat of X. Choose a ϑ-stable torus $T \subseteq G_\xi$. Then the flat is isomorphic to $A_X = T/(T \cap H)$. The space $\mathfrak{a} = \mathfrak{g}_\xi \cap \mathfrak{p} \cong \operatorname{Lie} A_X$ is called a Cartan subspace of $\mathfrak{p}$. It is known that every semisimple element of $\mathfrak{p}$ is conjugated by H to an element of $\mathfrak{a}$. This shows that all flats in X are conjugate to each other. Furthermore, W_X coincides with $N_G(T)/C_G(T)$, the little Weyl group of X (see e.g. [Kn1, p. 17/18]). This shows that the W_X-action on the flat is induced by $N_G(T)$. These are the facts which make symmetric varieties so accessible. In general, neither are the flats conjugated to each other, nor is the W_X-action on them induced by elements of G.

7 Central valuations

Let $\bar{X}$ be a normal embedding of X. The last step is to identify the closure $\bar{A}_{\bar{X}}$ of the generic flat. I can do that only in special cases. We recall some notion of [Kn4]. Let X be a G-variety and $K = k(X)$.

Definition. A valuation v of $k(X)$ with values in $\mathbb{Q}$ is called *central* if it is G-invariant and vanishes on all non-zero elements of K^B. Let $\mathcal{Z}(X)$ be the set of central valuations.

For spherical varieties K^B equals k. Hence in this case all $\mathbb{Q}$-valuations are central.

There is a short exact sequence

$$1 \longrightarrow (K^B)^\times \longrightarrow K^{(B)} \xrightarrow{f \mapsto \chi_f} \mathscr{X}(A_X) \longrightarrow 1.$$

By [Kn4, 3.6] every G-invariant valuation is uniquely determined by its restriction to $K^{(B)}$. The restriction of a central valuation factors through $\mathscr{X}(A_X)$. Hence, there is an injection $\mathscr{Z}(X) \hookrightarrow \mathscr{H}(X) := \mathrm{Hom}(\mathscr{X}(A_X), \mathbb{Q})$. Usually, we will identify $\mathscr{Z}(X)$ with its image. Note that $\mathscr{H}(X)$ is a $\mathbb{Q}$-vector space of dimension rk X.

7.1. Theorem [Kn4, 6.5] *The subset $\mathscr{Z}(X) \subseteq \mathscr{H}(X)$ is a finitely generated convex cone with non-empty interior.*

The geometric meaning of central valuations is expressed in the next theorem.

7.2. Theorem *Let X be a normal G-variety.*
1. *The valuation v_Z induced by a G-stable prime divisor $Z \subset X$ is central if and only if X and Z have the same complexity* [Kn4, 7.3].
2. *Let v be a non-trivial central valuation of X. Then there exists a smooth G-variety which is G-birational to X together with a G-stable prime divisor Z such that v_Z is a multiple of v* [Kn4, 4.4 and 7.2].

Consider $A = A_X$ as an A-variety by left translation. Then we obviously have $\mathscr{H}(A) = \mathscr{H}(X)$. Recall the classification of normal torus embeddings [TE, O]: The normal affine equivariant embeddings of A are classified by finitely generated strictly convex subcones $\mathscr{C} \subseteq \mathscr{H}(A)$. Let $\bar{A}(\mathscr{C})$ be the embedding determined by the cone $\mathscr{C}$. The orbits of $\bar{A}(\mathscr{C})$ corresponds to the faces of $\mathscr{C}$. At the moment, we are only interested in the case where $\mathscr{C} = \mathbb{Q}^+ v$ is a ray. Hence, $\bar{A}(v) = \bar{A}(\mathbb{Q}^+ v)$ has two orbits: The open one and one of codimension one. Because all rays are admitted we have $\mathscr{Z}(A) = \mathscr{H}(X)$.

7.3. Theorem *Let $X \hookrightarrow \bar{X}$ a normal equivariant embedding and $Z \subseteq \bar{X}$ a prime divisor such that $v = v_Z$ is central. Then for all $w \in W_X$ and generic $\hat{\alpha} \in \hat{T}_{\tilde{X}}$ there is a (unique) open A_X-embedding $\bar{A}(wv) \hookrightarrow \bar{A}_X = \bar{F}_{\hat{\alpha}}$ such that the closed orbit is mapped into Z.*

Proof. Because W_X acts on $\bar{A}_{\tilde{X}}$ we may assume $w = 1$. By shrinking $\bar{X}$ and X we may also assume that $\bar{X}$ is smooth and that $Z = \bar{X} \setminus X$.

Choose a transcendence basis $f_j \in k(\tilde{X})^{(B)}$ of $k(\tilde{X})^U$ and let $\hat{\psi}: \tilde{X} \setminus D \times \mathbf{A}_r^s \hookrightarrow \hat{C} \subseteq \hat{T}_{\tilde{X}}$ be the open embedding as in Sect. 4. Because $G \cdot \hat{C}$ is dense in $\hat{T}_{\tilde{X}}$ it suffices to consider flats $F_{\hat{\alpha}}$ with $\hat{\alpha}$ in the image of $\hat{\psi}$. We also fix $\underline{b} \in \mathbf{A}_r^s$.

The group G contains the structure group H of $\tilde{X} \to X$. Hence every B-divisor of $\tilde{X}$ is the preimage of a unique B-divisor of $\bar{X}$ which does not contain Z in its support. Let $\bar{D}(\underline{b})$ be the B-divisor corresponding this way to $D(\underline{b}) = \sum_j b_j [f_j]$. Let $D \subseteq \tilde{X}$ be the union of the supports of the $[f_j]$ and let $\bar{D} := \overline{p(D)} \subseteq \bar{X}$. Then we get the following commutative diagram:

$$\begin{array}{ccc}
\tilde{X} \setminus D & \xrightarrow{\psi(?, \underline{b})} & \hat{T}_{\tilde{X}} \\
\downarrow{\scriptstyle p} & & \downarrow{\scriptstyle \Phi} \\
\bar{X} \setminus \bar{D} & \xrightarrow{\psi_{D(b)}} & \mathfrak{a}^*
\end{array}$$

Observe, that $\bar{D}$ does not contain Z, i.e., Z intersects $\bar{X}\setminus\bar{D}$ non-trivially.

Fix $x_0 \in \tilde{X}\setminus D$ and let gL be the image of $\hat{\psi}(x_0, \underline{b})$ in G/L. Then $L(\underline{b}) := gLg^{-1}$ is the isotropy group of $\psi_{\bar{D}(\underline{b})}(p(x_0))$ which is equipped with a homomorphism $L(\underline{b}) \twoheadrightarrow A_X$. Let $\Sigma(\underline{b})$ be the fiber of $\psi_{\bar{D}(\underline{b})}$ through $p(x_0)$. Then, by the local structure theorem, this induces an isomorphism

$$P(X) \overset{L(\underline{b})}{\times} \Sigma(b) \xrightarrow{\sim} \bar{X}\setminus\bar{D}.$$

The group $L(\underline{b})$ acts on $\Sigma(\underline{b})$ only via its quotient $A = A_X$ and the orbits in $X \cap \Sigma(\underline{b})$ are just twisted flats. Hence, it suffices to prove the theorem for the A-orbit closures in $\Sigma(\underline{b})$.

Because A is a torus there is an A-stable affine subset $\Sigma_0 \subseteq \Sigma(\underline{b})$ such that $Z_0 := Z \cap \Sigma_0$ is non-empty and all orbits in Z_0 are closed (see e.g. [Kn4, 2.10]). Consider the categorical quotient $q: \Sigma_0 \twoheadrightarrow \Sigma_0/\!/A$. Now we use the main assumption, namely that v_Z is central, i.e., it vanishes on all B-invariant rational functions. Because $k(X)^B = k(\Sigma_0)^A$, this implies that $Z_0 \to \Sigma_0/\!/A$ is surjective and $k(\Sigma_0)^A = k(Z_0)^A = \operatorname{Quot} k[\Sigma_0/\!/A]$. Hence the generic fiber of q contains a dense orbit. This implies that the generic fiber is an affine A-embedding with a closed orbit of codimension one, namely the intersection with Z. Therefore, the induced valuation corresponds to $v = v_Z$. This shows that the generic orbit closure in $\Sigma(\underline{b})$ contains $\bar{A}(v)$ such that the closed orbit is in Z. □

Now we are in the position to prove the main result of this paper.

7.4. Theorem *The action of the little Weyl group W_X on $\mathscr{H}(X) = \operatorname{Hom}(\mathscr{X}(A_X), \mathbb{Q})$ is generated by reflections and $\mathscr{Z}(X)$ is one of its Weyl chambers. In particular, $\mathscr{Z}(X)$ is a simplicial cone and a fundamental domain of W_X.*

Proof. We know that $\mathscr{Z}(X)$ is a finitely generated cone with non-empty interior. Thus it suffices to show two things:

A: The reflections at faces of codimension one of $\mathscr{Z}(X)$ are contained in W_X.

B: Each W_X-orbit intersects $\mathscr{Z}(X)$ in at most one point.

A: This is a consequence of functorial properties. By [Kn4, 8.1.4], $\mathscr{Z}(\tilde{X})$ is the preimage of $\mathscr{Z}(X)$ in $\mathscr{H}(\tilde{x})$. By replacing X with $\tilde{X}$ we may assume that X is non-degenerate. Choose $v \in \mathscr{Z}(X)$ in the relative interior of a face of codimension one. Let $\bar{X}$ be a smooth G-variety, G-birational to X, which contains a prime divisor Z such that the valuation v_Z is a multiple of v. Let $N = N_{\bar{X}}(Z)$ be its normal bundle. Then $P(X) = P(Z) \supseteq P(N)$ (by e.g. [Kn4, 2.13, 2.6]). By [Kn4, 7.4] the equality $\mathscr{H}(N) = \mathscr{H}(X)$ holds. This implies $P(N) = P(X)$ and that N is non-degenerate. By the same theorem, $\mathscr{Z}(N)$ is the cone generated by $\mathscr{Z}(X)$ and $-v$. Hence $\mathscr{Z}(N)$ is a half space whose boundary $\mathscr{B}$ is the hyperplane spanned by the chosen face. From $\mathscr{Z}(N) \neq \mathscr{H}(X)$ follows $N \neq G \cdot N^U$ [Kn4, 8.5]. This implies that W_N is non-trivial [Kn1, 9.1].

By [Kn4, 8.2] there is a torus $H \subseteq \operatorname{Aut}^G N$ with $k(N)^B \subseteq k(N)^H$ and such that $\mathscr{B} = \operatorname{Hom}(\mathscr{X}(H), \mathbb{Q})$. Then Theorem 5.1 implies that W_N fixes $\mathscr{B}$ pointwise. Hence W_N consists of the identity and the reflection at $\mathscr{B}$.

There is a flat deformation $Y \to \mathbf{A}^1$ such that the generic fiber is $\bar{X}$ and the zerofiber is N (see e.g. [Fu, § 5.1]). Then we have $W_N \subseteq W_Y$ and $W_Y = W_X$ by [Kn1, 6.5, and 4.] respectively. This shows the claim.

B: Assume, there is $v \neq w \cdot v \in \mathscr{Z}(X)$, $w \in W_X$. Let $\bar{X}$ be G-birational to X containing two disjoint invariant divisors Z_1 and Z_2 whose valuations are proportional to v and wv, respectively. Then the closure $\bar{A}_{\bar{X}}$ of a generic twisted flat contains $\bar{A}(wv)$ whose closed orbit is both in Z_1 and in Z_2, a contradiction. □

In [Kn1] a little Weyl group has been defined for arbitrary G-varieties.

7.5. Corollary *The definition of W_X is independent of the choice of $\tilde{X}$ and coincides with that of* [Kn1].

Proof. W_X is uniquely determined by $\mathscr{Z}(X)$ whose definition does not depend on the choice of $\tilde{X}$. This shows the first claim. For the second let W'_X be the little Weyl group as defined in [Kn1]. We know already $W'_{\tilde{X}} = W_{\tilde{X}} = W_X$ because $\tilde{X}$ is non-degenerate (Lemma 3.4). Furthermore, $W'_X \subseteq W'_{\tilde{X}} = W_X$ by [Kn1, 6.5:1]. Finally, the whole reasoning of part A of the proof above applies also to W'_X. The only thing one has to observe is that statement and proof of Theorem 5.1.3 are also valid without the non-degeneracy assumption for the Weyl groups as defined in [Kn1]. Hence, W'_X contains all reflections at faces of $\mathscr{Z}(X)$, which shows $W_X = W_{\tilde{X}} \subseteq W'_X$. □

Here are some applications for an arbitrary normal G-variety X. Recall that the center of a valuation v is the largest closed subvariety Y with $v(\mathcal{O}_{X,Y}) \geqq 0$.

Definition. A *source* of X is a non-empty subvariety $Y \subseteq X$ which is the center of a central valuation.

7.6. Corollary *Let $\bar{A}_X$ be the closure of a generic twisted flat and $v \in \mathscr{H}(X)$. Then there is a morphism of embeddings $\bar{A}(v) \to \bar{A}_X$ if and only if the unique valuation $wv \in \mathscr{Z}(X)$, for $w \in W_X$, has a non-empty center in X. In this case the closed orbit of $\bar{A}(v)$ is mapped into this center.*

Proof. We may assume $w = 1$. Let $X \hookrightarrow X_1$ be an equivariant completion (see [Su]). Then v has a non-empty center Y in X_1. Choose a G-variety $\bar{X}$ birational to X which contains a divisor Z with $v_Z = v$ and such that there is a morphism $\bar{X} \to X_1$. This morphism maps Z dominantly to Y. Hence, there is a morphism $\bar{A}(v) \subseteq \bar{A}_{\bar{X}} \to \bar{A}_X$ if and only of $Y \cap X \neq \emptyset$. □

7.7. Corollary *Each G-variety X contains only a finite number of sources and $\bar{A}_X$ meets each one of them.*

Proof. The number of orbits in $\bar{A}_X$ is finite. □

7.8. Corollary 1. *The closure of a generic twisted flat is complete if and only if every valuation $v \in \mathscr{Z}(X)$ has a non-empty center in X.*

2. *The generic twisted flat is closed if and only if X contains no proper source.*

The next result has been proved by Brion for spherical varieties with an entirely different method (unpublished).

7.9. Corollary *Let X be a normal affine variety, containing a proper source. Then there exist a non-trivial G-stable non-negative grading of $k[X]$.*

Proof. The embedding $\bar{A}_X$ is affine. Hence its normalization is given by a cone $\mathscr{C} \subseteq \mathscr{H}(X)$. This cone must be W_X-stable. Because $\bar{A}_X$ meets Y it cannot be trivial. Hence, there is $v \neq 0$ in the relative interior of $\mathscr{C}$. But then $v_0 := \sum_{w \in W_X} wv$

is W_X-invariant and also in the interior of $\mathscr{C}$, hence non-zero and contained in $\mathscr{Z}(X) \cap -\mathscr{Z}(X)$. To this v_0 corresponds a one parameter subgroup $H \subseteq \mathrm{Aut}^G X$ by [Kn4, 8.2], which induces an invariant grading. The generic orbit of H is non-closed in the closure of a generic flat, hence in X, i.e., the grading is non-negative. □

Remark. The corollary is most useful in the spherical case: A normal affine non-homogeneous spherical variety is graded.

8 Toroidal sources

The reasoning in the preceding section allows various generalizations which we didn't include there in order to prove Theorem 7.4 as soon as possible. Let $Y \subseteq X$ be a G-stable subvariety. Then we denote by $\mathscr{F}(Y)$ the set of all prime divisors $D \subseteq X$ with $Y \subseteq D$, $B \cdot D = D$ but $G \cdot D \neq D$. See [Kn4] for the relevance of this notion.

8.1. Theorem *Let $X \hookrightarrow \bar{X}$ be an embedding and $Y \subseteq \bar{X}$ a source with $\mathscr{F}(Y) = \emptyset$. Then there is a $P(X)$-stable open subset $X_0 \subseteq \bar{X}$ with $X_0 \cap Y \neq \emptyset$ and*
(a) *The quotient $\Sigma_0 := X_0/P_u(X)$ exists and L acts on it only via A_X.*
(b) *For every $\underline{b} \in \mathbf{A}_r^s$ there is an isomorphism $P(X) \overset{L}{\times} \Sigma_0 \xrightarrow{\sim} X_0$ such that the generic A_X-orbits are mapped to twisted flats.*
(c) *Conversely, there is a non-empty open subset V of $\hat{T}_{\bar{X}}$ with: For every $\hat{\alpha} \in V$ there is a $\underline{b}$ such that the twisted flat $F_{\hat{\alpha}}$ is the image of an A_X-orbit as in* (b).

Proof. Same as the first part of the proof of Theorem 7.3, together with the following remark: Let $\bar{D}'(\underline{b})$ be the B-divisor of $\bar{X}$ which one obtains by omitting all G-stable components and let $\bar{D}'$ its support. Then by assumption $X_0 := \bar{X} \setminus \bar{D}'$ will intersect Y non-trivially and $\psi_{\bar{D}^\mp(\underline{b})}$ differs from $\psi_{\bar{D}(\underline{b})}$ only by translation with a character of $\mathfrak{g}$. □

With this theorem we are left with the study of the A_X-variety Σ_0. The best thing which can happen is when Σ_0 is isomorphic to $\bar{A} \times V$ with some torus embedding $\bar{A}$. Therefore, we define:

Definition. A source Y in a normal G-variety X is called *toroidal* if $\mathscr{F}(Y) = \emptyset$ and each G-stable prime divisor of X containing Y is central. Let $\mathscr{V}_X(Y) \subseteq \mathscr{Z}(X)$ be the set of valuations attached to these divisors.

Let $Y \subseteq X$ be a toroidal source. Then the local ring $\mathcal{O}_{X,Y}$ is uniquely determined by $\mathscr{V}_X(Y)$ [Kn4, 3.8].

8.2. Theorem *Let $X \hookrightarrow \bar{X}$ be a normal embedding and let $Y \subseteq \bar{X}$ be a toroidal source.*
(a) *There is a normal affine A_X-embedding $\bar{A}$ with closed orbit Z such that $\mathscr{V}_{\bar{A}}(Z) = \mathscr{V}_{\bar{X}}(Y)$ and which is contained in the closure of the generic flat.*
(b) *The closure $\bar{F}_{\hat{\alpha}}$ of a generic flat is transversal to Y, i.e., there is a smooth variety V, a point $v \in V$ and an open embedding $\bar{A} \times V \hookrightarrow \bar{X}$ such that $\bar{A} \times \{v\} \hookrightarrow \bar{F}_{\hat{\alpha}}$ and such that $Z \times V$ maps onto an open subset of Y.*

Proof. By Theorem 8.1 is suffices to study $Z_0 := (Y \cap X_0)/P_u(X)$ inside Σ_0. The proof is very similar to the second part of that of Theorem 7.3. Again, we may assume that Σ_0 is affine and that Z_0 consists of closed orbits. Because every B-stable divisor containing Y is central, it follows that every A_X-invariant rational function is defined in Z_0. This implies that the generic fiber of $q:\Sigma_0 \to \Sigma_0//A_X$ contains a dense orbit. Because principal bundles for a torus are locally trivial in the Zariski topology, there is a rational section σ of $\Sigma_0 \twoheadrightarrow \Sigma_0//A_X$, such that the image of σ is contained in the union of the open orbits. Then by Lemma 6.1 we may shrink Σ_0 such that it is isomorphic to $\bar{A} \times V$ where $\bar{A}$ is the generic fiber and $V = \Sigma_0//A_X$. From this all assertions follow. □

Now, we can use the theory of torus embeddings to conclude: Let $Y \subseteq X$ be a toroidal source. Then $\mathscr{V}_X(Y)$ is finite. Let $\mathscr{C}_X(Y) \subseteq \mathscr{H}(X)$ be the convex cone generated by $\mathscr{V}_X(Y)$. Then the elements of $\mathscr{V}_X(Y)$ apn exactly the extremal rays of $\mathscr{C}_X(Y)$. In particular, $\mathcal{O}_{X,Y}$ is uniquely determined also by $\mathscr{C}_X(Y)$. It is easy to see that if X is quasihomogeneous then a source Y is toroidal if and only if $\mathscr{F}(Y) = \emptyset$. The reason is that Y is contained under these assumptions in only a finite number of B-stable prime divisors.

8.3. Corollary *Let $X \hookrightarrow \bar{X}$ be a normal equivariant embedding such that all sources are toroidal. Let $\mathfrak{F}$ be the set of $\mathscr{C}_{\bar{X}}(Y) \subseteq \mathscr{H}(X)$ where Y runs through all sources of $\bar{X}$. Then the closure of a generic flat is normal and is given as an A_X-embedding by the fan $W_X \mathfrak{F}$.*

Next I want to improve Corollary 6.2. Remember the map $\hat{T}_{\bar{X}} \to \mathfrak{a}_{\bar{X}}^*$. *Let* $\mathfrak{a}_{\bar{X}}^{rr} \subseteq \mathfrak{a}_{\bar{X}}^r$ be the largest open subset over which all fibers are irreducible. This set is not empty (see proof of Lemma 3.4).

8.4. Theorem *Let $X \subseteq \bar{X}$ be a normal embedding and assume that for every source Y in $\bar{X}$ the set $\mathscr{F}(Y)$ is empty. Then an open subset $S' \subseteq \hat{T}_{\bar{X}}$ can be found having the properties of Corollary 6.2, and additionally that the image of $S' \to \mathfrak{a}_{\bar{X}}^r$ contains $\mathfrak{a}_{\bar{X}}^{rr}$.*

Proof. Let $Y \subseteq \bar{X}$ be a source and X_0 as in Theorem 8.1. Let $\bar{A}$ be the closure of a generic A_X-orbit in $X_0/P_u(X)$. It follows from that theorem that there is an open subset $S(Y) \subseteq \hat{T}_{\bar{X}}$ with $S(Y) \twoheadrightarrow \mathfrak{a}_{\bar{X}}^r$ such that $\bar{A} \hookrightarrow F_{\hat{\alpha}}$ for all $\hat{\alpha} \in S(Y)$. Now set $S' := \bigcap_{w,Y} wS(Y)$ where w runs through W_X and Y through all sources. Then S' has all required properties because the intersection of finitely many non-empty open subsets is non-empty on an irreducible variety. □

In general, the set S' will depend on the embedding $\bar{X}$ and no flat may be good for all $\bar{X}$. But let X be a homogeneous spherical variety with an embedding $X \hookrightarrow \bar{X}$. Then every valuation is central, hence every orbit of $\bar{X}$ is a source, and the condition of Theorem 8.4 means that no B-stable divisor of X contains a G-orbit in its closure. These embeddings are called toroidal (comp. [BL, 2.1]). Now we get really specific flats with a good behavior, independently of $\bar{X}$:

8.5. Corollary *Let $X = G/H$ be spherical and $\bar{X}$ a toroidal embedding. Let $S' \subseteq \hat{T}_{\bar{X}}$ be the preimage of $\mathfrak{a}_{\bar{X}}^{rr}$ and let $\bar{A}_{\bar{X}}$ be the closure of a generic twisted flat. Then the morphism $S' \times A_X \to S' \times X$: $(\hat{\alpha}, a) \mapsto (\hat{\alpha}, \zeta_{\hat{\alpha}}(a))$ extends to a closed embedding $S' \times \bar{A}_{\bar{X}} \to S' \times \bar{X}$.*

Proof. Because $\tilde{X}$ is also spherical, we can identify the parameter space $\mathbf{A}_r^s$ with $\mathfrak{a}_{\bar{X}}^*$. Then the existence of the open embedding

$$G \overset{P(X)}{\times} [\tilde{X} \backslash D \times \mathfrak{a}_{\bar{X}}^r] \hookrightarrow \hat{T}_{\bar{X}}$$

shows that each fiber of $\hat{\Psi}: \hat{T}_{\tilde{X}} \to \mathfrak{a}^r_{\tilde{X}}$ contains an open G-orbit. This implies that each fiber F of $\hat{\Phi}$: $S' \to G/L \times \mathfrak{a}^{rr}_{\tilde{X}}$ contains an open, hence dense A_X-orbit. The image of this dense orbit in $\tilde{X}$ is a flat. Because $\tilde{X}$ is homogeneous, this subspace is closed (Corollary 7.8). Hence all fibers F are homogeneous. Then the assertion follows from Theorem 8.4. □

9 Logarithmic and twisted cotangent bundles

In this section we discuss some variants of the cotangent bundle. Let me assume that $X=\tilde{X}$ is non-degenerate. Flats are the images in X of the A_X-orbits in $\hat{T}_X$ and we studied their closure in certain embeddings $\bar{X}$. It is convenient to have the same closure for the orbits themselves. This is possible by extending the bundles T^*_X and $\hat{T}_X$ to all of $\bar{X}$. Just taking $T^*_{\bar{X}}$ doesn't work, one reason being, that it is not a vector bundle if $\bar{X}$ is not smooth.

Assume again that every source of $\bar{X}$ is toroidal. Let $D \subset \bar{X}$ be the union of all central divisors. Then let $\Omega_{\bar{X}}[D]$ be the sheaf of differential forms with logarithmic singularities along D, i.e., it is locally generated by $\Omega_{\bar{X}}$ together with sections of the form $f^{-1}df$ where f is a rational function having its poles and zeros in D (see [O, 3.1]). Let $T^*_{\bar{X}}[D] = \mathrm{Spec}_{\bar{X}} S^* \Omega_{\bar{X}}[D]^\vee$ be the geometric realization of $\Omega_{\bar{X}}[D]$.

9.1. Theorem *Let X be non-degenerate and $X \hookrightarrow \bar{X}$ an equivariant normal embedding such that every source is toroidal.*
(a) *The moment map $T^*_X \to \mathfrak{g}^*$ extends to $T^*_{\bar{X}}[D] \to \mathfrak{g}^*$. In particular, $\hat{T}_{\bar{X}}[D]$ is defined as an irreducible component of $T^*_{\bar{X}}[D] \underset{\mathfrak{t}_*/W}{\times} \mathfrak{a}^r_X$.*
(b) *There is a non-empty G-stable open subset $\bar{X}_0 \subseteq \bar{X}$, meeting every source, such that $T^*_{\bar{X}}[D]$ is a vector bundle over $\bar{X}_0$.*
(c) *For $\hat{\alpha} \in \hat{T}_X$ generic, let $\bar{A}$ be the closure of the orbit $A_X\hat{\alpha}$ in $\hat{T}_{\bar{X}}[D]$. Then $\hat{\pi}: \bar{A} \to \bar{X}$ is a closed embedding, hence an isomorphism onto a flat.*

Proof. (a) Let $\xi \in \mathfrak{g}$ and $f \in \mathcal{O}_{\bar{X}}$ be an invertible function on some open subset $V \subseteq X$. Because D is G-stable we have $c_{D_0}(\xi_* f) \geqq v_{D_0}(f)$ for every component of D. Hence, $f^{-1}df(\xi_*) = f^{-1}\xi_* f$ regular along D, i.e., the moment map extends.

(b) Let $Y \subseteq \bar{X}$ be a source and let $\bar{A} := \bar{A}(\mathscr{C}(Y))$. Then there is a smooth variety V such that $V \times \bar{A}$ is isomorphic to an open subset of $\bar{X}$ which intersects Y non-trivially (Theorem 8.2). Let $D_0 := \bar{A} \backslash A_X$ the boundary. Then $V \times D_0$ corresponds to D. This reduces the assertion to the case of the torus embedding $\bar{A}$. Then $\Omega_{\bar{X}}[D_0]$ is spanned by $\chi^{-1}d\chi$ where χ is a character of A_X [O, Proposition 3.1]. This means that $T^*_{\bar{A}}[D_0] = \bar{A} \times \mathfrak{a}^*_X$ is just the trivial bundle, in particular a vector bundle.

(c) By the same reasoning it suffices to study $\bar{A}$. Then $\hat{T}_{\bar{A}}[D_0] = T^*_{\bar{A}}[D_0] = \bar{A} \times \mathfrak{a}^*_X$ proves the claim. □

The most important application is again for spherical varieties. A spherical variety is called toroidal if all orbits (= sources) are toroidal. These are classified by fans $\mathfrak{F}$ which are supported in $\mathscr{Z}(X)$ (see [Kn3] for details). Let $W_X\mathfrak{F}$ be the fan consisting of all cones $w\mathscr{C}$ where $w \in W_X$ and $\mathscr{C} \in \mathfrak{F}$.

9.2. Corollary *Let X be a non-degenerate toroidal spherical variety corresponding to a fan $\mathfrak{F}$. Consider the moment map on the logarithmic cotangent bundle*

$T_X^*[D] \to \mathfrak{g}^*$. *Then the fiber over any element of* $\mathfrak{a}^{rr}$ *is the disjoint union of varieties isomorphic to the* A_X*-embedding corresponding to the fan* $W_X \mathfrak{F}$.

Proof. It suffices to prove that for $\hat{\Phi}$: $\hat{T}_X[D] \to G/L \times \mathfrak{a}^*$. The proof of Corollary 8.5 showed that every fiber over $G/L \times \mathfrak{a}^{rr}$ is irreducible with dense A_X-orbit. Then Corollary 8.3 and Theorem 9.1 imply that the fiber is the torus embedding attached to $W_X \mathfrak{F}$. □

That the fibers are in general not connected is due to the fact that W_X may not equal the normalizer $W(\mathfrak{a}^*)$. The number of components is always equal to the index of W_X inside $W(\mathfrak{a}^*)$. The corollary is a weak version for spherical varieties of a theorem of Abeasis [Ab]. There, all fibers of the moment map for symmetric varieties over semisimple points are described.

Finally let me mention a slight reformulation of the theory. To make the theory work also for degenerate varieties, we applied our theorems to the cotangent bundle of an auxiliary space $\tilde{X}$. With a reformulation it is possible to do this the other way round, namely to keep X but to replace T_X^* by another bundle namely a twisted (or shifted) cotangent bundle introduced by Ginzburg and Kostant (see [BB, Sect. 2] for details). This has the advantage that the group H disappears (almost). Remember $H \subseteq G$ by definition and the short exact sequence

$$0 \to \mathfrak{a}_X^* \to \mathfrak{a}_{\tilde{X}}^* \to \mathfrak{h}^* \to 0.$$

Hence we have the diagram

$$\begin{array}{ccc} \hat{T}_{\tilde{X}} & \longrightarrow & T_{\tilde{X}}^* \\ \downarrow & & \downarrow{\scriptstyle q} \\ \mathfrak{a}_{\tilde{X}}^* & \longrightarrow & \mathfrak{h}^*. \end{array}$$

Both vertical arrows are H-invariant and H acts freely on the fibers (because it acts already freely on $\tilde{X}$). Choose $\lambda \in \mathfrak{h}^*$. Then one defines the *twisted cotangent bundle* as $T_X^\lambda := q^{-1}(\lambda)/H$ and analogously its polarized form $\hat{T}_X^\lambda$. It is easily verified that $T_X^\lambda \to X$ is a bundle which is locally isomorphic to T_X^*. Furthermore, the symplectic structure of $T_{\tilde{X}}^*$ induces one on T_X^λ (Hamiltonian reduction). Note, that G acts on T_X^λ via its quotient $\bar{G} = G/H$ (the group we started with). Similarly, on $\hat{T}_X^\lambda$ only $A_X = A_{\tilde{X}}/H$ is acting. We call the image in X of an A_X-orbit in $\hat{T}_X^\lambda$ a *flat with twist* λ.

The moment map on $T_{\tilde{X}}^*$ descends to a moment map $T_X^\lambda \to \bar{\mathfrak{g}}^*$. We call T_X^λ *non-degenerate* if λ is contained in the image of $\mathfrak{a}_{\tilde{X}}^{rr} \to \mathfrak{h}^*$. Then Theorem 8.4 implies

9.3. Theorem *Let* T_X^λ *be non-degenerate and* $X \hookrightarrow \bar{X}$ *an ambedding. Then the closure in* $\bar{X}$ *of a generic flat with twist* λ *is* W_X*-symmetric and contains* $\bar{A}(w\mathscr{C}(Y))$ *for all* $w \in W_X$ *and every toroidal source* $Y \subseteq \bar{X}$.

Finally, there is also a logarithmic-twisted version of this theorem, whose statement and proof I leave to the reader.

References

[Ab] Abeasis, S.: On as remarkable class of subvarieties of a symmetric variety. Adv. Math.

71, 113–129 (1988)
[BB] Borho, W., Brylinski, J-L.: Differential operators on homogeneous spaces. II. Relative enveloping algebras. Bull. Soc. Math. Fr. **117**, 167–210 (1989)
[Br] Brion, M.: Vers une généralisation des espaces symétriques. J. Algebra **134**, 115–143 (1990)
[BL] Brion, M., Luna, D.: Sur la structure locale des variétés sphériques. Bull. Soc. Math. Fr. **115**, 211–226 (1987)
[BLV] Brion, M., Luna, D., Vust, Th.: Espaces homogènes sphériques. Invent. Math. **84**, 617–632 (1986)
[Fu] Fulton, W.: Intersection Theory. (Ergebn. Math., 3. Folge, Bd. 2) Berlin Heidelberg New York: Springer 1984
[GS] Guillemin, V., Sternberg, S.: Symplectic techniques in physics. Cambridge: Cambridge University Press 1984
[Kn1] Knop, F.: Weylgruppe and Momentabbildung. Invent. Math. **99**, 1–23 (1990)
[Kn2] Knop, F.: Weylgruppe and äquivariante Einbettung. (Preprint 1990)
[Kn3] Knop, F.: The Luna-Vust theory of spherical embeddings. In: Ramanan, S. (ed.) Algebraic groups and its Applications, Hyderabad 1989, pp. 225–249. Madras: Manoj Prakashan 1991
[Kn4] Knop, F.: Über Bewertungen, welche unter einer reduktiven Gruppe invariant sind. Math. Ann. **295**, 333–363 (1993)
[KKV] Knop, F., Kraft, H., Vust, Th.: The Picard group of a G-variety. In: Kraft, H., Slodowy, P., Springer, T. (eds.) Algebraische Transformationsgruppen und Invariantentheorie. (DMV-Semin., vol. 13, pp. 77–87) Boston Basel Stuttgart: Birkhäuser 1989
[O] Oda, T.: Convex bodies and algebraic geometry. (Ergeb. Math., 3. Folge, Bd. 15) Berlin Heidelberg New York: Springer 1988
[Sp] Springer, T.: Linear algebraic groups. (Prog. Math., vol. 9) Boston Basel Stuttgart: Birkhäuser 1981
[Su] Sumihiro, H.: Equivariant completion. J Meth. Kyoto Univ. **14**, 1–28 (1974)
[TE] Kempf, G., Knudson, D., Mumford, D., Saint-Donat, B.: Toroidal Embeddings I. (Lect. Notes Math., vol. 339) Berlin Heidelberg New York: Springer 1973