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Classification of subfactors: the reduction to commuting squares

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Introduction

In [J1] Jones introduced the notion of index of a subfactor of a type II_1 factor. This concept, as well as the results in [J1], proved to be very deep and enlightening for numerous aspects of the theory of type II_1 factors. One of the surprising results in [J1] is the characterization of the possible values of indices of subfactors (less than 4) as being the same as the set of square norms of matrices with nonnegative integer entries (up to 4), that is $4\cos^2\frac{\pi}{n}$, $n \geqq 3$.

Among the operator algebraic problems arising from Jones' work the most important are the characterization of all real values > 4 that may occur as indices of subfactors $N \subset M$ with trivial relative commutant $N' \cap M = \mathbf{C}$ and the classification of subfactors having the same index when the ambient factor M is the hyperfinite II_1 factor. One way to approach these problems is by finite dimensional approximation. This means to construct finite dimensional subalgebras $B_n \subset M$ so that $B_n \uparrow M$, $A_n = B_n \cap N \uparrow N$. In addition the algebras $A_n \subset B_n$ should satisfy the so-called commuting square condition $(*)$ $E_{B_n} E_{A_{n+1}} = E_{A_n}$, E_B denoting the trace preserving conditional expectation onto B. The subfactors in [J1] do have this approximation property and in fact the corresponding sequences $A_n \subset B_n$ are uniquely determined by an initial commuting square

$$\begin{array}{ccc} B_0 & \subset & B_1 \\ \cup & & \cup \\ A_0 & \subset & A_1, \end{array}$$

the rest of the sequence being obtained by an iterative construction.

We prove in this paper a generating result for pairs of hyperfinite factors $N \subset M$ which will show in particular that all subfactors of index $[M:N]$ less than 4 arise in this way, from an initial commuting square of finite dimensional algebras, with inclusion matrices of norm $[M:N]^{1/2}$ and having dimensions and traces determined by these matrices. This result reduces the classification of such subfactors to the classification of some finite dimensional commuting squares and this in turn is a purely combinatorial problem. To describe in greater detail our results we'll recall some of the necessary background.

It has been noted by various people (Pimsner-Popa, Jones in 1983, Ocneanu 1984) that there are some (almost) canonical finite dimensional subalgebras $B_n \subset M$, associated to $N \subset M$, that satisfy ($*$): these are the relative commutants in M of the algebras in the Jones' tunnel of subfactors $M \supset N \supset N_1 \supset N_2 \supset \dots$, i.e. $B_i = N_i' \cap M$. At each step i, N_i (and thus B_i) are unique only up to unitary conjugacy by elements from the previous step N_{i-1} (cf. [PiPo1]). And the problem was posed on whether one can choose the tunnel N_i such that $N_i' \cap M$ generate M and $N_i' \cap N$ generate N, a result that would reduce the study of $N \subset M$ to a study of finite dimensional commuting squares.

Ocneanu was the first to realize that this generating problem may have a positive solution if $N' \cap M = \mathbf{C}$ and $\sup \dim \mathscr{Z}(N_i' \cap M) < \infty$, a condition he called finite depth (in fact an equivalent form of it). Since if $N \subset M \subset M_1 \subset M_2 \subset \dots$ is the Jones' canonical tower of factors associated to $N \subset M$ then the finite dimensional algebra $N_i' \cap M$ is antiisomorphic (and thus isomorphic) to $M' \cap M_{i+1}$, the condition $\sup \dim \mathscr{Z}(N_i' \cap M) < \infty$ is the same as $\sup \dim \mathscr{Z}(M' \cap M_i) < \infty$.

We prove in this paper the following generating result, without $N' \cap M = \mathbf{C}$ being assumed.

Theorem. *Let $N \subset M$ be a pair of hyperfinite II_1 factors with finite index. Assume the Jones' tower of factors $N \subset M \subset M_1 \subset M_2 \subset \dots$ satisfies $\sup \dim Z(M' \cap M_i) < \infty$. Then there is a choice of the tunnel of subfactors $M \supset N \supset N_1 \supset N_2 \dots$ such that $N_i' \cap M \uparrow M$, $N_i' \cap N \uparrow N$. Moreover if $\tilde{M} = \overline{\bigcup_i M_i}$ then the pairs $N \subset M$ and $M_1' \cap \tilde{M} \subset M' \cap \tilde{M}$ are antiisomorphic.*

The case $N' \cap M = \mathbf{C}$ of this result has been announced at various conferences during 1987–1988 by Ocneanu ([Oc]), however without presenting a proof since then.

The above theorem reduces the classification of subfactors with the property $\sup \dim \mathscr{Z}(M' \cap M_i) < \infty$ to the classification of sequences $\{N_i' \cap N \subset N_i' \cap M\}_i$. But these sequences are uniquely determined by an initial commuting square

$$\begin{array}{ccc} N_{i_0}' \cap M & \subset & N_{i_0+1}' \cap M\,, \\ \cup & & \cup \\ N_{i_0}' \cap N & \subset & N_{i_0+1}' \cap N\,, \end{array}$$

for some i_0 large enough, with i_0, the inclusions, the dimensions and the traces determined by some integral matrices of norm $[M:N]^{1/2}$. Moreover, since $[M:N] < 4$ implies the finite depth condition, we get a complete classification of subfactors of small index in terms of commuting squares:

Theorem. *The conjugacy class of a subfactor $N \subset M$ with finite depth of the hyperfinite II_1 factor M is uniquely determined by the isomorphism class of a* *commuting square of finite dimensional algebras canonically associated to the inclusion $N \subset M$. Moreover, the subfactors of index < 4 automatically satisfy the finite depth condition, so they are all classified by their corresponding commuting squares.*

So the problem remains to classify the corresponding commuting squares. In case the number is small (for instance $< 2 + \sqrt{5}$) their number can actually be estimated. Moreover, in the case $[M:N] < 4$ the analysis of the commuting squares has to do only with the simple matrices corresponding to the Dynkin diagrams A_n, D_n, E_6, E_7, E_8, (these are all possible matrices of norm < 2). One can actually obtain:

Corollary. *For each $n \geqq 3$, there are only finitely many subfactors (up to conjugacy) of the hyperfinite II_1 factor, with index $4\cos^2\frac{\pi}{n}$.*

In [Oc], Ocneanu gave an intrinsic characterization of the sequences $\{N_i' \cap N \subset N_i' \cap M\}_i$ in case $N' \cap M = \mathbf{C}$, describing them as group-like objects and announced a full list of such objects for $[M:N] < 4$. Thus a complete list of the subfactors of index < 4 of the hyperfinite II_1 factor can actually be given.

An important application of the case $N' \cap M \neq \mathbf{C}$ of the theorem is the classification up to outer conjugacy of (not necessarily outer) actions of finite groups G on the hyperfinite II_1 factor R and more generally of the classification up to conjugacy of the subgroups isomorphic to G in Aut R/int R (i.e. Jones' theorem in [J3]): this follows by taking a partition of the unity with $|G|$ projections p_g of equal trace in M, $R = p_e M p_e$ and by taking $N = \{\oplus_{g \in G} g(x) | x \in R = p_e M p_e\}$, where $g \in \mathrm{Aut}\, R$ is a lifting of $G \subset \mathrm{Aut}\, R/\mathrm{Int}\, R$. This application will be discussed in a more general setting in a sequel to this paper.

The proof of the theorem is based on the key observation that if $\sup \dim \mathscr{Z}(N_i' \cap M) < \infty$ then the algebras $B = N_i \vee (N_i' \cap M)$ satisfy $E_B(x) \geqq cx$, $x \in M_+$, with a constant $c > 0$ independent on i (see 3.8, 4.3). Since N_i are all hyperfinite (cf. [C]), $N_i \vee (N_i' \cap M)$ can be approximated by relative commutants $N_k' \cap M$, $k > i$ (see 4.4). Thus, by using the inequality one can choose recursively the tunnel N_i such that $R = \overline{\bigcup_i (N_i' \cap M)}$ satisfies $E_R(x) \geqq cx$, $x \in M_+$. By [PiPo1] this means R has finite index in M, $[M:R] \leqq c^{-1}$. The inequality also shows that given $x \in M_+$ and a choice of N_i up to j, there is a choice of N_i, $j < i \leqq k$, so that $N_k' \cap M$ contains a certain "percentage" of x. If this would be true for arbitrary x then the resulting R would equal M. If not, there would be $x \in M$ and j so that $x \perp uRu^*$ for all unitary elements u in N_j (4.6). But by the Rohlin type lemma in [Po4], used here in a form we call the linearization principle (5.1), one has span $uRu^* = \mathrm{span}\, N_j R N_j = M$. Thus $x = 0$ and the theorem is proved (4.9). Then Section 6 contains applications of the main theorem to the classification of subfactors.

The uniform bound from below $E_{N_i \vee (M_i' \cap M)}(x) \geqq cx$, $x \in M_+$, $i \geqq 0$, can be deduced from [Po3]. We included a complete and different proof here for the reader's convenience. The proof is a consequence of the uniform bound for local indices obtained in Section 3, which in turn follows by the general results on sequences of commuting squares in Section 2. Part of the algebraic considerations in Section 3 are of a similar type to some work in [W], [GHJ].

The proof of the main theorem in this paper was presented at the Annapolis Conference on "Knots and Operator Algebras" May 1988 and at the A.M.S. Summer Institute on "Operator Theory and Operator Algebras" August 1988.

1. Preliminaries

1.1. Jones' basic construction and the tower of factors. Let M be a finite von Neumann algebra with a normal finite faithful trace τ, $\tau(1) = 1$. Denote $\|x\|_2 = \tau(x^*x)^{1/2}$, for $x \in M$, and $L^2(M, \tau)$ the completion of M in the norm $\| \ \|_2$.

If $B \subset M$ is a von Neumann subalgebra of M then we denote by E_B the unique τ-preserving conditional expectation of M onto B. Also, we denote by e_B the orthogonal projection of $L^2(M, \tau)$ onto $L^2(B, \tau) \subset L^2(M, \tau)$. E_B, the projection e_B and the canonical conjugation J on $L^2(M, \tau)$, defined by $Jx = x^*$, are related by the properties:

1.1.1 (i) If $x \in M$ then $x \in B$ iff $[x, e_B] = 0$.
(ii) $e_B x e_B = E_B(x) e_B$, $x \in M$.
(iii) $[J, e_B] = 0$.

If M is a type II_1 factor and $N = B \subset M$ is a subfactor then the Jones' index of N in M, $[M:N]$, is the Murray and von Neumann coupling constant of N in its representation by left multiplication on $L^2(M, \tau)$. Thus, the finiteness of $[M:N]$ means that N' (or equivalently $JN'J$) is finite, and its value can then be recuperated as $\tau(e_N)^{-1}$, τ being the (unique) normalized trace on $JN'J$. The construction of the finite factor $JN'J$ is in fact a major tool in studying subfactors: it is called the Jones' *basic construction.* We list now some of its properties:

1.1.2. Assume $[M:N] < \infty$. Define $M_1 = JN'J$. Then M_1 is a type II_1 factor and we have:

(i) $M_1 = (M \cup \{e_N\})'' = \mathrm{span}\{x e_N y, x, y \in M\}$.
(ii) $[M_1 : M] = [M:N]$.
(iii) $\tau(e_N x) = [M:N]^{-1} \tau(x)$, $x \in M$, τ being the unique trace on M_1 with $\tau(1) = 1$.

Thus, starting from the initial inclusion $N \subset M$, the basic construction builds a new inclusion $M \subset M_1$, with the same index, with a projection $e_1 = e_N$ of trace $[\mathrm{M:N}]^{-1}$ implementing the conditional expectation (like in 1.1.1). We call e_1 the Jones' projection corresponding to $N \subset M$ and write the basic construction $M \subset^{e_1} M_1$.

We mention that in fact M_1 can be described abstractly: it is the unique II_1 factor that contains M and a projection e_1 so that M and e_1 generate M_1 and so that $e_1 x e_1 = E_N(x) e_1$, $x \in M$ (cf. [PiPo2]).

By iterating this construction we get a whole *tower of factors* $N \subset M \subset M_1 \subset M_2 \subset, \ldots,$ with projections $e_1, e_2, \ldots,$ so that $M_i \subset^{e_{i+1}} M_{i+1}$ is the basic construction for $M_{i-1} \subset M_i$. They satisfy:

1.1.3. (i) $e_{i+1}xe_{i+1} = E_{M_{i-1}}(x)e_{i+1}$, in particular $e_{i+1}e_ie_{i+1} = [M:N]^{-1}e_{i+1}$. Also, $e_ie_{i+1}e_i = [M:N]^{-1}e_i$.

(ii) $e_{i+1} \in M'_{i-1} \cap M_{i+1}$. In particular $e_ie_j = e_je_i$, for $|i-j| \geqq 2$.

(iii) $\tau(e_{i+1}x) = [M:N]^{-1}\tau(x)$, $x \in M_i$. In particular $\tau(e_{i+1}w) = [M:N]^{-1}\tau(w)$, for $w \in \mathrm{Alg}\{1, e_1, e_2, \ldots, e_i\}$.

1.2. The finite dimensional case. If $N \subset M$ are finite dimensional then we can still define $M_1 = JN'J$, which will be finite dimensional as well and will satisfy properties similar to 1.1:

1.2.1. $M_1 = (M \cup \{e_i\})'' = \mathrm{span}\{xe_Ny \mid x, y \in M\}$.

Moreover if $\tilde{M}$ is any algebra containing a copy of M and a nonzero projection e satisfying:

(a) $[N, e] = 0$;
(b) $x \in N$, $xe = 0$ implies $x = 0$;
(c) $exe = E_N(x)e$

then $\Sigma xe_Ny \mapsto \Sigma xey$ defines an isomorphism of M_1 onto the algebra $\mathrm{span}\{xey \mid x, y \in M \subset \tilde{M}\}$. This observation gives an abstract characterization of M_1 (see. e.g. [W]). We call $M \subset M_1$ with the projection e_1 the *algebraic basic construction* for $N \subset M$ and E_N.

By the definition of M_1 there is a natural isomorphism $z \mapsto JzJ$ between the centers of N and M_1. It can be alternatively described by: "JzJ is the unique element $z_1 \in \mathscr{Z}(M_1)$ satisfying $z_1e_N = ze_N$". This gives a natural identification between the sets of simple summands of N and M_1. If we denote by K this common set and by $A = (a_{kl})_{\substack{k\in K\\ l\in L}}$ the inclusion matrix of $N \subset M$ then the inclusion matrix of $M \subset M_1$ is A^t.

The case when M_1 has a trace which extends the trace of M and satisfies a property similar to 1.1. was clarified in [J1]:

1.2.2. Let $(s_k)_{k\in K}$ be the traces of the minimal projections of N and $(t_l)_{l\in L}$ those of M. Then the following conditions are equivalent.

1°. There is a trace τ_1 on M_1 such that $\tau_1|_M = \tau$ and $\tau_1(xe_N) = \tau_1(x)\tau_1(e_N)$, $x \in M$.

2°. $AA^t(s_k) = \lambda^{-1}(s_k)$ for some $\lambda > 0$.

3°. $A^tA(t_l) = \lambda^{-1}(t_l)$ for some $\lambda > 0$.

Moreover if the conditions hold true then the scalars λ of 2° and 3° satisfy $\lambda = \tau_1(e_N) = \|A\|^{-2}$, all irreducible components A_0 of A satisfy $\|A_0\| = \|A\|$ and (s_k) (resp. (t_l)) is a Perron-Frobenius eigenvector for AA^t (resp. A^tA).

Property 1° for the trace on M_1 is called the $\lambda(= \tau_1(e_N))$ Markov property. If the above conditions are satisfied then one can iterate the basic construction getting at each step the appropriate extension of traces satisfying the Markov property.

If $M \subset M_1$ satisfies 1.2.2 we simply call it the *basic construction* and write $M \subset^{e_N} M_1$.

1.3. The tunnel of subfactors. Jones also showed that given an inclusion of factors with finite index $N \subset M$, there is a subfactor $N_1 \subset N$ and a projection $e_0 \in M$ so that $N \subset^{e_0} M$ is the basic construction for $N_1 \subset N$. This fact is made more precise by the results in [PiPo1]. There is a unique (up to conjugacy by unitary elements $u \in N$) projection $e_0 \in M$ such that $E_N(e_0) = [M:N]^{-1}1$. And if $N_1 = \{e_0\}' \cap N$ then N_1 is a subfactor in N with the property that $N \subset^{e_0} M$ is the basic construction for $N_1 \subset N$ (i.e. with e_0 satisfying $e_0 x e_0 = E_{N_1}(x) e_0$, $x \in N$). In particular N_1 is unique up to conjugacy by unitary elements in N.

As before, we may iterate this construction k times and get a *tunnel of subfactors* up to k, $M \supset^{e_0} N \supset^{e_{-1}} N_1 \supset^{e_{-2}} N_2 \supset^{e_{-3}} \dots \supset^{e_{-k+1}} N_{k-1} \supset N_k$, with $[N_i:N_{i+1}] = [M:N]$, $e_{-i} x e_{-i} = E_{N_{i+1}}(x) e_{-i}$, $x \in N_i$, and $\tau(e_{-i}) = [M:N]^{-1}$, $0 \leqq i \leqq k-1$.

Note however that while the tower $N \subset M \subset^{e_1} M_1 \subset^{e_2} M_2 \subset$ (with the corresponding projections) is canonical and unique, the tunnel of subfactors

$$M \supset^{e_0} N \supset^{e_{-1}} N_1 \supset^{e_{-2}} N_2 \supset \dots$$

is unique only up to conjugacy by unitary elements, i.e. if $N_1, N_2, \dots, N_k$ and $e_0, e_{-1}, \dots, e_{-k+1}$ are chosen then $e_{-k} \in N_{k-1}$ and $N_{k+1} \subset N_k$ are unique up to conjugacy by a unitary element in N_k.

1.4. Other characterizations of $[M:N]$. There are two alternative descriptions of $[M:N]$ that have been given in [PiPo1]: $[M:N]^{-1}$ is the best constant λ for which the *basic inequality*, $E_N(x) \geqq \lambda x$, holds true for all $x \in M_+$. More precisely, if

$$\lambda(M, N) = \max\{\lambda \geqq 0 \mid E_N(x) \geqq \lambda x, x \in M_+\}$$

then $\lambda(M, N) = [M:N]^{-1}$. In addition, we saw in 1.3 that there is a projection e (unique up to unitary conjugacy by elements in N) such that $E_N(e) = \lambda(M, N)$. The basic inequality then shows that $\lambda(M, N)$ is the minimal scalar that can be the expected value on N of a nonzero projection in M (see more on this in [Po3]).

Moreover, the module dimension definition of $[M:N]$ can be made more precise: it was shown in [PiPo1] that there exists $\{m_i\}_i \subset M$, a finite set, such that $E_N(m_i^* m_j) = \delta_{ij} f_j$, with $f_j \in \mathscr{P}(N)$, $M = \sum_i m_i N$. A set $\{m_i\}_i$ like this is called an *orthonormal basis* of M over N. Any such set satisfies $\sum_i m_i m_i^* = [M:N] \cdot 1$. The projections f_j can be chosen to be all but possibly one equal to the identity.

Such an orthonormal basis obviously exists for finite dimensional pairs $N \subset M$ (and E_N) as well. It can be used to prove the abstract characterization of the basic construction in 1.2. Moreover if the conditions in 1.2.2 are satisfied then $\Sigma m_i m_i^* = \lambda^{-1}$ (λ as in 1.2.2).

2. Sequences of commuting squares

If M is isomorphic to the hyperfinite II_1 factor then a natural way to approach index problems for subfactors $N \subset M$ is by approximating the pair $N \subset M$ with pairs of finite dimensional algebras $B_n \subset M$ such that $B_n \uparrow M$, $B_n \cap N \uparrow N$ (see

[PiPo1, 3]). The natural condition to be imposed on the algebras B_n is then:

$$E_{B_n}E_N = E_{B_n \cap N} \quad (*)$$

This is called the *commuting square condition* ([Po1], [PiPo1], [Po2]). Note that if $B \subset M$ satisfies (*) then $\max\{\lambda \geqq 0 | E_{B \cap N}(x) \geqq \lambda x, x \in B_+\} \geqq [M:N]^{-1}$ (cf. [Po2]).

We will prove in this section some general properties for sequences of finite dimensional subalgebras satisfying the commuting square conditions with the Jones tower (or tunnel) of factors. We first consider two consecutive commuting squares.

2.1 Proposition. *Let $N \subset M \subset^{e_1} M_1 \subset^{e_2} M_2$ be two steps of the tower. Let $B_0 \subset M$, $B_1 \subset M_1$, $B_2 \subset M_2$ be finite dimensional subalgebras satisfying $E_{B_i}E_{M_{i-1}} = E_{B_{i-1}}$, $i = 1, 2$. Suppose $e_2 \in B_2$ and denote by p_2 the central support of e_2 in B_2. Then we have:*

1°. *$B_1 \ni x \mapsto xp_2 \in B_1 p_2$ is an isomorphism;*

2°. *$e_2B_2e_2 = B_0e_2$ and $B_2p_2 = \mathrm{Alg}(B_1p_2 \cup \{e_2\}) = span\ B_1e_2B_1$;*

3°. *$B_1 \simeq B_1p_2 \subset^{e_2} B_2p_2$ is the algebraic basic construction corresponding to $B_0 \subset B_1$;*

4°. *If we identify the sets of simple summands of B_0 and B_2p_2 via the equality $B_0e_2 = e_2B_2e_2$ (see 1.2.1) then the inclusion matrix for $B_1 \subsetneq B_2p_2$ is the transpose of the inclusion matrix for $B_0 \subset B_1$;*

5°. *If $(s_k)_k$ is the vector giving the traces of the minimal projections of B_0 then the traces of the minimal projections of B_2p_2 are given by $(\lambda s_k)_k$.*

Proof. Let $q_0 = \vee\{ue_2u^* | u \in \mathcal{U}(B_1)\}$. Note that $e_2 \leqq q_0 \leqq p_2$, $q_0 \in B_1' \cap B_2$ and that $B_1 \ni x \mapsto xq_0 \in B_1q_0$ in an isomorphism. Thus, since $e_2xe_2 = E_{B_0}(x)e_2$, $x \in B_1$, it follows that if we denote $B_2^- = \mathrm{span}\, B_1e_2B_1$ then the inclusion $B_1q_0 \subset^{e_2} B_2^-(= B_2^- q_0)$ is the algebraic basic construction for $B_0 \subset B_1$ (cf. 1.2). In particular the inclusion matrix for $B_1q_0 \subset B_2^-$ is the transpose of the inclusion matrix for $B_0 \subset B_1$.

We now show that $q_0 = p_2$ and that $B_2^- = B_2p_2$ and this will end the proof.

To do this we first show that $p_2 = q_0$. We have $q_0 = p \vee \{l(xe_2) | x \in B_1\}$ and $p_2 = \vee\{l(ye_2) | y \in B_2\}$, where $l(z)$ denotes the left support of an element z. But by [PiPo1] and by the commuting square condition, if $x = \lambda^{-1}E_{B_1}(ye_2)$ for some $y \in B_2$ then $x = \lambda^{-1}E_{M_1}(ye_2)$ and $xe_2 = ye_2$. Thus $l(xe_2) = l(ye_2)$ and $q_0 = p_2$. To prove that $B_2^-(= \mathrm{span}\ B_1e_2B_1)$ equals B_2p_2 it suffices to show that $e_2B_2e_2 = B_0e_2$. Indeed, since $p_2 = q_0 = \sum_i m_ie_2m_i^*$ for some $m_i \in B_1$, with $E_{B_0}(m_i^*m_j) = \delta_{ij}f_j, f_j \in \mathcal{P}(B_0)$ it would then follow that

$$B_2p_2 = p_2B_2p_2 = q_0B_2q_0 \subset \sum_{i,j} m_ie_2B_2e_2m_j^*$$

$$= \sum_{i,j} m_iB_0e_2m_j^* \subset \mathrm{span}\, B_1e_2B_1 \subset B_2p_2\,.$$

But $e_2B_2e_2 \subset e_2M_2e_2 = Me_2$, so that $e_2B_2e_2 = Xe_2$ for some $X \subset M$. Thus $X = \lambda^{-1}E_{M_1}(Xe_2) = \lambda^{-1}E_{M_1}(e_2B_2e_2) \subset B_1$, so that $X \subset B_1 \cap M = B_0$. This shows that $e_2B_2e_2 \subset B_0e_2 \subset e_2B_2e_2$, ending the proof of the proposition. Q.E.D.

From 2.1 we can now deduce the following property for a sequence of commuting squares:

2.2 Corollary. *Let $N \subset M \subset^{e_1} M_1 \subset^{e_2} M_2 \subset \ldots$ be the Jones' tower of factors and $B_i \subset M_i$, $i \geqq 0$, be finite dimensional subalgebras satisfying $E_{B_i} E_{M_{i-1}} = E_{B_{i-1}}$, $i \geqq 1$ and $e_i \in B_i$, for all $2 \leqq i < n$ (where $2 < n \leqq \infty$). Let K_i (respectively L_i) denote the sets of simple summands of B_{2i} (respectively B_{2i+1}), $n > i \geqq 0$, and identify K_i (resp. L_i) with a subset of K_{i+1} (resp. L_{i+1}) by using 2.1. Denote $K = \bigcup_i K_i$, $L = \bigcup_i L_i$. There exists a unique matrix of nonnegative integers $A = (a_{kl})_{k \in K, l \in L}$ with the properties.*

1°. *$(a_{kl})_{k \in K_i, l \in L_i}$ describes the inclusion $B_{2i} \subset B_{2i+1}$, for $i \geqq 0$, $2i + 1 < n$;*

2°. *$(a_{kl})^t_{k \in K_{i+1}, l \in L_i}$ describes $B_{2i+1} \subset B_{2i+2}$, $i \geqq 0$, $2i + 2 < n$;*

3°. *K_i, L_i satisfy $L_i = \{l \in L \mid \exists k \in K_i \text{ with } a_{kl} \neq 0\}$ and $K_{i+1} = \{k \in K \mid \exists l \in L_i \text{ with } a_{kl} \neq 0\}$.*

4°. *$\| A \|^2 \leqq [M:N]$ and A is irreducible iff $(a_{kl})_{k \in K_0, l \in L_0}$ is.*

Proof. The construction of K_i, L_i, K, L and $(a_{kl})_{kl}$, with 1°, 2°, follows by induction from 2.1. To prove 3°, let p', p'', q' be minimal central projections of B_{i+1}, B_{i+2}, B_{i-1} respectively, with $e_{i+1}p' = e_{i+1}q'$ and $p'p'' \neq 0$. We must show there exists some minimal central projection $q'' \in B_i$ such that $e_{i+2}p'' = e_{i+2}q''$ and $q''q' \neq 0$. Define $q'' = \lambda^{-1} E_{B_{i+1}}(p''e_{i+2})$. Since $e_{i+2}B_{i+2}e_{i+2} = B_i e_{i+2}$, q'' is a minimal central projection in B_i. Also $E_{B_{i+1}}(p'')p' = \alpha p'$ for some $\alpha \neq 0$. Thus $q'q'' = \lambda^{-1} q' E_{B_{i+1}}(p''e_{i+2})$ so that

$$\begin{aligned} p'e_{i+1}q'q''e_{i+1}p' &= \lambda^{-1} p'q' E_{B_{i+1}}(e_{i+1}p''e_{i+2}e_{i+1}) = p'q' E_{B_{i+1}}(e_{i+1}p'') \\ &= p'(q'e_{i+1}) E_{B_{i+1}}(p'') = \alpha p' e_{i+1} \neq 0 \,. \end{aligned}$$

Thus $q'q'' \neq 0$.

The irreducibility condition for A follows by 3°. Moreover by 2.9 in [PiPo3] we have $\| (a_{kl})_{k \in K_n, l \in L_n} \|^2 \leqq [M:N]$ for all n, so that $\| A \|^2 = \lim_n \| (a_{kl})_{k \in K_n, l \in L_n} \|^2 \leqq [M:N]$. Q.E.D.

2.3 Corollary. *If $\{B_i\}_{n > i \geqq 0}$ are finite dimensional subalgebras ($n \leqq \infty$) with $B_i \subset M_i$, for $i \geqq 0$, $E_{B_i} E_{M_{i-1}} = E_{B_{i-1}}$, for $i \geqq 1$, $e_i \in B_i$, for $i \geqq 2$, and if there exists $i_0 \geqq 0$ such that $\dim \mathscr{Z}(B_{i_0+2}) \leqq \dim \mathscr{Z}(B_{i_0})$, then $\{B_i\}_{n > i \geqq i_0}$ with the projections $e_i \in B_i$, $i \geqq i_0 + 2$, are obtained by iterating the basic construction starting from the inclusion $B_{i_0} \subset B_{i_0+1}$ and at each step the unique trace τ of $\cup M_i$ is a $\lambda = [M:N]^{-1}$-Markov trace. Thus, if $A = (a_{kl})_{kl}$ is the inclusion matrix of $B_{i_0} \subset B_{i_0+1}$ and $(s_k)_k$ are the traces of the minimal projections of B_{i_0} and $(t_l)_l$ those of B_{i_0+1} then*

1°. *The inclusions $B_{i_0} \subset B_{i_0+1} \subset B_{i_0+2} \ldots$ are given by A, A^t, $A, \ldots$*

2°. *The traces of minimal projections of B_{i_0+2j} are given by $(\lambda^j s_k)_k$ and those of B_{i_0+2j+1} by $(\lambda^j t_l)_l$.*

3°. *$\| A \|^2 = [M:N] = \lambda^{-1}$ and $\| A_0 \|^2 = [M:N]$ for any irreducible component A_0 of A.*

4°. *$(s_k)_k$ is a Perron-Frobenius eigenvector for AA^t and $(t_l)_l$ a Perron-Frobenius eigenvector for A^tA.*

Proof. Trivial by 2.1, 2.2, 1.2. Q.E.D.

As 2.3 shows, the existence of a sequence of commuting squares satisfying the hypothesis of 2.3 imposes restrictions on the index of N in M: we must have $[M:N] = \|A\|^2$. We now deduce that we "almost must have" $N' \cap M = C$.

2.4 Corollary. *Assume there exists a sequence of finite dimensional subalgebras $B_i \subset M_i$, $i \geqq 0$, satisfying the hypothesis of 2.3. Then $E_{N'\cap M}(e_0) = [M:N]^{-1}\,1$ for all Jones projections $e_0 \in M$ (i.e., projections e_0 for which $E_N(e_0) = [M:N]^{-1}1$).*

Proof. Let $B_{i_0} = \oplus_k M_{n_k \times n_k}(\mathbf{C})$. Then by [PiPo1] and 2.3 (see also [PiPo3]) we have

$$H(B_{i_0+2j+1}|B_{i_0+2j}) = \sum_{k,l} (a_{kl}(AA^t)^j\vec{n})_k t_l \lambda^j \ln \frac{(A^t(AA^t)^j\vec{n})_l(A\lambda^j\vec{t})_k}{((AA^t)^j\vec{n})_k(\lambda^j t_l)}\,.$$

Since $(AA^t)^j\vec{n}$ tends (up to renormalization) to the Perron-Frobenius eigenvector of AA^t, i.e. to a scalar multiple of $A\vec{t}$, it follows that

$$\lim_{j\to\infty} \frac{(A^t(AA^t)^j\vec{n})_l(A\vec{t})_k}{((AA^t)^j\vec{n})_k t_l} = \lambda^{-1}\,.$$

Thus, since $\sum_{k,l}(a_{kl}(AA^t)^j\vec{n})_k t_l \lambda^j = 1$, it follows that $\lim_{j\to\infty} H(B_{i_0+2j+1}|B_{i_0+2j}) = \ln\lambda^{-1}$. Since $M_{i_0} \subset M_{i_0+1}$ is isomorphic to a reduction of the pair $M_{i_0+2j} \subset M_{i_0+2j+1}$ by a projection in M_{i_0+2j} (cf. [PiPo1]) it follows that $H(M_{i_0+2j+1}|M_{i_0+2j}) = H(M_{i_0+1}|M_{i_0})$ for all $j \geqq 0$ (cf. e.g. 4.4 in [PiPo1]). Thus from the commuting square condition we get

$$H(B_{i_0+2j+1}|B_{i_0+2j}) \leqq H(M_{i_0+2j+1}|M_{i_0+2j}) = H(M_{i_0+1}|M_{i_0}) \leqq \ln\lambda^{-1}$$

which shows that $H(M_{i_0+1}|M_{i_0}) = \ln\lambda^{-1}$. By [PiPo2] this implies $H(M|N) = \ln\lambda^{-1}$ and $E_{N'\cap M}(e_0) = [M:N]^{-1}1$. Q.E.D.

If B_i, $i \geqq 0$, satisfy the conditions of 2.2, then $e_i \in B_i$ for all $i \geqq 2$ implies that $\overline{\bigcup_i B_i^w} \subset \overline{\bigcup_i M_i^w} = \tilde{M}$ is a type II_1 von Neumann algebra (since $\{e_i\}_i''$ is a type II_1 factor). In fact $\overline{\bigcup_i B_i^w}$ follows to be a factor under very general assumptions. We will prove this type of results elsewhere. For what we need the next easy consequence of 2.3 will do.

2.5 Corollary. *Let $\{B_i\}_{i\geqq 0}$ be like in 2.3 (with $n = \infty$), and assume the inclusion matrix of $B_0 \subset B_1$ is irreducible. Then $R = \overline{\bigcup_i B_i^w}$ is a type II_1 subfactor in $\overline{\bigcup_i M_i^w} = \tilde{M}$.*

Proof. Let $e^{k,i}, f^{l,i}$ be the minimal central projections of B_{2i} respectively B_{2i+1}. Arguing like in 2.4 (see also [PiPo3]) it follows that 1°. $\lim_{i\to\infty}\tau(e^{k,i}) = v_k^2$; 2°. $\lim_{i\to\infty}\tau(f^{l,i}) = w_l^2$; and 3°. $\lim_{i\to\infty}\tau(e^{k,i}f^{l,i}) = \lambda^{1/2}a_{kl}v_k w_l$, where $(v_k)_k, (w_l)_l$ are the Perron-Frobenius eigenvectors of AA^t, respectively A^tA normalized so that $\sum_k v_k^2 = 1$, $\sum_l w_l^2 = 1$, and $\lambda = \|A\|^{-2}$. Let $p \in \mathscr{P}(\mathscr{Z}(R))$ and $p_n \in \mathscr{P}(\mathscr{Z}(B_n))$ such that $\|p_n - p\|_2 \to 0$. In particular we have: 4°. $\|p_{2n}p_{2n+1} - p_{2n}\|_2 \to 0$. Let $E_n \subset K$, $F_n \subset L$ with $p_{2n} = \sum_{k\in E_n} e^{k,n}$, $p_{2n+1} = \sum_{l\in F_n} f^{l,n}$. Then 3°, 4° show that, for

n large enough, if $e^{k,n}f^{l,n} \neq 0$ (or equivalently $a_{kl} \neq 0$) for some $k \in E_n$, $l \in L$, then $l \in F_n$. Thus

$$F_n = \{l \in L \mid a_{kl} \neq 0 \text{ for some } k \in E_n\}$$

and $p_{2n+1} \geqq p_{2n}$, so in fact $p_{2n+1} = p_{2n}$ again by 3°, 4°. But this contradicts the irreducibility of A, unless $E_n = K$, $F_n = L$, $p_{2n} = 1$. Q.E.D.

We will refer to the property $\sup \dim \mathscr{Z}(B_n) < \infty$ for an increasing sequence of finite dimensional algebras $\{B_n\}_n$ by saying that it has *bounded* (or *finite*) *growth*.

3. Higher relative commutants and finite depth

Approximating a pair of hyperfinite factors $N \subset M$ with finite dimensional subalgebras satisfying the commuting square condition (*) is in fact quite difficult. However, there is a simple way of producing finite dimensional subalgebras of M satisfying (*) by taking the relative commutants in M of some subfactors $\tilde{N}$ of N. Indeed by [Po1] we then have $E_{\tilde{N}' \cap M} E_N = E_{\tilde{N}' \cap N}$. Also, if $[N : \tilde{N}] < \infty$ then $\tilde{N}' \cap M$ is finite dimensional. A possible choice for $\tilde{N}$ are the factors in the tunnel.

3.1 Proposition. *Let* $M \supset^{e_0} N \supset^{e_{-1}} N_1 \supset^{e_{-2}} N_2 \supset \dots$ (*respectively* $M \subset^{e_1} M_1 \subset^{e_2} M_2 \subset \dots$) *be a choice of the tunnel of factors* (*resp. the Jones tower*). *Then we have*

1°. $N_j' \cap M$, $M' \cap M_j$ *are finite dimensional.*

2°. $E_{N_j' \cap M} E_{N_i} = E_{N_j' \cap N_i}$, $E_{M_k' \cap M_j} E_{M_i} = E_{M_k' \cap M_i}$, *for* $k \leqq i \leqq j$.

3°. $e_0, e_{-1}, \dots, e_{-i+1} \in N_i' \cap M$, $e_2, e_3, \dots, e_j \in M' \cap M_j$.

Proof. By [J1] $N_i' \cap M_i$ are finite dimensional and by [Po1] we have the corresponding commuting squares. Q.E.D.

Although the tunnel of factors N_k is not uniquely determined (see 1.3) the resulting pair of algebras $\overline{\bigcup_i (N_i' \cap N)} \subset \overline{\bigcup_i (N_i' \cap M)}$ is actually unique up to isomorphism.

3.2 Proposition. 1°. *Let* $M \supset^{e_0} N \supset^{e_{-1}} N_1 \supset^{e_{-2}} N_2 \supset \dots$, $M \supset^{e_0^0} N \supset^{e_{-1}^0} N_1^0 \supset^{e_{-2}^0} N_2^0 \supset \dots$, *be two choices of the tunnel. Denote* $R = \overline{\cup (N_j' \cap M)}^w$, $R_0 = \overline{\bigcup_j (N_j' \cap N)}^w \subset R$, $R^0 = \overline{\bigcup_j (N_j^{0\prime} \cap M)}^w$, $R_0^0 = \overline{\bigcup_j (N_j^{0\prime} \cap N)}^w \subset R^0$. *Then there exists a trace presenting isomorphism of* R *onto* R^0 *carrying* R_0 *onto* R_0^0 *and* $N_j' \cap M$ *onto* $M' \cap M_{j+1, j \geqq 0}$.

2°. *Let* $N \subset M \subset^{e_1} M_1 \subset^{e_2} M_2 \subset \dots$ *be the tower of factors and* $\tilde{M} = \overline{\bigcup_n M_n}$, $R^1 = M' \cap \tilde{M}$, $R_0^1 = M_1' \cap \tilde{M}$. *Then there exists an antiisomorphism* $\Phi : \bigcup_j (N_j' \cap M) \to \bigcup_j (M' \cap M_{j+1})$ *with* $\Phi(N_j' \cap M) = M' \cap M_{j+1}$, $\Phi(N_j' \cap N) = M_1' \cap M_{j+1}$, $\Phi(e_{-j}) = e_{j+2}$. *Moreover if* $N \subset M$ *satisfies* $E_{N' \cap M}(e_0) = [M:N]^{-1} 1$ *then* Φ *is trace preserving and can be uniquely extended to a trace preserving antiisomorphism of* R *onto* R^1 *carrying* R_0 *onto* R_0^1.

Proof. 1°. By [PiPo1] (see 1.3) it follows that there exist unitary elements $u_i \in N_i^0$, $i \geqq 0$, such that $u_{i-1} \dots u_0 N_i u_0^* \dots u_{i-1}^* = N_i^0$, $i \geqq 1$. Define $\sigma : R \to R^0$ by $\sigma(x) = \lim_{i \to \infty} u_{i-1} \dots u_0 x u_0^* \dots u_{i-1}^*$. If $x \in N_i' \cap M$, then

$u_{i-1} \ldots u_0 x u_0^* \ldots u_{i-1}^* \in N_i^{0\prime} \cap M$ so that

$$u_n \ldots u_i(u_{i-1} \ldots u_0 x u_0^* \ldots u_{i-1}^*)u_i^* \ldots u_n^* = u_{i-1} \ldots u_0 x u_0^* \ldots u_{i-1}^* ,$$

for all $n > i - 1$. This shows that σ is a well defined trace preserving isomorphism. Obviously $\sigma(R_0) = R_0^0$.

2°. By [PiPo2] one can represent recursively each M_n on $L^2(M, \tau)$. Thus $\cup M_n \subset \mathscr{B}(L^2(M, \tau))$ and $N_i = JM'_{i+1}J$ will form a tunnel of subfactors in M. Since $JM'J = M$ we also have $J(M' \cap (\bigcup_i M_i))J = \bigcup_i (N'_i \cap M)$. Thus the reflection $\Phi(x) = JxJ$ gives the desired antiisomorphism, which, in case $E_{N' \cap M}(e_0) = \lambda$, is trace preserving by [PiPo2] and thus can be extended from $\overline{\bigcup_i (N'_i \cap M)}^w$ onto $M' \cap \tilde{M}$. Q.E.D.

3.3 Definition. The pair of algebras $R_0 = \overline{\bigcup_i (N'_i \cap N)} \subset \overline{\bigcup_i (N'_i \cap M)} = R$ is called the *core* of the inclusion $N \subset M$. Note that the isomorphism class of $R_0 \subset R$ doesn't depend on the choice of the tunnel.

Since the algebras $M' \cap M_i$ satisfy the conditions of 2.2. (cf. 3.1) we can now list the corresponding properties of these algebras:

3.4 Corollary. *Denote $B_i^0 = N'_{i-1} \cap M$ and $B_i = M' \cap M_i$. Then B_i^0 satisfy condition (*) in Section 2 and B_i, $i \geqq 0$, satisfy the hypothesis of 2.2 and they have the properties:*

3.4.1. *If p_{i+2} is the central support of e_{i+2} in B_{i+2} (resp. of e_{-i} in B_{i+2}^0) then $B_{i+2}p_{i+2} = \operatorname{span} B_{i+1}e_{i+2}B_{i+1}$ (resp. $B_{i+2}^0 p_{i+2} = \operatorname{span} B_{i+1}^0 e_{-i} B_{i+1}^0$) and $B_{i+1} \subsetneq^{e_{i+2}} B_{i+2}p_{i+2}$ (resp. $B_{i+1}^0 \simeq B_{i+1}^0 p_{i+2} \subset^{e_{-i}} B_{i+2}^0$) is the algebraic basic construction for $B_i \subset B_{i+1}$ (resp. $B_i^0 \subset B_{i+1}^0$).*

3.4.2. *Identify the sets of simple summands of B_{2i} and B_{2i}^0 by 3.2.2 and denote it by K_i, $i \geqq 0$, and those of B_{2i+1}, B_{2i+1}^0 by L_i, $i \geqq 0$. Via these identifications the embeddings $K_i \subset K_{i+1}$ (or $L_i \subset L_{i+1}$) defined in two ways by either the application $\mathscr{Z}(B_i) \ni q' \mapsto q'' \in \mathscr{Z}(B_{i+2}p_{i+2})$, with $q''e_{i+2} = q'e_{i+2}$, or by $\mathscr{Z}(B_i^0) \ni q' \mapsto q'' \in \mathscr{Z}(B_{i+2}^0 p_{i+2})$, with $q''e_{-i} = q'e_{-i}$, are the same.*

3.4.3. *Let $K = \bigcup_i K_i$, $L = \bigcup_i L_i$. Then $K_0 = \{k_0\}$ and there is a unique pointed matrix $A = (a_{kl})_{\substack{k \in K \\ l \in L}}$, $k_0 \in K$, such that $(a_{kl})_{\substack{k \in K_i \\ l \in L_i}}$ describes $B_{2i} \subset B_{2i+1}$ and $B_{2i}^0 \subset B_{2i+1}^0$ while $(a_{kl})^t_{\substack{k \in K_{i+1} \\ l \in L_i}}$ describes $B_{2i+1} \subset B_{2i+2}$ and $B_{2i+1}^0 \subset B_{2i+2}^0$. Moreover K_i, L_i satisfy*

$$L_i = \{l \in L \mid \exists k \in K_i \quad \text{with} \quad a_{kl} \neq 0\} ,$$
$$K_{i+1} = \{k \in K \mid \exists l \in L_i \quad \text{with} \quad a_{kl} \neq 0\} .$$

3.4.4. *A is an irreducible matrix and $\|A\|^2 \leqq [M:N]$.*

Proof. cf. 2.2 and 3.1, 3.2.2. Q.E.D.

3.5. Remark. The pointed matrix (A, k_0) associated to the sequences $N'_i \cap M$, $M' \cap M_i$ like in 3.4 coincides with the *standard matrix* defined in [Po3]. It is also equivalent to Ocneanu's principal graph for $N' \cap M = \mathbf{C}$. In fact for what we need in this section the definition and construction of A are not necessary; the result 3.8

which will be used in the proof of the main result in Section 4 can be derived directly from the general result 2.2.

We now mention the particularization to the case $B_i = M' \cap M_i$ of the results on sequences of commuting squares with $\sup \dim \mathscr{Z}(B_i) < \infty$ in Section 2. Following ([Oc]), if the sequence of higher relative commutants of $N \subset M$ has bounded growth, then we say that $N \subset M$ has finite depth. In fact this terminology is used in [Oc] for subfactors with $N' \cap M = \mathbf{C}$. We will use it here for subfactors $N \subset M$ without assuming the trivial relative commutant condition. We also introduce another terminology, which although less established seems better suited.

3.6 Definition. The inclusion $N \subset M$ has *finite depth* (or *finite growth*) if

$$\sup \dim \mathscr{Z}(N_i' \cap M) (= \dim \mathscr{Z}(M' \cap M_{i+1})) < \infty$$

(i.e. if $B_i = M' \cap M_i$ have bounded growth).

3.7 Corollary. *If $N \subset M$ has finite depth then:*

3.7.1. $E_{N' \cap M}(e_0) = [M:N]^{-1} 1$ *and the antiisomorphism of* 3.2.2 *is trace preserving.*

3.7.2. *There is an i_0 such that for all $i \geqq i_0$, $B_{i+1} \subset^{e_{i+2}} B_{i+2}$, $B_{i+1}^0 \subset^{e_{-i}} B_{i+2}^0$ are the basic constructions for $B_i \subset B_{i+1}$ respectively $B_i^0 \subset B_{i+1}^0$ and the unique trace τ of $\tilde{M}$ is a $\lambda = [M:N]^{-1}$ Markov trace on each basic construction. Thus the inclusions $B_{2i} \subset B_{2i+1}$, $B_{2i}^0 \subset B_{2i+1}^0$ are described by A and $B_{2i+1} \subset B_{2i+2}$, $B_{2i+1}^0 \subset B_{2i+2}^0$ by A^t, for $2i \geqq i_0$, A being the standard matrix defined in* 3.4.3.

3.7.3. *If p (resp. q) is a minimal projection of B_i^0 or B_i (resp. B_{i+2j}^0 or B_{i+2j}) which belongs to a simple summand labeled by the same k (if i is even) or l (if i is odd) then $\tau(q) = \lambda^j \tau(p)$.*

3.7.4. $\|A\|^2 = [M:N]$ *and for $2i \geqq i_0$ the traces of the minimal projections of B_{2i}, B_{2i}^0 give the Perron-Frobenius eigenvector for AA^t, A being the standard matrix.*

3.7.5. *The algebras $R = \overline{\bigcup_i (N_i' \cap M)}$ and $R \cap N_j = \overline{\bigcup_i (N_i' \cap N_j)}$, $j \geqq 0$, are type* II_1 *factors.*

Proof. By 2.3, 2.4 and 3.1, 3.2.2 we get 3.7.1–3.7.4. Moreover, by 2.5 and 3.2 it follows that $R = \overline{\bigcup_i (N_i' \cap M)}$ is a factor. Since $[M:N_j] < \infty$ and $E_{N_i' \cap M} E_{N_j} = E_{N_i' \cap N_j}$, by [PiPo3] it follows that the embedding matrix of $N_i' \cap N_j \subset N_i' \cap M$, $i \geqq j \geqq 0$, has square norm majorized by $[M:N]^{j+1}$. Thus, since $\sup \dim \mathscr{Z}(N_i' \cap M) < \infty$, $\sup \dim \mathscr{Z}(N_i' \cap N_j) < \infty$ as well. Thus $N_{j+1} \subset N_j$ has finite depth and $R \cap N_j = \overline{\bigcup_i (N_i' \cap N_j)}$ is a factor by the previous part. Q.E.D.

Corollary 3.7 yields an important property for subfactors with finite depth: there is a uniform bound for the local indices $[pMp : N_i p]$ of the inclusions $N_i \subset M$, independent of i and of the minimal projection p of $N_i' \cap M$. More precisely we have:

3.8 Theorem. *Assume $N \subset M$ is a subfactor of finite depth. Let $A = (a_{kl})_{k \in K, l \in L}$, $k_0 \in K$, be its standard matrix and let $(v_k)_{k \in K}$ be the Perron-Frobenius eigenvector of AA^t*

normalized so that $v_{k_0} = 1$. *Let* $i \geqq 0$ *and* p *be a minimal projection of* $N'_{2i-1} \cap M$ (*or* $M' \cap M_{2i}$) *which belongs to a simple direct summand labeled by* $k \in K$. *Then* $[pMp : N_{2i-1}p] = v_k^2$ (*respectively* $[pM_{2i}p : Mp] = v_k^2$). *Moreover if* q *is a minimal projection in* $N'_{2i} \cap M$ *or* $M' \cap M_{2i+1}$ *in the simple summand labeled by* $l \in L$ *then*

$$[qMq : N_{2i}q] = [M:N]^{-1}\left(\sum_k a_{kl} v_k\right)^2 \left(\textit{respectively}\,[qM_{2i+1}q : Mq] = [M:N]^{-1}\left(\sum_k a_{kl} v_k\right)^2\right).$$

Proof. Since $E_{N' \cap M}(e_0) = \lambda = [M:N]^{-1}$, by [PiPo2] and 3.1 it is sufficient to prove the statement for the tower. By [PiPo2] we then have $[pM_{2i}p : Mp] = [M_{2i} : M]\tau(p)^2 = \lambda^{-2i}\tau(p)^2$. Let $j_0 \geqq 0$ be so that $M' \cap M_{2j+1} \subset M' \cap M_{2j+2}$ is the basic construction for $M' \cap M_{2j} \subset M' \cap M_{2j+1}$ for $j \geqq j_0$. By 3.7 if $j \geqq 0$ and if p_k^{2j} is a choice of a minimal projection in the k-th summand of $M' \cap M_{2j}$ then $\tau(p_k^{2j}) = \lambda^{j-j_0}\tau(p_k^{2j_0})$. Thus

$$[p_k^{2j} M_{2j} p_k^{2j} : M p_k^{2j}] = \lambda^{-2j}\tau(p_k^{2j})^2 = \lambda^{-2j_0}\tau(p_k^{2j_0})^2 .$$

But by 3.7 $(\tau(p_k^{2j_0}))_{k \in K}$ is the Perron-Frobenius eigenvector for AA^t normalized by $\tau(p_{k_0}^{2j_0}) = \lambda^{j_0}$. Thus $[p_{k_0}^{2j} M_{2j} p_{k_0}^{2j} : M p_{k_0}^{2j}] = 1$ and $[p_k^{2j} M_{2j} p_k^{2j} : M p_k^{2j}]^{1/2}$ coincides with the k-th entry of the Perron-Frobenius eigenvector of AA^t which has the k_0 entry equal to 1.

Similarly if p_l^{2j+1} is a choice of a minimal projection in the l-th summand of $M' \cap M_{2j+1}$, $j \geqq 0$, then $(\tau(p_l^{2j+1}))_l = A^t(\tau(p_k^{2j+2}))_k$ and

$$[p_l^{2j+1} M_{2j+1} p_l^{2j+1} : M p_l^{2j+1}] = \tau(p_l^{2j+1})^2 \lambda^{-2j-1} = \tau(p_l^{2j_0+1})^2 \lambda^{-2j_0-1} .$$

Thus we get for $j \geqq j_0$

$$A([p_l^{2j+1} M_{2j+1} p_l^{2j+1} : M p_l^{2j+1}]^{1/2})_l = \lambda^{-j_0-1/2}(\tau(p_k^{2j_0}))_k = \lambda^{-1/2}([p_k^{2j_0} M_{2j_0} p_k^{2j_0} : M p_k^{2j_0}]^{1/2})_k .$$

This implies that $\lambda^{-1/2}([p_l^{2j+1} M_{2j+1} p_l^{2j+1} : M p_l^{2j+1}]^{1/2})_l$ is a Perron-Frobenius eigenvector for A^tA so that

$$\lambda^{1/2} A^t([p_k^{2j_0} M_{2j_0} p_k^{2j_0} : M p_k^{2j_0}]^{1/2})_k = ([p_l^{2j+1} M_{2j+1} p_l^{2j+1} : M p_l^{2j+1}]^{1/2})_l .$$

This ends the proof of the theorem. Q.E.D.

4. Finite depth and the generating property

As we mentioned before, a general strategy for approaching index problems for subfactors $N \subset M$ of the hyperfinite II_1 factor $M \simeq R$ is to approximate $N \subset M$ by a sequence of finite dimensional subalgebras satisfying the commuting square condition (*). From Section 2 we saw that the finite dimensional subalgebras $B_i^0 = N'_{i-1} \cap M$ are the first to be considered that satisfy $E_{B_i^0} E_N = E_{B_i^0 \cap N}$. Then the approximation with B_i^0 amounts to say that we may choose the tunnel N_i so that the core $R_0 = \overline{\bigcup_i (N'_i \cap N)} \subset \overline{\bigcup_i (N'_i \cap M)} = R$ coincides with $N \subset M$.

Approximating $N \subset M$ with the sequence of higher relative commutants is more rewarding than with other finite dimensional algebras, since the sequence of higher relative commutants is canonical.

4.1 Definition. We say that $N \subset M$ has the *generating property* if there exists a choice of the tunnel of subfactors N_i such that $N_i' \cap M \uparrow M$, $N_i' \cap N \uparrow N$. Note that since $E_{N_i' \cap M} E_N = E_{N_i' \cap N}$, $N_i' \cap M \uparrow M$ actually implies $N_i' \cap N \uparrow N$.

We will prove in this section that if M is the hyperfinite II_1 factor and $N \subset M$ has finite depth (i.e. $\sup \dim \mathscr{Z}(N_i' \cap M) < \infty$) then $N \subset M$ has the generating property. The key observation allowing us to prove this generating result is that the subalgebras $B = N_k \vee (N_k' \cap M)$ satisfy the basic inequality $E_B(x) \geqq cx$, $x \in M_+$, with c independent of k. This important device will be a simple consequence of 3.8. But first we need a simple formula which is implicit in [PiPo3].

4.2 Lemma. *Let $n_0, C_0 < \infty$. If $q_1, \ldots, q_n \in \mathscr{P}(M)$ is a partition of the unity in M, with $n \leqq n_0$, and if $P_i \subset q_i M q_i$ are subfactors with $[q_i M q_i : P_i] < C_0$, then $B = \sum_i P_i$ satisfies $E_B(x) \geqq (n_0 C_0)^{-1} x$, $x \in M_+$.*

Proof. Denote $B_1 = \sum_i q_i M q_i$. By [PiPo3] we have $E_{B_1}(x) \geqq n_0^{-1} x$, $x \in M_+$. Moreover if $y \in B_1$, $y = \Sigma y_i$, with $y_i \in q_i M q_i$ then $E_B(y) = \sum_i E_{P_i}(y_i)$. Thus if $y \geqq 0$ then by [PiPo1] (see 1.1) we have $E_B(y) \geqq \sum_i [q_i M q_i : P_i]^{-1} y_i \geqq C_0^{-1} y$. Altogether we get for $x \in M_+$, $E_B(x) = E_B E_{B_1}(x) \geqq C_0^{-1} E_{B_1}(x) \geqq (C_0 n_0)^{-1} x$. Q.E.D.

4.3 Theorem. *Suppose $N \subset M$ has finite depth. There exists $c > 0$ such that given any choice of the tunnel of subfactors $\{N_i\}_{i \geqq 1}$ we have*

$$E_{N_i \vee (N_i' \cap M)}(x) \geqq cx, \; x \in M_+ ,$$

for all $i \geqq 1$.

Proof. Let $\{q_j^i\}_j$ be the minimal central projections of $N_i' \cap M$. Then q_j^i are also the minimal central projections of $N_i \vee (N_i' \cap M)$. Thus $N_i \vee (N_i' \cap M) = \sum_j P_j^i$ with $P_j^i = (N_i \vee (N_i' \cap M)) q_j^i$ subfactors in $q_j^i M q_j^i$. Moreover if p is a minimal projection in $(N_i' \cap M) q_j^i$ then $p \in P_j^i$ and $N_i p = p(N_i \vee (N_i' \cap M))p = pP_j^i p$ so that

$$[q_j^i M q_j^i : P_j^i] = [pMp : pP_j^i p] = [pMp : N_i p] .$$

By 3.8 there is an upper bound on this indices,

$$\sup\{[pMp : N_i p] \mid i \geqq 0, p \text{ minimal projection in } N_i' \cap M\} = C_0 < \infty .$$

Thus $[q_j^i M q_j^i : P_j^i] \leqq C_0$ for all i, j. Moreover, since for each i the number of P_j^i's is bounded by $\sup \dim \mathscr{Z}(N_i' \cap M) = n_0 < \infty$, the statement follows by 4.2. Q.E.D.

If c is the uniform bound (from below) given by 4.3 then the above theorem easily yields the existence of a tunnel N_i so that $R = \overline{\bigcup_i (N_i' \cap M)}^w$ has finite index in M. This "almost" proves the generating property. To show all this let us first mention a consequence of the hyperfiniteness of M and of [J1].

4.4 Lemma. *Let $N \subset M$ be hyperfinite II_1 factors with finite index. Let $M \supset^{e_0} N \supset^{e_{-1}} N_1 \supset^{e_{-2}} \ldots \supset^{e_{-i+1}} N_{i-1} \supset N_i$ be a choice of the tunnel up to some i. Let $\varepsilon > 0$ and $F \subset N_i \vee (N_i' \cap M)$ a finite set. Then there exists $j \geqq i$ and a* *continuation of the tunnel* $\supset^{e_{-i}} N_i \supset^{e_{-i-1}} N_{i+1} \ldots \supset^{e_{-j+1}} N_{j-1} \supset N_j$ *such that* $x \in_\varepsilon N'_j \cap M$, $x \in F$.

Proof. Let $\{f^n_{r,s}\}_{r,s,n}$ be a matrix unit for $N'_i \cap M$, where n labels the summands of $N'_i \cap M$. Then $x = \Sigma x^n_{rs} f^n_{rs}$, with $x^n_{rs} \in N_i$. Since N_i is isomorphic to the hyperfinite II_1 factor (cf. [C]), by [J1] we may regard it as generated by a sequence of projections $\{e'_m\}_{m \geqq 1} \subset N_i$ satisfying Jones' axioms $e'_m e_{m \pm 1} e'_m = \lambda e'_m$, $[e'_m, e'_n] = 0$ for $|m - n| \geqq 2$, $\tau(e'_m) = \lambda$, where $\lambda = [M:N]^{-1}$. Thus for any $\delta > 0$ there is $p \geqq 1$ such that $x^n_{rs} \in_\delta \mathrm{Alg}\{1, e'_1, \ldots, e'_{p-1}\}$. Let $\supset^{e^0_{-i}} N_i \supset^{e^0_{-i-1}} N^0_{i+1} \ldots \supset^{e^0_{-i-p+1}} N^0_{i+p-1} \supset N^0_{i+p}$ be a continuation of the tunnel up to $i + p$. By [J1] there is a τ-preserving isomorphism between $\mathrm{Alg}\{1, e'_1, \ldots, e'_{p-1}\}$ and $\mathrm{Alg}\{1, e^0_{-i-1}, \ldots, e^0_{-i-p+1}\}$ sending e'_k to e^0_{-i-k}. Thus there is a unitary element in the ambient algebra N_i such that $u e^0_{-i-k} u^* = e'_k$, $1 \leqq k \leqq p-1$. Thus, if we put $j = i + p$ and $N_{i+k} = u N^0_{i+k} u^*$, $1 \leqq k \leqq p$, then N_{i+k} is a continuation of the tunnel such that $N'_j \cap M \supset \mathrm{Alg}\{1, e'_1, \ldots, e'_{p-1}\} \ni_\delta x^n_{rs}$ for all n, r, s (and all $x \in F$). Since $f^n_{rs} \in N'_i \cap M \subset N'_j \cap M$ and $x^n_{rs} \in_\delta N'_j \cap M$, if we take δ small enough we get $N'_j \cap M \ni_\varepsilon \Sigma x^n_{rs} f^n_{rs} = x$, $x \in F$. Q.E.D.

4.5 Corollary. *If* $N \subset M$ *has finite depth and* N, M *are hyperfinite* II_1 *factors then there is a choice of the tunnel* $\{N_i\}_{i \geqq 1}$ *such that the factor* $R = \overline{\bigcup_i (N'_i \cap M)}$ *has finite index in* M.

Proof. Let $c > 0$ be the constant in 4.3. Let $\{x_n\}_n \subset M_+$ be a sequence of elements dense in the unit ball of M_+ in the norm $\|\ \|_2$. By 4.3, 4.4 we may choose recursively numbers $k_1 < k_2 < \ldots < k_{n-1} < k_n \ldots$ and continuations of the tunnel $N_{k_{n-1}+1} \supset \ldots \supset N_{k_n}$ such that $\| E_{N'_{k_n} \cap M}(x_i) \|_2^2 \geqq {}_{1/n} c \| x_i \|_2^2$, $1 \leqq i \leqq n$. Indeed, suppose we achieved step $n - 1$. By 4.3 we have $y_i \overset{\mathrm{def}}{=} E_{N_{k_{n-1}} \vee (N'_{k_{n-1}} \cap M)}(x_i) \geqq c x_i$, thus $\| y_i \|_2^2 = \tau(y_i x_i) \geqq c\tau(x_i^2) = c \| x_i \|_2^2$, $1 \leqq i \leqq n$. By 4.4 we may choose $N_{k_{n-1}+1} \supset N_{k_{n-1}+2} \ldots \supset N_{k_n}$, for some $k_n > k_{n-1}$, so that $y_i \in_{1/n} N'_{k_n} \cap M$. Thus, since $N'_{k_n} \cap M$ and $N_{k_{n-1}} \vee (N'_{k_{n-1}} \cap M)$ form a commuting square (cf. [Po1]), we get $\| E_{N'_{k_n} \cap M}(x_i) \|_2^2 \geqq \| E_{N_{k_{n-1}} \vee (N'_{k_{n-1}} \cap M)} E_{N'_{k_n} \cap M}(x_i) \|_2^2$
$= \| E_{N'_{k_n} \cap M} E_{N_{k_{n-1}} \vee (N'_{k_{n-1}} \cap M)}(x_i) \|_2^2 = {}_{1/n} \| E_{N_{k_{n-1}} \vee (N'_{k_{n-1}} \cap M)}(x_i) \|_2^2 \geqq c \| x_i \|_2^2$.
If we take now $R = \overline{\bigcup_{n \geqq 1} (N'_{k_n} \cap M)^w} = \overline{\bigcup_{k \geqq 1} (N'_k \cap M)}$ then $\| E_R(x_i) \|_2^2 = \lim_n \| E_{N'_{k_n} \cap M}(x_i) \|_2^2 \geqq c \| x_i \|_2^2$ for all i. Thus $\| E_R(x) \|_2^2 \geqq c \| x \|_2^2$ for all $x \in M_+$, by density. Since R is a factor (cf. 3.7.5), by theorem 2.2 in [PiPo1] we get $[M:R] \leqq c^{-1}$. Q.E.D.

We now show that if we can choose the tunnel N_i so that $[M:R] < \infty$, with $R = \overline{\bigcup_i (N'_i \cap M)}$, then we can actually choose it so that $R = M$. We will prove this by contradiction. So we need to know what the failure of the generating property implies.

4.6 Lemma. *Let* $N \subset M$ *be so that there exists a choice of the tunnel* N_i, $i \geqq 1$, *with the property that* $R = \overline{\bigcup_i (N'_i \cap M)}$ *has finite index in* M. *Assume* $N \subset M$ *doesn't have the generating property. Then we have*

(1) $\forall \varepsilon > 0$, $\forall k_0 \geqq 1$, $\exists k_1 > k_0$, $\exists x \in M$, $\| x \|_2 = 1$, $\| x \| \leqq [M:R]^{1/2}$ *such that* $\| E_{u_1 R u_1^*}(x) \|_2 \leqq \varepsilon$, $\forall u_1 \in \mathscr{U}(N_{k_1})$.

Proof. Note first that the property (non 1) reads:

$$\text{(Non 1) "}\exists\varepsilon_0 > 0, \exists k_0 \geqq 1 \text{ such that } \forall k > k_0, \forall x \in M \text{ with } \|x\|_2 = 1,$$
$$\|x\| \leqq [M:R]^{1/2}, \exists u \in \mathcal{U}(N_k) \text{ such that } \|E_{uRu^*}(x)\|_2 > \varepsilon_0\text{"}.$$

Assume (Non 1), instead of (1), holds true. Let then $\{x_n\}_n \subset M$ be a sequence of elements satisfying $\|x_n\|_2 = 1$, $\|x_n\| \leqq [M:R]^{1/2}$, dense in the norm $\| \ \|_2$ in the set $\{x \in M \mid \|x\|_2 = 1, \|x\| \leqq [M:R]^{1/2}\}$. Let also $\{y_n\}_n \subset M_+$ be a sequence of elements dense in the norm $\| \ \|_2$ in M_+. We will find recursively $1 \leqq k_1 < k_2 < \dots$ and a choice N_i^0, $i \geqq 1$, of the tunnel such that

(i) $\|E_{N_{k_n}^{0\prime} \cap M}(x_n)\|_2 \geqq \varepsilon_0$;
(ii) $\|E_{N_{k_n}^{0\prime} \cap M}(y_n)\|_2^2 \geqq {}_{1/n}[M:R]^{-1}\|y_n\|_2^2$;

Suppose $k_1 < k_2 < \dots < k_{n-1}$ and $M \supset N \supset N_1^0 \supset \dots \supset N_{k_{n-1}}^0$ have been chosen. Let $u_0 \in \mathcal{U}(N)$ be such that $u_0 N_i^0 u_0^* = N_i$ for $i = 1, \dots, k_{n-1}$. Then

$$E_R(u_0 y_n u_0^*) \geqq [M:R]^{-1} u_0 y_n u_0^*,$$

so that there exists $k_n^0 > k_{n-1}$, $k_n^0 > k_0$ such that:

$$E_{N'_{k_n^0} \cap M}(u_0 y_n u_0^*) \geqq {}_{1/n}[M:R]^{-1} u_0 y_n u_0^*.$$

Thus we get

$$E_{u_0^* N_{k_n^0} u_0' \cap M}(y_n) = E_{u_0^*(N'_{k_n^0} \cap M)u_0}(y_n)$$
$$= u_0^* E_{N_{k_n^0}' \cap M}(u_0 y_n u_0^*) u_0 \geqq {}_{1/n}[M:R]^{-1} y_n.$$

Moreover, by (Non 1) applied to $x = u_0 x_n u_0^*$, there exists $u_1 \in \mathcal{U}(N_{k_n^0})$ such that

$$\|E_{u_1 R u_1^*}(u_0 x_n u_0^*)\|_2 > \varepsilon_0.$$

But then, again by the definition of R, there exists $k_n > k_n^0$ with:

$$\|E_{u_1(N'_{k_n} \cap M)u_1^*}(u_0 x_n u_0^*)\|_2 > \varepsilon_0.$$

Thus we also get:

$$\|E_{u_0^* u_1 N_{k_n} u_1^* u_0' \cap M}(x_n)\|_2 = \|E_{u_0^* u_1 (N'_{k_n} \cap M) u_1^* u_0}(x_n)\|_2 > \varepsilon_0.$$

But $u_0^* u_1 N_i u_1^* u_0 = N_i^0$, for $1 \leqq i \leqq k_{n-1}$, so that if we put $N_j^0 = u_0^* u_1 N_j u_1^* u_0$ for $k_{n-1} < j \leqq k_n$ then both (i) and (ii) will be satisfied (also, since $u_0^* u_1 N_{k_n^0} u_1^* u_0 = u_0^* N_{k_n^0} u_0$). This ends the proof of the existence of N_j^0.

Let $R^0 = \overline{\bigcup_i (N_i^{0\prime} \cap M)}$. Then by (ii), $\|E_{R^0}(y)\|_2^2 \geqq [M:R]^{-1}\|y\|_2^2$, $y \in M_+$, so that $[M:R^0] \leqq [M:R]$. Since $R^0 \neq M$, $[M:R^0] \geqq 2$ so by [PiPo1] there exists $m \in M$ with $\|m\|_2 = 1$, $\|m\| \leqq [M:R]^{1/2}$ such that $E_{R^0}(m) = 0$. But (i) shows that $\|E_{R^0}(m)\|_2 \geqq \varepsilon_0$, a contradiction. Q.E.D.

4.7 Corollary. *Let ω be a free ultrafilter on* **N**. *With the notations and under the hypothesis of* 4.6, *there exist $x = (x_n)_n \in M^\omega$, $\|x\|_2 = 1$, $\|x\| \leqq [M:R]^{1/2}$, and there exists $k_1 < k_2 < \dots$ such that $x \perp uR^\omega u^*$ for all unitary elements u in $\prod_{n\to\omega} N_{k_n}$.*

Proof. The construction of x_n follows by applying recursively 4.6. Q.E.D.

It turns out that if B_0, $B \subset M$ are von Neumann subalgebras of M then the span of uBu^* for u running in $\mathscr{U}(B_0)$ is almost a $B_0 - B_0$ bimodule. We will prove this technical result in the next section (see 5.1). Thus we get from 4.7:

4.8 Corollary. *With the notations and under the assumptions of* 4.6, 4.7, *there exists* $x \in M^\omega$, $\|x\|_2 = 1$ *such that* $x \perp (\prod_\omega N_{k_n}) R^\omega (\prod_\omega N_{k_n})$.

Proof. By 4.7 there exists $x \in M^\omega$, $\|x\|_2 = 1$, with

$$x \perp X \overset{\text{def}}{=} \operatorname{span}\left\{ uR^\omega u^* \mid u \in \mathscr{U}\left(\prod_\omega N_{k_n}\right)\right\}.$$

Also, by 4.4 we have $\prod_\omega (N_{k_n} \vee (N'_{k_n} \cap M)) \subset X$. Indeed, if $y = (y_n)_n \in \prod_\omega (N_{k_n} \vee (N'_{k_n} \cap M))$ then by 4.4 for each n there is a $w_n \in \mathscr{U}(N_{k_n})$ and $l_n > k_n$ such that

$$y_n \in_{1/n} w_n (N'_{l_n} \cap M) w_n^* \subset w_n R w_n^* .$$

Thus $y \in wR^\omega w^* \subset X$, where $w = (w_n)_n$. Thus 5.1 applies and we get $x \perp (\prod_\omega N_{k_n}) R^\omega (\prod_\omega N_{k_n})$. Q.E.D.

We can now end up the proof of the generating result.

4.9 Theorem. *Let M be the hyperfinite* II_1 *factor and $N \subset M$ a subfactor of finite index. If N has finite depth (i.e.* $\sup \dim \mathscr{Z}(M' \cap M_i) < \infty$ *for the Jones tower M_i) then $N \subset M$ has the generating property (i.e. there exists a choice of the tunnel N_i, $i \geqq 1$, so that* $\overline{\bigcup_i (N'_i \cap M)} = M$*). Moreover, if* $\tilde{M} = \overline{\bigcup_i M_i}$ *then there is an anti-isomorphism of M onto $M' \cap \tilde{M}$ carrying N onto $M'_1 \cap \tilde{M}$.*

Proof. Assume $N \subset M$ doesn't have the generating property. By 4.5 and 4.8 there exists $x \in M^\omega$, $x \neq 0$, $x \perp \overline{\operatorname{span}(\prod_\omega N_{k_n}) R^\omega (\prod_\omega N_{k_n})}$, where N_i is a choice of the tunnel such that $R = \overline{\cup (N'_i \cap M)}$ satisfies $[M:R] < \infty$.

Since $E_R E_{N_i} = E_{R \cap N_i}$, $[R : N_i \cap R] \leq [M : N_i]$ (cf. e.g. [Po2]). Moreover, since R contains the Jones projections in the tunnel, $e_0, e_{-1}, e_{-2}, \ldots$, it follows by [PiPo2] that R contains the Jones projection for $N_i \subset M$, i.e. a projection $f \in M$ with

$$E_{N_i}(f) = [M : N_i]^{-1} 1 .$$

Thus $f \in R$, $E_{N_i \cap R}(f) = E_{N_i}(f) = [M:N_i]^{-1} 1$, so that $[R : N_i \cap R] \geqq [M : N_i]$. Altogether $[M:N_i] = [R : N_i \cap R]$. But

$$[M:N_i][N_i : N_i \cap R] = [M : N_i \cap R] = [M:R][R : N_i \cap R]$$

(cf. [J1]) so that $[M:R] = [N_i : N_i \cap R]$. It follows that if $\{m^i_j\}_j$ is an orthonormal basis of N_i over $N_i \cap R$, then it is also an orthonormal basis of M over R, thus $\sum_j m^i_j R = M$. Put $m_j = (m_j^{k_n})_n \in M^\omega$. It follows that $\sum_i m_j R^\omega = M^\omega$. But $x \perp \sum_j m_j R^\omega$, a contradiction.

The rest follows by 3.2. Q.E.D.

Note that we also proved the following general result:

4.10 Theorem. *If $N \subset M$ are hyperfinite factors with finite index and if there is a choice of the tunnel N_k such that $R = \overline{\bigcup_k (N'_k \cap M)}$ is a factor and satisfies $E_R(x) \geqq cx$, for all $x \in M^+$ and some $c > 0$, then $N \subset M$ has the generating property.*

Proof. By the duality result 1.5 in [PiPo1] it follows that if $\overline{\cup (N'_k \cap M)}^w$ is a factor then $\overline{\cup (N'_k \cap N_1)}^w$ is also a factor, and more generally $\overline{\bigcup_k (N'_k \cap N_{2i+1})}^w$ are all factors for $i \geqq 0$. Moreover since $\overline{\cup (N'_{2i+1} \cap M)}^w = \overline{\cup (N'_k \cap M)}^w$, it follows that $N \subset M$ has the generating property iff $N_1 \subset M$ has the generating property. Altogether this shows that in the above statement we may assume $R_i = \overline{\cup (N'_k \cap M)}^w \cap N_i = \overline{\bigcup_i (N'_k \cap N_i)}^w$ are all factors. But then the proof of 4.9 shows that $N \subset M$ has the generating property. Q.E.D.

5. Proof of the linearization principle

We prove here the linearization principle that we used in the proof of Theorem 4.9. The essence of this linearization principle is the noncommutative Rohlin type theorem in [Po4].

5.1 Theorem. *Let M be a finite von Neumann algebra with a finite faithful trace τ. Let $B \subset M$ be a von Neumann subalgebra and $S \subset M$ a vector subspace. Then*

$$(\operatorname{span}\{uSu^* \mid u \in \mathscr{U}(B)\} + B \vee B' \cap M)^- = (\operatorname{span} BSB + B \vee B' \cap M)^- ,$$

the closures being taken in $L^2(M, \tau)$.

Proof. We clearly have the inclusion $\subset$. Let $\xi \in L^2(M, \tau)$ with $\langle uyu^*, \xi \rangle = 0$, $\langle b, \xi \rangle = 0$ for all $u \in \mathscr{U}(B)$, $y \in S$, $b \in B \vee B' \cap M$. Let $h_i \in B_h$, $i \leqq i \leqq n$. Pur $f: \mathbf{C}^n \to \mathbf{C}$,

$$f(\lambda_1, \lambda_2, \ldots, \lambda_n) = \langle \exp(\lambda_1 h_1) \exp(\lambda_2 h_2) \ldots \exp(\lambda_n h_n) y \exp(-\lambda_n h_n) \ldots \exp(-\lambda_1 h_1), \xi \rangle .$$

Then f is analytic on $\mathbf{C}^n$ and vanishes on $(i\mathbf{R})^n$. Thus f vanishes everywhere on $\mathbf{C}^n$. Since any invertible element in B is a product of two elements of the form $\exp(\lambda h)$, with $\lambda \in \mathbf{C}$, $h \in B_h$, we obtain that $\langle sys^{-1}, \xi \rangle = 0$, for any $y \in S$ and any invertible element s in B. Let $p \perp q$ be projections in B and $s \in B$ invertible. Denote $s_1(t) = tp + (1-p)$, $s_2(t) = t(1-q) + q$. Observe that $s_1(t)^{-1} = s_1(t^{-1})$, $s_2(t^{-1}) = s_2(t^{-1})$ and that

$$\lim_{t \to \infty} \left\| \frac{1}{t^2} s_2(t) s_1(t) sys^{-1} s_1(t)^{-1} s_2(t)^{-1} - psys^{-1} q \right\| = 0 .$$

It follows that $\langle psys^{-1}q, \xi \rangle = 0$. We now deduce that $\langle zyz^*, \xi \rangle = 0$ for all $z \in B$. It is clearly sufficient to prove this when spec(z^*z) is finite. So let $z = au$, where $a = \Sigma \alpha_i e_i$, with $\alpha_i \geqq 0$, $\Sigma e_i = 1$, $e_i \in \mathscr{P}(B)$, and $u \in \mathscr{U}(B)$. By [Po4] there is a refinement $f_j \in B$ of the e_i's so that

$$\left\| \sum_j f_j \xi f_j - e_{\sum_i f_i (B' \cap M)}(\xi) \right\|_2$$

is small (the result in [Po4] is stated for $B' \cap M \subset B$ but the general case used here has exactly the same proof). But since $\sum_j f_j(B' \cap M) \subset B \vee B' \cap M$ and $e_{B \vee B' \cap M}(\xi) = 0$, $e_{\Sigma f_j(B' \cap M)}(\xi) = 0$ as well. This shows that we can make $\| \sum_i f_j \xi f_j \|_2$ as small as we please, with $f_j \in \mathscr{P}(B)$ refining the e_i's.

Since $\langle f_i u y u^* f_j, \xi \rangle = 0$, for $i \neq j$, it follows that

$$\langle zyz^*, \xi \rangle = \langle auyu^*a, \xi \rangle = \left\langle \sum f_j auyu^*af_j, \xi \right\rangle$$
$$= \left\langle auyu^*a, \sum_j f_j \xi f_j \right\rangle = \left\langle zyz^*, \sum_j f_j \xi f_j \right\rangle .$$

By the Cauchy-Schwartz inequality

$$|\langle zyz^*, \xi \rangle| \leqq \| zyz^* \|_2 \left\| \sum_j f_j \xi f_j \right\|_2$$

and for fixed z, y we can make this last term as small as we want. This shows that

$$\langle zyz^*, \xi \rangle = 0, z \in B, y \in S .$$

The standard polarization trick then implies: $\{z_1 y z_2, \xi\rangle = 0$, for all $z_1, z_2 \in B$, $y \in S$. Q.E.D.

6. Applications to the classification of subfactors

The generating type results reduce the classification (up to conjugacy) of the subfactors of the hyperfinite II_1 factor to the classification (up to trace preserving isomorphism) of the corresponding sequences of inclusions $\{N_i' \cap N \subset N_i' \cap M\}_i$ (or $\{M_1' \cap M_i \subset M' \cap M_i\}_i$). Moreover under the finite depth assumption all the information on such a sequence is contained in an initial commuting square:

$$\begin{array}{ccc} N'_{i_0} \cap M & \subset & N'_{i_0+1} \cap M , \\ \cup & & \cup \\ N'_{i_0} \cap N & \subset & N'_{i_0+1} \cap N , \end{array}$$

where i_0, the inclusions, the dimensions and the traces are all determined by the standard matrices of $N \subset M$ and $N_1 \subset N$. To deduce this we return to the general setting of Section 2.

The next result was first noted by Jones and by Pimsner-Popa, independently ([JPP]).

6.1 Lemma. *Let*

$$\begin{array}{ccc} B_0 & \subset_A & B_1 \\ C \cup & & \cup D \\ B_0^1 & \subset_B & B_1^1 \end{array}$$

be a commuting square of finite dimensional algebras with the matrices A, B, C, D describing the corresponding inclusions. Assume $B_0 \subset B_1$, $B_0^1 \subset B_1^1$ satisfy the equivalent conditions of 1.2.2 *and* $\|A\| = \|B\|$. *If* $B_0 \subset B_1 \subset^{e_2} B_2 \subset^{e_3} B_3 \subset \ldots$ *is the* *iterated basic construction for* $B_0 \subset B_1$, τ *is the* $\lambda = \|A\|^{-2}$*-Markov trace on* $\cup B_i$ *and if we denote* $B_{i+1}^1 = \operatorname{span} B_i^1 e_{i+1} B_i^1$, $i \geqq 1$, *then* B_{i+1}^1 *has support one in* B_{i+1}, $B_0^1 \subset B_1^1 \subset^{e_2} B_2^1 \subset^{e_3} B_3^1 \subset \dots$ *give the iterated basic construction for* $B_0^1 \subset B_1^1$, τ *is a* $\lambda = \|B\|^{-2}$*-Markov trace on* $\cup B_i^1$ *and we have the sequence of commuting squares*

$$\begin{array}{ccccccccc} B_0 & \subset_A & B_1 & \subset^{e_2}_{A^t} & B_2 & \subset^{e_2}_{A} & B_3 & \subset & \dots \\ \cup & & \cup & & \cup & & \cup & & \\ B_0^1 & \subset_B & B_1^1 & \subset^{e_2}_{B^t} & B_2^1 & \subset^{e_3}_{B} & B_3^1 & \subset & \dots \end{array}$$

Moreover, with the usual identifications of the centers of B_i, *respectively* B_i^1, *with the sets of indices of the matrices A, respectively B, the inclusion matrix for* $B_{2i}^1 \subset B_{2i}$ *is C and that of* $B_{2i+1}^1 \subset B_{2i+1}$ *is D,* $i \geqq 0$, *that is, in Wenzl's terminology* ([*W*]), *the sequence of commuting squares is periodic with period two.*

Proof. Let $B_0^1 \subset B_1^1 \subset^{\tilde{e}_2} \tilde{B}_2^1$ be the basic construction for $B_0^1 \subset B_1^1$ with a $\lambda = \|B\|^{-2}$-Markov trace τ_1 on $\tilde{B}_2^1$ satisfying $\tau_1(\tilde{e}_2 x) = \lambda\tau(x)$, $x \in B_1^1$, where τ is the Markov trace on B_2 (restricted to B_1^1). Define an isomorphism σ from $\tilde{B}_2^1$ onto B_2^1 by $\sigma(\Sigma x\tilde{e}_2 y) = \Sigma x e_2 y$. Since $\tau_1 \circ \sigma = \tau$ and $\tau_1(1) = 1$ it follows that $\tau_1(\sigma(1)) = 1$ so that $\sigma(1) = 1$ and thus B_2^1 has support 1 in B_2.

To show that the inclusion matrix for $B_2^1 \subset B_2$ is the same as for $B_0^1 \subset B_0$ note that $e_2 \in ((B_2^1)' \cap B_2)'$ and $e_2 \in ((B_0^1)' \cap B_0)'$. Note also that, since e_2 has support 1 in B_2^1, the map $(B_2^1)' \cap B_2 \ni z \mapsto z e_2$ is an isomorphism and that $(B_0^1)' \cap B_0 \ni z \mapsto z e_2$ is an isomorphism as well. Since

$$e_2((B_2^1)' \cap B_2)e_2 = (e_2 B_2^1 e_2)' \cap e_2 B_2 e_2 = (B_0^1 e_2)' \cap B_0 e_2 = ((B_0^1)' \cap B_0)e_2$$

it follows that there is a unique isomorphism $\varphi: (B_2^1)' \cap B_2 \to (B_0^1)' \cap B_0$ determined by $\varphi(z)e_2 = z e_2$. But this isomorphism, when restricted to $\mathscr{Z}(B_2)$, $\mathscr{Z}(B_2^1)$, gives the usual identifications between $\mathscr{Z}(B_2)$ and $\mathscr{Z}(B_0)$, respectively $\mathscr{Z}(B_2^1)$ and $\mathscr{Z}(B_0^1)$. Moreover the dimensions of $((B_2^1)' \cap B_2)q'q''$ for minimal projections q' in $\mathscr{Z}(B_2)$, q'' in $\mathscr{Z}(B_2^1)$, give the multiplicities in the inclusion matrix of $B_2^1 \subset B_2$, which will therefore coincide with that of $B_0^1 \subset B_0$. Q.E.D.

The next result will not be used here but it may be useful for constructing examples of subfactors. Related results on periodic sequences of commuting squares can be found in [W].

6.2 Corollary. *Let*

$$\begin{array}{cccc} & B_0 & \subset_A & B_1 & \\ C & \cup & & \cup & D \\ & B_0^1 & \subset_B & B_1^1 & \end{array}$$

be like in 6.1 *and let*

$$\begin{array}{ccccccccc} B_0 & \subset_A & B_1 & \subset^{e_2}_{A^t} & B_2 & \subset^{e_s}_{A} & B_3 & \subset & \dots \\ \cup & & \cup & & \cup & & \cup & & \\ B_0^1 & \subset_B & B_1^1 & \subset^{e_2}_{B^t} & B_2^1 & \subset^{e_3}_{B} & B_3^1 & \subset & \dots \end{array}$$

be the corresponding iterated sequence of commuting squares. Let $M = \overline{\bigcup_n B_n}^{w}$, $N = \overline{\bigcup_n B_n^1}^{w}$. *Then*

1°. *M is a factor iff A is irreducible*

2°. *N is a factor iff B is irreducible*

3°. *If N, M are factors then*

(a) $[M:N] = \|C\|^2 = \|D\|^2$

(b) $H(M|N) = \ln[M:N]$ *so that* $E_{N'\cap M}(e_0) \in \mathbb{C}$ *for the Jones projection* $e_0 \in M$.

Proof. 1°, 2° follow by Section 2 and 3° (a) is a particular case of 1.5 in [W]. Then 3°, (b) follows by [PiPo1] and, again, by [W]. Q.E.D.

6.3 Lemma. *If* $N \subset M$ *has finite depth then* $N_i \subset N_{i-1}$ *and* $M_{i-1} \subset M_i$ *have finite depth for all* $i \geq 1$. *Moreover the standard matrices for* $N_{2i+1} \subset N_{2i}$, $M_{2i} \subset M_{2i+1}$ *coincide with that of* $N_1 \subset N$ *and the standard matrices of* $N_{2i} \subset N_{2i-1}$, $M_{2i-1} \subset M_{2i}$ *coincide with that of* $N \subset M$.

Proof. If C_i is the matrix of the inclusion $N_i' \cap N \subset N_i' \cap M$ then by 2.9 in [PiPo3] we have $\|C_i\|^2 \leq [M:N]$. Since $\sup \dim \mathscr{Z}(N_i' \cap M) < \infty$ it follows that $\sup \dim \mathscr{Z}(N_i' \cap N) < \infty$. Thus $N_1 \subset N$ has finite depth. The rest follows by 1.8 in [PiPo1]. Q.E.D.

The previous lemma and 6.1 show that in the finite depth case the sequence of commuting squares

$$\begin{array}{ccccccc} M' \cap M_{2i_0} & \subset & M' \cap M_{2i_0+1} & \subset & M' \cap M_{2i_0+2} & \subset & \dots \\ \cup & & \cup & & \cup & & \\ M_1' \cap M_{2i_0} & \subset & M_1' \cap M_{2i_0+1} & \subset & M_1' \cap M_{2i_0+2} & \subset & \dots \end{array}$$

for i_0 large enough comes by iterating the basic construction for the first commuting square, like in 6.1. The next result clarifies what are the inclusion matrices.

6.4 Lemma. *Assume* $N \subset M$ *has finite depth and let* $A = (a_{kl})_{k \in K, l \in L}$, $k_0 \in K$, $A^1 = (a_{k^1 l^1})_{k^1 \in K^1, l^1 \in L^1}$, $k_0^1 \in K^1$, *be the standard matrices of* $N \subset M$ *respectively* $M \subset M_1$. *Let* M_n *be the Jones' tower of factors and* $i_0 \geq \operatorname{card} K$, $\operatorname{card} L^1 + 1$. *Then the commuting square*

$$\begin{array}{ccc} M' \cap M_{2i_0} & \subset_A & M' \cap M_{2i_0+1} \\ \cup & & \cup \\ M_1' \cap M_{2i_0} & \subset_{A^{1t}} & M_1' \cap M_{2i_0+1} \end{array}$$

has the corresponding inclusion matrices and satisfies the conditions of 6.1. *Moreover there are trace preserving isomorphisms between the pairs* $M_1' \cap M_{2i_0} \subset M' \cap M_{2i_0}$ *and* $M' \cap M_{2i_0-1} \subset M' \cap M_{2i_0}$ *and respectively* $M_1' \cap M_{2i_0+1} \subset M' \cap M_{2i_0+1}$ *and* $M_1' \cap M_{2i_0+1} \subset M_1' \cap M_{2i_0+2}$. *Via these identifications the inclusion matrices for* $M_1' \cap M_{2i_0} \subset M' \cap M_{2i_0}$ *and* $M_1' \cap M_{2i_0+1} \subset M' \cap M_{2i_0+1}$, *in the commuting square, are* A^t *respectively* A^1, *in particular* $\operatorname{card} L = \operatorname{card} L^1$.

Proof. The first part is clear by 6.3. The last part follows by observing that if J_{i_0}, J_{i_0+1} are the canonical conjugations in $L^2(M_{i_0}, \tau)$ respectively $L^2(M_{i_0+1}, \tau)$ then

$J_{i_0}(M' \cap M_{2i_0}) J_{i_0} = M' \cap M_{2i_0}$, $J_{i_0}(M'_1 \cap M_{2i_0}) J_{i_0} = M' \cap M_{2i_0-1}$ and $J_{i_0+1}(M'_1 \cap M_{2i_0+1}) J_{i_0+1} = M'_1 \cap M_{2i_0+1}$, $J_{i_0+1}(M' \cap M_{2i_0+1}) J_{i_0+1} = M'_1 \cap M_{2i_0+2}$. But by 6.3, 3.2, the corresponding applications $J_{i_0} \cdot J_{i_0}$, $J_{i_0+1} \cdot J_{i_0+1}$ are trace preserving. Since the algebras are finite dimensional it follows that there exist the required trace preserving isomorphisms as well. Q.E.D.

We mention that the use of the reflections $J_i \cdot J_i$ as antiautomorphisms of $M' \cap M_{2i}$ was discovered by M. Pimsner and the author and, independently, by Ocneanu ([Oc]). Moreover in ([Oc]) the reflections and the commuting square conditions are suitably used to give a complete intrinsic description of the sequence $\{M'_1 \cap M_i \subset M' \cap M_i\}$.

6.5 Definition. If $N \subset M$ has finite depth and $A = (a_{kl})_{\substack{k \in K \\ l \in L}}$, is its standard matrix and if $i_0 = \max\{\operatorname{card} K, \operatorname{card} L + 1\}$ then

$$\begin{array}{ccc} M' \cap M_{2i_0} & \subset & M' \cap M_{2i_0+1} \\ \cup & & \cup \\ M'_1 \cap M_{2i_0} & \subset & M'_1 \cap M_{2i_0+1} \end{array}$$

is called the *canonical commuting square* associated with $N \subset M$.

We say that two commuting squares

$$\begin{array}{ccc} B^0_{0i} & \subset & B^0_{1i} \\ \cup & & \cup \\ B^1_{0i} & \subset & B^1_{1i} \end{array}$$

$i = 1, 2$, are isomorphic iff there is a trace preserving isomorphism of B^0_{11} onto B^0_{12} carrying B^i_{j1} onto B^i_{j2}, $i, j = 0, 1$.

We can now derive the classification result for subfactors, in terms of classification of their canonical commuting squares.

6.6 Theorem. *Let $N \subset M$, $N^0 \subset M^0$ be pairs of hyperfinite type* II_1 *factors with finite depth. Then $N \subset M$ and $N^0 \subset M^0$ are isomorphic iff their associated canonical commuting squares are isomorphic. Thus, the subfactors of finite depth of the hyperfinite type* II_1 *factor are completely classified by their canonical commuting squares.*

Proof. If the commuting squares are isomorphic then their corresponding iterated sequences will be isomorphic by 6.1, i.e. there will be a trace preserving $*$-isomorphism of $\cup(M' \cap M_n)$ onto $\bigcup_n (M^{0\prime} \cap M^0_n)$ carrying $M' \cap M_n$ onto $M^{0\prime} \cap M^0_n$, $M'_1 \cap M_n$ onto $M^{0\prime}_1 \cap M^0_n$. Taking $M_0 = \overline{\cup(M' \cap M_n)}^w$, $N_0 = \overline{\cup(M'_1 \cap M_n)}^w$, $M^0_0 = \overline{\bigcup_n (M^{0\prime} \cap M^0_n)}^w$, $N^0_0 = \overline{\bigcup_n (M^{0\prime}_1 \cap M^0_n)}^w$, it follows that $N_0 \subset M_0$ is isomorphic to $N^0_0 \subset M^0_0$. By Theorem 4.9 it follows that $N \subset M$ is isomorphic to $N^0 \subset M^0$. The converse implication is trivial. Q.E.D.

As the previous theorem shows, classifying the subfactors of finite depth amounts to the investigation of the corresponding commuting squares. By 6.4 the algebras, the traces and the horizontal inclusions involved are completely determined by the standard (pointed!) matrices of $N \subset M$, $M \subset M_1$. Moreover the vertical inclusions are determined by the same matrices up to some possible permutations of the sets of indices. These matrices and the permutations must satisfy some very strong conditions of matching the traces and the dimensions: for instance by 6.4 the Perron Frobenius eigenvectors of $A^t A$ and $A^{1t} A^1$ must coincide (via the identification of L, L^1 resulting from 6.4). Also the dimensions should satisfy $\tilde{A}(A^1)^t (A^1 A^{1t})^{i-1} v_0^1 = (AA^t)^i v_0$, where $v_0 = (\delta_{k_0 k})_k$, $v_0^1 = (\delta_{k_0^1 k^1})_{k^1}$ and $\tilde{A} = (\tilde{a}_{kl^1})_{\substack{k \in K \\ l^1 \in L^1}}$ with $\tilde{a}_{kl^1} = a_{kl}$, $l^1 \to l$ being the identification of L with L^1 given by 6.4.

In many situations these conditions impose $A^1 = A$. This can be easily seen for $\| A \|^2 = [M:N] < 4$, where one has to analyse only the matrices coming from the Dynkin diagrams A_n, D_n, E_6, E_7, E_8 and in fact even for $\| A \|^2 < 2 + \sqrt{5}$. Then direct simple computations can be used to deduce that if

$$\begin{array}{ccc} & B_0 \subset_A B_1 & \\ A^t & \cup \qquad \cup & A \\ & B_0^0 \subset B_1^0 & \\ & A^t & \end{array}$$

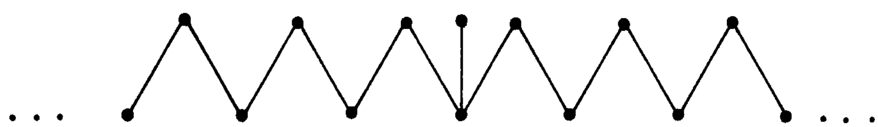
is a commuting square with $A = A_n$, D_n, E_6, E_7, E_8 then there will be no small perturbations of B_1^0 to still satisfy the commuting square condition with B_0, B_0^0, unless it is of the form $vwB_1^0 w^* v^*$ for some unitary elements $w \in B_0$, $v \in B_0^1 \cap B_1$, with $wB_0^0 w^* = B_0^0$. Thus the number of commuting squares in 6.5 with given matrices is bounded. But if $[M:N] < 4$ then $N \subset M$ automatically has finite depth and $N' \cap M = \mathbf{C}$ (as first noted by Jones). Indeed if $[M:N] < 4$ and $N \subset M \subset^{e_1} M_1 \subset^{e_2} M_2 \subset \ldots$ is the Jones' tower then $e_1 \vee e_2 \vee \ldots \vee e_n = 1$ (cf. [J1]) so $\mathrm{Alg}\{e_1, \ldots, e_{n-1}\} e_n \ \mathrm{Alg}\{e_1, \ldots, e_{n-1}\}$ has support 1. Thus $N' \cap M_{n-1} e_n N' \cap M_{n-1}$ has support 1 so $N \subset M$ has finite depth by Section 3. Thus Theorem 6.6 completely classifies the subfactors of index less than 4 in terms of their associated commuting squares, coming from their higher relative commutants picture, which in turn, by the above discussion, are finite in number. In fact, in case $\| A \|^2 < 4$, the above observation on the combinatorial problem of estimating the number of canonical commuting squares is superceded by the complete list of such objects corresponding to indices less than 4 announced in [Oc]. We use in the next corollary only the finiteness of the number of canonical commuting squares of small indices resulting from the above remarks, without however giving further details on the proof.

6.7 Corollary. *Up to conjugacy, there are only finitely many subfactors N of index $4\cos^2 \frac{\pi}{n}$ in the hyperfinite II_1 factor M, for each $n \geqq 3$, and all have finite depth. Moreover if the subfactor N has only the Jones projections $e_1, \ldots, e_n$ in its relative commutant in M_n then $N \subset M$ coincides with the Jones' subfactor of index $4\cos^2 \frac{\pi}{n}$.*

Note that the last part of the corollary is just a simple consequence of 4.9 and of Jones' result ([J1]) that if $[M:N] = 4\cos^2\frac{\pi}{n}$ and $N' \cap M_n = \mathrm{Alg}\{e_1, \ldots, e_n\}$ then $N' \cap M_k = \mathrm{Alg}\{e_1, \ldots, e_k\}$ for all k. Thus $M_1' \cap \tilde{M} \subset M' \cap \tilde{M}$ coincides with the Jones' pair of factors of index $4\cos^2\frac{\pi}{n}$ which is antiisomorphic to itself and thus it is isomorphic to $N \subset M$ by 4.9.

We mention that one can prove under very general assumptions that the commuting square problem has only finitely many solutions (up to isomorphism), when the dimensions of the algebras involved are fixed.

As concerning the index > 4, the case $< 2 + \sqrt{5}$ is still tractable, since the matrices of such small norm can actually be computed. One can show for instance that only finitely many of them may produce commuting squares like in 6.4. A first result of this type was proved by M. Pimsner and the author (1983 unpublished) for finite matrices tending to the infinite matrix:



In fact to show this it is sufficient to consider the infinite matrix itself describing the matricial inclusion of type II_1 algebras with atomic centers like in [PiPo3], where the problem becomes quite easy. The same arguments can be used to settle the case of the other infinite matrices (and thus accumulation points) of square norm less than $2 + \sqrt{5}$.

This shows that there can be only finitely many subfactors of finite depth and index between 4 and $2 + \sqrt{5}$ in the hyperfinite II_1 factor. Note however that although by [PiPo1] subfactors of index $< 2 + \sqrt{5}$ satisfying $E_{N' \cap M}(e_0) \in \mathbf{C}$ automatically satisfy $N' \cap M = \mathbf{C}$, the finite depth condition doesn't follow automatically.

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