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Inventiones mathematicae

On the cone structure at infinity of Ricci flat manifolds with Euclidean volume growth and quadratic curvature decay

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0. Introduction

Background

Let M^n be a complete riemannian manifold with metric g, such that

$$\text{Ric}_{M^n} \geqq 0\,, \tag{0.1}$$

$$\text{Vol}(B_r(p)) \geqq \Omega r^n \quad (\Omega > 0) \tag{0.2}$$

Fix $p \in M^n$ and let $r_j \to \infty$. It follows from Gromov's compactness theorem [GLP] that the sequence of pointed rescaled manifolds, $(M, p, r_j^{-2} g)$, has a subsequence which converges in the pointed Gromov-Hausdorff topology to a length space, M_∞, which might, a priori depend on the sequence, $\{r_j\}$ and the choice of convergent subsequence.

By Bishop's inequality, $\text{Ric}_{M^n} \geqq 0$ implies

$$\text{Vol}(B_r(p))/r^n \downarrow\,. \tag{0.3}$$

If one grants that the volume behaves continuously in the limit, it follows that M_∞ is a volume cone,

$$\text{Vol}(B_r(p_\infty)) = \Omega r^n\,. \tag{0.4}$$

Consideration of the Riccati equation along a geodesic suggests the following stronger statement which is proved in [CC2].

Theorem 0.5. ([CC2)] *M_∞ is a metric cone.*

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It turns out that Theorem 0.5 is actually somewhat easier to prove than the continuity of the volume under Gromov-Hausdorff limits. The latter is proved in [CC3].

In [BKN], a proof of Theorem 0.5 is given which works in the special case in which $M_\infty \setminus p_\infty$ is smooth, $\mathrm{Ric}_{M^n} \geqq 0$ and the sub-convergence of $(M^n, r_j^{-2}g, p)$ is in a sufficiently strong topology.

A basic question concerning M_∞ is whether or not it is unique, i.e. is M_∞ the same up to isometry for all $\{r_j\}$ and all convergent subsequences? An example of Perelman shows that even if one imposes the additional condition of quadratic curvature decay, this need not be the case. Note that since any cone which is a smooth riemannian manifold outside the singular point has quadratic curvature decay this condition is natural in our context.

Let R denote the curvature tensor.

Example 0.6. ([P]). There exist complete metrics on R^4 satisfying (0.1), (0.2) and

$$|R(x)| \leqq cr^{-2} \quad (r = \overline{x, p}) \tag{0.7}$$

for which uniqueness fails, although all M_∞ are indeed metric cones.

In this paper, we consider the situation in which (0.1) is strengthened to

$$\mathrm{Ric}_{M^n} \equiv 0 \tag{0.8}$$

and (0.7) is replaced by the apparently weaker condition,

$$\int_{B_{2r}(p)\setminus B_r(p)} |R|^{n/2} \leqq \Lambda \quad (\Lambda \text{ independent of } r)\,. \tag{0.9}$$

In fact, (0.9) turns out to imply (0.7), given (0.2), (0.8).

Again in this case, Theorem 0.5 follows from the argument of [BKN], and in particular, $M_\infty = C(N^{n-1})$ is *Ricci flat*. Our basic concern is with the question of *uniqueness*.

Statement of Results

Recall that the condition that $C(N^{n-1})$ is Ricci flat is equivalent to $\mathrm{Ric}_{\tilde g} - (n-2)\tilde g = 0$, where $\tilde g$ is the metric on the cross-section, N^{n-1}. The *linearized deformation equation* is, by definition, the equation on h gotten by putting $h = \tilde g_0'$, in

$$\frac{d}{du}(\mathrm{Ric}_{\tilde g_u} - (n-2)\tilde g_u)|_{u=0} = 0\,. \tag{0.10}$$

Definition 0.11. The cone, $C(N^{n-1})$, is *integrable* if every solution of the linearized deformation equation arises from a curve of metrics, $\tilde g_u$, satisfying

$$\mathrm{Ric}_{\tilde g_u} - (n-2)\tilde g_u = 0\,. \tag{0.12}$$

Let $A_{c,d}(\underline{p})$ denote $\{(r,x) \in C(N^{n-1}) | c < r < d\}$.

Theorem 0.13. *If* (0.2), (0.8), (0.9) *hold and some tangent cone* $M_\infty = C(N^{n-1})$ *is integrable, then* M_∞ *(equivalently* $\tilde{g}$*) is unique. Moreover, for some compact set* C*, there is a diffeomorphism,* ϕ*, from the annulus* $A_{c,\infty}(\underline{p}) \subset C(N^{n-1})$ *to* $M^n \setminus C$*, such that*

$$
\begin{aligned}
|\phi^* g - g_0| &= |\phi^* g - (dr^2 + r^2 \tilde{g})| \\
&= O(r^{-\beta})
\end{aligned}
\tag{0.14}
$$

for some $\beta > 0$.

The issue of the optimal rate of convergence is dealt with later in this introduction.

Now consider the case in which M^{2k} is Kähler. Then it follows that $C(N^{2k-1})$ is Kähler as well. Let J be the almost complex structure. Observe that $r\frac{\partial}{\partial r}, J\left(r\frac{\partial}{\partial r}\right) := \frac{\partial}{\partial \theta}$, generate a $\tilde{\mathbb{C}}^*$ action, where $\tilde{\mathbb{C}}^*$ is the universal covering group of $\mathbb{C}^* = \mathbb{C} \setminus 0$. Call a Kählerian cone *standard* if the $\tilde{\mathbb{C}}^*$ action descends to a $\mathbb{C}^*$ action. In this case, $C(N^{n-1})$ is actually a complex cone with a complex base, N^{2k-1}/S^1, which might be an orbifold ($S^1 = e^{2\pi i\theta}, 0 \leqq \theta \leqq 2\pi$). Note that $\frac{\partial}{\partial \theta}$ is a real holomorphic Killing field. Moreover, if the dimension of the space of real holomorphic Killing fields is 1, then of necessity, all orbits of $\frac{\partial}{\partial \theta}$ are closed (and $C(N^{2k-1})$ can be shown to be standard).

In the complex case, there is a notion of *complex integrability* for cones which plays a role analogous to that of *integrability* for real Ricci flat cones; see Definition 8.1.

Theorem 0.15. *Let* M^{2k} *be Kähler and assume that for some tangent cone,* $C(N^{2k-1})$*, the dimension of the space of holomorphic Killing fields is 1. Then* $C(N^{2k-1})$ *is complex integrable and hence, unique.*

Theorem 0.16. *Under the assumptions of Theorem* 0.15, *if*

i) $b^{1,1}(N^{2k-1}) = 1$, *then* $\beta \geqq 2k$ *(*β *as in* (0.14)*).*

ii) If $k \neq 3$*, the complex structure converges to that of* $C(N^{2k-1})$ *at the rate* r^{-2k}.

As a consequence of ii) above, M^{2k} can be complex analytically compactified. This will be dealt with in [CT], where the more general case, $\mathrm{Ric}_{M^{2k}} \geqq 0$, is also treated.

In dim 3 some additional assumptions are required for part ii) of Theorem 0.16. Probably they always are satisfied. This will be discussed elsewhere.

We mention that for $k \geqq 2$, there exist nonstandard Kählerian cones with positive Ricci curvature.

Conjecture 0.17. *All Ricci flat Kählerian cones are standard.*

In any case, ultimately it may be possible to prove Theorems 0.15, 0.16 without the hypothesis that the dimension of the space of holomorphic Killing fields is 1.

Relation to previous results

If condition (0.9) is strengthened to

$$\int_{M^n} |R|^{n/2} < \infty, \tag{0.18}$$

then we are in the so-called asymptotically locally Euclidian or ALE case. Here $C(N^{n-1})$ is known to be unique and flat. Thus, $C(N^{n-1}) = R^n/\Gamma$, where Γ is a finite group of isometries on R^n, acting freely outside the origin; [An], [BKN]. Moreover, as is shown in [BKN], (0.14) hold with $\beta \geqq n-1$. see [BKN].

The argument of [BKN] uses inequalities based on (0.18), as well as some curvature identities which hold only in very special cases. Thus, their argument does not generalize to our case where (0.18) fails. Nonetheless, as explained below, our method permits us to show

$$\beta \geqq n\,; \tag{0.19}$$

see Theorem 5.103. By relaxing (0.18) to (0.9), we bring in many more examples, including a large family of Kählerian ones constructed in [TY] (see also [BK] for special cases with extra technical assumptions). Theorems 0.15 and 0.16 provide at least a partial converse to the construction of [TY], a complete converse if one can remove the condition that the dimension of the space of holomorphic Killing fields is 1.

Of course it is natural to ask whether Theorem 0.13 continues to hold if the hypothesis of integrability is omitted. Indeed, the discussion of Simon [S1], [S2], where the analogous question of the uniqueness of asymptotic limits is treated for *variational problems* (including those for minimal surfaces and harmonic maps) suggests that we should expect uniqueness of $\tilde{g}$ without any additional assumptions. However, without the integrability hypothesis, we should not expect convergence at the rate $r^{-\beta}$, but only at the slower rate $\frac{1}{|\log r|^{-\beta'}}$.

Our approach has considerable overlap with that of Simon, although in our situation (unlike his) suitable monotonicity inequalities are not known and the equation is degenerate elliptic i.e. we must get rid of the action of the diffeomorphism group on the space of metrics. The existence of suitable monotonicity inequalities would provide one method for treating the nonintegrable case. We point out that for minimal surfaces, the integrable case was first treated by Allard-Almgren, [AA].

Sketch of the proof

In proving Theorem 0.13, we begin by considering metric g which is sufficiently close in the appropriate scaled topology to a Ricci flat cone metric, g_0, on an arbitrarily large (possibly semi-infinite) annulus in $C(N^{n-1})$. We show that there exists a diffeomorphism, η, close to the identity in the appropriate scaled topology, such that $\eta^* g$ is divergence free with respect to g_0. For the application, it is crucial the degree of closeness required for the conclusion to hold is *independent of the size of the annulus.* Since $C(N^{n-1})$ is noncompact, this result is not quite standard and in certain exceptional cases, we must actually use a slightly modified notion of divergence. This technical point is mostly ignored for the remainder of this section.

Next we study the linearized deformation equation, gotten by putting $h = g_0'$ in

$$\frac{d}{du}(\mathrm{Ric}_{g_u})|_{u=0} = 0 \, , \tag{0.20}$$

for *divergence free* symmetric bilinear forms on the cone, $(C(N^{n-1}), g_0)$. By using separation of variables, we observe that the general solution of (0.20) can be written as a sum of solutions which are of three different types,

a) growth, like $r^{\beta}(\beta > 0)$,
b) decay, like $r^{-\beta}(\beta > 0)$,
c) radially parallel.

The existence of tangent cones tells us that our solution to the nonlinear equation, (M^n, g), lies as close as we like to *some* cone for r sufficiently large. Therefore we must show that:

i) the behavior of solutions to the nonlinear equation which lie sufficiently close to a cone can be modeled on that of solutions of the linearized equation;
ii) for (M^n, g) as above, the influence of the solutions of types a) and c) is negligible.

If we take i) for granted, ii) can be seen roughly as follows.

In the presence of the integrability hypothesis, the contribution from the radially parallel solutions, c), can be subtracted off (i.e. removed) by changing the reference cone. *We emphasize that when integrability does not hold, it is these solutions which cause all the trouble.*

For r sufficiently large, a given growth solution, a), lies at a definite distance from *any* cone. Thus, if its influence were felt at all, this would force (M^n, g) to eventually lie at a definite distance from *any* cone as well. But this contradicts the existence of tangent cones.

Note that since a growth solution can start out arbitrarily small, we have *no a priori control* over the size of the annulus required for it to grow to a definite size. This is why we must consider *arbitrarily large* annuli when constructing η such that $\eta^* g$ is divergence free.

Point i) seems almost clear, provided we are considering an annulus of a *fixed size* over which the norm of our solution does not vary too much. But, as noted above, we must in actuality, deal with annuli which are *arbitrarily large*. As in [S1], [S2], we use an argument based on consideration of *three adjacent* annuli which (up to scaling) have a *fixed* size. In this context, if i) failed to hold for solutions, $\eta^* g$, lying *arbitrarily close* to our conical metric g_0, then by an argument based on rescaling and elliptic estimates, we would obtain a solution of the *linearized* equation with the wrong behavior; a contradiction.

In the divergence free case, (0.10) can be written as $\tilde{\square}\tilde{h} = 0$, where $\tilde{h} = \tilde{g}_0'$ and $\tilde{\square}$ is a second order elliptic operator on N^{n-1}. The admissible values for the number β in (0.14), are determined by the spectrum of $\tilde{\square}$.

In general, $\tilde{\square}$ has a finite number of negative eigenvalues, and $\tilde{\square} \geqq 0$ actually implies $\beta \geqq (n-2)$. In the ALE case we show that in fact $\beta \geqq n$. If M^{2k} is Kähler, under the assumptions of Theorem 0.15, we show that $\tilde{\square} \geqq 0$, by reducing matters to the complex base, on which the corresponding statement is essentially known. Actually, in the Kähler case, we show $\beta \geqq 2k$.

Outline of the paper

The remainder of the paper is divided into eight sections as follows.

1. Existence of tangent cones
2. The operator $\delta_{g_0} L_X g_0$ on a cone
3. Reduction to the (modified) divergence free case
4. The linearized equation
5. The integrable case
6. A special result in the nonintegrable case
7. The linearized equation in the Kähler case

Appendix A sharp decay estimate.

8. Complex integrability and the Kähler case.

In Section 1, we observe that a slight extension of the argument of [BKN] proves the existence of tangent cones in our situation. We state this in a more precise form which is required for the argument sketched above, see Proposition 1.50.

In Sections 2 and 3 we reduce matters to the (modified) divergence free case. In Section 3 (see Theorem 3.1), we apply the implicit function theorem, based on the analysis of the relevant linearized operator carried out in Section 2; see Theorem 2.68. This enables us to state a version of Proposition 1.50 which holds for the (modified) divergence free gauge; see Proposition 3.24. Such a result is required for Section 5.

In Section 4, we write out the linearized deformation equation, (0.11), on a Ricci flat cone, and using separation of variables, exhibit its solutions; see Proposition 4.65. The specific applications to Section 5 are explained at the end of Section 4.

In Section 5, we prove our main result in the integrable case, Theorem 0.13. In addition, we give an essentially sharp result on the rate of convergence, Theorem 5.78.

In Section 6 (which for the present paper represents something of a digression) assuming uniqueness of the tangent cone, we prove logarithmic convergence in the most nondegenerate nonintegrable case

Sections 7 and 8 deal with the Kähler case. In Section 7, we show for a standard Ricci flat cone that (modulo diffeomorphisms) all bounded infinitesimal deformations come either from deformations of Kähler class and decay quadratically, or come from infinitesimal deformations of the complex structure on the base and decay at least like $r^{-(2k-\varepsilon)}$.

In Section 8, we complete the proof of Theorem 0.15 and Theorem 0.16.

As mentioned above, in a future publication, [CT], we will study the Kähler case in the presence of (0.2), (0.5) when assumption (0.1), $\mathrm{Ric}_{M^n} \equiv 0$, is weakened to $\mathrm{Ric}_{M^n} \geqq 0$. We will show that in this situation, the complex structure still behaves in asymptotically conical fashion and that one can still construct a complex analytic compactification.

We mention that for the analysis of Section 7, we need the following basic eigenvalue estimate which does not seem to be too well known; see however [EM] and [DNP] p.48. (We are indebted to Michael Taylor and McKenzie Wang for providing these references).

Let $(X^{n-1}, \tilde{g})$ satisfy

$$\mathrm{Ric}_{X^{n-1}} \geqq (n-2)\tilde{g}\,. \tag{0.21}$$

Then the smallest eigenvalue, $\underline{\mu}_1$, of the Laplacian on coclosed 1-forms satisfies

$$\underline{\mu}_1 \geqq 2(n-2)\,, \tag{0.22}$$

with equality holding precisely for 1-forms which are dual to Killing fields and which are pointwise eigenvectors of the Ricci tensor with eigenvalue, $n-2$. The proof of (0.22) is very short; see Theorem 7.6.

A final point of notation. Throughout the paper, Ω, Λ, will always denote the constants in (0.2),(0.9) respectively.

1 Existence of tangent cones

In this section we prove the existence of tangent cones, $(C(N^{n-1}), g_0)$. In fact, we require a more precise result, Proposition 1.50, which asserts the following. Given $\varepsilon > 0$, there is a tangent cone, $C(N^{n-1})$ and a gauge, $\phi\colon A_{R,R'}(\underline{p}) \to M^n$ such that in a suitable scaled norm,

i) $\phi^* g$ is ε-close to g_0, somewhere on $A_{R,R'}(\underline{p})$,

ii) $\phi^* g$ is ε-almost radially parallel with respect to g_0,

iii) $\phi^* g$ is ε-almost divergence free with respect to g_0.

Moreover, either

iv) near $r = R' < \infty, \phi^* g$ is a definite distance, χ, independent of ε, away from g_0,

or

v) $R' = \infty$.

In Proposition 3.24 we will show that in iii), "ε-almost divergence free" can be replaced by "divergence free" (provided we use the modified divergence, δ_t). In Section 5, this formulation will help us, to rule out possibility iv) and to obtain our main result.

Let $|T|_{k,\alpha,\tilde{g}}$ denote the $C^{k,\alpha}$ norm of tensor field, T, on a riemannian manifold, $(X^m, \tilde{g})$, where the norm and covariant derivatives are computed with respect to a fixed riemannian metric, $\tilde{g}$. Note that if $\beta: X^m \to X^m$ is a diffeomorphism and if, $\beta_\sharp(T)$ denotes the tensor field on the domain corresponding to a given tensor field on the range, then

$$|\beta_\sharp(T_2) - \beta_\sharp(T_1)|_{k,\alpha;\beta^*\tilde{g}} = |T_2 - T_1|_{k,\alpha;\tilde{g}} \tag{1.1}$$

In particular, if β is an isometry, the two norms in (1.1) coincide.

If $\tilde{g}_1, \tilde{g}_2$ satisfy say $|g_2 - g_1|_{0,0;g_1} < \frac{1}{2}$, then for any T of type (p,q),

$$|T|_{k,\alpha;\tilde{g}_2} \leqq C_{p,q}|T|_{k,\alpha;g_1}|g_2 - g_1|_{k,\alpha;\tilde{g}_1} . \tag{1.2}$$

We begin with some notation.

Let $\{[X_i^m, \tilde{g}_i]\}$ be a sequence of isometry classes of riemannian manifolds diffeomorphic to a fixed riemannian manifold $(X^m, \tilde{g})$. We say that $\{[X_i^m, g_i]\}$ converges in the $C^{k,\alpha}$ topology if there exists $(X^m, \tilde{g})$ and diffeomorphisms, $\beta_i: X^m \to X_i^m$, of class $C^{k+1,\alpha}$, such that

$$\lim_{i\to\infty} |\beta_i^* \tilde{g}_i - \tilde{g}|_{k,\alpha,g} = 0 . \tag{1.3}$$

If $(C(N^{n-1}), g_0)$ is a metric cone, then the metric, g_0, is given by, $g_0 = dr^2 + r^2\tilde{g}$, where $\tilde{g}$ is the metric on N^{n-1}.

Proposition 1.4. *The collection* $[N^{n-1}, \tilde{g}]$ *such that the associated cone,* $(C(N^{n-1}), g_0)$, *satisfies* (0.2), (0.8), (0.9) *for fixed* Ω, Λ, *is compact in the* $C^{k,\alpha}$-*topology for all* k, α. *In particular, for* $\underline{x} \in A_{1,2}(\underline{p})$, *inj rad*$_{\underline{x}} \geqq \varepsilon(n, \Omega, \Lambda)$

Proof The conical structure, together with (0.9) implies that $(N^{n-1}, \tilde{g})$ satisfies

$$\int_{N^{n-1}} |\tilde{R}|^{n/2} \leqq C(n, \Omega, \Lambda). \tag{1.5}$$

Since $n > \dim N^{n-1}, \mathrm{Vol}(N^{n-1}) \geqq n\Omega, \mathrm{diam}(N^{n-1}) \leqq \pi$, the claim is an obvious consequence of the compactness theorems proved in [A1],[G],[Y].

Recall that the results of [A1],[G],[Y] referred to above, proceed by bounding from below, the $C^{k,\alpha}$-harmonic radius, r_H, of $\tilde{g}$ (equivalently, g_0).

On a cone, it is also convenient to introduce norms which take into account the scalings of the cone. Let $\psi_u: C(N^{n-1}) \to C(N^{n-1})$ be defined by $\psi_u(x,r) = (ux, r)$ be defined by $\psi_u(x,r) = (ux, r)$. Put

$$A_u = \overbrace{(\psi_{u^{-1}})_* \otimes \cdots \otimes (\psi_{u^{-1}})_*}^{p} \otimes \overbrace{(\psi_u^* \otimes \cdots \otimes \psi_u^*)}^{q} . \tag{1.6}$$

Let T be a tensor field of type (p,q). Define the pointwise norm $|T(u,x)|_{k,\alpha;0}$, at $r = u$, by

$$(1.7) \qquad |T(u,x)|_{k,\alpha;0} = |u^{p-q} A_u T(1,x)|_{k,\alpha;0} ,$$

where the norm on the right-hand side is the $C^{k,\alpha}$-norm, with respect to g_0 at $(1,x)$. Put

$$(1.8) \qquad |T|_{k,\alpha;0} = \sup_{(u,x)} |T(u,x)|_{k,\alpha;0}$$

Then it is easy to check that

$$(1.9) \qquad |T|_{k,\alpha;0} \leqq c ,$$

is equivalent to

$$(1.10) \qquad |(\nabla^0)^i T(r,x)| \leqq c(k) r^{-i}, \quad i = 0,\ldots,k$$

$$(1.11) \qquad |(\nabla^0)^k T(r,x)|_{0,\alpha} \leqq c(k) r^{-(k+\alpha)} ,$$

where the norm on the left-hand side of (1.12) is the α-Hölder norm and ∇^0 is the riemannian connection of g_0.

If T is defined for all r sufficiently large and for all such r satisfies (1.10), we write

$$(1.12) \qquad T \in \mathscr{T}^{p,q}_{k,\alpha,0} .$$

More generally, for $\ell \in R$, write

$$(1.13) \qquad T \in \mathscr{T}^{p,q}_{k,\alpha;\ell}$$

if and only if

$$(1.14) \qquad r^{-\ell} T \in \mathscr{T}^{p,q}_{k,\alpha;0} .$$

Thus for example, the function r itself satisfies, for all k, α

$$(1.15) \qquad r \in \mathscr{T}^{0,0}_{k,\alpha;1}$$

For $T \in \mathscr{T}^{p,q}_{k,\alpha;\ell}$, we put

$$(1.16) \qquad |T|_{k,\alpha;\ell} = |r^{-\ell} T|_{k,\alpha;0}$$

Let $p \in M^n$ and let ρ denote the distance function from p.

Theorem 1.17. *Let (M^n, g) satisfy* (0.2), (0.8), (0.9). *For all $0 < \varepsilon << L$ and k, α, there exists $c(g, \varepsilon, L, k, \alpha)$, such that if*

$$(1.18) \qquad R \geqq c(g, \varepsilon, L, k, \alpha) ,$$

there is a cone, $(C(N^{n-1}), g_0)$, satisfying (0.2), (0.8), (0.9) *and an imbedding,*

$$(1.19) \qquad \phi\colon A_{R,LR}(\underline{p}) \to A_{R,LR}(p)$$

such that

$$|\phi^* g - g_0|_{k,\alpha;0} < \varepsilon \,, \tag{1.20}$$

$$|\rho \circ \phi - r|_{0,\alpha;1} < \varepsilon \,. \tag{1.21}$$

Remark 1.22. Of course the metric, g_0, on $C(N^{n-1})$, or equivalently, the metric $\tilde{g}$ on N^{n-1}, might a priori depend on the particular number $R \geqq c(g, \varepsilon, L, k, \alpha)$. Indeed, this issue is our main concern.

Corollary 1.23. *For some compact set $C, M^n \setminus C$ is diffeomorphic to $(R, \infty) \times N^{n-1}$.*

Corollary 1.24. (Existence of tangent metric cones). *Given a sequence $\varepsilon_i \to 0$, $L_i \to \infty$, and annuli, $A_{R_i,L_iR_i}(p)$ as in Theorem* 1.18, *there is a subsequence $(\varepsilon_j, L_j, A_{R_i,L_iR_i}(p))$ such that for suitably chosen ϕ_i as in* (1.20), *the metric, $\tilde{g}$, on N^{n-1} can be chosen independently of j.*

Proof of Theorem 1.18. If instead of (0.9) we assumed

$$|R| \leqq \frac{c}{r^2} \tag{1.25}$$

(where in (1.26) R denotes the curvature tensor) then Theorem 1.18 would follow by repeating the argument of [BKN] (see also [A1], [AC], [CC2]). Given the more general assumption (0.9), the argument requires only minor modifications. Indeed, by the convergence theorem of [A2], the rescaling argument of [BKN] (see also [A], [AC]) yields a space, M^n_∞, which is both a volume cone and a Ricci flat manifold with at most a finite number of singular points on any annulus, $A_{r_1,r_2}(\underline{p}), 0 < r_1 < r_2 < \infty$. Moreover, the singularities are all of orbifold type. A straightforward modification of the argument in [BKN] shows that if $\partial B_{r_1}(\underline{p})$ is free of singular points, then the metric behaves conically along all radial geodesics passing through $\partial B_{r_1}(\underline{p})$. If the singular set is nonempty certain of these geodesics will end in a singular point. However, this possibility is easily seen to be incompatible with the volume cone structure. It follows that there are no singular points. Thus M_∞ is a Ricci flat cone, $C(N^{n-1})$, where N^{n-1} carries an Einstein metric, $\tilde{g}$, satisfying (0.10).

For the construction of the gauges described at the beginning of this section, we will need some standard facts concerning the existence of divergence free gauges in the compact case [E]. We now recall these facts and introduce some notation.

Let $(X^{n-1}, \tilde{g})$ denote an arbitrary compact riemannian manifold (we use the notation $\tilde{g}$ since in the application, N^{n-1} will play the role of X^{n-1}).

Let $\delta_{\tilde{g}}$ denote divergence with respect to the metric $\tilde{g}$.

For $k \in \mathbb{Z}^+, 0 < \alpha < 1$ there exists $0 < \tau_1(\tilde{g}, k, \alpha) < \tau_2(\tilde{g}, k, \alpha)$ and $c(\tilde{g}, k, \alpha) \geqq 1$ with the following properties.

For each $\tilde{g}_1$, with

$$(1.26)\qquad |\tilde g_1-\tilde g|_{k,\alpha;\tilde g}<\tau_1(\tilde g,k,\alpha)$$

there is a diffeomorphism, $\eta_{\tilde g_1}:X^{n-1}\to X^{n-1}$, of class $C^{k+1,\alpha}$, such that

$$(1.27)\qquad \delta_{\tilde g}(\eta^*_{\tilde g_1}\tilde g_1)=0\,.$$

Moreover, the map, $\tilde g_1\to\eta_{\tilde g_1}$, satisfies

$$(1.28)\qquad |\eta_{\tilde g_1}-\eta_{\tilde g_2}|_{k+2,\alpha;\tilde g}\leqq c(\tilde g,k,\alpha)|\tilde g_1-\tilde g_2|_{k,\alpha;\tilde g}\,.$$

and if Id denotes the identity map,

$$(1.29)\qquad |\eta_{\tilde g_1}-Id|_{k+2,\alpha;\tilde g}\leqq c(\tilde g,k,\alpha)|\delta_{\tilde g}(\tilde g_1)|_{k-1,\alpha;\tilde g}\,.$$

Finally, if ζ is a diffeomorphism such that $\delta_{\tilde g}(\zeta^*\tilde g_1)=0$ and

$$(1.30)\qquad |\zeta^*\tilde g_1-\tilde g|_{k,\alpha;\tilde g}<\tau_2(\tilde g,k,\alpha)\,,$$

then for some isometry, ι, of $\tilde g$,

$$(1.31)\qquad \iota^*\eta^*_{\tilde g_1}\tilde g_1=\zeta^*\tilde g_1\,.$$

It is clear that τ_1,τ_2 above can be chosen such that if $\tilde g_1$ satisfies (1.27), then

$$(1.32)\qquad |\eta^*_{\tilde g_1}\tilde g_1-\tilde g|_{k,\alpha;\tilde g}<\tau_2(\tilde g,k,\alpha)\,.$$

Now, fix a metric, $\tilde g_0$, on X^{n-1}. We can assume that $\tau_1(\tilde g_0,k,\alpha)$ satisfies

$$(1.33)\qquad \tau_1(\tilde g_0,k,\alpha)<\frac{1}{8}\tau_2(\tilde g_0,k,\alpha)\,.$$

By (1.2), (1.27), we can and will assume that τ_2 is so small that

$$(1.34)\qquad |\tilde g_1-\tilde g_0|_{k,\alpha;\tilde g_0}\leqq\frac{\tau_2}{8},\quad j=1,2$$

$$(1.35)\qquad |\tilde g_2-\tilde g_1|_{k,\alpha;\tilde g_1}\leqq\frac{\tau_2}{8}\,,$$

implies

$$(1.36)\qquad |\tilde g_2-\tilde g_1|_{k,\alpha;\tilde g_0}\leqq 2|\tilde g_2-\tilde g_1|_{k,\alpha;\tilde g_1}\,.$$

Lemma 1.37. *Let $\tilde g_i$ be a sequence of metrics on $X^{n-1}, 0\leqq i<N\leqq\infty$, such that*

$$(1.38)\qquad |\tilde g_{i+1}-\tilde g_i|_{k,\alpha;\tilde g_i}<\varepsilon<\frac{\tau_2}{8},\quad i+1<N\,.$$

Then there exists $N'\leqq N$ and for $0\leqq i<N'$, diffeomorphisms, $\beta_i:X^{n-1}\to X^{n-1}$, such that $\beta_0=Id$ and

$$(1.39)\qquad \delta_{\tilde g_0}(\beta_i^*\tilde g_i)=0\,,$$

(1.40) $$|\beta_i^*\tilde{g}_i - \beta_{i-1}^*\tilde{g}_{i-1}|_{k,\alpha;\tilde{g}_0} < c(\tilde{g}_0,k,\alpha)\varepsilon$$

Moreover, if $N' < N \leqq \infty$,

(1.41) $$|\beta_{N'-1}^*\tilde{g}_{N'-1} - \tilde{g}_0|_{k,\alpha;\tilde{g}_0} \geqq \tau_1 - 2\varepsilon$$

Proof For $i < N' \leqq N$ we define by induction the sequence of diffeomorphisms,

(1.42) $$\beta_1 = \eta_{\beta_0^*\tilde{g}_1} = \eta_{\tilde{g}_1}\ ,$$

(1.43) $$\beta_2 = \eta_{\beta_1^*\tilde{g}_2}\beta_1\ ,$$

and in general,

(1.44) $$\beta_i = \eta_{\beta_{i-1}^*\tilde{g}_i}\beta_{i-1}\ .$$

Here, $\eta_{\beta_{i-1}^*\tilde{g}_i}$ is the diffeomorphism in (1.28) defined with respect to the metric $\tilde{g}_0$ ($= \tilde{g}$ of (1.28)) and $N' \leqq N$ is the largest (extended) integer such that for $i < N'$

(1.45) $$|\beta_{i-1}^*g_i - g_0|_{k,\alpha;\tilde{g}_0} < \tau_1\ .$$

By the definition of τ_1, it follows that β_i is well defined for $i < N'$ and by definition, relation (1.40) holds. Moreover, if $N' < N$, then by definition,

(1.46) $$|\beta_{N'-1}^*\tilde{g}_{N'} - g_0|_{k,\alpha;\tilde{g}_0} \geqq \tau_1\ .$$

Using (1.1), (1.39) together with (1.35)–(1.37), we have for all $i \leqq N'$,

(1.47) $$|\beta_{i-1}^*\tilde{g}_i - \beta_{i-1}^*\tilde{g}_{i-1}|_{k,\alpha;\tilde{g}_0} < 2\varepsilon$$

If (1.42) fails, then (1.47) (for $i = N'$) contradicts (1.46). Similarly (1.47) together with (1.29), (1.30) give (1.41).

Before proceeding to the main result of this section, we make a definition: If $U \subset C(N^{n-1}, g_0)$ and $\gamma_j\colon U \to C(N^{n-1}), j = 1,2$, put for each $(u,x) \in U$

(1.48) $$\rho_{g_0}(\gamma_1(u,x),\gamma_2(u,x))_{k,\alpha;1} = \rho_{g_0}(\psi_{u}^{-1}\gamma_1\psi_u, \psi_{u}^{-1}\gamma_2\psi_u)_{k,\alpha}\ ,$$

where ψ_u is as in (1.7). Also, put

(1.49) $$\rho_{g_0}(\gamma_1,\gamma_2)_{k,\alpha;1} = \sup_{(u,x)\in U} \rho_{g_0}(\gamma_1(u,x),\gamma_2(u,x))_{k,\alpha;1}\ ,$$

Here $\rho_{k,\alpha}$ is the k,α distance from γ_1, γ_2 defined with respect to the metric g_0.

Fix $0 < \varepsilon << \chi << 2$, where $\chi < c_1(n,c,\Lambda,k)$, and c, Λ are as in (0.2), (0.9).

Proposition 1.50. *Let M^n satisfy* (0.2), (0.8), (0.9). *Then there exists a Ricci flat tangent cone,* $(C(N^{n-1}), g_0)$, *satisfying* (0.2), (0.8), (0.9), *and* $R < R'$ *with*

$$\frac{R'}{R} > 2 \tag{1.51}$$

and an imbedding,

$$\phi: A_{R_1,R'}(\underline{p}) \to A_{R,R'}(p)\,, \tag{1.52}$$

such that

$$|\phi^* g - g_0|_{k,\alpha;0} < \chi\,, \tag{1.53}$$

$$\min_{A_{R,R'}(\underline{p})} |\phi^* g - g_0|_{k,\alpha;0} < \varepsilon\,, \tag{1.54}$$

$$|\nabla^0_{\partial/\partial r}(\phi^* g)|_{k-1,\alpha;-1} < \varepsilon\,, \tag{1.55}$$

$$|\delta_{g_0}(\phi^* g)|_{k-1,\alpha;-1} < \varepsilon\,. \tag{1.56}$$

Moreover, either $R' < \infty$ *and for some* $c_2(n, \Omega, \Lambda, k) > 0$,

$$|\phi^* g - g_0|_{R,\alpha;g_0} > c_2\chi \quad on \quad A_{(1/2)R',R'}(\underline{p}) \tag{1.57}$$

or

$$R' = \infty\,, \tag{1.58}$$

$$\lim_{r\to\infty} |\nabla^0_{\partial/\partial r}(\phi^* g)(r))|_{k,\alpha;-1} = 0\,, \tag{1.59}$$

$$\lim_{r\to\infty} |\delta_{g_0}(\phi^* g)(r)|_{k,\alpha;-1} = 0\,, \tag{1.60}$$

and for some sequence, $r_i \to \infty$,

$$\lim_{r_i\to\infty} |\phi^* g(r_i) - g_0(r_i)|_{k,\alpha;0} = 0\,. \tag{1.61}$$

Proof Fix $0 < \varepsilon_1 < \varepsilon$ and $L > 0$ sufficiently large, to be determined later. Choose R so large that for each annnulus, $A_{R_1,LR_1}(p)$, with $R_1 \geqq R$, there exists a Ricci flat cone and a map ϕ as in (1.20).

For $0 \leqq i < \infty$, put

$$a_i = \frac{(1+2L)^i}{3^i} R\,, \tag{1.62}$$

$$I_i = (a_i, La_i)\,, \tag{1.63}$$

and note that for $i \neq j, I_i \cap I_j$ is nonempty only for $j = i-1, i+1$. Also, $I_{i-1} \cap I_i \cap I_{i+1} = \emptyset$.

For $0 \leqq i < \infty$, choose tangent cones $(C(N^{n-1}), g_i)$ and imbeddings

$$\phi_i : A_{a_i, La_i}(\underline{p}) \to A_{a_i, La_i}(p) \tag{1.64}$$

satisfying (1.21) with ε replaced by ε_1. Since an isometry between annular domains in cones is of the form $(r,x) \to (r,qx)$, for some isometry q of the cross-section, we see that there exist diffeomorphisms, $\lambda_{i+1} : N^{n-1} \to N^{n-1}$ such that on $A_{a_i, La_i}(\underline{p}) \cap A_{a_{i+1}, La_{i+1}}(\underline{p})$,

$$\rho_{g_i}((Id, \lambda_{i+1}^{-1})\phi_{i+1}^{-1}\phi_i, Id)_{k,\alpha;1} < c\delta(\varepsilon_1), \tag{1.65}$$

$$|\lambda_{i+1}^* \tilde{g}_{i+1} - \tilde{g}_i|_{k,\alpha;\tilde{g}_i} < \delta(\varepsilon_1) \tag{1.66}$$

where $\delta(\varepsilon)$ is an increasing function, with $\lim_{\varepsilon \to 0} \delta(\varepsilon) = 0$.

Put

$$\underline{\tilde{g}}_i = \lambda_1^* \cdots \lambda_i^* \tilde{g}_i . \tag{1.67}$$

Then $\{\underline{\tilde{g}}_i\}$, satisfies (1.39) of Lemma 1.38, we have with ε replaced by $\delta(\varepsilon_1)$. Thus, for χ sufficiently and β_i, N' as in Lemma 1.38, we have

$$\underline{\underline{\tilde{g}}}_i = \beta_i^* \underline{\tilde{g}}_i, \quad i < N' \tag{1.68}$$

$$\delta_{\tilde{g}_0}(\underline{\underline{\tilde{g}}}_i) = 0 . \tag{1.69}$$

Moreover, if we put

$$\underline{\underline{\phi}}_i = \beta_i \lambda_1 \cdots \lambda_i , \tag{1.70}$$

then the $\underline{\underline{\phi}}_i$ satisfy

$$\rho_{\underline{\underline{g}}_i}(\underline{\underline{\phi}}_{i+1}^{-1} \underline{\underline{\phi}}_i, Id)_{k,\alpha;1} < \delta(\varepsilon_1) . \tag{1.71}$$

Now by a standard argument (compare [C], [AC]) we can slightly modify the $\underline{\underline{\phi}}_i$ such that on the annuli say $A_{(1/5)La_i,(4/5)La_i}(\underline{p})$, the modified $\underline{\underline{\phi}}_i$ fit together to define a map

$$\phi : A_{(1/5)LR,(4/5)La_{N'-1}}(\underline{p}) \to M^n . \tag{1.72}$$

Let L be sufficiently large and ε_1 be sufficiently small. Then it is clear from Lemma 1.38 that (1.53)–(1.57) will hold in the case $N' < N = \infty$ of that lemma, and that (1.58)–(1.61) can be arranged in case $N' = N = \infty$.

2 The operator $\delta_{g_0} L_X g_0$ on a cone

In this section we do the linear analysis which is required for the construction of divergence free gauges given in Section 3.

Let $(C(N^{n-1}), g_0)$ be a metric cone.

Let γ be as in (1.48), (1.49). Write $\gamma \in \mathscr{D}_{k,\alpha;1}$ if γ is defined for all (r,x) with r sufficiently large and for some c,

$$\rho_{g_0}(\gamma, Id)_{k,\alpha;1} \leqq c\,. \tag{2.1}$$

Note that for the usual action of diffeomorphisms on tensors,

$$\mathscr{D}_{k+1,\alpha;1}\colon \mathscr{T}^{p,q}_{k,\alpha;\ell} \to \mathscr{T}^{p,q}_{k,\alpha;\ell}\,. \tag{2.2}$$

Let g be defined on $A_{c,d}(\underline{p}) \subset C(N^{n-1})$. We ask whether $|g - g_0|_{k,\alpha;0}$ sufficiently small, implies the existence of $\eta_g \in \mathscr{D}_{k+1,\alpha;1}$, with $\rho_{g_0}(\eta_g, Id)_{k,\alpha;1}$ small, such that

$$\delta_{g_0}(\eta_g^* g) = 0\,. \tag{2.3}$$

For the application, it is important that the required smallness of $|g - g_0|_{k,\alpha;0}$ should be independent of the size of d/c, where we assume say

$$\frac{d}{c} \geqq 2\,. \tag{2.4}$$

The implicit function theorem approach to this question necessitates that we analyze the linearized equation

$$\delta_{g_0} L_X g_0 = -\delta_{g_0} h\,, \tag{2.5}$$

where X is a vector field, L_X denotes Lie derivative and h is a symmetric bilinear form. More precisely, the standard approach makes use of $H_X(1,\cdot)$, where $H_X(t,\cdot)$ is the flow generated by X. Relation (2.2) and the following lemma (whose straightforward proof we omit), suggests that we require $X \in \mathscr{T}^{1,0}_{k+1,\alpha;1}$ in (2.5).

Lemma 2.6. *If $X \in \mathscr{T}^{1,0}_{k+1,\alpha;1}$ is defined on an annulus, $A_{c,\infty}(\underline{p}) \subseteq C(N^{n-1})$ and is tangent to the boundary, (c, N^{n-1}), then*

$$H_X(t,\cdot) \in \mathscr{D}_{k+1,\alpha;1}\,. \tag{2.7}$$

Note that since in the application of (2.5), we will take $h = g - g_0$, where $g \in \mathscr{T}^{0,2}_{k,\alpha;0}$, we will have

$$\delta_{g_0} h \in \mathscr{T}^{0,1}_{k-1,\alpha;-1}\,. \tag{2.8}$$

Using (2.12), (2.13) below, it is easy to check that

$$\delta_{g_0} L_X g_0\colon \mathscr{T}^{0,1}_{k+1,\alpha;1} \to \mathscr{T}^{1,0}_{k-1,\alpha;-1}\,. \tag{2.9}$$

Thus, we ask if there is a bounded *right* inverse, $(\delta_{g_0} L_X g_0)^{-1} : \mathscr{T}^{1,0}_{k-1,\alpha;-1} \to \mathscr{T}^{0,1}_{k+1,\alpha;1}$.

It turns out that if no such inverse exists, then there exists

$$X \in \mathscr{T}^{1,0}_{k+1,\infty;1} \tag{2.10}$$

such that $\delta_{g_0} L_X g_0 = 0$ and X has precisely linear growth.

The existence of such an X corresponds to the existence of certain particular eigenvalues, μ, for the Laplacians on co-closed 0 and 1-forms of N^{n-1} (including the eigenvalue $\mu = 0$ on functions, which is always present); see (2.37)–(2.39).

In the cases (2.38), (2.39), the gauge condition must actually be modified, to obtain the existence of η_g.

For (2.37), which corresponds to the existence of a Killing field, a generalization of (2.4) and Lemma 2.6 holds. Thus, in this case, no modification if the gauge condition is required; see (3.3)–(3.9).

Recall that

$$\begin{aligned} L_X g_0(Y,Z) &= g_0(\nabla_Y X, Z) + g_0(Y, \nabla_Z X)\,, \\ &:= \nabla^{sym} X^*(Y,Z)\,, \end{aligned} \tag{2.11}$$

$$\begin{aligned} \delta_{g_0} L_X g_0(y) &= \sum_i g_0(\nabla_{e_i} \nabla_{e_i} X, Y) + g_0(\nabla_{e_i} \nabla_Y X, e_i) \\ &= g_0(\nabla^* \nabla X, Y) + g_0(\mathrm{Ric}(X), Y) + g_0(\operatorname{grad}\operatorname{div} X, Y)\,. \end{aligned} \tag{2.12}$$

Here X^* denotes the 1-form dual to X. Using Bochner's formula, we see that (2.5) is equivalent to

$$(d^*d + 2dd^* - 2\mathrm{Ric}_{g_0})X^* = \delta_{g_0} h\,. \tag{2.13}$$

Because our ultimate interest is in the case $\mathrm{Ric}_{g_0} \equiv 0$, for convenience we will study the operator $(d^*d + 2dd^*)$ for g_0 arbitrary. Then we can directly adapt the discussion of [C].

Let $\tilde{d}, \tilde{d}^*$ denote the operations of exterior differentiation and its adjoint, for forms on the cross-section, N^{n-1}. Then letting prime denote differentiation with respect to r, the operators dd^*, d^*d on 1-forms of $C(N^{n-1})$ are given by

$$\begin{aligned} d\tilde{d}^*(\eta(r,x) + \kappa(r,x)\,dr) &= r^{-2}\tilde{d}\tilde{d}^*\eta - \tilde{d}\kappa' - (n-1)r^{-1}\tilde{d}\kappa \\ &\quad + (r^{-2}\tilde{d}^*\eta' - 2r^{-3}\tilde{d}^*\eta)\,dr \\ &\quad + (-\kappa'' - (n-1)r^{-1}\kappa' + (n-1)r^{-2}\kappa)\,dr\,, \end{aligned} \tag{2.14}$$

$$d^*d(\eta(r,x)+\kappa(r,x)\,dr) \;=\; \begin{aligned}[t] &-\eta'' - (n-3)r^{-1}\eta' + r^{-2}\tilde{d}^*\tilde{d}\eta \\ &+\tilde{d}\kappa' + (n-3)r^{-1}\tilde{d}\kappa \\ &-r^{-2}\tilde{d}^*\eta'\,dr \\ &+r^{-2}\tilde{d}^*\tilde{d}\kappa\,dr\,. \end{aligned} \tag{2.15}$$

For the above see [C], p. 586.

Observe that any 1-form can be written as an infinite sum of forms of the following types and that $d^*d + 2dd^*$ preserves the types.

$$f(r)\,\phi(x) \qquad (\deg\phi = 1) \tag{2.16}$$

$$k(r)\,\tilde{d}\phi(x) + \ell(r)r^{-1}\,\phi(x)\,dr \quad (\deg\phi = 0) \tag{2.17}$$

Here,

$$\tilde{d}^*\phi = 0\,, \tag{2.18}$$

$$\tilde{d}^*\tilde{d}\phi = \mu\phi\,. \tag{2.19}$$

It will turn out that the kernel representing the Green's function for $(2d^*d + dd^*)$ does not decay fast enough to be defined on the subspace of $\mathscr{T}^{0,1}_{k,\alpha;-1}$ corresponding to (2.16), (2.17), unless μ is greater than the values in (2.37)–(2.39). Thus we will begin by solving (2.5) directly by variation of parameters, when the right-hand side is a *finite* linear combination of forms as in (2.16), (2.17).

Put

$$r = e^t\,. \tag{2.20}$$

After performing (2.20) and the substitutions $f(e^t) = p(t), p'(t) = q(t)$, etc. in order to reduce to a first order system, we get a system of the form

$$U' - AU = e^{2t}K\,. \tag{2.21}$$

Here A is a matrix of constants, of size 2×2 for (2.16) and 4×4 for (2.17). The vector valued function K is bounded in t provided the inhomogeneous term in the original equation is in $\mathscr{T}^{0,1}_{k,\alpha;-1}$. This follows from the fact that for ϕ and i-form,

$$|\phi(x)|_{r=u} = u^{-i}|\phi(x)|_{r=1}\,. \tag{2.22}$$

Suppose that for an invertible matrix, V, with constant entries and a diagonal matrix of constants, Λ, we have,

$$M' - AM = 0\,, \tag{2.23}$$

where

$$M = Ve^{\Lambda t}\,. \tag{2.24}$$

Here the columns of V and corresponding entries of Λ are eigenvectors and corresponding eigenvalues of A.

Put

$$\Lambda = \begin{pmatrix} \lambda_1 & & O \\ & \ddots & \\ O & & \lambda_N \end{pmatrix}, \tag{2.25}$$

$$V^{-1}K = \begin{pmatrix} R_1(t) \\ \vdots \\ R_N(t) \end{pmatrix}. \tag{2.26}$$

Define

$$\int e^{(2-\lambda_j)s} R_j(s)\,ds = \begin{cases} \int_0^t e^{(2-\lambda_j)s} R_j(s)\,ds & \lambda_j \leqq 2 \\ -\int_t^\infty e^{(2-\lambda_j)s} R_j(s)\,ds & \lambda_j > 2 \end{cases} \tag{2.27}$$

and set

$$\int e^{-(\Lambda-2I)s}\, V^{-1}K\,ds = \begin{pmatrix} \int e^{(2-\lambda_1)s} R_1(s)\,ds \\ \vdots \\ \int e^{(2-\lambda_N)s} R_N(s)\,ds \end{pmatrix}. \tag{2.28}$$

Then

$$U = M(t)\int e^{(\Lambda-2I)} V^{-1} K(s)\,ds \tag{2.29}$$

is a solution of (2.21).

Clearly if for all j,

$$\lambda_j \neq 2\,, \tag{2.30}$$

then

$$\left|\frac{d^j}{ds^j}(K(e^s))\right| \leqq C, \quad 0 \leqq j \leqq k \tag{2.31}$$

implies

$$\left|\frac{d^j}{dt^j}U\right| \leqq C_k\, C\, e^{2t}, \quad 0 \leqq j \leqq k \tag{2.32}$$

for some constant,

$$C_k = C_k(\max|\lambda_j - 2|^{-1}, \det V^{-1})\,, \tag{2.33}$$

(with corresponding $C^{k,\alpha}$ bounds). In the same way, for the original equation,

$$(d^*d + 2dd^*)X^* = v\,, \tag{2.34}$$

we have, more generally, the following.

Lemma 2.35. *Let v be a finite linear combination of forms as in* (2.16) *or* (2.17) *such that $\lambda_j \neq 2 + \rho$ holds for the equation gotten by the substitution $r = e^t$. Then $v \in \mathscr{T}^{0,1}_{k-1,\alpha;-1-\rho}$ implies that the solution corresponding to* (2.29) *satisfies*

$$X^* \in \mathscr{T}^{0,1}_{k+1,\alpha;1-\rho}\,. \tag{2.36}$$

Until Section 5, we will be primarily concerned with the case $\rho = 0$ corresponding to (2.30).

Let μ be as in (2.19). Using (2.14), (2.15) it is easy to check that the condition in (2.30) is violated precisely when for (2.16)

$$\mu = 2(n-2)\,, \tag{2.37}$$

and for (2.17)

$$\mu = 0\,, \tag{2.38}$$

or

$$\mu = 2n\,. \tag{2.39}$$

It is also easy to check that in just these cases, condition (2.10) is violated. More precisely, in these cases, $d^*d + 2dd^*$ is not surjective from $\mathscr{T}^{0,1}_{k+1,\alpha;1}$ to $\mathscr{T}^{0,1}_{k-1,\alpha;-1}$, and (2.29) leads to a solution satisfying $(r\log r)^{-1}X^* \in \mathscr{T}^{0,1}_{k+1,\alpha;0}$. In the cases (2.38), (2.39), this does not suffice for our purposes.

The 1-forms corresponding to (2.37), (2.38), (2.39), which violate (2.10) are

$$r^2\phi\,, \tag{2.40}$$

$$r\,dr\,, \tag{2.41}$$

$$r^2\,d\phi + 2r\,\phi \tag{2.42}$$

(see [C], p. 586). The case (2.41), in which the form is dual to the infinitesimal homothety, $r\frac{\partial}{\partial r}$, always occurs.

To obtain condition (2.30), we replace the operator $\delta(=\delta_{g_0})$ by an operator δ_t defined as follows. In forms as in (2.17) corresponding to (2.37), (2.38),

$$\delta_t = \delta - t i_{r^{-1}\partial/\partial r}\,. \tag{2.43}$$

Here, $t \in R$ and i denotes interior product. In the direct sum of the remaining spaces of forms in (2.16), (2.17), we put $\delta_t = \delta$. If X is dual to the 1-form $X^* = \eta + \kappa\, dr$, then by a straightforward computation,

$$i_{r^{-1}\partial/\partial r} L_X g_0 = (r^{-1}\eta' - 2r^{-2}\eta + r^{-1}\tilde{d}\kappa) + 2r^{-1}\kappa'\, dr\,. \tag{2.44}$$

By using (2.14), (2.15), it is easy to check that (2.9) and (2.30) hold for the modified operator,

$$\delta_t L_X g_0 \quad (t \neq 0)\,. \tag{2.45}$$

Alternatively, the forms in (2.38), (2.39) are *not* in the kernel of the operator in (2.44). Hence they are not in the kernel of the operator in (2.45) either.

We mention that for the application to Ricci flat manifolds in Section 5, the modified gauge condition defined by $\delta_t = 0$ will turn out to be an intermediate technical device. We will ultimately be able to control such solutions in the standard gauge corresponding to $t = 0$.

The 1-form in (2.40) is dual to a Killing field, $\underline{x}$, on $C(N^{n-1})$. Equivalently, $\delta\phi = 0, \mu = 2(n-2), \mathrm{Ric}_{N^{n-1}} \equiv (n-2)$, implies that ϕ is dual to a Killing field, $\underline{x}(1,x)$ on N^{n-1}. *Since* $L_{\underline{x}} g_0 \equiv 0, r^2\phi$ *will be in the kernel of the operator corresponding to* (2.45), *no matter what the gauge condition.* In this case as is easily checked, (2.29) leads to a solution, X^*, such that (on say $A_{1,\infty}(\underline{p})$)

$$(r\log r)^{-1} X \in \mathscr{T}^{1,0}_{k+1,\alpha;0}\,; \tag{2.46}$$

compare (1.15). However, since X generates a 1-parameter group of *isometries*, it will turn out that a generalization of (2.4) and Lemma 2.6 holds for X^* as in (2.46) in this case. Thus, we can work with the usual gauge condition, $\delta g = 0$, on the corresponding space of forms.

We have not yet discussed boundary conditions, nor have we dealt with infinite sums of forms in (2.16), (2.17).

For μ as in (2.19), put

$$\alpha = \frac{2 + 2i - n}{2}\,, \tag{2.47}$$

$$\nu = \sqrt{\alpha^2 + \mu}\,, \tag{2.48}$$

$$a^{\pm} = \alpha \pm \nu\,. \tag{2.49}$$

Here $i = 1$ in (2.16) and $i = 0$ in (2.17).

The solutions corresponding to (2.16) are

$$r^{a^{\pm}}\phi \tag{2.50}$$

The solutions corresponding to (2.17) are of two types,

$$r^{a^{\pm}}\, d\phi + a^{\pm} r^{a^{\pm}-1}\, dr \wedge \phi\,, \tag{2.51}$$

(2.52) $$2r^{a^{\pm}+2}\,d\phi + a^{\mp}r^{a^{\pm}+1}\,dr \wedge \phi$$

(compare (2.52) and (4.11)–(4.20)). Note that by subtracting off appropriate multiples of the $(-)$-solutions in (2.50)–(2.52) we can modify the solution corresponding to (2.29) to obtain a solution, u, satisfying *absolute* boundary conditions

(2.53) $$(u)^N = 0\,,$$

(2.54) $$(du)^N = 0\,,$$

such that (2.35) continues to hold. Here $(\ )^N$ denotes normal component.

The above remarks apply essentially unchanged to the perturbed operator in (2.45). The only difference is that the values $a^{\pm}$ change slightly (depending on the size of t).

For infinite sums of forms as in (2.16), (2.17) we must bring in the Green's function in order to solve (2.5).

We will discuss explicitly only annuli of the form $A_{c,\infty}(\underline{p})$. The general case $A_{c,d}(\underline{p})$ is handled similarly. Alternatively, one can use a simple argument involving a cutoff function to deduce results from the case $d = \infty$, which suffice for the eventual application in Section 5.

We consider first, the case (2.16). For arbitrary sums of forms as in (2.16) the Green's function on $A_{c,\infty}(\underline{p})$, with absolute boundary conditions at c is given by

(2.55) $$\sum \frac{1}{v_j}\left(r_1^{a_j^+} - c^{2v_j} r_1\right) r_2^{a_j^-}\,\phi_j \otimes \phi_j, \quad r_1 \leqq r_2$$

(see [C]). Observe that since the volume element on $C(N^{n-1})$ is

(2.56) $$r^{n-1}\,dr \wedge \omega_{n-1}\,,$$

where ω_{n-1} is the volume form on N^{n-1}, in order even to be able to apply a kernel $f(r_1,x_1,r_2,x_2)$ to a form of linear decay, r^{-1}, we should have for fixed r_1,

(2.57) $$|f(r_1,x_1,r_2,x_2)| \leqq c\,r^{1-n-\epsilon}\,,$$

so that

(2.58) $$\int_1^{\infty} r_2^{1-n-\epsilon} r_2^{n-1} r_2^{-1}\,dr_2 < \infty\,.$$

It follows from (2.22), (2.49) that (2.57) will hold for the sum in (2.55) *provided* we drop the terms with $\mu \leqq 2(n-2)$. Thus, let $\underline{G}$ be the sum in (2.55) with the restriction $\mu > 2(n-2)$. Then it follows that $\underline{G}$ inverts $d^*d + 2dd^*$ on the direct sum of the spaces of forms in (2.9) corresponding to $\mu > 2(n-2)$.

The discussion of [C], Section 6, can be adapted immediately to show that the symmetric kernel $\underline{G}$ satisfies

$$(2.59)\quad |\underline{G}(r_1,x_1,r_2,x_2)| \leqq C\cdot \mathrm{dist}((r_1,x_1),(r_2,x_2))^{2-n},\quad \frac{1}{2}\leqq\frac{r_1}{r_2}\leqq 1$$

$$(2.60)\quad |\underline{G}(r_1,x_1,r_2,x_2)| \leqq C(r_1r_2)^{(2-n)/2}, h(r_1/r_2),\quad \frac{r_1}{r_2}\leqq\frac{1}{2}$$

where $h(s)$ is smooth for $s>0$ and as $s\to 0$,

$$(2.61)\quad h(s)=O(s^{\varepsilon+n/2}).$$

Here $\varepsilon>0$ is given by

$$(2.62)\quad \varepsilon=\min_{\mu>2(n-2)} \nu-\frac{n}{2}$$

Suppose

$$(2.63)\quad k=\sum k_j(r)\phi_j,\quad \mu_j>2(n-2)$$

satisfies

$$(2.64)\quad |k|\leqq\omega r^{-1},$$

Then if $c\leqq\frac{1}{2}r_1$,

$$|\underline{G}(k)|\leqq\int_c^{1/2r_1}\int_N|\underline{G}||k|r_2^{n-1}\,dr_2+\int_{1/2r_1}^{2r_1}\int_N|\underline{G}||k|r_2^{n-1}\,dr_2$$

$$(2.65)\quad +\int_{2r_1}^{\infty}|\underline{G}||k|r_2^{n-1}\,dr_2\,.$$

If $\frac{1}{2}r_1\leqq c$, the first two integrals get replaced by an integral from c to $2r_1$ and the estimates which follow undergo an obvious modification.

Putting $s=r_1/r_2$ and using (2.60), (2.61), we see that the third integral is bounded by

$$(2.66)\quad \omega Cr_1\int_0^{1/2}h(s)s^{1-n/2}\,ds\leqq\omega cr_1\,.$$

In the second integral in (2.65), we put $s=r_2/r_1$ and use the fact that the singularity in (2.59) is integrable to obtain the bound

$$(2.67)\quad \omega Cr_1\,.$$

The first integral in (2.65) is handled similarly and we again get the bound ωCr_1 (for say $c\geqq 1$).

Similarly, one can obtain estimates on derivatives of the form (1.11), (1.12), by using the more precise parametrix of [C].

Finally, as above, one can adapt the discussion of [C] to handle 1-forms which are sums of those as in (2.17), with $\mu_j > 2n$.

The results obtained in this section can be summarized as follows.

On $A_{c,d}(\underline{p})$ (with say $d/c > 2$) let $\mathscr{V}_1 \subset \mathscr{T}^{0,1}_{k+2,\alpha;-1}$ denote the space of 1-forms in (2.16) for $\mu = 2(n-2)$. Let $\mathscr{W}_1$ denote the corresponding space of vector fields satisfying (2.46), whose dual 1-forms satisfy absolute boundary conditions at (c, N^{n-1}). Let $\mathscr{V}_2 \subset \mathscr{T}^{0,1}_{k,\alpha;-1}$ denote the direct sum of the 1-forms as in (2.16), (2.17) for the remaining values of μ, and let $\mathscr{W}_2 \subset \mathscr{T}^{1,0}_{k+2,\alpha;1}$ denote the corresponding space of vector fields whose duals satisfy absolute boundary conditions at (c, N^{n-1}).

Theorem 2.68. *For $t \neq 0$, there exists a bounded operator*

$$(\delta_t L_X g_0)^{-1} : \mathscr{V}_1 \oplus \mathscr{V}_2 \to \mathscr{W}_1 \oplus \mathscr{W}_2 \tag{2.69}$$

such that $(\delta_t L_X g_0)(\delta_t L_X g_0)^{-1}$ is the identity on $\mathscr{T}^{0,1}_{k,\alpha;-1}$.

In the same way we have the following; compare Lemma 2.35. Fix $\rho > 0$. Let $\underline{\mathscr{W}} \subset \mathscr{T}^{0,1}_{k+2,\alpha;1}$ denote the space of 1-forms satisfying absolute boundary conditions. Let the $\{\lambda_j\}$ be as in Lemma 2.35.

Theorem 2.70. *If $\lambda_j \neq 2 + \rho$, there exists a bounded operator*

$$(\delta_{g_0} L_X g_0)^{-1} : \mathscr{T}^{1,0}_{k,\alpha;-1-\rho} \to \underline{\mathscr{W}}, \tag{2.71}$$

such that $(\delta_{g_0} L_X g_0)(\delta_{g_0} L_X g_0)^{-1}$ is the identity on $\mathscr{T}^{1,0}_{k,\alpha;-1-\rho}$.

3 Reduction to the divergence free case

In this section we construct δ_t-free gauges and use this construction to sharpen Proposition 1.50 (see Proposition 3.24).

By Theorem 2.68, for $t \neq 0$, the linear operator $(\delta_t L_X g_0)^{-1}$ is bounded. Fix such a number t. In what follows let $0 < \alpha < 1$ and $d/c \geqq 2$.

Theorem 3.1. *There exists $\kappa(t,k)$ such that if $(C(N^{n-1}), g_0)$ is a Ricci flat cone and g is a metric on $A_{c,d}(\underline{p}) \subset C(N^{n-1})$ such that*

$$|g - g_0|_{k,\alpha;0} < \kappa(t,k) \tag{3.2}$$

then there exists a diffeomorphism, $\eta : A_{c,d}(\underline{p}) \to A_{c,d}(\underline{p})$ such that

$$\eta^* g \in \mathscr{T}^{0,2}_{k,\alpha;0} \tag{3.3}$$

and

$$\delta_t(\eta^* g - g_0) = 0\,. \tag{3.4}$$

Moreover, if

$$|\delta_t(g)|_{k-1,\alpha;-1} < \varepsilon\,, \tag{3.5}$$

then

$$|\eta^* g - g|_{k,\alpha;0} < \delta(\varepsilon)\,, \tag{3.6}$$

where $\delta(\varepsilon)\downarrow$ *and*

$$\lim_{\varepsilon\to 0}\delta(\varepsilon) = 0\,. \tag{3.7}$$

Proof With the notation of Theorem 2.68, let $Y = (Y_1, Y_2) \in \mathscr{W}_1 \oplus \mathscr{W}_2$. Thus, $Y_1 = f(r)X$, with X a Killing field for the metric g_0. We denote by K_Y, the diffeomorphism

$$K_Y = H_{Y_1}(1, H_{Y_2}(1,\cdot))\,. \tag{3.8}$$

Here, $H_{Y_t}(t,\cdot)$ is the flow generated by Y_t. Hence,

$$\begin{cases} \frac{\partial H_{Y_t}(t,\cdot)}{\partial t} = Y_t(H_{Y_t}(t,\cdot)) \\ H_{Y_t}(0,\cdot) = \mathrm{Id} \end{cases} \tag{3.9}$$

The main point is to check that the flow $H_{Y_1}(t,\cdot)$ has properties which, for our purposes, are just as good as those of $H_{Y_2}(t,\cdot)$; see (3.14). In this connection recall that vector fields in $\mathscr{W}_1$ can grow more rapidly than those in $\mathscr{W}_2$; see (2.46).

It follows from (3.9) that

$$H_{Y_1}(t,\cdot) = H_X(tf(r),\cdot)\,. \tag{3.10}$$

Since the Killing field X is tangent to N^{n-1}, for fixed t, r, the map $H_X(tf(r),\cdot)$ is an isometry of the corresponding cross-section. It follows that for some constant C,

$$|\widetilde{\nabla}^r H_X(f(r),\cdot)|_{k+1,\alpha;0} \leqq C\,. \tag{3.11}$$

Here $\widetilde{\nabla}^r$ denotes the covariant derivative for the induced metric on the cross-section (r, N^{n-1}). On the other hand,

$$\frac{\partial}{\partial r} H_X(f(r),\cdot) = f'(r)X(H_X f(r),\cdot))\,, \tag{3.12}$$

and by the definition of $\mathscr{W}_1$, for some constant, C,

$$\left|\frac{\partial}{\partial r} H_X(f(r),\cdot)\right|_{k+1,\alpha;0} < C\,, \tag{3.13}$$

Then by induction, one easily shows that

$$(3.14) \qquad |\nabla H_X(f(r),\cdot)|_{k+1,\alpha;0} \leqq C_k \,.$$

This is the analog of Lemma 2.6.

There is a well defined map,

$$(3.15) \qquad B\colon (\mathscr{W}_1 \oplus \mathscr{W}_2) \times \mathscr{T}^{0,2}_{k+1,\alpha;0} \to \mathscr{V}_1 \oplus \mathscr{V}_2 = \mathscr{T}^{0,1}_{k,\alpha;-1}$$

given by

$$(3.16) \qquad B(Y_1, Y_2, g) = \delta_t(K_Y^* g)$$

In fact, it follows from (3.14) that

$$(3.17) \qquad H_{Y_1}^* g \in \mathscr{T}^{0,2}_{k+1,\alpha;0} \,.$$

Since $Y_2 \in \mathscr{T}^{1,0}_{k+2,\alpha,1}$, it follows that $K_{Y_2}^* g \in \mathscr{T}^{0,2}_{k+1,\alpha,0}$. Thus

$$(3.18) \qquad B(Y_1, Y_2; g) \in \mathscr{T}^{0,1}_{k,\alpha;-1} \,,$$

and, the map B is well-defined.

It is easy to show that B is differentiable at $(Y_1, Y_2, g) \in (\mathscr{W}_1 \oplus \mathscr{W}_2) \times \mathscr{T}^{0,2}_{k+1,\alpha;0}$. Indeed,

$$(3.19) \qquad |\delta_t(K_Y^* g) - \delta_t(g_0) - \delta_t L_{Y_1+Y_2} g_0|_{k-1,\alpha;-1} = O(|Y|^2)$$

where $|Y|$ is computed with respect to the direct sum norm on $\mathscr{W}_1 \oplus \mathscr{W}_2$. We have

$$(3.20) \qquad D_{(Y_1,Y_2)} B(0,0;g_0) = \delta_t L_{Y_1+Y_2} g_0$$

and by Theorem 2.68, the inverse

$$(3.21) \qquad (\delta_t L_{Y_1+Y_2} g_0)^{-1}\colon \mathscr{V}_1 \oplus \mathscr{V}_2 \to \mathscr{W}_1 \oplus \mathscr{W}_2$$

is bounded. Therefore, by the implicit function theorem, for $|g - g_0|_{k+1,\alpha,0}$ sufficiently small, there is a diffeomorphism

$$(3.22) \qquad \eta = K_Y$$

such that (3.3) holds. The second part of Theorem 3.1, (3.5)–(3.7) is an obvious consequence of the proof of the implicit function theorem.

The following remark is important for the sharp decay estimate proved in Section 5.

Remark 3.23. Suppose that in fact $|g - g_0|_{k,\alpha;-\rho}$ is sufficiently small for $\rho > 0$. Then it is clear that subject to obvious modifications in the relevant norms, δ_t can be replaced by $\delta_0 = \delta_{g_0}$ in Theorem 3.1; see Theorem 2.70. For instance, the space in (3.3) gets replaced by $\mathscr{T}^{0,1}_{k,\alpha;-\rho}$.

Now we can strengthen Proposition 1.50. Let the notation be as in that proposition.

Proposition 3.24. *Let M^n satisfy* (0.2), (0.8), (0.9). *Then for any* $\varepsilon, \chi > 0, \varepsilon << \chi$, *there exists a Ricci flat tangent cone,* $(C(N^{n-1}), g_0)$, *satisfying* (0.2), (0.8), (0.9), *and* $R < R'$ *with*

$$\frac{R'}{R} > 2 \tag{3.25}$$

and an imbedding,

$$\phi : A_{R,R'}(\underline{p}) \to A_{R,R'}(p) , \tag{3.26}$$

such that

$$|\phi^* g - g_0|_{k+1,\alpha;0} < \chi , \tag{3.27}$$

$$\min_{A_{R,R'}(\underline{p})} |\phi^* g - g_0|_{k+1,\alpha;0} < \varepsilon , \tag{3.28}$$

$$| \nabla^0_{\partial/\partial r} (\phi^* g)|_{k,\alpha;-1} < \varepsilon . \tag{3.29}$$

$$\delta_t(\phi^* g - g_0) = 0 . \tag{3.30}$$

Moreover, either $R' < \infty$ *and for some* $c_2(n, \Omega, \Lambda, k) > 0$,

$$|\phi^* g - g_0|_{k+1,\alpha;0} > c_2 \chi \quad \text{on} \quad A_{(1/2)R',R}(\underline{p}) \tag{3.31}$$

or

$$R' = \infty . \tag{3.32}$$

Proof It is clear from the proof of Proposition 1.50 that we can replace δ_{g_0} by δ_t in (1.56). Then Proposition 3.24 follows from Proposition 1.50, so modified, and Theorem 3.1 (see in particular (3.5), (3.6)).

Remark 3.33. The reason that we had to drop the counterparts of (1.59)–(1.61) in stating Proposition 3.24, stems from the nature of the operator in (2.27) for the case $\lambda_j < 2$ (such λ_j might exist). However, this will not cause any serious problems in Section 5.

4 The linearized equation

In this section we derive the explicit form of the linearized deformation equation on a Ricci flat cone for divergence zero symmetric bilinear forms, h, and exhibit its solutions. A basic reference here is [B], Chapter 12. We also need to consider the modified divergence condition of Section 2.

If ζ_u is a 1-parameter family of diffeomorphisms, the metrics $\zeta_u^* g_0$ are all Ricci flat. Hence, symmetric tensors of the form $L_X g_0$ are automatically solutions of the linearized equation, (4.1), and vector fields, X, such that $\delta L_X g_0 = 0$ (as studied in Section 2) generate divergence free solutions of (4.3). Note that the component of a symmetric tensor field which involves dr is of the form $dr \boxtimes Y^*$ (where $\boxtimes$ denotes symmetric tensor product). Hence, the space of mixed components is isomorphic to the space of vector fields. Thus, it is reasonable to hope that the mixed component of any divergence free solution of (4.3) agrees with the mixed component of some $L_X g_0$, with $\delta L_X g_0 = 0$. A precise result to this effect is stated in Proposition 4.65.

Proposition 4.65 is required for the sharp results on the rate of convergence; see Theorem 5.78. The qualitative properties of solutions of the linearized equation (see Corollary 4.86) are used in the proof of Theorem 0.13.

For any Ricci flat metric, the linearized deformation equation is

$$(\nabla \nabla^* - 2\delta^*\delta - 2\overset{\circ}{R})h - \operatorname{Hess} \operatorname{tr} h = 0 . \tag{4.1}$$

Here * denotes adjoint and tr denotes trace. The adjoint and the trace are computed with respect to the background metric. Also, $R(x,y) = [\nabla_x, \nabla_y] - \nabla_{[x,y]}$ and

$$\overset{\circ}{R} h(x,y) = -\sum h(R(x,e_i)y, e_i) . \tag{4.2}$$

Thus (4.1) coincides with [B], (12.28′) although our definitions of R and $\overset{\circ}{R}$ are the negatives of those used in [B]; (1.1) and p. 52.

In case $\delta h = 0$, (4.1) reduces to

$$(\nabla^* \nabla - 2\overset{\circ}{R})h - \operatorname{Hess} \operatorname{tr} h = 0 . \tag{4.3}$$

Before looking at (4.3) for the case of cones, we begin with some general remarks.

Note that

$$\operatorname{tr} \nabla^* \nabla h = \nabla^* \nabla \operatorname{tr} h . \tag{4.4}$$

Also, if $\operatorname{Ric}_{g_0} = \lambda g_0$ is constant, it is easy to check that

$$\operatorname{tr}(\overset{\circ}{R} h) = \lambda \operatorname{tr} h . \tag{4.5}$$

Hence, if $\operatorname{Ric}_{g_0} \equiv 0$, taking the trace in (4.3) gives

$$2\,\nabla^*\nabla\,\operatorname{tr} h = 0\,. \tag{4.6}$$

Now putting $h = \hat{h} + \frac{1}{n}(\operatorname{tr} h)g_0$, and

$$\square = (\nabla^*\nabla - 2\overset{\circ}{R})\,, \tag{4.7}$$

we see that relation (4.3) is equivalent to (4.6), together with

$$\square \hat{h} = \operatorname{Hess}\operatorname{tr} h\,. \tag{4.8}$$

As noted above, since the condition of being Ricci flat is invariant under diffeommorphisms, it is clear that for any $X, L_X g_0$ satisfies (4.1). Also

$$\operatorname{tr}(L_X g_0) = 2\delta X\,. \tag{4.9}$$

Thus if

$$\delta L_X g_0 = 0\,, \tag{4.10}$$

or equivalently, $(d^*d + 2dd^*)X^* = 0$, then X satisfies (4.3) and by (4.6), $2d^*X^*$ is harmonic.

To find a solution of (4.9) with $2\delta X = u$, a prescribed harmonic function, we can attempt to proceed as follows. Let v satisfy

$$\Delta v = \frac{1}{2}u\,. \tag{4.11}$$

Then

$$2d^*(dv) = u\,, \tag{4.12}$$

$$dd^*(dv) = \frac{1}{2}du\,. \tag{4.13}$$

Also, since u is harmonic,

$$d^*(du) = 0\,. \tag{4.14}$$

Suppose θ, ψ are 2-forms satisfying

$$d^*\theta = \frac{1}{4}du\,, \tag{4.15}$$

$$\Delta\psi = \theta\,, \tag{4.16}$$

$$d\psi = d\theta = 0\,. \tag{4.17}$$

Then

$$(d^*d + 2dd^*)(dv - d^*\psi) = 0\,, \tag{4.18}$$

$$2d^*(dv - d^*\psi) = u\,, \tag{4.19}$$

and we can put

$$X = (dv - d^*\psi)^*\,. \tag{4.20}$$

If the underlying manifold is a Ricci flat cone, then v, θ, ψ as above exist and $dv - d^*\psi$ is a form of the type occurring in (2.52). It is neither closed nor coclosed (unlike the forms in (2.50), (2.51) which are both closed and coclosed). Thus, for Ricci flat cones, any solution of (4.3) can be written as $h + L_X g_0$, with X as in (4.20) and h satisfying $\delta h = 0$ and

$$\Box h = 0\,, \tag{4.21}$$

$$\operatorname{tr} h = 0\,. \tag{4.22}$$

We now concentrate on (4.21), (4.22) for h with $\delta h = 0$.

Let $\widetilde{\nabla}^r$ denote the riemannian connection with respect to the induced metric on $(r, N^{n-1}) \subset C(N^{n-1})$. Let $P: T(1, N^{n-1}) \to T(r, N^{n-1})$ denote the identification of tangent bundles induced by parallel translation along radial geodesics. Put

$$\underline{\widetilde{\nabla}} = P\widetilde{\nabla}^1 P^{-1}\,, \tag{4.23}$$

Then

$$\widetilde{\nabla}^r = r^{-1}\underline{\widetilde{\nabla}}\,. \tag{4.24}$$

Also, let $\widetilde{\nabla}^r, \tilde{\delta}^r, \widetilde{\operatorname{tr}}^r$ (= trace) be defined with respect to the induced metric on (r, N^{n-1}) and let $\tilde{d}^r$ denote exterior differentiation on (r, N^{n-1}). Then if

$$\begin{aligned} \underline{\widetilde{\nabla}} &= P\widetilde{\nabla}^1 P^{-1}\,, \\ \underline{\tilde{\delta}} &= P\tilde{\delta}^1 P^{-1}\,, \\ \underline{\tilde{d}} &= P\tilde{d}^1 P^{-1}\,, \\ \underline{\widetilde{\operatorname{tr}}} &= P\widetilde{tr}^1 P^{-1}\,, \end{aligned} \tag{4.25}$$

we have

$$\begin{aligned} \tilde{\Delta}^r &= r^{-2}\underline{\tilde{\Delta}}\,, \\ \tilde{\delta}^r &= r^{-1}\underline{\tilde{\delta}}\,, \\ \tilde{d}^r &= r^{-1}\underline{\tilde{d}}\,, \\ \widetilde{\operatorname{tr}}^r &= \underline{\widetilde{\operatorname{tr}}}\,. \end{aligned} \tag{4.26}$$

Let e be tangent to (r, N^{n-1}) and let e^* be its dual 1-form. Let η be a 1-form such that

$$\eta\left(\frac{\partial}{\partial r}\right) = 0\,, \tag{4.27}$$

$$\nabla_{\partial/\partial r}\,\eta \equiv 0\,. \tag{4.28}$$

Then

$$\nabla_e\,\eta = r^{-1}(\underline{\widetilde{\nabla}}_e\eta - \eta(e)\,dr)\,, \tag{4.29}$$

$$\nabla_e\,dr = r^{-1}e^*\,, \tag{4.30}$$

$$\nabla_{\partial/\partial r}\,\frac{\partial}{\partial r} = \nabla_{\partial/\partial r}\,dr = 0\,. \tag{4.31}$$

Let $\frac{\partial}{\partial r}, e_1, \ldots, e_{n-1}$, be a local orthonormal basis satisfying

$$\widetilde{\nabla}_{e_i} e_j = 0 \tag{4.32}$$

at some fixed point, (r, x), and

$$\nabla_{\partial/\partial r}\,e \equiv 0\,. \tag{4.33}$$

Then at the point where (4.32) holds,

$$\sum_i \nabla_{e_i}\,e_i = -(n-1)r^{-1}\frac{\partial}{\partial r}\,. \tag{4.34}$$

Put

$$\omega_1 \boxtimes \omega_2 = \omega_1 \otimes \omega_2 + \omega_2 \otimes \omega_1\,. \tag{4.35}$$

We now compute the expression in (4.3) for $h = f(r)\eta_1 \boxtimes \eta_2$, with η_1, η_2 as in (4.27), (4.28).

$$\begin{aligned} \sum_i \nabla_{e_i}\nabla_{e_i}(\eta_1 \boxtimes \eta_2) &= \sum_i \nabla_{e_i}\nabla_{e_i}\,\eta \boxtimes \eta_2 \\ &\quad + \nabla_{e_i}\,\eta_1 \boxtimes \nabla_{e_i}\,\eta_2 + \eta_1 \boxtimes \nabla_{e_i}\nabla_{e_i}\,\eta_2 \end{aligned} \tag{4.36}$$

$$\begin{aligned} \nabla_{e_i}\nabla_{e_i}\,\eta_j &= r^{-1}\,\nabla_{e_i}\,(\underline{\widetilde{\nabla}}_{e_i}\eta_j - \eta_j(e_i)\,dr) \\ &= r^{-2}(\underline{\widetilde{\Delta}}\eta_j - 2\underline{\widetilde{\delta}}\eta_j\,dr - \eta_j)\,, \end{aligned} \tag{4.37}$$

$$\begin{aligned} \sum \nabla_{e_i}\,\eta_1 \boxtimes \nabla_{e_i}\,\eta_2 &= r^{-2}\Big(\sum_i \underline{\widetilde{\nabla}}_{e_i}\eta_1 \boxtimes \underline{\widetilde{\nabla}}_{e_i}\eta_2 - \underline{\widetilde{\nabla}}_{\eta_1^*}\eta_2 \boxtimes dr \\ &\quad - \underline{\widetilde{\nabla}}_{\eta_2^*}\eta_1 \boxtimes dr + \underline{\widetilde{\mathrm{tr}}}(\eta_1 \boxtimes \eta_2)\,dr \otimes dr\Big)\,, \end{aligned} \tag{4.38}$$

$$\nabla_{\partial/\partial r}\nabla_{\partial/\partial r}(f\eta_1 \boxtimes \eta_2) = f''\eta_1 \boxtimes \eta_2\,, \tag{4.39}$$

$$-\nabla_{e_i} e_i(f\eta_1 \boxtimes \eta_2) = r^{-1}f'\eta_1 \boxtimes \eta_2\,. \tag{4.40}$$

Thus,

$$\begin{aligned}\nabla^*\nabla\,(f\eta_1 \boxtimes \eta_2) &= (-f'' - (n-1)r^{-1}f' + r^{-2} + (\underline{\tilde{\nabla}}^*\underline{\tilde{\nabla}} + 2))\eta_1 \boxtimes \eta_2\\ &\quad + 2r^{-2}f\underline{\tilde{\delta}}(\eta_1 \boxtimes \eta_2)\boxtimes dr\\ &\quad - 2r^{-2}f\underline{\widetilde{\mathrm{tr}}}(\eta_1 \boxtimes \eta_2)\,dr \otimes dr\,.\end{aligned} \tag{4.41}$$

(See (4.53) below for $\underline{\tilde{\delta}}(\eta_1 \boxtimes \eta_2)$.)

Next observe that on a cone,

$$\left\langle R(\ ,\ ),\frac{\partial}{\partial r}\right\rangle = 0 \tag{4.42}$$

(for all choices of the remaining arguments) and that for x, y, z tangent to (r, N^{n-1}),

$$R(x,y)z = \widetilde{R}^r(x,y)z + r^{-2}(\langle x,z\rangle y - \langle y,z\rangle x)\,. \tag{4.43}$$

Then

$$\overset{\circ}{R}(f\eta_1 \boxtimes \eta_2) = r^{-2}f(\underline{\overset{\circ}{\tilde{R}}}\eta_1 \boxtimes \eta_2 + \eta_1 \boxtimes \eta_2 - \underline{\widetilde{\mathrm{tr}}}(\eta_1 \boxtimes \eta_2)\underline{\tilde{g}})\,. \tag{4.44}$$

Now let B be a sum of forms of type $\eta_1 \boxtimes \eta_2$. Then by (4.41), (4.44),

$$\begin{aligned}\square(fB) &= (-f'' - (n-1)r^{-1}f' + r^{-2}f\tilde{\square})B\\ &\quad + 2r^{-2}f\underline{\widetilde{\mathrm{tr}}}(B)\underline{\tilde{g}} + 2r^{-2}f\underline{\tilde{\delta}}(B)\boxtimes dr\\ &\quad - 2r^{-2}f\underline{\widetilde{\mathrm{tr}}}(B)\,dr \boxtimes dr\,.\end{aligned} \tag{4.45}$$

Now consider $h = k(r)\tau \boxtimes dr$ with τ as in (4.27), (4.28). Then

$$\sum_i \nabla_{e_i}(k\tau)\boxtimes \nabla_{e_i} dr = r^{-2}k(\underline{\tilde{\nabla}}^{sym}\tau - \tau \boxtimes dr)\,, \tag{4.46}$$

where $\underline{\tilde{\nabla}}^{sym}\tau = L_{\tau^*}\tilde{g}$, is the symmetrized covariant derivative,

$$\underline{\tilde{\nabla}}^{sym}\tau = \sum_i \tilde{\nabla}_{e_i}\tau \boxtimes e_i^*\,. \tag{4.47}$$

Also,

$$\sum_i \nabla_{e_i}\nabla_{e_i} dr = -(n-1)r^{-2}\,dr\,. \tag{4.48}$$

Since, by (4.43), $\overset{\circ}{R}h = 0$ in this case, with (4.37) we get

$$\begin{aligned}\square(k\tau \boxtimes dr) &= -2r^{-2}k\underline{\widetilde{\nabla}}^{sym}\tau \\ &\quad + \{[-k'' - (n-1)r^{-1}k' + r^{-2}k(\underline{\widetilde{\nabla}}^*\underline{\widetilde{\nabla}} + n + 2)]\tau\} \boxtimes dr \\ &\quad + 4r^{-2}k\underline{\tilde{\delta}}\tau\, dr \otimes dr\,. \end{aligned} \tag{4.49}$$

Last, consider $h = \ell(r)\,\phi(x)\,dr \otimes dr$. Then

$$\sum_i \nabla_{e_i}(\ell\phi\, dr) \otimes \nabla_{e_i} dr = r^{-2}\ell(\phi\underline{\tilde{g}} + \underline{\tilde{d}}\phi \boxtimes dr)\,, \tag{4.50}$$

and with (4.48) we get

$$\begin{aligned}\square(\ell\phi\, dr \otimes dr) &= -2r^{-2}\ell\,\phi\,\underline{\tilde{g}} - 2r^{-2}\ell\underline{\tilde{d}}\phi \boxtimes dr \\ &\quad + (-\ell'' - (n-1)r^{-1}\ell' \\ &\quad + r^{-2}\ell(\underline{\widetilde{\nabla}}^*\underline{\widetilde{\nabla}} + 2(n-1))\phi\, dr \otimes dr\,. \end{aligned} \tag{4.51}$$

Next we compute δh for h as in the three cases above.

$$\begin{aligned}\delta(f\eta_1 \boxtimes \eta_2) &= f\sum_i \nabla_{e_i}\eta_1(e_i)\eta_2 + \eta_1(e_i)\nabla_{e_i}\eta_2 \\ &\quad + \nabla_{e_i}\eta_2(e_i)\eta_1 + \eta_2(e_i)\nabla_{e_i}\eta_1\,. \end{aligned} \tag{4.52}$$

Thus, for B as in (4.45)

$$\delta(fB) = r^{-1}f\underline{\tilde{\delta}}(B) - r^{-1}f\underline{\widetilde{\operatorname{tr}}}(B)\,dr\,. \tag{4.53}$$

$$\begin{aligned}\delta(k\tau \boxtimes dr) &= k\sum_i \nabla_{e_i}\tau(e_i)\,dr + \tau(e_i)\nabla_{e_i} dr + (\nabla_{e_i} dr)(e_i)\tau + k'\tau\,, \\ &= k'\tau + nr^{-1}k\tau + r^{-1}k(\underline{\tilde{\delta}}\tau)\,dr\,. \end{aligned} \tag{4.54}$$

$$\begin{aligned}\delta(\ell\phi\, dr \otimes dr) &= \sum_i \ell\phi(\nabla_{e_i} dr)(e_i)\,dr + \ell'\phi\, dr\,, \\ &= (\ell' + (n-1)r^{-1}\ell)\phi\, dr \end{aligned} \tag{4.55}$$

Now we consider a solution of (4.3), which is a sum of the three types previously considered and satisfies in addition,

$$\delta h = 0\,, \tag{4.56}$$

$$\operatorname{tr} h = f\,\underline{\widetilde{\operatorname{tr}}}(B) + \ell\phi = 0\,. \tag{4.57}$$

Then we easily find that (4.3) is equivalent to

$$(4.58)\quad (-f'' - (n-1)r^{-1}f' + r^{-2}f\widetilde{\Box})B = 4r^{-2}\ell\underline{\tilde{g}} + 2r^{-2}k\underline{\widetilde{\nabla}}^{sym}\tau\,,$$

$$(4.59)\quad (-k'' - (n+1)r^{-1}k' + r^{-2}k(\underline{\widetilde{\nabla}}^{*}\underline{\widetilde{\nabla}} - (n-2)))\tau = 2r^{-2}\ell\tilde{d}\phi\,,$$

$$(4.60)\quad (-\ell'' - (n+3)r^{-1}\ell' + r^{-2}\ell(\underline{\widetilde{\nabla}}^{*}\underline{\widetilde{\nabla}} - 2n))\phi = 0\,,$$

$$(4.61)\quad r^{-1}f\underline{\tilde{\delta}}B + nr^{-1}k\tau + k'\tau = 0\,,$$

$$(4.62)\quad \ell' + nr^{-1}\ell + r^{-1}k\underline{\tilde{\delta}}\tau = 0\,.$$

At this point we emphasize that in reconciling the results of the present section with those of Section 2, we must bear in mind that in Section 2 we used polar coordinates in order to trivialize the tangent bundle in the radial direction, while in the present section we used parallel translation in the radial direction for this purpose; compare e.g. (2.22), (4.25), (4.26).

Let ϕ in (4.60) be as in (2.18), (2.19). Then it is easy to check that for $a^{\pm}$ as in (2.49),

$$(4.63)\quad r^{a^{\pm}-2}\phi$$

are the two solutions of (4.60). The corresponding solution of the full system (4.57)–(4.62) is $L_X g_0$ where X^* is the dual of the form in (2.51).

Similarly, $L_X g_0$ with X^* the dual of the form in (2.50) is the solution of (4.57)–(4.62) with $\ell\phi = 0$ and τ playing the role of ϕ in (2.50).

Note that the forms (2.42) give rise to the radially parallel solutions of (4.57)–(4.62). The form in (2.41) gives the radially parallel solution ($h = g_0$) of (4.3) with $\operatorname{tr} h = 2$.

Remark 4.64. Corresponding to (2.41), we get $L_X g = 0$, since in this case X is a Killing field. Thus, we do not obtain a solution to (4.58)–(4.62). Note that in this instance, $dr \boxtimes \tau$ *does* define a radially parallel solution of (4.58)–(4.60) (equivalently of (4.3)) which, however, does *not* have divergence zero. In fact, even if we were to introduce a tangential component B, (4.61) could not be satisfied since $\underline{\tilde{\delta}}B$ is orthogonal to the space of Killing fields.

The computations of this section can now be summarized as follows.

Proposition 4.65. *If $-(1-n/2)^2$ is not an eigenvalue of $\widetilde{\Box}$, then every solution h of* (4.1) *satisfying $\delta h = 0$, can be written uniquely as a sum*

$$(4.66)\quad h = L_X g_0 + \sum_i r^{b_i^{\pm}} B_i$$

where each B_i satisfies

$$(4.67)\quad (-f'' - (n-1)r^{-1}f' + r^{-2}f\widetilde{\Box})B_i = 0\,.$$

(4.68) $$tr\, B_i = 0\,,$$

(4.69) $$\delta B_i = \tilde{\delta} B_i = 0\,.$$

If $-(1-n/2)^2$ is an eigenvalue of $\tilde{\Box}$, then a solution of the form $r^{1-n/2} \log r B$ must be included on the right hand side of (4.36).

Remark 4.70. Let us note that $\tilde{\Box} \geqq 0$ implies that for b^- as in (4.66),

(4.71) $$b^- \leqq 2-n\,.$$

Proposition 4.65 has several consequences which are used in Section 5. Before stating these, we introduce some notation.

For fixed r, let us define a symmetric bilinear form on tensor fields of $C(N^{n-1})$ over (r, N^{n-1}) by

(4.72) $$\langle\langle h_1, h_2\rangle\rangle = r^{-(n-1)} \int_{(r,N^{n-1})} \langle h_1, h_2\rangle\, d\,\mathrm{vol}_{N^{n-1}}\,,$$

where

(4.73) $$\langle h_1, h_2\rangle\,,$$

denotes the usual pointwise inner product.

Now consider the modified equation

(4.74) $$(\nabla^*\nabla - 2t\delta^* i_{r^{-1}\partial/\partial r} - 2\overset{\circ}{R})h - \text{Hess tr}\, h := \Box_t h - \text{Hess tr}\, h = 0$$

It will suffice from now on to assume that $|t|$ is small. As in Proposition 4.65 we find that solutions occur in pairs of the form

(4.75) $$r^{\beta_i^{\pm}} T_i$$

or exceptionally,

(4.76) $$r^{\beta_i^-} T_i,\ r^{\beta_i^-} \log r T_i\,,$$

where T_i is a symmetric bilinear form with $\{T_i\}$ orthonormal with respect to $\langle\langle\ ,\ \rangle\rangle$ and

(4.77) $$\nabla_{\partial/\partial r} T_i \equiv 0\,.$$

Corollary 4.78. *For t sufficiently small, the only radially parallel solutions of* (4.74) *are of the form fB, where f is a constant function,*

(4.79) $$\tilde{\Box} B = 0\,,$$

and B satisfies (4.68), (4.69).

Proof For $t \neq 0$, the solutions which are perturbations of those in (2.41), (2.42) are no longer radially parallel (compare (2.45)). Since it is clear that for t sufficiently small, the solutions of (4.74) with $\delta_t h$ are small perturbations of those in (4.67), our claim follows.

Note that since the operators $\tilde{\Box}$, $\underline{\tilde{\nabla}}^* \underline{\tilde{\nabla}} - (n-2)$ and $\underline{\tilde{\nabla}}^* \underline{\tilde{\nabla}} - 2n$, which appear on the left-hand sides of (4.58) are not necessarily positive semidefinite, we might not have $\beta_i^+ \geqq 0$ for some finite number of β_i^+ in (4.75) (see also (4.76)). For such values of i we get solutions of the following types.

(4.80) $$(c_i r^{\beta_i^-} + d_i r^{\beta_i^+}) T_i, \quad \beta_i^- < \beta_i^+ < 0$$

(4.81) $$(c_i r^{\beta_i^-} + d_i r^{\beta_i^+} \log r) T_i, \quad \beta_i^+ = \beta_i^\pm$$

(4.82) $$(c_i r^{\beta_i^-} + d_i r^{\beta_i^+}) T_i, \quad \mathrm{Re}\, \beta_i^\pm < 0, \quad \beta_i^- = \overline{\beta_i^+}$$

In (4.82), $\mathrm{Im}\, \beta_i^\pm \neq 0$.

We will group the solutions in (4.80)–(4.82) together with the remaining - solutions and call a linear combination of such solutions a $\downarrow$ solution. The remaining + solutions will be called $\uparrow$ solutions. Thus, any solution, h, of (4.74) can be decomposed as

(4.83) $$h = h_\uparrow + h_\downarrow + h_0 \,,$$

where h_0 represents the radially parallel component.

For our purposes in Section 5 it will be necessary to specify a precise sense in which the solutions, $h_\downarrow$, are norm decreasing.

Define the modified L_2-norm of h over an annulus, $A_{a,b}(\underline{p})$, by

(4.84) $$|||h|||_{a,b} = \int_a^b \|h\|^2 r^{-1}\, dr \,,$$

where $\|h\|^2$ is defined as in (4.72).

For β_i as in (4.75), (4.76), put

(4.85) $$\beta = \min_{\beta_i \neq 0} |\beta_i| \,.$$

Corollary 4.86. *Given* $0 < \beta' < \beta$, *there exists* ℓ *such that for all* $a > 0$ *and* $L \geqq \ell$

(4.87) $$|||h_\uparrow|||_{La, L^2 a} \geqq L^{\beta'} |||h_\uparrow|||_{a, La} \,,$$

(4.88) $$|||h_\downarrow|||_{La, L^2 a} \leqq L^{-\beta'} |||h_\downarrow|||_{a, La} \,.$$

Proof Since $\{T_i\}$ is orthonormal, if we did not have to take into account the solutions in (4.80)–(4.82), this would be obvious from (4.75), (4.77). The solutions in (4.80)–(4.82) are handled by the following proposition whose elementary proof we omit; compare however [S1], [S2].

Proposition 4.89. *Let $f_\iota(r)$ denote any of the functions in* (4.80)–(4.82) *(where c_ι, d_ι are arbitrary). Given $0 < \beta' < -\beta_\iota^+$, there exists $\ell = \ell(\beta_\iota^-, \beta_\iota^+, \beta') > 0$, such that for $a > 0$ and $L \geqq \ell$,*

$$\int_{La}^{L^2 a} f_\iota^2(r) r^{-1}\, dr \leqq L^{-2\beta'} \int_{a}^{La} f_\iota^2(r) r^{-1}\, dr . \tag{4.90}$$

5 The integrable case

In this section, by using an argument from [S1], [S2] we prove Theorem 0.13, which asserts uniqueness of the tangent cone under the assumption that some tangent cone is integrable. By a small extension of the argument we obtain an essentially sharp estimate for the rate of convergence, $r^{-\beta}$, in terms of the spectrum of the operator, $\tilde{\Box}$, on the cross section.

Consider a gauge as in Proposition 3.24. Thus $\phi^* g$ is a Ricci flat metric over an annulus, $A_{R,R'}(\underline{p}) \subset C(N^{n-1})$, for some tangent cone, $(C(N^{n-1}), g_0)$. Moreover, $\delta_t(\phi^* g - g) = 0$.

Let g_1 be a second Ricci flat metric defined on $A_{R,R'}(\underline{p})$, satisfying

$$\delta_t(g_1 - g_0) = 0 . \tag{5.1}$$

In the application, g_1 will be a suitable Ricci flat cone metric,

$$g_1 = dr^2 + r^2 \tilde{g}_1 . \tag{5.2}$$

If we put

$$h = \phi^* g - g_1 , \tag{5.3}$$

then

$$\delta_t h = 0 . \tag{5.4}$$

Subject to (5.4), the equation

$$Ric_{g_1+h} - Ric_{g_1} = 0 , \tag{5.5}$$

is a nonlinear elliptic equation on h, which, for h small, we can view as a perturbation of the linearized equation (4.74). More precisely, we have

Lemma 5.6.

$$(5.7)\qquad Ric_{g_1+h} - Ric_{g_1} = \Box_t h - Hess\, tr\, h + F(g_0, g_1, h)$$

where

$$|F(g_0,g_1,h)|_{k,\alpha;0} \leqq C\{|g_1 - g_0|_{k,\alpha;0} + |h|_{k,\alpha;0}]|\nabla^2 h|_{k-2,\alpha;-2}$$
$$(5.8)\qquad + [|\nabla(g_1 - g_0)|_{k-1,\alpha;-1} + |\nabla h|_{k-1,\alpha;-1}]|\nabla h|_{k-1,\alpha;-1}\}$$

Proof In local coordinates the Christoffel symbols of $g_1 + h$ are given by

$$(5.9)\qquad \Gamma^i_{jk} = \frac{1}{2}(g_1 + h)^{i\ell}(h_{\ell j,k} + h_{\ell k,j} - h_{jk,\ell})$$

where

$$(5.10)\qquad ((g_1 + h)^{i\ell}) = ((g_1 + h)_{ij})^{-1}$$

Note also that

$$(5.11)\qquad Ric(g_1 + h)_{ij} = \Gamma^k_{ij;k} - \Gamma^k_{ik;j} + \Gamma^k_{ij}\Gamma^\ell_{k\ell} - \Gamma^k_{i\ell}\Gamma^\ell_{jk}$$

From this, (5.8) follows in a straightforward manner.

Fix an annulus $A_{a,b}(\underline{p})$. The modified L_2-norm of h over $A_{a,b}(\underline{p})$ is defined as in (4.84). Note that if

$$(5.12)\qquad q = a^{-2}\psi_a^*(h)\,,$$

with ψ_a as in (1.6), then

$$(5.13)\qquad |||h|||_{a,La} = |||q|||_{1,L}$$

The basic elliptic estimate for this section is the following.

Lemma 5.14. *There is a small number, $\chi = \chi(n,\Omega,\Lambda)$, such that if $|g_1 - g_0|_{k,\alpha,0} \leqq \chi$, then for any solution h of (5.7) with $|h|_{k,\alpha,0} \leqq \chi$,*

$$(5.15)\qquad |h(\frac{3}{2}a)|_{k,\alpha;0} \leqq c(n,\Omega,\Lambda,k)|||h|||_{a,2a}$$

Proof By (5.13), it suffices to consider the case, $a = 1$, for which the result follows from standard elliptic theory; see [GT], Chapter 6.

In order to reduce the verification of assertions about sufficiently small solutions of the nonlinear equation, (5.5), to the verification of corresponding assertions about the linear equation, (4.74), we will use an argument by contradiction based on the following Lemma 5.18.

Let $\{\phi_i\}$ be a sequence of gauges as in Proposition 3.24 relative to a fixed tangent cone, $(C(N^{n-1}), g_0)$, such that for the sequence of constants, χ_i, of that proposition,

$$\lim_{i\to\infty} \chi_i = 0\,. \tag{5.16}$$

Let g_i be a sequence of Ricci flat cones such that

$$\delta_t(g_i - g_0) = 0\,, \tag{5.17}$$

$$|g_i - g_0|_{k,\alpha;0} < \chi_i\,. \tag{5.18}$$

The norm in (5.16) is the uniform norm. Put

$$h_i = \phi_i^* g - g_i\,. \tag{5.19}$$

Fix $L > 0$. For $j = 0, 1, 2$, consider the annuli

$$A_{L^j a_i, L^{j+1} a_i}(\underline{p}) \subset A_{R_i, R_i'}(\underline{p}) \tag{5.20}$$

Here R_i, R_i' are as in Proposition 3.24. Put

$$q_i = a_i^{-2} \psi_{a_i}^*(h_i)\,. \tag{5.21}$$

Lemma 5.22. *If for some fixed $c > 0$,*

$$c|||h_i|||_{La_i, L^2 a_i} \geqq |||h_i|||_{a_i, La_i} + |||h_i|||_{L^2 a_i, L^3 a_i}\,, \tag{5.23}$$

then for

$$k_i = |||q_i|||^{-1}_{L, L^2} q_i\,, \tag{5.24}$$

there is a convergent subsequence, in $\mathscr{T}^{0,2}_{k,\alpha;0}$,

$$k_{i_j} \to k\,, \tag{5.25}$$

on any annulus, $A_{b,c}(\underline{p})$, with $a < b < c < La$. Moreover, k satisfies

$$\square_t k - Hess\, tr\, k = 0\,, \tag{5.26}$$

$$\delta_t k = 0\,. \tag{5.27}$$

Proof In view of (5.15) relation (5.25) follows from standard compactness results. Then (5.26) is a direct consquence of (5.8), (5.16), (5.20). Finally, (5.27) is clear.

In the next lemma, we isolate a property of solutions of (4.74). By means of Lemma 5.18, we will show that it holds for sufficiently small solutions of (5.5) as well.

Let k be an arbitrary solution of (4.74) satisfying

$$\delta_t k = 0\,. \tag{5.28}$$

As in (4.83), we write

$$k = k_{\uparrow} + k_{\downarrow} + k_0 \,, \tag{5.29}$$

where for β as in (4.86), (4.87) we have

$$\beta = \beta(t) > 0 \,. \tag{5.30}$$

Lemma 5.31. *Given* $0 < \beta' < \beta$, *there exists* L *such that if*

$$|||k|||_{L,L^2} \geqq L^{\beta'} |||k|||_{1,L} \,, \tag{5.32}$$

then

$$|||k|||_{L^2,L^3} \geqq L^{\beta'} |||k|||_{L,L^2} \,, \tag{5.33}$$

and if

$$|||k|||_{L^2,L^3} \leqq L^{-\beta'} |||k|||^2_{L,L^2} \,, \tag{5.34}$$

then

$$|||k|||_{L,L^2} \leqq L^{-\beta'} |||k|||_{1,L} \,. \tag{5.35}$$

Moreover, if

$$k_0 = 0 \,, \tag{5.36}$$

then at least one of (5.33), (5.35) *holds* (*whether or not at least one of* (5.32), (5.34) *holds*).

Proof If (5.32) holds, then

$$\begin{aligned} &(|||k_{\uparrow}|||^2_{L,L^2} + |||k_{\downarrow}|||^2_{L,L^2} + |||k_0|||^2_{L,L^2}) \geqq L^{2\beta'} \\ &(|||k_{\uparrow}|||^2_{1,L} + |||k_{\downarrow}|||^2_{1,L} + |||k_0|||^2_{1,L}) \end{aligned} \tag{5.37}$$

Since k_0 is radially parallel and we can assume that (4.87) holds, this gives

$$|||k_{\uparrow}|||^2_{L,L^2} \geqq (L^{2\beta'} - 1)(|||k_{\downarrow}|||^2_{L,L^2} + |||k_0|||^2_{L,L^2}) \,. \tag{5.38}$$

Together with (5.29), we get

$$\begin{aligned} |||k|||^2_{L^2,L^3} &\geqq L^{2\beta} |||k_{\uparrow}|||^2_{L,L^2} \,, \\ &\geqq L^{2\beta} \frac{L^{2\beta'} - 1}{L^{2\beta'}} |||k|||^2_{L,L^2} \,. \end{aligned} \tag{5.39}$$

Then, by taking L so large that

$$L^{2(\beta-\beta')} \geqq \frac{L^{2\beta'}}{L^{2\beta'}-1}, \tag{5.40}$$

the first assertion follows. The proof of the second assertion is entirely similar.

Finally, if (5.36) holds then

$$|||k_\uparrow|||_{L,L^2} \geqq \frac{1}{2}|||k|||_{L,L^2} \tag{5.41}$$

implies

$$\begin{aligned} |||k|||^2_{L^2,L^3} &\geqq \frac{L^{2\beta}}{4}|||k|||^2_{L,L^2}, \\ &\geqq L^{2\beta'}|||k|||^2_{L,L^2}, \end{aligned} \tag{5.42}$$

provided

$$L^{2(\beta-\beta')} \geqq 4. \tag{5.43}$$

Similarly,

$$|||k_\downarrow|||_{L,L^2} \geqq \frac{1}{2}|||k|||_{L,L^2}, \tag{5.44}$$

implies

$$|||k|||^2_{L,L^2} \leqq L^{-2\beta'}|||k|||^2_{1,L}, \tag{5.45}$$

provided (5.43) holds. This suffices to complete the proof.

We now return to the situation of Proposition 3.24 and we continue to assume (5.1)–(5.4). We assume in addition, that

$$|g_1 - g_0|_{k,\alpha,0} < \chi. \tag{5.46}$$

Let $\beta' > 0$, $L > 0$ be as in Lemma 5.31. For $j = 0, 1, 2$, let

$$A_{L^j a, L^{j+1} a}(\underline{p}) \subset A_{R,R'}(\underline{p}). \tag{5.47}$$

Over, $A_{L^j a, L^{j+1} a}(\underline{p})$ let π denote orthogonal projection on the subspace, $\ker(\square_t - \text{Hess}\, tr)|A_{La,L^2 a}(\underline{p})$, with respect to the inner product defining $|||\ |||_{La,L^2 a}$. As in (5.28), put

$$(\pi h) = (\pi h)_\uparrow + (\pi h)_\downarrow + (\pi h)_0. \tag{5.48}$$

Finally, let c, Λ be as in (0.2), (0.9).

Proposition 5.49. *There exists* $\chi(n,\Omega,\Lambda) > 0$, *such that if* $\chi < \chi(n,c,\Lambda)$, *then if*

$$|||h|||_{La,L^2a} \geqq L^{\beta'}|||h|||_{a,La}\,, \tag{5.50}$$

then

$$|||h|||_{L^2a,L^3a} \geqq L^{\beta'}|||h|||_{La,L^2a}\,, \tag{5.51}$$

and if

$$|||h|||_{L^2a,L^3a} \leqq L^{-\beta'}|||h|||_{La,L^2a}\,, \tag{5.52}$$

then

$$|||h|||_{La,L^2a} \leqq L^{-\beta'}|||h|||_{a,La}\,, \tag{5.53}$$

Moreover, if

$$(\pi h)_0 = 0\,, \tag{5.54}$$

at least one of (5.51), (5.53) holds.

Proof In view of Proposition 1.6, it suffices to prove the claim for $\chi < \chi(N^{n-1}, g_0)$.

Note that the inequalities in Proposition 5.49 hold if and only if they hold when h is multiplied by a nonzero constant.

Assume there exists a sequence of gauges, ϕ_i, and solutions, h_i, for which the constants χ_i, satisfy $\lim_{i\to\infty}\chi_i = 0$, but none of the assertions the proposition fails. Then, using the rescaling construction of Lemma 5.22 we produce a solution of (4.74) with a property contradicting the corresponding assertion of Lemma 5.31. This suffices to complete the proof.

Remark 5.55. The first part of Proposition 5.49 says roughly that if h starts to grow at a definite rate then it continues to grow at at least that rate. Similarly, if somewhere h decays at a definite rate, then previously it decays at at least that rate.

The last essential preliminary that is required for the proof of Theorem 0.13 is the assertion that g_1 can be chosen such that $h = \phi^* g - g_1$ satisfies (5.54), $(\pi h)_0$. At this point the *integrability condition* enters.

Lemma 5.56. *Let* $(C(N^{n-1}), g_0)$ *be a tangent cone which is integrable. Then if* $\chi < \chi(n,\Omega,\Lambda,L)$, *for any annulus,* $A_{a,La}(\underline{p}) \subset A_{R,R'}(\underline{p})$, *there is a Ricci flat cone metric,* g_1, *satisfying* (5.1), *such that* (5.54) *holds. Moreover, if*

$$|||\phi^* g - g_0|||_{La,L^2a} \leqq \chi\,, \tag{5.57}$$

then

$$|||g_1 - g_0|||_{La,L^2a} \leqq 2|||\pi(\phi^* g - g_0)|||_{La,L^2a}\,. \tag{5.58}$$

Proof The integrability assumption implies that the set of metrics, $\underline{\tilde{g}}$, satisfying

(5.59) $$\mathrm{Ric}_{\underline{\tilde{g}}} = 0\,,$$

(5.60) $$\tilde{\delta}_{\tilde{g}_0}\underline{\tilde{g}} = 0\,,$$

has a natural smooth manifold structure near $\tilde{g}_0$; see [B]; chapter 12E. Here we have put $g_0 = dr^2 + r^2\tilde{g}_0$. Let $\mathscr{U}$ be a sufficiently small Euclidean neighborhood of $\tilde{g}_0$. The tangent space to $\mathscr{U}$ at $\tilde{g}_0$ is naturally identified with

(5.61) $$\mathscr{K} = \{B \in \ker\tilde{\Box}\,|\,\tilde{\delta}B = 0,\ \mathrm{tr}\,B = 0\}\,.$$

Let B_i be an orthonormal basis for $\mathscr{K}$ with respect to the natural inner product. The map Ψ: $\mathscr{U} \to \mathscr{K}$ defined by

(5.62) $$\Psi(\underline{\tilde{g}}) = \sum_i \langle \underline{\tilde{g}}\,, B_i\rangle B_i$$

is smooth. Moreover, with the above identifications, it is easy to see that the differential of Ψ is the identity map. Thus, our claim follows from the implicit function theorem together with (5.15).

Proof of Theorem 0.13. Consider a gauge as in Proposition 3.24. Take $\varepsilon << \chi << \chi(n,\Omega,\Lambda)$ as in Proposition 5.49. Let L be as in Proposition 5.49.

First we show that by choosing χ, ε sufficiently small, we can guarantee that $R' = \infty$.

By (3.29), (3.31) together with (5.15) we see that there exists $c_3 = c_3(n,\Omega,\Lambda,L,\varepsilon)$, such that

(5.63) $$|||\phi^*g - g_0|||_{L^{-1}R',R'} \geqq c_3\chi\,.$$

By (3.28) and (3.29), there is a $j \geqq 10$, such that

(5.64) $$\frac{1}{100}c_3\chi \leqq |||\phi^*g - g_0|||_{L^{-(j+1)}R',L^{-j}R'} \leqq \frac{1}{80}c_3\chi\,.$$

We can assume that χ is so small that we can apply Lemma 5.56 to obtain g_1, such that if $h = \phi^*g - g_1$, then over $A_{L^{(j+1)}R',L^{-j}R'}(\underline{p})$, $(\pi h)_0 = 0$ and

$$|||g_1 - g_0|||_{L^{-(j+1)}R',L^{-j}R'} \leqq 2|||\phi^*g - g_0|||_L - (j+1)_{R',L^{-j}R'} \leqq \frac{1}{40}C_3\chi\,.$$
(5.65)

Then (5.64) and (5.65) imply

(5.66) $$|||\phi^*g - g_1|||_{L^{-1}R',R'} \geqq \frac{1}{10}c_3\chi\,.$$

If (5.50) holds, then Proposition 5.49 implies that for $i \leqq j$,

(5.67) $$|||h|||_{L^{-i}R',L^{-(i-1)}R'} \geqq L^{\beta'}|||h|||_{L^{-(i+1)}R',L^{-i}R'},$$

where $h = \phi^*g - g_1$. But from (3.29) and Lemma 5.14 we easily get

$$(5.68)\qquad |||h|||_{L^{-2}R',L^{-1}R'} + c(n)\varepsilon(\log L)^{1/2} \geqq |||h|||_{L^{-1}R',R'},$$

which, for ε sufficiently small, contradicts (5.66), (5.67). Therefore, (5.50) does not hold. Similarly, one can show that (5.52) does not hold, either. Then, since $(\pi h)_0 = 0$, by using the last statement in Proposition 5.44, we get a contradiction. Thus, $R' = \infty$.

Now consider a sequence of annuli $A_{L^i a, L^{i+1}a}(\underline{p})$, $i = 0, 1, 2, \cdots$, and a corresponding sequence, g_i, such that for

$$(5.69)\qquad h_i = \phi^* g - g_i\,,$$

we have,

$$(5.70)\qquad (\pi h_i)_0 = 0\,.$$

By reasoning as above, we can assume that (5.53) holds for all i.

After passing to a subsequence, we can assume that for some Ricci flat cone, g_∞,

$$(5.71)\qquad \lim_{j\to\infty} |g_{i_j} - g_\infty|_{k,\alpha';0} = 0 \qquad (\alpha' < \alpha)\,.$$

If it is not the case that

$$(5.72)\qquad \lim_{j\to\infty} |||h_{i_j}|||_{L^{i_j}a, L^{i_j+1}a} = 0\,,$$

then as above, using (5.51) and the implication, (5.50) implies (5.51), we contradict (3.27).

Thus, (5.72) holds and by using Proposition 5.49 inductively and standard elliptic estimates, (see Lemma 5.14) we find that for some $c > 0$,

$$(5.73)\qquad |\phi^* g - g_\infty|_{k,\alpha;0} \leqq c r^{-\beta'}\,.$$

Finally, since g_0 is a tangent cone, by (1.31), (1.32), we find that

$$(5.74)\qquad g_\infty = g_0\,.$$

This completes the proof of Theorem 0.13.

The decay estimate proved so far is not optimal. We now show how to improve this estimate to obtain one which is essentially optimal.

To begin with, by Theorem 0.13 and Remark 3.23, after modifying ϕ by a suitable diffeomorphism, we can assume that for some $\beta' > 0$,

$$(5.75)\qquad \phi^* g \in \mathscr{T}^{0,2}_{k,\alpha;-\beta'}\,,$$

$$(5.76)\qquad \delta(\phi^* g) = 0\,.$$

In (5.75), (5.76) we continue to denote the modified metric by $\phi^* g$.

Let $r^{b^{\pm}} B$ be as in (4.66) and put

$$b = \min_{b^\pm < 0} |b^\pm| \tag{5.77}$$

Theorem 5.78. *For all* $0 < b' < b$, *there exists a gauge,* $\phi_{b'}$, *such that*

$$|\phi_{b'}^* g - g_0|_{k,\alpha;0} = O(r^{-b'})\,, \tag{5.79}$$

$$\delta \phi_{b'}^* g = 0\,. \tag{5.80}$$

Moreover, if $r^b B$ *is a solution of* (4.67), (4.68) *and* $r^b \log rB$ *is not a solution, then* b' *can be replaced by* b *in* (5.79) *and* (5.80).

Proof Let $\phi^* g$, g_0, β be as in Theorem 0.13. Put $h = \phi^* g - g_0$. Then by (5.8) we have

$$\Box h - \operatorname{Hess}\operatorname{tr} h = F(h) \tag{5.81}$$

where

$$F(h) \in \mathscr{T}^{0,2}_{k-2,\alpha;-2\beta'-2} \tag{5.82}$$

As in Section 2 (see in particular (2.20)–(2.33) and Theorem 2.69), we can find h_3 satisfying

$$\Box h_3 - \operatorname{Hess}\operatorname{tr} h_3 = F(h) \tag{5.83}$$

and

$$(\log r)^{-1} h_3 \in \mathscr{I}^{0,2}_{k,\alpha,-2\beta'}\,. \tag{5.84}$$

It follows that

$$(\Box - \operatorname{Hess}\operatorname{tr})(h - h_3) = 0\,. \tag{5.85}$$

As in (4.75), (4.76), we can write

$$h = \sum_1^\infty f_i(r) T_i(x)\,. \tag{5.86}$$

Here the $T_i(x)$ are radially parallel symmetric bilinear forms as in (4.58)–(4.60) (some of which may involve dr) which are orthonormal with respect to the inner product in (4.72).

Then as in Section 4 we can find h_2 such that for fixed subset, say $A \subset Z^+$,

$$h_2 = \sum_{i \in A} f_i(r) T_i(x) = \sum_{i \in A} c_i r^{e_i^-} T_i\,, \tag{5.87}$$

(where e.g. $e_i^- = a_i^- - 2$ as in (4.63)).

$$(\Box - \operatorname{Hess}\operatorname{tr})(h - h_2 - h_3) = 0\,, \tag{5.88}$$

and, at worst,

$$(\log r)^{-1} h_1 \in \mathscr{T}^{0,2}_{k,\alpha;-\min\{2\beta',b\}} \tag{5.89}$$

where $h_1 = h - h_2 - h_3$. Thus,

$$h = h_1 + h_2 + h_3 \tag{5.90}$$

and in (5.87),

$$c_i \neq 0, \tag{5.91}$$

implies

$$\min\{2\beta', b\} > |e_i^-| \geqq \beta' , \tag{5.92}$$

where b is as in (5.77).

By Corollary 4.65 and Theorem 2.70, we can choose A such that

$$h_2 = L_X g_0 , \tag{5.93}$$

where

$$X \in \mathscr{T}^{1,0}_{k+1,\alpha;1-\beta'} , \tag{5.94}$$

Let K_X be the diffeomorphism generated by taking the flow of X to time 1. Then it follows easily from (5.92) and (5.93) that

$$K_X^*(g_0) - L_X g_0 \in \mathscr{T}^{0,2}_{k,\alpha;-\min\{2\beta',b\}} ; \tag{5.95}$$

compare (2.2), Lemma 2.6 and (3.19). Similarly, in view of (5.84), (5.89), (5.93) and (5.94),

$$(\log r)^{-1}(K_X^* g_0 - \phi^* g) \in \mathscr{T}^{0,2}_{k,\alpha;-\min\{2\beta' b\}} \tag{5.96}$$

this implies

$$(\log r)^{-1}(k_{-X}^* \phi^* g - g_0) \in \mathscr{T}^{0,2}_{k,\alpha,-\min\{2\beta',b\}} \tag{5.97}$$

If $2\beta' \geqq b$, then the theorem is proved. If $2\beta' < b$, we redefine h to be $K_{-X}^* \phi^* g - g_0$. This new h still satisfies (5.81) and (5.82). Thus we can proceed as above with β' replaced by $2\beta' - \varepsilon$ for some sufficiently small $\varepsilon > 0$.

By an obvious induction we can complete the proof.

We close this section by explaining the relation of our results to those of [BKN] in the ALE case. In this case the tangent cone, $C(N^{n-1})$ is of the form R^n/Γ. Thus, any solution of (4.3) on $C(N^{n-1})$ can be lifted to a solution h, on R^n. In view of Theorem 5.78, in studying the optimal rate of decay we can restrict attention to those h with $\operatorname{tr} h = \delta h = 0$.

Since $\operatorname{tr} h = 0$, the components of h are just harmonic functions. Thus, the rate of decay is at least r^{2-n}, since this is the Green's function i.e. the homogeneous, decay solution which decays most slowly. However, such a homogeneous solution is of the form

$$r^{2-n}T\,, \tag{5.98}$$

where

$$\nabla T = 0\,. \tag{5.99}$$

Moreover, (5.99) implies

$$i_{\partial/\partial r}T \not\equiv 0\,, \tag{5.100}$$

and as an easy consequence,

$$\delta(r^{2-n}T) \not\equiv 0\,. \tag{5.101}$$

It follows that for homogeneous decay solutions satisfying $\operatorname{tr} h = \delta h = 0$,

$$|h(r)| \leqq cr^{1-n}\,, \tag{5.102}$$

since this is the rate of decay for homogeneous harmonic functions, which, apart from the Green's function decay most slowly. However, the existence of such a function corresponds to $\mu = n-1$ in (2.47)–(2.50) (where $i = 0$). By Obata's Theorem, this can only happen if $C(N^{n-1}) = R^n$. Then, clearly M^n is isometric to R^n as well. Thus, with Theorem 5.78 we get (compare [BKN]).

Theorem 5.103. *If M^n is an ALE space, then $b = n$.*

6 A special result in the nonintegrable case

In this section, we show that if a manifold (M^n, g) satisfying (0.2), (0.8), (0.9) has a unique tangent cone which (as opposed to being integrable) is *maximally nonintegrable* in a suitable sense, then (M^n, g) converges to its tangent cone at the rate $O\left(\frac{1}{\log r}\right)$.

Let $(C(N^{n-1}), g_0)$ be a Ricci-flat metric cone, In this section, we will write $\tilde{\mathscr{L}}$ for the operator $\tilde{\square}$–Hess tr, since this operator occurs repeatedly. Also, where previously we wrote $\phi^* g$, here we just write g.

Consider a one-parameter family of cone metrics $g_s = dr^2 + r^2\tilde{g}_s$, and the formal expansion

$$\tilde{g}_s = \tilde{g}_0 + sh_1 + s^2h_2 + \cdots \tag{6.1}$$

Then at least formally, the Einstein equation $\operatorname{Ric}(\tilde{g}_s) = (n-2)\tilde{g}_s$ reduces to a sequence of recursive equations,

$$\tilde{\mathscr{L}}h_i = E_i(h_i, \cdots, h_{i-1}), \qquad i = 1, 2, \cdots \tag{6.2}$$

where $\tilde{\mathscr{L}}$ is computed with respect to $\tilde{g}_0 = \tilde{g}$. Here, $E_1 = 0$, and E_i is a polynomial in the $h_1, h_2, \cdots, h_{i-1}$ and their derivatives up to second order. In fact, it is well-known that $(C(N^{n-1}), g_0)$ is integrable if and only if for any h_1 in $\ker\tilde{\mathscr{L}}$, one can solve $(6.2)_i$ inductively for h_2, $h_3, \cdots$; see [B], Section 12F. Equivalently, on N^{n-1},

$$(6.3) \qquad E_i(h_1, \cdots, h_{i-1}) \perp \ker \tilde{\mathscr{L}} \qquad i = 2, 3, \cdots$$

Definition 6.4. We say $(C(N^{n-1}), g_0)$ is maximally nonintegrable if there exists $h_1 \in \ker\tilde{\mathscr{L}}$ such that for all $h \neq 0, h \in \ker \tilde{\mathscr{L}}$

$$(6.5) \qquad \int_{N^{n-1}} \langle E_2(h), h_1 \rangle > 0$$

In case $\dim\ker\tilde{\mathscr{L}} = 1$, the maximal nonintegrability simply means that for $h \in \ker\tilde{\mathscr{L}}$ $h \neq 0$,

$$(6.6) \qquad \int_N (E_2(h), h) \neq 0 .$$

Clearly, a maximally nonintegrable cone is locally rigid. The purpose of this section is to prove

Theorem 6.7. *Let $(C(N^{n-1}), g_0)$ be a maximally nonintegrable Ricci-flat cone, and g be a complete Ricci-flat metric on $C(N^{n-1})$ with $\delta_t g = 0$. Suppose that g converges to g_0 as $r \to \infty$. Then for all k, α, there is a constant C, possibly depending on g_0, such that*

$$(6.8) \qquad |g - g_0|_{k,\alpha;0}(r) \leqq \frac{C}{\log r} \quad for \quad r > 1 .$$

Remark 6.9. We believe that the estimate in (6.8) is optimal for a maximally nonintegrable cone and a general Ricci-flat metric g which is asymptotic to g_0.

Remark 6.10. It should be possible to generalize the arguments in the proof of Theorem 6.7 to prove the uniqueness of tangent cones for a complete Ricci flat manifold in case some tangent cone $(C(N^{n-1}), g_0)$ satisfies the following. The set,

$$(6.11) \qquad EI = \{h \in \ker\tilde{\mathscr{L}} | E_2(h) \perp \ker\tilde{L}, \ \|h\|_{L^2(N)} = 1\} ,$$

is a smooth submanifold in $S_1\ker\tilde{\mathscr{L}} := \{h \in \ker\tilde{\mathscr{L}} | \ \|h\|_{L^2(N)} = 1\}$. Moreover, any h in EI is integrable, and E_2 satisfies a nondegeneracy condition analogous to that in Definition 6.4, in directions normal to EI.

In proving Theorem 6.7, it is more convenient to work with the cylindrical coordinates $(t, x) \in R^+ \times N^{n-1} = C(N^{n-1})$, where $t = \log r$. For any object T in polar coordinates, we denote by T^c the correspondence of T in cylindrical coordinates. For instance, we have

$$g^c = \frac{1}{r^2} g, \quad g_0^c = \frac{1}{r^2} g_0 = dt^2 + \tilde{g}\,, \tag{6.12}$$

$$h := g - g_0, \quad h^c := g^c - g_0^c\,, \tag{6.13}$$

Also, if we put

$$\mathscr{P}^c h^c = \mathrm{Ric}_{r^2(g_0^c + h^c)} - \mathrm{Ric}_{r^2 g_0^c} = 0\,, \tag{6.14}$$

$$|\mathscr{P}^c h^c - \mathscr{L} h^c|_{C^{k,\alpha}} \leqq c\{|h^c|_{C^{k,\alpha}} |(\nabla^c)^2 h^c|_{C^{k-2,\alpha}} + |\nabla^c h^c|^2_{C^{k-1,\alpha}}\} \tag{6.15}$$

where

$$\mathscr{L} h^c = (\Box_t - \mathrm{Hess\,tr})(r^2 h^c)\,. \tag{6.16}$$

Fix $u > 0$. For L as in Lemma 5.31. Set

$$\begin{aligned} A_i &= \{(t,x) |\, \log u + (i-1)\log L < t < \log u + i \log L\} \\ &:= \{(t,x) | t \in I_i\} \end{aligned} \tag{6.17}$$

Also, put

$$|||h^c|||_i^2 = \int_{A_i} |h^c|^2 d\mathrm{vol}_{g_0^c}\ .\,, \tag{6.18}$$

and

$$\|h^c(t)\|^2 = \int_{N^{n-1}} |h^c(t)|^2\, d\mathrm{vol}_{\tilde{g}_t}\,. \tag{6.19}$$

Finally, put

$$\dot{h}^c = \nabla^c_{\partial/\partial t}\, h^c\,. \tag{6.20}$$

Let β' be as in Lemma 5.31 and let $0 < \theta < 1$. The following lemma can be proved by arguments just like those which were used in Section 5; see also [S1], Lemma 3.3.

Lemma 6.21. *There exists* $\chi_0 = \chi_0(n, \theta, \beta', L)$ *such that if* h^c *satisfies* (6.14) *and*

$$|h^c|_{C^{2,1/2}} < \chi < \chi_0\,, \tag{6.22}$$

then

$$0 \neq |||h^c|||_{i+2} \geqq e^{\beta' L} |||h^c|||_{i+1}\,, \tag{6.23}$$

implies

$$|||h|||_{i+3} \geqq c^{\beta' L} |||h^c|||_{i+2}\,, \tag{6.24}$$

and

$$0 \neq |||h^c|||_{i+3} \leqq c^{-\beta' L} |||h^c|||_{i+2}\,, \tag{6.25}$$

implies

$$\||h^c\||_{i+2} \leqq e^{-\beta' L} \||h^c\||_{i+1} . \tag{6.26}$$

If neither (6.24) *nor* (6.26) *holds, then for all* $t_1, t_2, t \in I_{i+2}$,

$$\|h^c(t_2)\| \leqq (1+\theta)\|h^c(t_1)\| , \tag{6.27}$$

$$\|\dot{h}^c(t)\| \leqq \theta\|h^c(t)\| . \tag{6.28}$$

Moreover, if in this case we write $h^c = h_1^c + h_2^c$, *where for every* $t \in I_{i+2}$, $h_1^c \in$ *ker*$\tilde{\mathscr{L}}$ *and* h_2^c *is orthogonal to ker*$\tilde{\mathscr{L}}$, *then*

$$\|h_2^c(t)\| \leqq \theta\|h_1^c(t)\| . \tag{6.29}$$

Note that in the last statement in Lemma 6.21, for each t we regard elements of $\ker\tilde{\mathscr{L}}$ as symmetric tensor fields on $C(N^{n-1}) = (0,\infty)\times N^{n-1}$, via the natural identification.

Proof of Theorem 6.7. By standard elliptic estimates, it will suffice to prove (6.8) for the case $k=2$, $\alpha = 1/2$.

Since g converges to g_0, by standard elliptic estimates, given $0 < \delta < < \psi_0$, there exists $t_0 > 0$, such that for any $t \geqq t_0$,

$$|h^c(t)|_{C^{2,1/2}} < \delta . \tag{6.30}$$

Define A_i as in (6.17) using t_0 in place of u_0. Thus, using the implication, (6.22) implies (6.23) and induction, it follows that the conclusion (6.23) cannot hold.

Next suppose that there is a sequence $\{I_{i_j}\}_{j\geqq 1}$ with $\lim i_j = \infty$ for which (6.26) holds. Then by using (6.26) inductively, we get for all $i \geqq 1$,

$$\||h^c\||_i \leqq e^{-\beta R}\||h^c\||_{i-1} \leqq \cdots \leqq e^{-\beta i R}\||h^c\||_1 . \tag{6.31}$$

Hence, h^c decays exponentially. Equivalently, h decays at the order $r^{-\beta}$ and Theorem 6.7 and follows.

Therefore, we can assume that after changing t_0 if necessary, (6.27)–(6.29) hold for all $t_1 t_2$, $t \geqq t_0$, and $|t_1 - t_2| \leqq L$.

Differentiating the equation (6.13) with respect to t, we obtain (with obvious notation)

$$0 = \mathscr{L}_{g_0^c}\dot{h}^c - b_1^c \cdot \nabla^2 \dot{h}^c - b_2^c \cdot \nabla \dot{h}^c - b_3^c \cdot \dot{h}^c \tag{6.32}$$

where for some constant, C,

$$\max\{|b_1^c|_{C^{2,1/2}}, |b_2^c|_{C^{1,1/2}}, |b_3^c|_{C^{0,1/2}}\} \leqq C|h^c|_{C^{2,1/2}} . \tag{6.33}$$

As a consequence, we can repeat the argument of Lemma 6.21 with h^c replaced by $\dot{h}^c$. Thus, all conclusions of that lemma hold for $\dot{h}^c$ as well.

Now, if $\|\dot{h}^c(t)\|$ decays exponentially, so does $\|h^c(t)\|$. But this contradicts the above assumption on $\|h^c(t)\|$. Therefore, by applying Lemma 6.21 to $\dot{h}^c$, we get that for all $t_1, t_2, t \geqq t_0$ and $|t_1 - t_2| \leqq L$,

$$\|\ddot{h}^c(t)\| \leqq \theta \|\dot{h}^c(t)\| , \tag{6.34}$$

$$\|\dot{h}^c(t_1)\| \leqq (1+\theta)\|\dot{h}^c(t_2)\| . \tag{6.35}$$

As in (6.29), we write

$$h^c = h_1^c + h_2^c , \tag{6.36}$$

where

$$\tilde{\mathscr{L}} h_1^c = 0 , \tag{6.37}$$

$$h_2^c \perp \ker \tilde{\mathscr{L}} . \tag{6.38}$$

Then equation (6.14) is of the form

$$\mathscr{L}_{g_0^c} h_2^c = Q(h_1^c) + \ddot{h}_1^c + b \cdot \dot{h}_1 + \tilde{a}_1 \cdot \nabla^2 h_2^c + \tilde{a}_2 \cdot \nabla h_2^c + \tilde{a}_3 \cdot h_2^c \tag{6.39}$$

(again with obvious notation) where b is a constant tensor, $Q(h_1^c)$ depends only on h_1^c,

$$|Q(h_1^c)|_{C^{0,1/2}} \leqq C|h_1^c|^2_{C^{2,1/2}} , \tag{6.40}$$

and

$$\max \leqq \{|\tilde{a}_1|_{C^{2,1/2}}, |\tilde{a}_2|_{C^{1,1/2}}, |\tilde{a}_3|_{C^{0,1/2}}\} \leqq C|h^c|_{C^{2,1/2}} . \tag{6.41}$$

By (6.27), (6.28), (6.30), (6.34), together with standard elliptic estimates, we find that for $t \geqq t_0 + 1$,

$$|h_2^c(t)|_{C^{2,1/2}} \leqq C(\theta + \delta)\|h_1^c(t)\| . \tag{6.42}$$

By (6.29) together with standard elliptic estimates, to prove Theorem 6.7, it suffices to show that

$$\|h_1^c(t)\| \leqq \frac{C}{t} . \tag{6.43}$$

In this connection, note that (6.42) is actually a qualitative improvement of (6.29). It is used below in verifying (6.43).

By the maximal nonintegrability assumption, there exists $k \in S_1 \ker \tilde{\mathscr{L}}$ such that (6.5) holds with k in place of h_1.

Put

$$f(t) = \int_{N^{n-1}} \langle h^c, k \rangle . \tag{6.44}$$

Then it follows from (6.14), that

$$\ddot{f}(t) + \mu \dot{f}(t) = \int_{N^{n-1}} \langle E_2(h_1^c), k\rangle + \int_{N^{n-1}} O \leqq \left(|h_1^c|^3 + |h_2^c|^2\right) \tag{6.45}$$

where $\mu > 0$. By using (6.42) and the choice of k, we get

$$\begin{aligned} \ddot{f}(t) + \mu \dot{f}(t) &\geqq c\|h_1^c(t)\|^2 \,, \\ &\geqq c f(t)^2 \,, \end{aligned} \tag{6.46}$$

where c is a positive constant. However, (6.34) implies

$$|\ddot{f}(t)| \leqq \theta |\dot{f}(t)| \,. \tag{6.47}$$

Thus, if $\theta < \mu/2$, then for $t \geqq t_0 + 1$,

$$\frac{\mu}{2} \dot{f}(t) \geqq c f(t)^2 > 0 \,. \tag{6.48}$$

Since $\lim_{t \to \infty} f(t) = 0, f(t) < 0$ for $t \geqq t_0 + 1$. Dividing both sides of (6.48) by $f^2(t)$ and integrating over the interval $[t_0 + 1, t]$, we get

$$\frac{\mu}{2}(-f(t)^{-1} + f(t_0+1)^{-1}) \geqq (t - t_0 - 1) \,. \tag{6.49}$$

Equivalently,

$$|f(t)| = -f(t) \leqq \frac{1}{\frac{2c}{\mu}(t - t_0 - 1) - f(t_0 + 1)^{-1}} = O\left(\frac{1}{t}\right) \,. \tag{6.50}$$

If $\dim \ker \tilde{\mathscr{L}} = 1$ and then $\|h_1^c(t)\| = |f(t)|$, the theorem is proved.

In the general case, it is easy to see that for any $\underline{k}$ in a small neighborhood of k,

$$\int_{N^{n-1}} \langle E_2(h), \underline{k}\rangle > 0 \quad \text{on} \quad S_1 \ker \tilde{\mathscr{L}} \,. \tag{6.51}$$

Then the above argument shows that for all $t \geqq t_0 + 1$,

$$\left|\int_N \langle h_1^c, \underline{k}\rangle_{\tilde{g}}(t)\, d\mathrm{vol}_{\tilde{g}}\right| \leqq \frac{C}{t} \,. \tag{6.52}$$

From this, it is easy to deduce that for $t \geqq t_0 + 1$

$$\|h_1^c(t)\| \leqq \frac{C}{t} \,. \tag{6.53}$$

Thus, the theorem follows.

7. The linearized equation in the Kähler case

In this section and the next we consider the case in which M^{2k} is a Kähler manifold of complex dimension k. Then if $C(N^{2k-1})$ is a tangent cone as in Section 1, it is clear that $C(N^{2k-1})$ admits a natural Ricci flat Kähler structure. Our aim in this section is to discuss the rate of decay of infinitesimal Kähler Einstein deformations, and the rate of decay of infinitesimal deformations of the complex structure. We also derive the results necessary for the proof given in Section 8, that (given the assumptions of Theorem 0.15) tangent cones are complex integrable. To do this, we show that matters can be reduced to considerations on the *complex base* of $C(N^{2k-1})$.

Let J denote the complex structure on a Ricci flat Kähler cone, $C(N^{2k-1})$. Thus, $\nabla J = 0$. If we put

$$\frac{\partial}{\partial\theta} = J\left(r\frac{\partial}{\partial r}\right) , \tag{7.1}$$

$$(r^{-1}\frac{\partial}{\partial\theta})^* = \Theta , \tag{7.2}$$

then,

$$J(dr) = -\Theta . \tag{7.3}$$

Let $r^{b^\pm}B$ be as (4.66). In showing that matters can be reduced to considerations on the complex base, a major step is to show that $B\left(\frac{\partial}{\partial\theta}, \cdot\right) = 0$ provided $-2k < b^\pm \leqq 0$. This is accomplished somewhat indirectly, as we now explain.

According to [B] p. 363, if we define the action of J on symmetric bilinear forms by

$$Jh(v,w) = h(Jv,Jw) , \tag{7.4}$$

then ker $\square$ is J invariant. The results of this section are obtained by analyzing the action of the *involution* J in our situation.

Let us put

$$\ker_0\square = \{h \in \ker\square | \operatorname{tr} h = 0,\ \delta h = 0\} . \tag{7.5}$$

In fact, $\ker_0\square$ is properly contained in $\ker\square$. However we will show that the action of J preserves the subspace of $\ker_0\square$ that we are interested in studying i.e. the one spanned by those $h \in \ker_0\square$ which are homogeneous satisfy $cr^{-2k} < |h| \leqq c$, and for which additional integrability conditions, (7.63)–(7.66), hold. In the application to Theorems 0.15 and 0.16, the integrability conditions will automatically be satisfied.

Additionally, we will show that the space spanned by the homogeneous solutions of the form $L_X g_0$ in (4.66), where $cr^{-2k} < |L_X g_0| \leqq c$, is J-invariant. But, as is easy to check, the decomposition in (4.66) is orthogonal for the J-invariant inner product in (4.70). Thus, the space of solutions $r^{b^\pm}B$, with

$2k < b^{\pm} \leqq 0$, is J-invariant as well. Since by definition, $B\left(\frac{\partial}{\partial r}, \cdot\right) = 0$, we get $B\left(\frac{\partial}{\partial \theta}, \cdot\right) = 0$.

In order to analyze the action on solutions of the form $\nabla^{sym} X^* = L_X g_0$, the following eigenvalue estimate is required; compare [EM], [DNP].

Theorem 7.6. *Let X^{n-1} be compact, with*

$$Ric_{X^{n-1}} \geqq (n-2)g . \tag{7.7}$$

Then the first eigenvalue, μ_1, of the Laplacian on coclosed 1-forms satisfies

$$\underline{\mu}_1 \geqq 2(n-2) . \tag{7.8}$$

Equality holds if and only if ψ is dual to a Killing field which is also an eigenvector of Ric with eigenvalue $(n-2)$.

Proof. If ψ is an arbitrary 1-form, the Bochner Weitzenbock formula gives

$$|d\psi|^2 + |\delta\psi|^2 = |\nabla\psi|^2 + \langle \mathrm{Ric}\psi, \psi\rangle . \tag{7.9}$$

Write

$$\nabla\psi = \nabla^S \psi + \nabla^{SS}\psi . \tag{7.10}$$

where the superscripts denote symmetric and skew symmetric parts respectively. (Thus, in the notation of Section 4, $\nabla^S = \frac{1}{2}\nabla^{sym}$.) Under the usual identification between 2-forms and skew symmetric 2-tensors,

$$\frac{1}{2} d\psi \to \nabla^{SS}\psi . \tag{7.11}$$

Note however, that this identification is *not norm preserving* for the norms in (7.9), (7.10), and in fact,

$$\frac{1}{2}|d\psi|^2 = |\nabla^{SS}\psi|^2 . \tag{7.12}$$

Then by (7.9),

$$\frac{1}{2}|d\psi|^2 + |\delta\psi|^2 = |\nabla^S\psi|^2 + \langle \mathrm{Ric}\psi, \psi\rangle , \tag{7.13}$$

from which our claim immediately follows.

Recall, that the harmonic functions on a cone can be written as a sum of functions of the form

$$r^a\phi , \tag{7.14}$$

where

$$\tilde{\Delta}\phi = \mu\phi \tag{7.15}$$

and a is as in (2.47)–(2.49). In our case, $\mathrm{Ric}_{N^{2k-1}} = 2(k-1)\tilde{g}$ and by Obata's theorem, $\mu > 0$ implies

$$\mu \geqq 2k - 1 \tag{7.16}$$

$$a^{-} \leqq 1 - 2k\,, \tag{7.17}$$

with equality only for $N^{2k-1} = S^{2k-1}$.

The harmonic 1-forms on $C(N^{2k-1})$ can be written as sums of forms of the following three types. Namely

$$d(r^{a^{\pm}}\phi) \quad (\phi \in \Lambda^0) \tag{7.18}$$

where ϕ is as in (7.14);

$$r^{a^{\pm}}\phi \quad (\phi \in \Lambda^1) \tag{7.19}$$

where

$$\tilde{\delta}\phi = 0\,, \tag{7.20}$$

$$\tilde{\Delta}\phi = \mu\phi\,; \tag{7.21}$$

and last of all,

$$r^{a^{\pm}+2}d\phi + a^{\mp}r^{a^{\pm}+1}\,dr \wedge \phi \quad (\phi \in \Lambda^0) \tag{7.22}$$

where ϕ is in (7.14).

Lemma 7.23. *Let ψ be a homogeneous harmonic 1-form on $C(N^{2k-1})$, such that, $|\psi| \leqq cr$, for $r \geqq 1$. Then either*

$$|\psi| \leqq cr^{1-2k} \tag{7.24}$$

where equality holds if and only if $\psi|(1, N^{2k-1})$ is dual to a Killing field on N^{2k-1}, or

$$c \leqq |\psi| \leqq cr \quad (r \geqq 1) \tag{7.25}$$

If N^{2k-1} is not isometric to the unit sphere, S^{2k-1}, then for some $\varepsilon > 0$, the form in (7.25) actually satisfies

$$cr^{\varepsilon} \leqq |\psi| \leqq cr\,. \tag{7.26}$$

Lemma 7.27. *If ψ is homogeneous and for some $\varepsilon > 0$,*

$$c \leqq |\psi| \leqq cr^{1-\varepsilon} \quad (r \geqq 1) \tag{7.28}$$

then ψ is as in (7.18) and hence exact. Moreover, if

$$|\psi| = cr \quad (r \geqq 1) \tag{7.29}$$

then ψ *is a sum of exact forms as in* (7.18) (*the case* $\mu = 4k$) *and* (7.22) (*the case* $\mu = 0$) *and forms as in* (7.19) *which are dual to Killing fields.*

In the case of (7.22), ϕ *is the constant function and the harmonic* 1*-form is a multiple of*

$$r\,dr = d\left(\frac{1}{2}r^2\right) \tag{7.30}$$

Given Theorem 7.6 and (7.17), the proofs of the above two lemmas follow from the definitions, (2.47)–(2.49), by inspection.

In view of Lemmas 7.23 and 7.27, the following lemma has obvious implications for our discussion.

Let ψ be a 1-form on an arbitrary Kähler manifold.

Lemma 7.31. *If*

$$d(J\psi) = 0\,, \tag{7.32}$$

then

$$(\nabla^{sym}\,\psi)^H = 0\,. \tag{7.33}$$

If $J\psi$ *is dual to a Killing field*

$$(\nabla^{sym}\,\psi)^{SH} = 0\,. \tag{7.34}$$

Proof. If (7.32) holds, then

$$\nabla\, J\psi = \frac{1}{2}\,\nabla\,(J\psi)^{sym} \tag{7.35}$$

and

$$\begin{aligned} \nabla_{JX}\,\psi(JY) &= \nabla_{JX}\,J\psi(Y)\,,\\ &= \nabla_Y\,J\psi(JX)\,,\\ &= -\,\nabla_Y\,\psi(X)\,, \end{aligned} \tag{7.36}$$

which clearly implies (7.33).

If $J\psi$ is dual to a Killing field,

$$\nabla^{sym}\,J\psi = 0 \tag{7.37}$$

and

$$\begin{aligned} \nabla_{JX}\,\psi(JY) &= \nabla_{JX}\,J\psi(Y)\,,\\ &= -\,\nabla_Y\,J\psi(JX)\,,\\ &= \nabla_Y\,\psi(X)\,, \end{aligned} \tag{7.38}$$

which implies (7.34).

Let $\mathscr{V} \oplus \mathscr{W}$ the decomposition of the space of harmonic 1-forms of $C(N^{2k-1})$ satisfying (7.25) into exact forms and forms dual to Killing fields fixing the singular point, as guaranteed by Lemma 7.27. Let $\pi: \mathscr{V} \oplus \mathscr{W} \to \mathscr{V}$ be the projection. From Lemma 7.23, Lemma 7.27 and Lemma 7.31 we immediately obtain

Corollary 7.39. *The subspace of* $\ker_0\Box$ *spanned by those* $\nabla^{sym}\psi$ *which are homogeneous and which for some* $\varepsilon > 0$, *satisfy*

$$cr^{\varepsilon-2k} < |\nabla^{sym}\psi| \leqq c\,, \tag{7.40}$$

is J*-invariant.*

If for some $\varepsilon > 0$,

$$cr^{-2k} < |\nabla^{sym}\psi| \leqq cr^{-\varepsilon}\,, \tag{7.41}$$

then $\nabla^{sym}\psi$ *is skew hermitian.*

If ψ *is homogeneous and has linear growth, then*

$$(\nabla^{sym}\psi)^H = 0 \tag{7.42}$$

if and only if

$$\psi \in \pi(J\mathscr{V})\,. \tag{7.43}$$

Moreover,

$$(\nabla^{sym}\psi)^{SH} = 0 \tag{7.44}$$

if and only if

$$\psi \in \pi(J\mathscr{W})\,. \tag{7.45}$$

We now show that subspace of $\ker_0\Box$ consisting of those h, satisfying $cr^{-2k} < |h| \leqq c$ and the integrability conditions, (7.63)–(7.66), is also J-invariant. To this end, we recall the characterizations of the hermitian and skew hermitian elements of $\ker_0\Box$; [B], p. 362.

Let

$$h = h^H + h^{SH} \tag{7.46}$$

denote the decomposition of h into its hermitian and skew hermitian parts, corresponding to the $+1$, -1, eigenspaces for the action of J. If A is an endomorphism and $k(\,,\,)$ is a bilinear form we put

$$k \circ A(x,y) = k(x, Ay)\,. \tag{7.47}$$

Then for h as above,

$$h = h^H\,, \tag{7.48}$$

if and only if $h \circ J$ is skew symmetric i.e. $h \circ J$ is a 2-form, if and only if $h \circ J = (h \circ J)^H$. Also, if we define I by

$$g \circ I = h \circ J\,, \tag{7.49}$$

we can regard I as a 1-form with values in tangent vectors. Then

$$h = h^{SH} \tag{7.50}$$

if and only if for all $X, I(X) = J(I(JX))$, in other words, if and only if I is the *real part* of a unique $T^{1,0}$ valued form of type (0,1). Here we use the splitting of the tangent bundle into its holomorphic and antiholomorphic parts.

With this terminology we have for all k,

$$\delta(k \circ J) = \delta k \circ J\,. \tag{7.51}$$

Moreover, if $\Delta = d\delta + \delta d$, then $h^H \in \ker \Box$ if and only if

$$h^H \circ J \in \ker \Delta \tag{7.52}$$

On the other hand,

$$\delta h^{SH} = -J\bar{\partial}^* I \tag{7.53}$$

and if $d = \partial + \bar{\partial}$, then $h^{SH} \in \ker \Box$ if and only if

$$I \in \ker(\bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial})\,. \tag{7.54}$$

At this point, we require some additional assumptions on those $h \in \ker \Box$ satisfying $\delta h = 0$ which we consider. These assumptions hold automatically for the solutions required in our applications.

Let $h \in \ker \Box$, with $\delta h = 0$, be of the form

$$\lim_{\varepsilon \to 0} \frac{g_\varepsilon - g}{\varepsilon} = h\,, \tag{7.55}$$

where $g_\varepsilon \to g, (\nabla^\varepsilon)^i R^\varepsilon \to \nabla^i R, i = 0, 1, \ldots$. Here, g_ε is Ricci flat and Kähler with respect to the complex structure, J_ε, where $J_\varepsilon \to J$.

Assume in addition that

$$\lim_{\varepsilon \to 0} \frac{J_\varepsilon - J}{\varepsilon} = \dot{J} \tag{7.56}$$

exists.

Note that $h = L_X g_0$ (as in (4.66)) automatically implies (7.55), (7.56). Thus, given $h \in \ker \Box$ with $\delta h = 0$ satisfying (7.55), (7.56) we can write

$$h = h_0 + L_X g_0 \tag{7.57}$$

where $h_0 \in \ker_0 \Box$ satisfies (7.55), (7.56). Therefore, since the action of J preserves trace, in what follows (and for the application) it will suffice to assume $h (= h_0) \in \ker_0 \Box$.

As in (7.10), let $A = A^S + A^{SS}$ denote the decomposition of the endomorphism A into its symmetric and skew symmetric parts. Note that

$$JJ^{SS} + \dot{J}^{SS}J = (J\dot{J}^{SS} + \dot{J}^{SS}J)^S \tag{7.58}$$

$$J\dot{J}^S + \dot{J}^S J = (J\dot{J}^S + \dot{J}^S J)^{SS} \tag{7.59}$$

Thus, from

$$J\dot{J} + \dot{J}J = 0\,, \tag{7.60}$$

it follows that

$$J\dot{J}^S + \dot{J}^S J = 0\,, \tag{7.61}$$

$$J\dot{J}^{SS} + \dot{J}^{SS}J = 0\,. \tag{7.62}$$

Recall that $\omega_\varepsilon = g_\varepsilon(J_\varepsilon)$ is the Kähler form of g_ε. Since $\dot{\omega}$ is skew symmetric, we get

$$\begin{aligned} -\dot{\omega} &= h \circ J + g \circ \dot{J} \\ &= h^H \circ J + g \circ \dot{J}^{SS}\,, \end{aligned} \tag{7.63}$$

which corresponds to the decomposition,

$$\dot{\omega} = \dot{\omega}^H + \dot{\omega}^{SH} \tag{7.64}$$

together with

$$h^{SH} \circ J + g \circ \dot{J}^S = 0\,. \tag{7.65}$$

Of course we have in addition,

$$d\dot{\omega} = 0\,, \tag{7.66}$$

since for all ε,

$$d\omega_\varepsilon = 0\,. \tag{7.67}$$

Lemma 7.68. *On* $C(N^{2k-1})$, *let* ω *satisfy* (7.63)–(7.66). *If*

$$|\dot{\omega}| \leqq c \tag{7.69}$$

then

$$\dot{\omega}^H = \beta_1 + \beta_2\,, \tag{7.70}$$

where

$$\Delta\beta_j = 0\,, \tag{7.71}$$

$$\delta\beta_1 = 0\,, \tag{7.72}$$

$$|\beta_2| \leqq cr^{-2k}. \tag{7.73}$$

Proof. So as not to disrupt the continuity of the presentation, here we will verify (7.73) with the exponent $-2k$ replaced by $2-2k$. The shaper statement will be proved in the Appendix at the end of this section. There we will actually show that $\dot{\omega}$ itself is harmonic (see (7A.18)) and is the sum of a radially parallel harmonic form and one which decays no slower than cr^{-2k}.

Let β_2 denote the sum of the homogeneous components of the harmonic form $\dot{\omega}^H$ whose norms are bounded by cr^{2-2k} $(r \geqq 1)$. Put $\beta_1 = \dot{\omega} - \beta_2$. It will suffice to observe that in our situation every homogeneous harmonic form, β, on $C(N^{2k-1})$, with $cr^{2-2k} \leqq |\beta| \leqq c$, satisfies $\delta\beta = 0$.

The harmonic 2-forms on a cone are of four types; see [C] Section 3. Two of these are analogous to those appearing in (7.18), (7.19) and hence, are of divergence zero. The remaining two types are of the form

$$r^{a^{\pm}+2}\,d\phi + a^{\mp}r^{a^{\pm}+1}\,dr\wedge\phi \quad (\phi \in \Lambda^1) \tag{7.74}$$

$$r^{a^{\pm}+1}\,dr\wedge d\phi \quad (\phi \in \Lambda^0) \tag{7.75}$$

By definition $\mu > 0$ and hence $a^+ > 0$ in (7.75). Thus, for $a = a^+$, the form in (7.75) is not bounded in norm for $r \geqq 1$. Similarly, by Theorem 7.6 (or already by Bochner's Theorem) $a^+ > 0$ in (7.74). Hence, for $a = a^+$, the form in (7.74) is not bounded in norm for $r \geqq 1$.

On the other hand, by (7.17), $a = a^-$ in (7.75) implies $a \leqq 1-2k$. By (7.24), $a = a^-$ in (7.74) implies $a \leqq 2-2k$. This completes the proof.

The following proposition is an immediate consequence of (7.72), (7.73).

Proposition 7.76. *If $h \in ker_0\Box$ is homogeneous, satisfies (7.63)–(7.66) and*

$$cr^{\iota-2k} < |h| \leqq c \quad (r \geqq 1) \tag{7.77}$$

then

$$Jh \in ker_0\Box\,. \tag{7.78}$$

Equivalently, $h^H, h^{SH} \in ker_0\Box$ and satisfy (7.63)–(7.66).

Now we can easily verify the J-invariance of the subspace of $\ker_0\Box$, spanned by the $r^{b^{\pm}}B$, satisfying (7.63) – (7.66), and (7.77).

Now let $\langle\langle\,,\rangle\rangle$ be the inner product defined in (4.72). Note that clearly,

$$\langle\langle h_1, h_2\rangle\rangle = \langle\langle Jh_1, Jh_2\rangle\rangle\,. \tag{7.79}$$

Lemma 7.80. *On the subspace of* $\ker_0 \square$ *satisfying* (7.63)–(7.66) *and* (7.77), *the orthogonal complement of the subspace spanned by the bilinear forms,* $\nabla^{sym}\psi$, *is the subspace spanned by those* $r^{b^{\pm}}B$ *such that*

$$B\left(\frac{\partial}{\partial r},\cdot\right)=0\,, \tag{7.81}$$

$$tr\,B=0\,, \tag{7.82}$$

$$\tilde{\delta}B=0\,. \tag{7.83}$$

Proof. If $\langle X,\frac{\partial}{\partial r}\rangle\equiv 0$, $\nabla_{\partial/\partial r}X=0$, then (as in (4.29), (4.30)), the tangent component of

$$\nabla^{sym}\left(f(r)X+u(r,x)\frac{\partial}{\partial r}\right) \tag{7.84}$$

is

$$f(r)\widetilde{\nabla}^{sym}X+u(r,x)r^{-1}\tilde{g}\,. \tag{7.85}$$

It is clear from (7.85) that the subspaces in question are orthogonal. By Proposition 4.65 they also span.

Given B as above, write

$$B=B^b+\Theta\boxtimes\tau+u\Theta\otimes\Theta\,, \tag{7.86}$$

where,

$$B^b\left(\frac{\partial}{\partial r},\cdot\right)=B^b\left(\frac{\partial}{\partial\theta},\cdot\right)=0\,, \tag{7.87}$$

$$\tau\left(\frac{\partial}{\partial r}\right)=\tau\left(\frac{\partial}{\partial\theta}\right)=0\,. \tag{7.88}$$

Let the bilinear form, $\nabla^{sym}\psi$, be as in Corollary 7.39.

Corollary 7.89. *If* $h\in\ker_0\widetilde{\square}$ *is homogeneous and satisfies* (7.63)–(7.66) *and* (7.77), *then*

$$h=\nabla^{sym}\psi+r^{b^{\pm}}B\,, \tag{7.90}$$

where B *satisfies* (7.86)–(7.88) *and*

$$B=r^{b^{\pm}}B^b\,. \tag{7.91}$$

Proof. This is an immediate consequence of (4.66), Corollary 7.39, Proposition 7.76, (7.79) and Lemma 7.80.

As a consequence of the discussion so far, we also obtain a homological description of the Hermitian symmetric elements of $\ker_0\Box$ having slow decay.

Let $T^b N^{2k-1} \subset TN^{2k-1}$ denote the subspace orthogonal to $\frac{\partial}{\partial\theta}$.

Theorem 7.92. *Let $h = h^H \in \ker_0\Box$ satisfy* (7.63)–(7.66) *and* (7.77). *Then*

$$h = \nabla^{sym}\, \psi + B \tag{7.93}$$

where $J\psi$ is dual to a Killing field and

$$\beta = B \circ J \tag{7.94}$$

satisfies

$$d\beta = \delta\beta = 0 \tag{7.95}$$

$$\beta\left(\frac{\partial}{\partial r}, \cdot\right) = \beta\left(\frac{\partial}{\partial\theta}, \cdot\right) = 0\,, \tag{7.96}$$

$$\beta | T^b N^{2k-1} \text{ is of type (1,1)} \tag{7.97}$$

$$\nabla_{\partial/\partial r}\,(r^2\beta) = 0 \tag{7.98}$$

$$L_{\partial/\partial\theta}\beta = 0\,. \tag{7.99}$$

Proof. In view of Corollary 7.89 it suffices to characterize those B such that $B = B^b$ and $B = B^H$. By (7.62), (7.63)

$$\beta = B \circ J = -\omega^H\,, \tag{7.100}$$

where $\dot\omega$ satisfies (7.66). From Lemma 7.27, together with considerations of homogeneity, it follows that we can assume that either

$$\dot\omega\left(\frac{\partial}{\partial r}, \cdot\right) = 0 \tag{7.101}$$

$$\nabla_{\partial/\partial r}\,(r^2\dot\omega) = 0 \tag{7.102}$$

or

$$\dot\omega = d(r^2\phi) \tag{7.103}$$

where $r^2\phi$ as in (7.19) is dual to a Killing field (compare the discussion after (7.166)).

In the former case, (7.101) together with $\dot\omega^H\left(\frac{\partial}{\partial\theta},\right)=0$ implies (7.96). The J-invariance of the space of forms satisfying (7.96), (7.98) implies that we can assume that (7.97) holds. Moreover, since

$$L_{\partial/\partial\theta}\beta = i_{\partial/\partial\theta}\, d\beta + d i_{\partial/\partial\theta}\beta\,, \tag{7.104}$$

(7.99) holds as well.

In case (7.103) holds, the conclusion follows from Lemma 7.31 (see (7.34)). This completes the proof.

We now consider the remaining case of skew Hermitian elements of $\ker_0\Box$, satisfying (7.63)–(7.66) and (7.77).

As mentioned after (7.50), if h is skew Hermitian it can be regarded as the real part of a $T^{1,0}$ valued (0,1)-form. The integrability condition corresponding to (7.63), (7.64), (7.65) is the condition that this form is $\bar\partial$-closed. An equivalent way of expressing this condition is the following.

Let $\dot J$ be as in (7.56), and regard $\dot J$ as a real 1-form with values in tangent vectors. Let $d\dot J$ be the exterior derivative of this 1-form. Here, we use the riemannian connection on the coefficient bundle (in this case, the tangent bundle) in defining the operator d. Let $(d\dot J)^{SH}$ be the sum of the type (2,0) and (0,2) components of the 2-form $d\dot J$ i.e. the skew Hermitian part with respect to the first two slots.

Lemma 7.105.

$$(d\dot J)^{SH} = (d\dot J^S)^{SH} = (d\dot J^{SS})^{SH} = 0\,. \tag{7.106}$$

Proof. Differentiating (7.1) gives

$$\nabla\dot J = J\dot\nabla - \dot\nabla J\,. \tag{7.107}$$

Thus,

$$\begin{aligned} d\dot J(X,Y) &= \nabla_X\dot J(Y) - \nabla_Y\dot J(X)\\ &= J(\dot\nabla_X Y) - \dot\nabla_X(JY) - J(\dot\nabla_Y X) + \dot\nabla_Y(JX)\\ &= \dot\nabla_Y JX - \dot\nabla_X JY\,, \end{aligned} \tag{7.108}$$

where we have used

$$\dot\nabla_X Y - \dot\nabla_Y X = 0\,, \tag{7.109}$$

which follows from the fact that riemannian connections have torsion zero. Letting SH denote the skew Hermitian part in the first two slots, we get

$$\begin{aligned} 2(d\dot J)^{SH}(X,Y) &= \dot\nabla_Y JX - \dot\nabla_X JY + \dot\nabla_{JY}X - \dot\nabla_{JX}Y,\\ &= 0\,, \end{aligned} \tag{7.110}$$

where again we use (7.107). Replacing J by J^S in (7.107) and the relations which follow, gives

$$(7.111) \qquad (dJ^S)^{SH} = 0\,,$$

which easily implies (7.106).

The condition, (7.106) imposes an additional constraint on elements of the form $r^{-c}B$, with $B = B^b = B^{SH}$; see Proposition 7.131. We now proceed to derive this constraint.

Let $B = B^b$ and put

$$(7.112) \qquad \mathcal{J}^H B(X,Y) = \frac{1}{2}(B(JX,Y) - B(X,JY))\,,$$

$$(7.113) \qquad \mathcal{J}^{SH} B(X,Y) = \frac{1}{2}(B(JX,Y) + B(X,JY))\,.$$

Then $\mathcal{J}^H$, $\mathcal{J}^{SH}$ define *almost complex structures* on the Hermitian, respectively skew Hermitian, bilinear forms. Moreover,

$$(7.114) \qquad \mathcal{J}^H B^{SH} = \mathcal{J}^{SH} B^H = 0\,,$$

so that $\mathcal{J} = \mathcal{J}^H + \mathcal{J}^{SH}$ is an almost complex structure on the space of all B as above.

Note that if

$$(7.115) \qquad \left\langle Y, \frac{\partial}{\partial\theta} \right\rangle = \left\langle Z, \frac{\partial}{\partial\theta} \right\rangle = 0$$

and

$$(7.116) \qquad \left[\frac{\partial}{\partial\theta}, Y\right] = 0\,,$$

then

$$(7.117) \qquad \begin{aligned} \langle \tilde{\nabla}_{\partial/\partial\theta} Z, Y \rangle &= \frac{\partial}{\partial\theta}\langle Z, Y\rangle + \langle JZ, Y\rangle \\ &= \left\langle \left[\frac{\partial}{\partial\theta}, Z\right] + JZ, Y \right\rangle . \end{aligned}$$

Also, if Z_1, Z_2 satisfy (7.80), then

$$(7.118) \qquad \left\langle \tilde{\nabla}_{Z_1} Z_2, \frac{\partial}{\partial\theta} \right\rangle = \langle Z_1, JZ_2\rangle\,.$$

Hence

$$(7.119) \qquad \left\langle [Z_1, Z_2], \frac{\partial}{\partial\theta} \right\rangle = 2\langle Z_1, JZ_2\rangle\,.$$

From the above we easily find that if τ is a 1-form such that

$$\tau\left(\frac{\partial}{\partial r}\right)=\tau\left(\frac{\partial}{\partial \theta}\right)=0\,, \tag{7.120}$$

$$\nabla_{\partial/\partial r}\,\tau=0\,, \tag{7.121}$$

then at $r=1$,

$$\nabla\,\tau=\nabla^{b}\,\tau-J\tau\otimes\Theta+\Theta\otimes(L_{\partial/\partial\theta}\tau-J\tau)-\tau\otimes dr \tag{7.122}$$

Now let

$$B=B^{b}=B^{SH} \tag{7.123}$$

and

$$\nabla_{\partial/\partial r}\,B=0\,. \tag{7.124}$$

Locally we can write

$$r^{-c}B=r^{-c}\sum_i\tau_i\otimes\tau_i\,, \tag{7.125}$$

where τ_i are 1-forms satisfying (7.120), (7.121). From (7.122), we obtain at $r=1$,

$$\begin{aligned}\nabla\,(\tau\otimes\tau)\;=\;&\nabla^{b}\,(\tau\otimes\tau)\\&+[-J\tau\otimes\Theta+\Theta\otimes(L_{\partial/\partial\theta}\tau-J\tau)-\tau\otimes dr]\otimes\tau\\&+[-J\tau\otimes\tau\otimes\Theta+\Theta\otimes\tau\otimes(L_{\partial/\partial\theta}\tau-J\tau)-\tau\otimes\tau\otimes dr]\end{aligned} \tag{7.126}$$

from which we find that the component of $\nabla\,(\tau\otimes\tau)$ which involves Θ in the first two slots is

$$\Theta\wedge\left[L_{\partial/\partial\theta}\tau\otimes\tau+\tau\otimes(L_{\partial/\partial\theta}\tau-J\tau)\right]\,, \tag{7.127}$$

while the component involving dr in these slots is

$$dr\wedge(\tau\otimes\tau)\,. \tag{7.128}$$

Thus, the component involving Θ of in the first two slots, of $d^{SH}\Sigma\tau_i\otimes\tau_i$ is

$$\frac{1}{2}\Theta\wedge(L_{\partial/\partial\theta}-2\mathcal{J}^{H})\sum_i\tau_i\otimes\tau_i=\frac{1}{2}\Theta\wedge L_{\partial/\partial\theta}\sum_i\tau_i\otimes\tau_i \tag{7.129}$$

where we have used (7.114), and $B=B^{SH}$. Similarly, it follows that for B as in (7.123), (7.124), the component of $d^{SH}(r^{-c}B)$ involving Θ in the first two slots is (at $r=1$)

$$\frac{1}{2}\Theta\wedge(L_{\partial/\partial\theta}-c\mathcal{J}^{SH})B \tag{7.130}$$

From (7.113) together with (7.65) we now conclude

Proposition 7.131. *Let $B = B^b = B^{SH}$ and let $r^{-c}B$ satisfy*

$$d^{SH}(r^{-c}B) = 0\,. \tag{7.132}$$

Then

$$L_{\partial/\partial\theta}B = c\mathcal{J}^{SH}B\,. \tag{7.133}$$

Recall that we have made no assumption concerning the closedness of the orbits of the vector field $\frac{\partial}{\partial\theta}$. Nonetheless, for V a sufficiently small open set about $x \in N^{2k-1}$, we can consider the space whose points are the components $(V \cap \mathcal{O})^0$, of $(U \cap \mathcal{O})$ where $\mathcal{O}$ ranges over the orbits whose intersection with U is nonempty. This local quotient space V inherits a well defined Kähler-Einstein structure.

According to [B], p.363 the linearized deformation operator on V, when restricted to skew Hermitian deformations, can be identified with the operator $\bar\partial^*\bar\partial + \bar\partial\bar\partial^*$; compare (7.54). Let us denote the horizontal lift of this operator to N^{2k-1} by $\Box^b$ and let $\bar\partial^b, (\bar\partial^*)^b$ denote the horizontal lifts of the operators $\bar\partial, \bar\partial^*$. Then, as is standard,

$$\int_{N^{2k-1}} \langle \Box^b B, B\rangle = \int_{N^{2k-1}} \langle \bar\partial^b B, \bar\partial^b B\rangle + \langle (\bar\partial^*)^b B, (\bar\partial^*)^b B\rangle \geqq 0\,. \tag{7.134}$$

Proposition 7.135. *Let $B = (B^b)^{SH}$ and let $r^{-c}B$ be as in Proposition 7.131. Let $r^{-c}B \in ker_0\Box$ and $\widetilde{\Box}B = \mu B$. Assume that the complex dimension, k, is $\geqq 4$. Then if $c \geqq 0$, in fact*

$$c = 0\,, \tag{7.136}$$

$$L_{\partial/\partial\theta}B = 0\,, \tag{7.137}$$

$$\Box^b B = 0\,. \tag{7.138}$$

Proof. Let τ satisfy (7.120), (7.121). Then

$$\widetilde{\nabla}_{\partial/\partial\theta}\tau = L_{\partial/\partial\theta}\tau - J\tau\,, \tag{7.139}$$

$$\begin{aligned} \widetilde{\nabla}_{\partial/\partial\theta}\widetilde{\nabla}_{\partial/\partial\theta}\tau \otimes \tau &= \left(L_{\partial/\partial\theta}L_{\partial/\partial\theta}\tau - 2JL_{\partial/\partial\theta}\tau - \tau\right) \boxtimes \tau \\ &\quad + 2(L_{\partial/\partial\theta}\tau - J\tau) \otimes (L_{\partial/\partial\theta}\tau - J\tau) \\ &= L_{\partial/\partial\theta}L_{\partial/\partial\theta}(\tau \otimes \tau) - 2(\tau \otimes \tau - J\tau \otimes J\tau) \\ &\quad - 2L_{\partial/\partial\theta}(\tau \otimes \tau)(J, \cdot\,) - 2L_{\partial/\partial\theta}(\tau \otimes \tau)(\,\cdot\,, J) \end{aligned} \tag{7.140}$$

Thus,

$$\widetilde{\nabla}_{\partial/\partial\theta}\widetilde{\nabla}_{\partial/\partial\theta}B = (L_{\partial/\partial\theta}L_{\partial/\partial\theta} - 4L_{\partial/\partial\theta}\mathscr{J} - 4)B\,. \tag{7.141}$$

Similarly, using (7.122) we get

$$\widetilde{\nabla}_{e_i}\tau = \nabla^b_{e_i}\tau - J\tau(e_i) \tag{7.142}$$

$$\begin{aligned}&\widetilde{\nabla}_{e_i}\widetilde{\nabla}_{e_i}(\tau\otimes\tau)\\ &\quad= [\nabla^b_{e_i}\nabla^b_{e_i}\tau - 2J(\nabla^b_{e_i}\tau)(e_i)\Theta - J\tau(e_i)(Je_i)^*] \boxtimes \tau\\ &\quad\quad+2(\nabla^b_{e_i}\tau - J\tau(e_i)\Theta)\otimes(\nabla^b_{e_i}\tau - J\tau(e_i)\Theta)\,,\end{aligned} \tag{7.143}$$

from which it follows that

$$\sum_i \widetilde{\nabla}_{e_i}\widetilde{\nabla}_{e_i}B = \Delta^b B - 2\delta^b\mathscr{J}B \boxtimes \Theta - 2B \tag{7.144}$$

Finally, if we define R^b to be the horizontal lift of the curvature tensor on the base, by a straightforward computation,

$$\overset{\circ}{\tilde{R}}B = \overset{\circ}{R}{}^b - 3B\,. \tag{7.145}$$

If, for the moment, we grant (7.145) and use (7.141), (7.144), then from the definition of $\widetilde{\Box}$ (see (4.2), (4.7)) we get

$$\widetilde{\Box}B = \Box^b B - (L_{\partial/\partial\theta}L_{\partial/\partial\theta} - 4L_{\partial/\partial\theta}\mathscr{J})B - 2\delta^b\mathscr{J}B \boxtimes \Theta \tag{7.146}$$

However, since

$$\operatorname{div}^b \mathscr{J}B = 0\,, \tag{7.147}$$

we get

$$\widetilde{\Box}B = \Box^b B - (L_{\partial/\partial\theta}L_{\partial/\partial\theta} - 4L_{\partial/\partial\theta}\mathscr{J})B\,. \tag{7.148}$$

Also,

$$\widetilde{\Box}B = \mu B\,, \tag{7.149}$$

and with Proposition 7.131,

$$\mu = \underline{\mu} + c^2 - 4c\,, \tag{7.150}$$

where by (7.134),

$$\underline{\mu} \geqq 0\,. \tag{7.151}$$

On the other hand, it follows from (4.58) that

$$-c = \alpha \pm \sqrt{\alpha^2 + \mu} \tag{7.152}$$

where

$$\alpha = \frac{2-2k}{2}\,; \tag{7.153}$$

compare (2.47)–(2.50). From (7.152) we get

$$c^2 + 2\alpha c = \mu\,. \tag{7.154}$$

Combining this with (7.150) and using (7.151) gives

$$2(\alpha+2)c = \underline{\mu} \geqq 0\,. \tag{7.155}$$

However, $k \geqq 4$ implies

$$\alpha + 2 < 0\,, \tag{7.156}$$

and hence by (7.147),

$$c \leqq 0\,. \tag{7.157}$$

This gives (7.130)–(7.132).

We now return to the computation of $\overset{\circ}{\widetilde{R}}B$. Recall (see (4.2)) that

$$\begin{aligned} \overset{\circ}{\widetilde{R}}B(x,y) &= -\sum_{i=1}^{2k-2} B(\widetilde{R}(x,e_i)y,e_i) \\ &\quad - B\left(\widetilde{R}\left(x,\frac{\partial}{\partial\theta}\right)y,\frac{\partial}{\partial\theta}\right) \\ &= -\sum_{i=1}^{2k-2} B(\widetilde{R}(x,e_i)y,e_i) \\ &= -\sum_{i,j}\langle\widetilde{R}(x,e_i)y,e_j\rangle B(e_j,e_i) \end{aligned} \tag{7.158}$$

If $x, y \in T^bN^{2k-1}$, our claim is a direct consequence of O'Neill's formula for the curvature of riemannian submersions, together with (7.120).

If $y = \frac{\partial}{\partial\theta}$,

$$\left\langle \widetilde{R}(x,e_i)\frac{\partial}{\partial\theta},e_i\right\rangle = \left\langle R(x,e_i)\frac{\partial}{\partial\theta},e_j\right\rangle + \left\langle x\wedge e_i,\frac{\partial}{\partial\theta}\wedge e_j\right\rangle, \tag{7.159}$$

where as usual, R is the curvature tensor of $C(N^{2k-1})$. Then

$$\begin{aligned} R(\tau,e_i)\frac{\partial}{\partial\theta} &= R(x,e_i)J\left(\frac{\partial}{\partial r}\right) \\ &= J\left[R(\tau,e_i)\frac{\partial}{\partial r}\right] \\ &= 0\,. \end{aligned} \tag{7.160}$$

Also,

$$\left\langle x \wedge e_i, \frac{\partial}{\partial\theta} \wedge e_j \right\rangle = \begin{cases} 0, & x \neq \frac{\partial}{\partial\theta} \text{ or } i \neq j \\ 1, & x = \frac{\partial}{\partial\theta}, i = j \end{cases} \tag{7.161}$$

Thus,

$$\overset{\circ}{\tilde{R}} B\left(\frac{\partial}{\partial\theta}, x\right) = 0 \quad (x \in T^b N^{2k-1}) \tag{7.162}$$

$$\overset{\circ}{\tilde{R}} B\left(\frac{\partial}{\partial\theta}, \frac{\partial}{\partial\theta}\right) = \operatorname{tr} B = 0\,. \tag{7.163}$$

This completes the proof.

In the exceptional case $k = 2$, the real cross-section, N^3 of $C(N^3)$ of necessity has constant curvature. Then it follows that we are essentially reduced to the ALE case considered in [BKN]. Further details, together with the case by case discussion (compare [T]) required if $k = 3$ will be given elsewhere.

Finally, we consider homogeneous skew symmetric infintesimal deformations of complex structure which satisfy the decay condition

$$cr^{\varepsilon-2k} < |\dot{J}^{SS}| \leqq c \quad (r \geqq 1) \tag{7.164}$$

As in (7.57) it is easy to reduce our considerations to the case $\operatorname{tr} h = 0$.

By (7.63), every such deformation corresponds to the skew Hermitian part of a closed and coclosed 2-form, $\beta = -\dot{\omega}$. By Lemma 7.68 (see also Remark 7A.49), the form β is either of type 1, $r^{a^+}\phi, (\phi \in \Lambda^2)$ or type 2, $d(r^{a^+}\phi), (\phi \in \Lambda^1)$.

In the former case, since β is closed,

$$\beta = \phi \tag{7.165}$$

$$\tilde{d}\phi = \tilde{\delta}\phi = 0\,. \tag{7.166}$$

In the latter (type 2) case, we consider the decomposition $\mathscr{V} \oplus \mathscr{W}$ of the harmonic 1-forms which occurred in Lemma 7.48, Corollary 7.56. It is easy to check that when we apply d, the roles of the forms are the reverse of what they are when we apply ∇^{sym}. Thus, the conclusion is that $d(J\mathscr{V})$ is just the space of type 2 skew Hermitian 2-forms, while $d(J\mathscr{W})$ is space of type 2 Hermitian 2-forms.

Therefore, in the type 2 case, it suffices to assume $\dot{J} = \dot{J}^{SS}$ and that

$$d(JV)^*(X,Y) = g(X,\dot{J}Y)\,, \tag{7.167}$$

for some Killing field V. Then, choosing $\nabla X = \nabla Y = 0$ at $y \in C(N^{2k-1})$, we have

$$\begin{aligned} g(X,(L_V J)(Y)) &= g(X, \nabla_V JY - \nabla_{JY} V - J \nabla_V Y + J \nabla_Y X), \\ &= g(X, J \nabla_Y V - \nabla_{JY} V), \\ &= g(\nabla_Y V, X) + g(\nabla_X V, JY), \\ &= d(JV^*)(X,Y), \\ &= g(X, \dot{J} Y). \end{aligned} \tag{7.168}$$

Thus,

$$\dot{J} = L_V J . \tag{7.169}$$

Therefore, the type 2 skew symmetric infinitesimal deformations are inessential.

Since as we have already seen, the space of harmonic 2-forms of type 2 is J-invariant, by arguing as in Lemma 7.80, it follows that the same holds for the 2-forms of type 1. Thus, for such a form β, (as in (7.165), (7.166)) we have

$$\beta\left(\frac{\partial}{\partial\theta}, \cdot\right) = 0 . \tag{7.170}$$

By (7.104) together with $d\beta = 0$, as in (7.99), we get

$$L_{\partial/\partial\theta}\beta = 0 \tag{7.171}$$

as well.

Thus, for the case in which $Y = N^{2k-1}/S^1$ is a Kahler-Einstein *manifold* (necessarily of positive Ricci curvature) by a well known theorem of Bochner, [GH], Y admits no nonvanishing holomorphic p-forms. Since β is a sum of forms of type (2,0) and (0,2), this gives

$$\beta = 0 . \tag{7.172}$$

In fact, as in (7.134), we find that this holds in the general case as well.

Thus we get

Theorem 7.173. *All homogeneous skew symmetric infinitesimal complex deformations satisfying* (7.164) *are as in* (7.169), *and hence, are inessential.*

Appendix: A sharp decay estimate

Recall that in proving Lemma 7.68, we only proved a weakened version of (7.73), in which the exponent $-2k$ replaced by $2 - 2k$. Here we prove (7.73) itself.

Throughout this Appendix we assume that (7.56) and its consequences (7.63)–(7.66) hold. We also assume

$$\operatorname{tr} h = 0 . \tag{7A.1}$$

In the following lemma the underlying manifold can be an arbitrary Kähler manifold.

Lemma 7A.2.

$$\delta J \dot{\omega} = 0 . \tag{7A.3}$$

Proof: Since

$$\delta(h \circ J) = 0 , \tag{7A.4}$$

we have

$$\delta(-\dot{\omega}) = \sum_i g(e_i, \nabla_{e_i} \dot{J}) \tag{7A.5}$$

Fix a point p. Let $\nabla e_i = 0, \nabla Y = 0$ at p, where Y is a vector field and $\{e_i\}$ is a local orthonormal frame field. Then

$$\begin{aligned} \sum_i g(e_i, \nabla_{e_i} \dot{J}(Y)) &= \sum_i g(e_i, J \dot{\nabla}_{e_i} Y) - \dot{\nabla}_{e_i} JY) \\ &= -\frac{1}{2}\sum_i \{\nabla_{e_i} h(Y, Je_i) + \nabla_Y h(e_i, Je_i) - \nabla_{Je_i} h(e_i, Y)\} \\ & -\frac{1}{2}\sum_i \{\nabla_{e_i} h(JY, e_i) + \nabla_{JY} h(e_i, e_i) - \nabla_{e_i} h(e_i, JY)\} , \end{aligned} \tag{7A.6}$$

where we have used the standard formula for the variation of the riemannian connection, together with (7.107). Since $\operatorname{tr} h = 0$, the fifth term on the right hand side of (7A.6) vanishes. Also, since h is symmetric, the fourth and sixth terms cancel. Using the substitution, $e_i \to Je_i$, we see that the second term vanishes and the first and third terms are each equal to $\delta(h \circ J)$. Thus,

$$\begin{aligned} \delta(-\dot{\omega}) &= 2\delta(h \circ J) , \\ &= 2\delta(-\dot{\omega}^H) , \end{aligned} \tag{7A.7}$$

from which our claim immediately follows.

Corollary 7A.8.

$$d \Delta \dot{\omega} = 0 \tag{7A.9}$$

$$\delta \Delta \dot{\omega} = 0 \tag{7A.10}$$

$$\Delta \dot{\omega} = \Delta \dot{\omega}^{SH} \tag{7A.11}$$

Proof: Equations (7A.9), (7A.11) follow from (7.66), and (7.63) together with (7.52) (and $J\Delta = \Delta J$). Also, by (7.52), (7.170),

$$
\begin{aligned}
\delta\Delta\dot{\omega} &= -\Delta\delta\dot{\omega}\,, \\
&= 2\Delta\delta(h\circ J)\,, \\
&= 2\delta\Delta(h\circ J)\,, \\
&= 0\,. \qquad \text{(7A.12)}
\end{aligned}
$$

If we now restrict attention to the case $C(N^{2k-1})$, it follows that $\Delta\dot{\omega}$ is a sum of forms of type 2, as in (2.51), and possibly a type 1 form as in (7.170). However, as we have already observed, there are no such type 1 forms in our situation (see (7.172)).

Proposition 7A.13. *On $C(N^{2k-1})$ a skew Hermitian type 2 harmonic form,*

$$r^a d\phi + ar^{a-1}dr\wedge\phi\,, \qquad \text{(7A.14)}$$

satisfies

$$a \geqq \frac{k}{k-1}\,. \qquad \text{(7A.15)}$$

Since $|\dot{\omega}| \leqq c$ implies $|\dot{\Delta}\omega| \leqq cr^{-2}$ (see (7.55)) we immediately get

Corollary 7A.16. *On $C(N^{2k-1})$, if $|\dot{\omega}| \leqq c$ then*

$$\Delta\dot{\omega} = 0\,. \qquad \text{(7A.17)}$$

To prove Proposition 7A.13, we need the following integral formula. Let Z be a vector field on N^{2k-1} with

$$\left\langle Z, \frac{\partial}{\partial\theta}\right\rangle = 0\,. \qquad \text{(7A.18)}$$

Let $(\nabla Z)^b$ denote the restriction of the bilinear form $\langle\nabla Z,\rangle$ to the sub bundle, $T^bN^{2k-1} \subset TN^{2k-1}$ and let $\{e_i\}$ be a local orthonormal frame field for T^bN^{2k-1}.

Proposition 7A.19.

$$
\begin{aligned}
\int_{N^{2k-1}} |(\nabla Z)^b|^2 &= \int_{N^{2k-1}} \Big\{ \langle(\tilde{d}\tilde{d}^* + \tilde{d}^*\tilde{d})Z^*, Z^*\rangle \\
&\qquad - (2k-1)|Z|^2 - \left|\left[\frac{\partial}{\partial\theta}, Z\right] + JZ\right|^2 \Big\} \qquad \text{(7A.20)}
\end{aligned}
$$

Proof:

$$
\begin{aligned}
\sum_{i,j}\langle\tilde{\nabla}_{e_i}Z, e_j\rangle^2 &= |\tilde{\nabla}Z|^2 - \sum_{i,j}\langle\tilde{\nabla}_{\frac{\partial}{\partial\theta}}Z, e_i\rangle^2 \\
&\quad + \left\langle\tilde{\nabla}_{e_i}Z, \frac{\partial}{\partial\theta}\right\rangle - \left\langle\tilde{\nabla}_{\frac{\partial}{\partial\theta}}Z, \frac{\partial}{\partial\theta}\right\rangle^2 \qquad \text{(7A.21)}
\end{aligned}
$$

Since

$$\left\langle \tilde{\nabla}_{\frac{\partial}{\partial\theta}} Z, \frac{\partial}{\partial\theta} \right\rangle = 0 , \tag{7A.22}$$

by using (7.117), (7.118) and Bochner's formula, the lemma easily follows.

Proof of Proposition 7A.13. Since $dr\Lambda\Theta$ is Hermitian symmetric, it follows that

$$\phi\left(\frac{\partial}{\partial\theta}\right) = 0 . \tag{7A.23}$$

Also, from

$$J\left(\Theta \wedge \frac{\partial\phi}{\partial\theta}\right) = -adr \wedge \phi \tag{7A.24}$$

we get

$$\frac{\partial\phi}{\partial\theta} = L_{\frac{\partial}{\partial\theta}}\phi, = -aJ\phi . \tag{7A.25}$$

Since $\frac{\partial}{\partial\theta}$ is Killing, putting $\phi = Z^*$, we have

$$\left[\frac{\partial}{\partial\theta}, Z\right] = -aJZ , \tag{7A.26}$$

Now

$$(\tilde{d}\tilde{d}^* + \tilde{d}^*\tilde{d})\phi = \mu\phi \tag{7A.27}$$

and

$$a^2 - 2\alpha a - \mu = 0 , \tag{7A.28}$$

where

$$\alpha = 2 - k ; \tag{7A.29}$$

see (2.47)-(2.51). Since the expression (7A.20) is nonnegative, the proposition follows easily from (7A.26)-(7A.28) and (7A.20).

By the part of the proof of Lemma 7.68 which was completed in Section 7, in proving (7.73), we can certainly assume $|\dot{\omega}| = O(1)$. Thus, by (7.66) and (7A.17), ω is the sum of type 2 harmonic form as in (as in (7A.15)) and of type 4 harmonic forms (as in (7.75)). However, from Lemma 7.23, Lemma 7.27 it follows that for a type 2 harmonic form whose norm decays to zero at ∞, we must have $a = a^-$. Then as consequence of (7.17), the rate of decay is no slower than cr^{-2k}. So for our purposes, we can assume that the homogeneous components of $\dot{\omega}$ whose rate of decay is slower than r^{-2k}, are all of type 4 i.e. of the form $r^{a+1}dr \wedge d\phi$.

We now show that these components actually vanish. In view of the absence of type 2 harmonic forms in the above range, it follows from (7A.3) that for each type 4 component, $J(r^{a+1}dr\wedge d\phi)$ is of type 1 i.e. is as in (2.50). To complete the proof of (7.73), we must show that this additional condition implies $a^- \leqq -2k$, or equivalently,

$$\mu \geqq 4k\,. \tag{7A.30}$$

Since $dr\wedge\Theta$ is J-invariant, it follows from what has just been observed that

$$\frac{\partial\phi}{\partial\theta} = 0\,. \tag{7A.31}$$

Thus, $Z = grad\,\phi$ is tangent to T^bN^{2k-1}, and satisfies

$$\left[\frac{\partial}{\partial\theta}, Z\right] = 0\,. \tag{7A.32}$$

Unfortunately (unlike what took place earlier) (7A.20), (7A.32) do not yield (7A.30). For this, we will require an additional integral formula.

Proposition 7A.33. *Let Z be tangent to T^bN^{2k-1}. Then*

$$\int_{N^{2k-1}} \langle(\nabla Z)^b, J(\nabla Z)^b\rangle = \int_{N^{2k-1}}\left\{2k|Z|^2 + (2k-2)\left\langle\left[\frac{\partial}{\partial\theta}, Z\right], JZ\right\rangle\right\}. \tag{7A.34}$$

Proof. We have

$$\begin{aligned}\langle\nabla_{e_i} Z, e_j\rangle\langle\tilde{\nabla}_{Je_i}Z, Je_j\rangle &= e_i(\langle Z,e_j\rangle\langle\tilde{\nabla}_{Je_i}Z, Je_j\rangle) - \langle Z,e_j\rangle\langle\tilde{\nabla}_{e_i}\tilde{\nabla}_{Je_i}Z, Je_j\rangle\\ &\quad - \langle Z,e_j\rangle\langle\tilde{\nabla}_{Je_i}Z, \tilde{\nabla}_{e_i}Je_j\rangle\end{aligned} \tag{7A.35}$$

Since

$$\langle Z,e_j\rangle\langle\tilde{\nabla}_{Je_i}Z, Je_j\rangle = J(\nabla Z)^b(e_i, Z)\,, \tag{7A.36}$$

$$\sum_j\langle Z,e_j\rangle\langle\tilde{\nabla}_{Je_i}Z, Je_i\rangle = \sum_j -\langle\tilde{\nabla}_{JZ}JZ, e_i\rangle\,, \tag{7A.37}$$

it follows that the first term on the right hand side of (7A.35) is a divergence. Hence its integral vanishes.

The second term on the right hand side of (7A.35) equals

$$-\frac{1}{2}\sum_i\langle\tilde{R}(e_i, Je_i)Z, JZ\rangle - \frac{1}{2}\langle\tilde{\nabla}_{[e_i,Je_i]}Z, JZ\rangle\,. \tag{7A.38}$$

By the Jacobi identity,

$$-\frac{1}{2}\sum_i\langle\tilde{R}(e_i,Je_i)Z,JZ\rangle = \sum_i\langle\tilde{R}(Z,e_i)Je_i,JZ\rangle$$
$$= \sum_i\langle R(Z,e_i)Je_i,JZ\rangle - \langle Z,Je_i\rangle\langle e_i,Jz\rangle + \langle Z,JZ\rangle\langle e_i,Je_i\rangle$$
$$(7A.39) \qquad = \sum_i - \langle R(Z,e_i)e_i,Z\rangle + |Z|^2$$

where R denotes the curvature tensor of $C(N^{2k-1})$. Since $C(N^{2k-1})$ is a Ricci flat cone, the quantity in (7A.35) becomes

$$= \left\langle R\left(Z,\frac{\partial}{\partial\theta}\right)\frac{\partial}{\partial\theta},Z\right\rangle + |Z|^2 ,$$
$$= -\left\langle R\left(Z,\frac{\partial}{\partial\theta}\right)\frac{\partial}{\partial r},JZ\right\rangle + |Z|^2 ,$$
$$(7A.40) \qquad = |Z|^2 .$$

Also, since

$$(7A.41) \qquad [e_i,Je_i] = \widetilde{\nabla}_{e_i}Je_i - \widetilde{\nabla}_{Je_i}e_i, = -2\frac{\partial}{\partial\theta} ,$$

we get

$$-\frac{1}{2}\sum_i\langle\widetilde{\nabla}_{[e_i,Je_i]}Z,JZ\rangle = (2k-2)\langle\widetilde{\nabla}_{\partial/\partial\theta}Z,JZ\rangle ,$$
$$(7A.42) \qquad = (2k-2)\left(|Z|^2 + \left\langle\left[\frac{\partial}{\partial\theta},Z\right],JZ\right\rangle\right) .$$

Thus, the second term on the right-hand side of (7A.35) becomes

$$(7A.43) \qquad (2k-1)|Z|^2 .$$

Finally, using (7.118), we find that the third term on the right hand side of (7A.35) equals

$$(7A.44) \qquad |Z|^2 .$$

By (7A.38),(7A.41), (7A.43), (7A.44), the lemma follows.

Since the group generated by $\frac{\partial}{\partial\theta}$, acts by isometries which preserve T^bN^{2k-1} and commute with $J|T^bN^{2k-1}$, the space of vector fields, Z, tangent to T^bN^{2k-1} can be decomposed as a direct sum of subspaces, $\mathscr{Z}_\lambda$ (for a certain countable set of λ) such that $z \in \mathscr{Z}_\lambda$ if and only if

$$(7A.45) \qquad \left[\frac{\partial}{\partial\theta},Z\right] = \lambda JZ .$$

Corollary 7A.46. *Let $Z \in \mathscr{Z}_\lambda$. Then*

$$\int_{N^{2k-1}} |((\nabla Z)^b)^H|^2 = \frac{1}{2}\int_{N^{2k-1}} \langle(\tilde{d}\tilde{d}^* + \tilde{d}^*\tilde{d})Z^*, Z^*\rangle - ((4-2k)\lambda + \lambda^2)|Z|^2 \tag{7A.47}$$

$$\int_{N^{2k-1}} |((\nabla Z)^b)^{SH}|^2 = \frac{1}{2}\int_{N^{2k-1}} \langle(\tilde{d}\tilde{d}^* + \tilde{d}^*\tilde{d})Z^*, Z^*\rangle - (4k + 2k\lambda + \lambda^2)|Z|^2 \tag{7A.48}$$

Now we observe that (7A.32) and (7A.48) immediately yield (7A.30). As previously noted, this suffices to complete the proof of (7.73).

Remark 7A.49. Note that we have actually shown that $\dot{\omega}$ itself is the sum of a radially parallel harmonic form and one which decays at a rate no slower that r^{-2k}.

8. Complex integrability and the Kähler case

In this section we prove Theorem 0.15 and Theorem 0.16.

Recall that in the Kähler case, solutions of the linearized equation which arise as rescaled limits of the nonlinear equation, $\mathrm{Ric}_{g+h} \equiv 0$, satisfy the integrability conditions, (7.63) – (7.66). Also, for $k \neq 3$, it follows from Theorem 7.92, and Proposition (7.135) that the radially parallel solutions which satisfy these integrability conditions and which are annihilated by δ_t ($t \neq 0$) satisfy $B = (B^b)^{SH}$, as well as (7.137), (7.138).

As in Theorems 0.15 and 0.16, we assume that the dimension of the space of holomorphic Killing fields on $(C(N^{2k-1}), g_0)$ is 1. This implies in particular that $C(N^{n-1})$ is a *standard* complex cone, with complex base, $Y = N^{2k-1}/S^1$, which might be an orbifold.

Before proceeding further, we will recall some relevant definitions pertaining to orbifolds.

Let (Y, g^b) be a compact Kahler Einstein orbifold, $Y = N^{2k-1}/S^1$. Each point of y has a neighborhood of the form, U/Γ, where Γ is a finite group acting by biholomorphisms, and for some imbedding, $\phi : U \to C^{k-1}$, $\phi \circ \Gamma \circ \phi^{-1} \subset U(k-1)$, the unitary group of C^{k-1}. The triple, (U, Γ, ϕ) is called a *local uniformizing chart*.

Denote by $\mathrm{Sing}(Y)$ the set of singular points of Y. By definition, the Kähler metric, g^b, is a metric on $Y \setminus \mathrm{Sing}(Y)$ such that for any local uniformizing chart, (U, Γ, ϕ), the pullback extends to a smooth metric on U. Similarly one can introduce tensors of arbitrary type on Y, where in actuality, all computations are performed with equivariant objects on U.

Below, we will use some facts which, at least in the case of smooth manifolds, are well known. Their extension to the orbifold case is straightforward, given the above remarks.

The skew Hermitian solutions, B, as above, correspond to orbifold symmetric infinitesimal deformations of the complex strucure of Y; compare Section 7.

Definition 8.1. The cone $(C(N^{2k-1}), g_0)$, is *complex integrable*, if every such infinitesimal deformation is tangent to a curve of Kähler Ricci flat orbifold metrics on Y.

Proof of Theorem 0.15 By using an obvious variant of the discussion of Section 5, it is clear that to prove uniqueness of the tangent cone in the Kahler case, it suffices to verify that some tangent cone, is complex integrable. We now proceed with the verification under the assumption that the dimension of the space of holomorphic Killing fields on $(C(N^{2k-1}), g_0)$ is 1. As a consequence, the orbifold Y admits *no* holomorphic Killing field, or equivalently, since Y is Kahler Einstein, no holomorphic field whatsoever.

Let B as above determine a $T_Y^{1,0}$-valued (0,1)-form on Y as in (7.39), (7.40). We denote this form by ϕ. Thus, ϕ determines an infinitesimal orbifold deformation of complex structure.

We now observe that the obstruction space for the problem of orbifold deformation of the complex structure on Y is trivial; compare [B], p. 350.

Lemma 8.2. *The cohomology group* $H^2(Y, T_Y^{1,0})$ *vanishes.*

Proof. By the Serre duality theorem,

$$
\begin{aligned}
H^2(Y, T_Y^{1,0}) &= H^{k-2}(Y, \Omega_Y^{(k,0)}[(T_Y^{1,0})^*]) , \\
&= H^{k-2}(Y, \Omega_Y^{1,0}(K_Y)) , \qquad (8.3)
\end{aligned}
$$

where K_Y is the canonical line bundle over Y, i.e. $K_Y = \Lambda^k(T_Y^{1,0})^*$. Since the first Chern class, $c_1(Y)$ is represented by a positive (1,1)-form, by the Kodaira vanishing theorem, (see [GH])

$$
H^{k-2}(Y, \Omega_Y^{1,0}(K_Y)) = 0 . \qquad (8.4)
$$

It follows from Lemma 8.2 that for ϕ as above, there is a smooth family of integrable orbifold almost complex structures, J_t, on Y, with

$$
J_0 = J , \qquad (8.5)
$$

$$
\dot{J}_0 = \phi . \qquad (8.6)
$$

It remains to show that the Kähler-Einstein orbifold metric, g^b, on Y can be deformed in the direction, B. In other words, to find a smooth family of Kähler-Einstein orbifold metrics on (Y, J_t).

Let g_t^b be a smooth family of Kähler orbifold metrics on (Y, J_t) with Kähler form ω_t^b, such that $g_0^b = g^b$. Consider the complex Monge-Ampere equations

$$(8.7) \qquad (\omega_t^b + \partial_t^b \bar{\partial}_t^b \phi_t)^k = e^{f_t - 2k\phi_t} (\omega_t^b)^k$$

subject to the condition

$$(8.8) \qquad \omega_t^b + \partial_t^b \bar{\partial}_t^b \phi_t > 0 ,$$

where f_t is defined on (Y, J_t) by

$$(8.9) \qquad \mathrm{Ric}(g_t^b) - 2k\omega_t = \partial_t^b \bar{\partial}_t^b f_t ,$$

Here $\mathrm{Ric}(g_b^t)$ denotes the Ricci form of g_t^b, and

$$(8.10) \qquad \int_Y (e^{f_t} - 1)\omega_t^b)^k = 0 .$$

Note that $f_0 = 0$ since $g_0^b = g^b$ is Kähler-Einstein.

If (8.7), (8.8) has a solution ϕ_t, then we can produce a Kähler-Einstein orbifold metric, g_t by defining its Kähler form to be

$$(8.11) \qquad \omega_t^b + \partial_t^b \bar{\partial}_t^b \phi_t .$$

Therefore, the complex integrability is equivalent to the solvability of (8.9) for t small.

Theorem 8.12. *If Y has no nonvanishing holomorphic vector field then* (8.7), (8.8) *is solvable for t small.*

Proof. By the Implicit Function Theorem, it suffices to show that the linearization of (8.7) at $t = 0$ is invertible. Differentiating (8.7) with respect to ϕ at $t = 0$, we obtain

$$(8.13) \qquad D_\phi E(\psi)|_{(0,0)} = \Delta_{g^b}\psi - 2k\psi ,$$

where for ε small, E is the operator

$$(8.14) \qquad E : (-\varepsilon, \varepsilon) \times C_t^{2,1/2}(Y) \to C^{0,1/2}(Y) ,$$

given by

$$(8.15) \qquad E(t, \phi) = \log \frac{(\omega_t^b + \partial_t \bar{\partial}_t \phi_t)^k}{(\omega_t^b)^k} - f_t + 2k\phi .$$

Here

$$(8.16) \qquad C_t^{2,1/2}(Y) = \{\phi \in C^{2,1/2}(Y) | \omega + \partial_t \bar{\partial}_t \phi > 0\}$$

Note that $E(t, \phi) \equiv 0$ if and only if ϕ satisfies (8.7).

We claim that $D_\phi E|_{(0,0)}$ has no nontrivial elements in its kernel. In fact, if

$$(8.17) \qquad D_\phi E(\psi)|_{(0,0)} = 0\,,$$

then by the Bochner identity, $g^{i\bar{j}}\psi_{,\bar{j}}$ is a holomorphic vector field on Y, the existence of which contradicts our assumption. This suffices to complete the proof of Theorem 8.12 and hence, of Theorem 0.15 as well.

Proof of Theorem 0.16. As mentioned in the introduction, the discussion of the complex analytic compactification will be deferred to [CT]. Statement i) follows from Theorem 7.93 and Proposition 7.135, by the argument used to prove Theorem 5.78. The part of statement ii) concerning the rate of convergence of the complex structure follows similarly from Proposition 7.135 and Theorem 7.173.

Example 8.18. According to [N], [T], the Fermat hypersurfaces of degree d in CP^n, where $\frac{n}{2}+1 < d < n+1$, admit Kähler Einstein metrics with $c_1 > 0$. For $k \geqq 3$, these admit no holomorphic fields. Thus, the complex cones on these varieties provide explicit examples to which our results apply.

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