AN EXACT ANALYTICAL SOLUTION OF KEPLER'S EQUATION

C. E. SIEWERT

Dept. of Nuclear Engineering, North Carolina State University, Raleigh, N.C., U.S.A.

and

E. E. BURNISTON

Dept. of Mathematics, North Carolina State University, Raleigh, N.C., U.S.A.

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Abstract. Complex-variable analysis is used to develop an exact solution to Kepler's equation, for both elliptic and hyperbolic orbits. The method is based on basic properties of canonical solutions to appropriately posed Riemann problems, and the final results are expressed in terms of elementary quadratures.

1. Introduction

Kepler's equation is, of course, basic to studies of celestial mechanics and consequently has been of interest for several centuries. For elliptic orbits, the equation is normally written as

$$e \sin E = E - M\,, \tag{1}$$

whereas for hyperbolic orbits the form is

$$e \sinh F = F + N\,. \tag{2}$$

In Equation (1), we presume that the eccentricity $e \in (0,1)$ and the mean anomaly $M \in [0,2\pi]$ are given and thus consider the equation transcendental in E, the eccentric anomaly. In Equation (2), we view $e > 1$ and the hyperbolic mean anomaly $N > 0$ as given and thus seek the hyperbolic eccentric anomaly F.

To solve Equation (1), we first let

$$E = M + (e/z)\,, \qquad \omega = (1/e)\,, \quad \text{and} \quad \xi = (M/e) \tag{3}$$

and subsequently consider the equivalent problem of seeking the zeros of

$$\Lambda(z) = 1 + \xi z - \omega z \sin^{-1}(1/z) \tag{4}$$

in an appropriately cut plane. We note that each branch of

$$\sin^{-1}\frac{1}{z} = k\pi + (-1)^k \left[\frac{\pi}{2} - i \log\left[f(z) + \frac{1}{z}\right]\right], \qquad k = 0, \pm 1, \pm 2, \ldots, \tag{5}$$

with $\log z$ denoting the principal branch of the log-function,

$$f(z) = \sqrt{\frac{1}{z^2} - 1}\,, \quad \text{and} \quad f(\infty) = i\,, \tag{6}$$

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is analytic in the plane cut from -1 to 1 along the real axis. It therefore follows that any zero $z_{k\alpha} \notin (-1, 1)$ of

$$\Lambda_k(z) = 1 + \xi z - \omega z\left[k\pi + (-1)^k \frac{\pi}{2} - i(-1)^k \log\left[f(z) + \frac{1}{z}\right]\right],$$
$$k = 0, \pm 1, \pm 2, \dots, \qquad (7)$$

will yield, by way of Equations (3), a solution of Kepler's equation, in the elliptic form. We note that only the branch corresponding to $k=1$ in Equation (5) is such that $\sin^{-1}(-z) = -\sin^{-1} z$.

In a similar manner, it follows that any zero $z_{k\alpha}$ of

$$\hat{\Lambda}_k(z) = 1 + \zeta z - \omega z\left[k\pi + (-1)^k \frac{\pi}{2} - i(-1)^k \log\left[f(z) + \frac{1}{z}\right]\right],$$
$$k = 0, \pm 1, \pm 2, \dots, \qquad (8)$$

can be used with

$$F = -N + i\frac{e}{z}, \qquad \omega = \frac{1}{e}, \quad \text{and} \quad \zeta = i\frac{N}{e} \qquad (9)$$

to yield a solution to Kepler's equation, in the hyperbolic form. Here we use the circumflex superscript to denote quantities basic to the solution of Equation (2).

2. Basic Analysis

Our previously reported procedure (Siewert and Burniston, 1972; Burniston and Siewert, 1972) for solving a class of transcendental equations is based on the fact that if an appropriate Riemann problem can be formulated, then the solution(s) of the considered transcendental equation can be expressed in terms of a canonical solution of that Riemann problem.

From Equation (7), we find the boundary values of $\Lambda_k(z)$, as z approaches the branch cut $[-1, 1]$ from above $(+)$ and below $(-)$, to be

$$\Lambda_k^{\pm}(t) = 1 + t[\xi - \omega\pi\Delta(k)] + (-1)^k\omega\frac{\pi}{2}|t| \mp i(-1)^k\omega t C(t), \qquad (10)$$

where we have let

$$\Delta(k) = k + (-1)^k \qquad (11)$$

and

$$C(t) = \ln\left[f(t) + \frac{1}{|t|}\right]. \qquad (12)$$

We now introduce

$$\Omega_k(z) = \Lambda_k(z)\,\Lambda_k(-z) \qquad (13)$$

and consider the Riemann problem defined by the boundary condition

$$\Phi_k^+(t) = G_k(t)\,\Phi_k^-(t), \qquad t \in (0, 1), \qquad (14)$$

where

$$G_k(t) = \frac{\Omega_k^+(t)}{\Omega_k^-(t)} = \exp\left[2i \arg \Omega_k^+(t)\right]. \tag{15}$$

Here we seek a function $\Phi_k(z)$ which is analytic in the plane cut from 0 to 1 along the real axis and nonvanishing in the finite plane. Since $G_k(t)$ is continuous (Muskhelishvili, 1953) and nonvanishing for $t\in(0,1)$, the desired canonical solution can be written as

$$\Phi_k(z) = (1-z)^{-\aleph_k} \exp\left[\frac{1}{\pi}\int_0^1 \arg \Omega_k^+(t) \frac{dt}{t-z}\right], \tag{16}$$

with $\arg \Omega_k^+(0)=0$. In addition the index $\aleph_k$ is such that $2\pi\aleph_k$ is the change in the argument of $G_k(t)$ as t varies from 0 to 1.

We now restrict our efforts to seeking *real* solutions of Equation (1), for $e\in(0,1)$ and $M\in[0,2\pi]$, and thus in view of Equations (3) can consider in Equation (7) only those values of k for which the resulting $\Lambda_k(z)$ will have real zeros. It is interesting to observe that the relevant values of k depend on the eccentricity and the mean anomaly. The argument principle (Ahlfors, 1953) can be used to determine, for a fixed $e\in(0,1)$ and $M\in[0,2\pi]$, the number of zeros of $\Omega_k(z)$ in the cut plane. In general, this number can be expressed as $2(\aleph_k+1)$. We find, upon considering *all* k, that $\aleph_k$ can be -1, 0, 1, or 2. The case $\aleph_k=-1$ is not interesting since the corresponding $\Omega_k(z)$ does not have any zeros in the cut plane. For $\aleph_k=0$, we find that $\Omega_k(z)$ has two real zeros in the cut plane, and thus we must consider all such possibilities; however, since we are seeking only the real solutions of Equation (1), we can dismiss all cases for which $\aleph_k=1$. We conclude that we must also include all possibilities for which $\aleph_k=2$.

For the allowed $e\in(0,1)$ and $M\in[0,2\pi]$, we can summarize our conclusions regarding the appropriate values of k and $\aleph_k$ in terms of the three open regions

$$R_1 : e < \frac{\pi}{2} - M,$$

$$R_2 : e > \frac{\pi}{2} - M \quad \text{and} \quad e > M - \frac{3\pi}{2},$$

$$R_3 : e < M - \frac{3\pi}{2},$$

such that

$$\{e, M\}\in R_1 \Rightarrow k = 1 \quad \text{and} \quad \aleph_1 = 2,$$
$$\{e, M\}\in R_2 \Rightarrow k = 0 \quad \text{and} \quad \aleph_0 = 0,$$
$$\{e, M\}\in R_3 \Rightarrow k = 3 \quad \text{and} \quad \aleph_3 = 2.$$

Of course, Equation (1) can be solved immediately for any of the special values of $\{e, M\}$.

Observing that the considered $\Omega_k(z)$ is such that

$$\Omega_k(z) = \Omega_k(-z) \quad \text{and} \quad \Omega_k(z) = \overline{\Omega_k(\bar{z})}, \tag{17}$$

where the bar denotes the complex conjugate, we conclude that $\Omega_k(z)\,\Phi_k^{-1}(-z)$ is also a solution to the Riemann problem defined by Equation (14). It therefore follows that

$$\Omega_k(z)\,\Phi_k^{-1}(-z) = \Phi_k(z)\,P_k(z), \tag{18}$$

where $P_k(z)$ is a polynomial in z. Since $\Phi_k(z)$ is nonvanishing in the finite plane, we deduce that

$$P_k(z) = B_k^2 \prod_{\alpha=1}^{\aleph_k+1} [z_{k\alpha}^2 - z^2], \tag{19}$$

with

$$B_k = \xi - \omega\pi\Delta(k), \tag{20}$$

and we can therefore write Equation (18) as

$$\Omega_k(z) = \Phi_k(z)\,\Phi_k(-z)\,B_k^2 \prod_{\alpha=1}^{\aleph_k+1} [z_{k\alpha}^2 - z^2]. \tag{21}$$

If we now consider the hyperbolic form of Kepler's equation, we find analogously to the foregoing, that

$$\hat{\Omega}_1(z) = \hat{\Lambda}_1(z)\,\hat{\Lambda}_1(-z) \tag{22}$$

can be factored in the manner

$$\hat{\Omega}_1(z) = \hat{\Phi}_1(z)\,\hat{\Phi}_1(-z)\,\zeta^2 \prod_{\alpha=1}^{\hat{\aleph}_1+1} [z_{1\alpha}^2 - z^2], \tag{23}$$

where

$$\hat{\Phi}_1(z) = (1-z)^{-\hat{\aleph}_1} \exp\left[\frac{1}{\pi}\int_0^1 \arg\hat{\Omega}_1^+(t)\,\frac{dt}{t-z}\right], \tag{24}$$

with $\arg\hat{\Omega}_1^+(0)=0$. In regard to Equation (2), we note that only the case $k=1$ in Equation (8) need be considered, since we seek only the real solution of Kepler's equation; however, we find again that the index $\hat{\aleph}_1$ depends on the parameters, e and N. In terms of the open regions defined by

$$\hat{R}_1:\ e\cosh N > \frac{\pi}{2},$$

$$\hat{R}_2:\ e\cosh N < \frac{\pi}{2},$$

we find that

$$\{e, N\} \in \hat{R}_1 \Rightarrow \hat{\aleph}_1 = 0,$$
$$\{e, N\} \in \hat{R}_2 \Rightarrow \hat{\aleph}_1 = 2.$$

We prefer to consider separately those special values of e and N for which

$$\{e, N\} \in \hat{R}_s \Rightarrow e\cosh N = \frac{\pi}{2}, \tag{25}$$

since here the boundary values $\hat{\Omega}_1^+(t)$ have zeros on the cut $[-1,1]$. It follows that, for $\{e, N\} \in \hat{R}_s$, the argument of the coefficient

$$\hat{G}_1(t) = \frac{\hat{\Omega}_1^+(t)}{\hat{\Omega}_1^-(t)}, \tag{26}$$

in the Riemann problem defined by

$$\hat{\Phi}_1^+(t) = \hat{G}_1(t)\,\hat{\Phi}_1^-(t), \qquad t \in (0, 1), \tag{27}$$

is discontinuous at

$$t = t_0 = \frac{2e}{\pi}. \tag{28}$$

It thus follows immediately that Equation (2) admits the solutions

$$F = -N \pm i\frac{\pi}{2}, \qquad \{e, N\} \in \hat{R}_s, \tag{29}$$

but, of course, we seek a real solution. A canonical solution of equation (27) can be written as

$$\hat{\Phi}_1(z) = (t_0 - z)^{-1} \exp\left[\frac{1}{\pi}\int_0^1 \arg \hat{\Omega}_1^+(t)\,\frac{\mathrm{d}t}{t - z}\right], \qquad \{e, N\} \in \hat{R}_s, \tag{30}$$

with $\arg \hat{\Omega}_1^+(0) = 0$. It is important to note here, for $\{e, N\} \in \hat{R}_s$, that $\arg \hat{\Omega}_1^+(t)$ has a discontinuity of π at $t = t_0$, so that $\arg \hat{\Omega}_1^+(1) = 0$. Equation (30) can now be used to factor $\hat{\Omega}_1(z)$ in the manner

$$\hat{\Omega}_1(z) = \hat{\Phi}_1(z)\,\hat{\Phi}_1(-z)\,\zeta^2\,[t_0^2 - z^2]\,[z_{11}^2 - z^2], \qquad \{e, N\} \in \hat{R}_s. \tag{31}$$

3. Explicit Solutions

Having established the required formalism, we are now able to solve Equation (1) almost immediately. Since our explicit solution is most concise for $\{e, M\} \in R_2$, we consider that case first. Noting for this case that $k = 0$ and $\aleph_0 = 0$, we can solve Equation (21) to obtain

$$z_{01}^2 = z^2 + \Omega_0(z)\,[B_0^2 \Phi_0(z)\,\Phi_0(-z)]^{-1}, \qquad \{e, M\} \in R_2, \tag{32}$$

which can be evaluated at any convenient z to yield $\pm z_{01}$, respectively the real zeros of $\Lambda_0(z)$ and $\Lambda_0(-z)$. If we now let $z = iy$, y real, then Equations (7) and (13) can be used to yield

$$e^2 \Omega_k(iy) = \left[e + (-1)^k y \ln\left[\sqrt{\frac{1}{y^2} + 1} + \frac{1}{y}\right]\right]^2 + y^2\,[M - \pi \Delta(k)]^2. \tag{33}$$

We also wish to define

$$E_k(iy) = \exp\left[-\frac{1}{\pi}\int_0^1 t \arg \Omega_k^+(t) \frac{dt}{t^2 + y^2}\right], \tag{34}$$

with

$$\arg \Omega_k^+(t) = \tan^{-1}\left[\frac{2(-1)^{k+1} t C(t)\left[e + (-1)^k \frac{\pi}{2}|t|\right]}{\left[e + (-1)^k \frac{\pi}{2}|t|\right]^2 - t^2[M - \pi\Delta(k)]^2 - t^2 C^2(t)}\right] \tag{35}$$

and arg $\Omega_k^+(0)=0$. Equation (32) can now be used with Equations (3) to establish the desired solution of Equation (1)

$$E = M - e(M - \pi)\left[e^2\Omega_0(iy) E_0^2(iy) - y^2(M - \pi)^2\right]^{-1/2}, \quad \{e, M\} \in R_2. \tag{36}$$

Different choices of y in Equation (36) can, of course, alter the computational merits of that result; the choice $y=0$ yields

$$E = M - (M - \pi)\exp\left[\frac{1}{\pi}\int_0^1 \arg\Omega_0^+(t)\frac{dt}{t}\right], \qquad \{e, M\} \in R_2, \tag{37}$$

or, if we let y tend to infinity in Equation (36), we obtain the equivalent solution

$$\boxed{E = M - e(M - \pi)\left[(e + 1)^2 - (M - \pi)^2 \frac{2}{\pi}\int_0^1 t \arg\Omega_0^+(t)\,dt\right]^{-1/2}, \quad \{e, M\} \in R_2.} \tag{38}$$

We must now consider the remaining cases $\{e, M\} \in R_1$ or R_3. For both of these cases the appropriate $\aleph_k$ is 2, and thus we choose to evaluate Equation (21) at three distinct points, say $z=iy$, $y=\alpha$, β and γ, to obtain

$$F_k(i\alpha) = [z_{k1}^2 + \alpha^2][z_{k2}^2 + \alpha^2][z_{k3}^2 + \alpha^2], \tag{39a}$$

$$F_k(i\beta) = [z_{k1}^2 + \beta^2][z_{k2}^2 + \beta^2][z_{k3}^2 + \beta^2], \tag{39b}$$

and

$$F_k(i\gamma) = [z_{k1}^2 + \gamma^2][z_{k2}^2 + \gamma^2][z_{k3}^2 + \gamma^2], \tag{39c}$$

for $k=1$ or 3. Here

$$F_k(iy) = \frac{e^2\Omega_k(iy)[1 + y^2]^2 E_k^2(iy)}{[M - \pi\Delta(k)]^2}. \tag{40}$$

Equations (39) can be reduced by elimination to yield a cubic equation in $z_{k\alpha}^2$. The resulting cubic equation can, naturally, be solved analytically; some care must be exercised, however, in selecting the correct value of $z_{k\alpha}$ to be used with Equations (3) to yield the real solution of Equation (1). We find that the desired solution of Equation (1) can be expressed as

$$E = M + e(2-k)\left[S_{1k}(\alpha,\beta,\gamma) + S_{2k}(\alpha,\beta,\gamma) - \tfrac{1}{3}A_{2k}(\alpha,\beta,\gamma)\right]^{-1/2}, \quad \{e, M\} \in R_k, \quad k = 1 \text{ or } 3, \tag{41}$$

where

$$S_{jk}(\alpha,\beta,\gamma) = \left[D_k(\alpha,\beta,\gamma) - (-1)^j\left[D_k^2(\alpha,\beta,\gamma) + Q_k^3(\alpha,\beta,\gamma)\right]^{1/2}\right]^{1/3}, \tag{42}$$

with

$$D_k(\alpha,\beta,\gamma) = \tfrac{1}{6}\left[A_{1k}(\alpha,\beta,\gamma)\,A_{2k}(\alpha,\beta,\gamma) - 3A_{0k}(\alpha,\beta,\gamma)\right] - \left[\tfrac{1}{3}A_{2k}(\alpha,\beta,\gamma)\right]^3 \tag{43}$$

and

$$Q_k(\alpha,\beta,\gamma) = \tfrac{1}{3}A_{1k}(\alpha,\beta,\gamma) - \left[\tfrac{1}{3}A_{2k}(\alpha,\beta,\gamma)\right]^2. \tag{44}$$

In addition,

$$A_{0k}(\alpha,\beta,\gamma) = \alpha^2\beta^2\gamma^2 + \beta^2\gamma^2(\beta^2-\gamma^2)\,TF_k(i\alpha) + \gamma^2\alpha^2(\gamma^2-\alpha^2)\,TF_k(i\beta) + \alpha^2\beta^2(\alpha^2-\beta^2)\,TF_k(i\gamma), \tag{45a}$$

$$A_{1k}(\alpha,\beta,\gamma) = \alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2 + (\beta^4-\gamma^4)\,TF_k(i\alpha) + (\gamma^4-\alpha^4)\,TF_k(i\beta) + (\alpha^4-\beta^4)\,TF_k(i\gamma), \tag{45b}$$

and

$$A_{2k}(\alpha,\beta,\gamma) = \alpha^2 + \beta^2 + \gamma^2 + (\beta^2-\gamma^2)\,TF_k(i\alpha) + (\gamma^2-\alpha^2)\,TF_k(i\beta) + (\alpha^2-\beta^2)\,TF_k(i\gamma), \tag{45c}$$

where

$$T = \left[(\alpha^2-\beta^2)(\beta^2-\gamma^2)(\gamma^2-\alpha^2)\right]^{-1}. \tag{46}$$

The choice of α, β, and γ in Equation (41) can, of course, alter the accuracy of a calculational scheme based on that result; a convenient choice is $\alpha=0$, $\beta=1$, and $\gamma=2$, for which Equations (45) reduce to

$$A_{0k}(0,1,2) = -F_k(0), \tag{47a}$$

$$A_{1k}(0,1,2) = 4 - \tfrac{5}{4}F_k(0) + \tfrac{4}{3}F_k(i) - \tfrac{1}{12}F_k(2i), \tag{47b}$$

and

$$A_{2k}(0,1,2) = 5 - \tfrac{1}{4}F_k(0) + \tfrac{1}{3}F_k(i) - \tfrac{1}{12}F_k(2i). \tag{47c}$$

Equations (47) can be used to write an analytical solution of Equation (1) as

$$E = M + e(2-k)\left[S_{1k}(0,1,2) + S_{2k}(0,1,2) - \tfrac{1}{3}A_{2k}(0,1,2)\right]^{-1/2}, \quad \{e, M\} \in R_k, \quad k = 1 \text{ or } 3. \tag{48}$$

In order to establish, for $\{e, M\} \in R_1$ or R_3, a solution to Equation (1) equivalent in form to Equation (38), we now replace α, β, and γ in Equation (41) by αt, βt, and γt and subsequently observe the limit as t tends to infinity to obtain

$$\boxed{E = M + e(2-k)\left[S_{1k} + S_{2k} - \tfrac{1}{3}A_{2k}\right]^{-1/2}, \quad \{e, M\} \in R_k, \quad k = 1 \text{ or } 3}\,, \tag{49}$$

where

$$S_{jk} = [D_k - (-1)^j [D_k^2 + Q_k^3]^{1/2}]^{1/3}, \tag{50}$$

$$D_k = \tfrac{1}{6}[A_{1k}A_{2k} - 3A_{0k}] - [\tfrac{1}{3}A_{2k}]^3, \tag{51}$$

and

$$Q_k = \tfrac{1}{3}A_{1k} - [\tfrac{1}{3}A_{2k}]^2. \tag{52}$$

To complete the solution given by Equation (49), we list the explicit results

$$A_{0k} = \tfrac{4}{3}I_{1k}^3 + 4I_{1k}I_{3k} + 2I_{5k} + J(k)[M - \pi\Delta(k)]^{-2}, \tag{53a}$$

$$A_{1k} = 2I_{1k}^2 + 2I_{3k} + [-2(e-1)^2 I_{1k} + \tfrac{1}{3}(e-1)][M - \pi\Delta(k)]^{-2}, \tag{53b}$$

and

$$A_{2k} = 2I_{1k} - (e-1)^2 [M - \pi\Delta(k)]^{-2}, \tag{53c}$$

where

$$J(k) = -2(e-1)^2 [I_{1k}^2 + I_{3k}] + (e-1)[\tfrac{2}{3}I_{1k} + \tfrac{3}{20}] - \tfrac{1}{36} \tag{54}$$

and

$$I_{\alpha k} = \frac{1}{\pi}\int_0^1 t^\alpha \arg\Omega_k^+(t)\,\mathrm{d}t - \frac{2}{\alpha+1}. \tag{55}$$

Equations (36; 37; 38) and (41; 48; 49) are our final solutions of the elliptic form of Kepler's equation. Although we have restricted our attention to those values of e and M of physical interest, we wish to emphasize that our method can be used for all values, real or complex. In the same vein, although we have sought only the real solutions of Equation (1), the method clearly is not restricted to real solutions; in fact, even for $\{e, M\} \in R_1$ or R_3 the complex solutions of Equation (1) deriving from the branches corresponding to $k=1$ and $k=3$ are immediately available from two of the complex solutions of Equations (39).

We now wish to solve Equation (2), the form of Kepler's equation appropriate to hyperbolic orbits. Except for those special values of e and N for which $\{e, N\} \in \hat{R}_s$, the solutions required here follow almost immediately from the previous considerations. First of all, for $\{e, N\} \in \hat{R}_1$, Equation (23) for $z = iy$, y real, yields

$$F = -N + eN[e^2\hat{\Omega}_1(iy)\hat{E}_1^2(iy) + N^2y^2]^{-1/2}, \qquad \{e, N\} \in \hat{R}_1, \tag{56}$$

after Equations (9) are invoked. Here

$$e^2\hat{\Omega}_1(iy) = \left[e - y\ln\left[\sqrt{\frac{1}{y^2} + 1} + \frac{1}{y}\right]\right]^2 - N^2y^2, \tag{57}$$

$$\hat{E}_1(iy) = \exp\left[-\frac{1}{\pi}\int_0^1 t\arg\hat{\Omega}_1^+(t)\frac{\mathrm{d}t}{t^2+y^2}\right], \tag{58}$$

and

$$\arg\hat{\Omega}_1^+(t) = \tan^{-1}\left[\frac{2tC(t)\left[e - \frac{\pi}{2}|t|\right]}{\left[e - \frac{\pi}{2}|t|\right]^2 + t^2[N^2 - C^2(t)]}\right], \tag{59}$$

with $\arg \hat{\Omega}_1^+(0)=0$. We can now set $y=0$, or let y tend to infinity, in Equation (56) to obtain two equivalent solutions of Equation (2):

$$F = -N + N \exp\left[\frac{1}{\pi}\int_0^1 \arg \hat{\Omega}_1^+(t)\frac{\mathrm{d}t}{t}\right], \qquad \{e, N\}\in \hat{R}_1 \tag{60}$$

and

$$\boxed{F = -N + eN\left[(e-1)^2 + N^2\frac{2}{\pi}\int_0^1 t \arg \hat{\Omega}_1^+(t)\,\mathrm{d}t\right]^{-1/2}, \qquad \{e, N\}\in \hat{R}_1} \,. \tag{61}$$

If we now consider $\{e, N\}\in \hat{R}_2$, then, in the established way, we can evaluate Equation (23) at three distinct points, say $z=i\alpha$, $z=i\beta$, and $z=i\gamma$, with α, β, and γ real, and solve the resulting equations to find

$$F = -N + e\left[\hat{S}_{11}(\alpha,\beta,\gamma) + \hat{S}_{21}(\alpha,\beta,\gamma) - \tfrac{1}{3}\hat{A}_{21}(\alpha,\beta,\gamma)\right]^{-1/2}, \qquad \{e, N\}\in \hat{R}_2, \tag{62}$$

where

$$\hat{S}_{j1}(\alpha,\beta,\gamma) = \left[\hat{D}_1(\alpha,\beta,\gamma) - (-1)^j\left[\hat{D}_1^2(\alpha,\beta,\gamma) + \hat{Q}_1^3(\alpha,\beta,\gamma)\right]^{1/2}\right]^{1/3}, \tag{63}$$

$$\hat{D}_1(\alpha,\beta,\gamma) = \tfrac{1}{6}\left[\hat{A}_{11}(\alpha,\beta,\gamma)\hat{A}_{21}(\alpha,\beta,\gamma) - 3\hat{A}_{01}(\alpha,\beta,\gamma)\right] - \left[\tfrac{1}{3}\hat{A}_{21}(\alpha,\beta,\gamma)\right]^3, \tag{64}$$

and

$$\hat{Q}_1(\alpha,\beta,\gamma) = \tfrac{1}{3}\hat{A}_{11}(\alpha,\beta,\gamma) - \left[\tfrac{1}{3}\hat{A}_{21}(\alpha,\beta,\gamma)\right]^2. \tag{65}$$

In addition,

$$\hat{A}_{01}(\alpha,\beta,\gamma) = -\alpha^2\beta^2\gamma^2 - \beta^2\gamma^2(\beta^2-\gamma^2)\,T\hat{F}_1(i\alpha) - \gamma^2\alpha^2(,^2-\alpha^2)\,T\hat{F}_1(i\beta) - \alpha^2\beta^2(\alpha^2-\beta^2)\,T\hat{F}_1(i\gamma), \tag{66a}$$

$$\hat{A}_{11}(\alpha,\beta,\gamma) = \alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2 + (\beta^4-\gamma^4)\,T\hat{F}_1(i\alpha) + (\gamma^4-\alpha^4)\,T\hat{F}_1(i\beta) + (\alpha^4-\beta^4)\,T\hat{F}_1(i\gamma), \tag{66b}$$

and

$$\hat{A}_{21}(\alpha,\beta,\gamma) = -\alpha^2 - \beta^2 - \gamma^2 - (\beta^2-\gamma^2)\,T\hat{F}_1(i\alpha) - (\gamma^2-\alpha^2)\,T\hat{F}_1(i\beta) - (\alpha^2-\beta^2)\,T\hat{F}_1(i\gamma), \tag{66c}$$

where

$$\hat{F}_1(iy) = -\frac{e^2}{N^2}\hat{\Omega}_1(iy)\left[1+y^2\right]^2\hat{E}_1^2(iy). \tag{67}$$

Again, the choice $\alpha=0$, $\beta=1$, and $\gamma=2$ yields

$$F = -N + e\left[\hat{S}_{11}(0,1,2) + \hat{S}_{21}(0,1,2) - \tfrac{1}{3}\hat{A}_{21}(0,1,2)\right]^{-1/2}, \qquad \{e, N\}\in \hat{R}_2, \tag{68}$$

with

$$\hat{A}_{01}(0,1,2) = \hat{F}_1(0), \tag{69a}$$

$$\hat{A}_{11}(0,1,2) = 4 - \tfrac{5}{4}\hat{F}_1(0) + \tfrac{4}{3}\hat{F}_1(i) - \tfrac{1}{12}\hat{F}_1(2i), \tag{69b}$$

and

$$\hat{A}_{21}(0, 1, 2) = -5 + \tfrac{1}{4}\hat{F}_1(0) - \tfrac{1}{3}\hat{F}_1(i) + \tfrac{1}{12}\hat{F}_1(2i). \tag{69c}$$

If we now let α, β, and γ tend to infinity, as previously, in Equation (62), we find

$$\boxed{F = -N + e[\hat{S}_{11} + \hat{S}_{21} - \tfrac{1}{3}\hat{A}_{21}]^{-1/2}, \qquad \{e, N\} \in \hat{R}_2}, \tag{70}$$

where

$$\hat{S}_{j1} = [\hat{D}_1 - (-1)^j [\hat{D}_1^2 + \hat{Q}_1^3]^{1/2}]^{1/3}, \tag{71}$$

$$\hat{D}_1 = \tfrac{1}{6}[\hat{A}_{11}\hat{A}_{21} - 3\hat{A}_{01}] - [\tfrac{1}{3}\hat{A}_{21}]^3, \tag{72}$$

and

$$\hat{Q}_1 = \tfrac{1}{3}\hat{A}_{11} - [\tfrac{1}{3}\hat{A}_{21}]^2. \tag{73}$$

In addition,

$$\hat{A}_{01} = -\tfrac{4}{3}\hat{I}_{11}^3 - 4\hat{I}_{11}\hat{I}_{31} - 2\hat{I}_{51} + \hat{J}(1)\,N^{-2}, \tag{74a}$$

$$\hat{A}_{11} = 2\hat{I}_{11}^2 + 2\hat{I}_{31} - [-2(e-1)^2\hat{I}_{11} + \tfrac{1}{3}(e-1)]\,N^{-2} \tag{74b}$$

and

$$\hat{A}_{21} = -2\hat{I}_{11} - (e-1)^2\,N^{-2}, \tag{74c}$$

where

$$\hat{J}(1) = -2(e-1)^2[\hat{I}_{11}^2 + \hat{I}_{31}] + (e-1)[\tfrac{2}{3}\hat{I}_{11} + \tfrac{3}{20}] - \tfrac{1}{36} \tag{75}$$

and

$$\hat{I}_{\alpha 1} = \frac{1}{\pi}\int_0^1 t^\alpha \arg\hat{\Omega}_1^+(t)\,\mathrm{d}t - \frac{2}{\alpha+1}. \tag{76}$$

Finally we need consider the special case $\{e, N\} \in \hat{R}_s$. If we solve Equation (31), with $z = iy$, for z_{11} and use that result with Equations (9), we find

$$F = -N + eN[e^2\hat{\Omega}_1(iy)\,\hat{E}_1^2(iy) + N^2y^2]^{-1/2}, \qquad \{e, N\} \in \hat{R}_s. \tag{77}$$

We can now set $y = 0$, or let y tend to infinity in Equation (77) to obtain

$$F = -N + N\exp\left[\frac{1}{\pi}\int_0^1 \arg\hat{\Omega}_1^+(t)\,\frac{\mathrm{d}t}{t}\right], \qquad \{e, N\} \in \hat{R}_s, \tag{78}$$

or

$$\boxed{F = -N + eN\left[(e-1)^2 + N^2\frac{2}{\pi}\int_0^1 t\arg\hat{\Omega}_1^+(t)\,\mathrm{d}t\right]^{-1/2}, \qquad \{e, N\} \in \hat{R}_s}. \tag{79}$$

Though Equations (77), (78), and (79) are the same as those derived for $\{e, N\} \in \hat{R}_1$, we wish to reiterate that for $\{e, N\} \in \hat{R}_s$ the resulting $\arg\hat{\Omega}_1^+(t)$ has a discontinuity of π at $t = t_0$, so that $\arg\hat{\Omega}_1^+(1) = 0$, $\{e, N\} \in \hat{R}_1$ *or* $\hat{R}_s$. Equations (56; 60; 61), (62; 68; 70), and (77; 78; 79) are our final solutions of the hyperbolic form of Kepler's equation. Although we have restricted our attention to those values of e and N of physical interest and have sought only real solutions of Equation (2), the method is not limited to either real parameters or real solutions.

A Gaussian quadrature integration procedure has been used to evaluate numerically all of our explicit solutions, for numerous cases, and accuracy to within ten significant figures was achieved, quite straightforwardly.

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