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On Nevanlinna's second main theorem in projective space

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Summary. We first prove a theorem concerning higher order logarithmic partial derivatives for meromorphic functions of several complex variables. Then we show the best nature of the second main theorem in Nevanlinna theory under two different assumptions of non-degeneracy of meromorphic mappings $f : \mathbb{C}^n \to \mathbb{P}^m$ for arbitrary positive integers n and m. Moreover, we derive a upper bound of the error term in the second main theorem for meromorphic mappings of finite order. Finally, we demonstrate the sharpness of all upper bounds in our main theorems.

1 Introduction

The most striking result in Nevanlinna theory is the second main theorem, which is an inequality relating two leading quantities, one is the characteristic function that measures the rate of growth of a function or map, the other is the counting function that tells the size of the preimages of points or sets, by means of an error term. Among all classical applications of the second main theorem, only the growth order of the error term was taken into account. However, motivated by P. Vojta's dictionary [17] between Nevanlinna theory and Diophantine Approximations in number theory, people, see [10] and [19] for instance, have started to find a precise form of the second main theorem because of the analog between the Siegel-Roth-Schmidt Theorem in number theory and the second main theorem in Nevanlinna theory. There are many different versions of the second main theorem in Nevanlinna theory, but the closest analog to number theory is the second main theorem for holomorphic curves, or more generally for maps such that the dimension of the domain is less than the dimension of the target. Historically speaking, this case was first studied by Ahlfors using higher order osculating curves, and by H. Cartan using logarithmic derivative lemmas when the dimension of the domain is one. Then Ahlfors' method was extended by many authors to the case when the dimension of the domain is greater than one and less than the dimension of the image space. Cartan's method was extended by A. Vitter [16], B. Shiffman [13], and S. Lang [8] in different contexts. Another approach towards the second main theorem is the negative curvature method, which started from J. Carlson and P. Griffiths [2] and was extended by many authors. However, no one had attempted to find a precise form of the second main theorem until Lang raised the question of a best possible error term in [9]. Other questions on this matter have been investigated by the author and others, e.g. see [15] and [21].

In 1990, Lang [11] found the best nature of the upper bound of the second main theorem by improving Pit-Mann Wong's method [18] in the equidimensional case. Later, W. Cherry [3] extended Lang's results to the case when the dimension of the domain is not less than the dimension of the image space under the assumption that the image of the map contains a non-empty open set in the image space, i.e. the map is non-degenerate. In this case, this definition of non-degenerate is quite natural. Thus the study of maps which decrease dimension can be reduced to that of equidimensional map. However, this definition of non-degenerate does not make any sense if the dimension of the domain is less than the dimension of the image space. This observation leads people to introduce a weaker non-degeneracy assumption: the image of the map is not contained in any hyperplane. We call a map with this property a linearly non-degenerate map. Under this weaker assumption, the proof of the second main theorem is much more difficult since people have to cope with the associated maps and higher order derivatives. Lately, Wong and W. Stoll [19] have obtained an estimate on the second main theorem for linearly non-degenerate meromorphic maps in the non-equidimensional case.

Let ψ and ϕ be increasing functions in $\mathbb{R}^+$ with

$$\int_e^\infty \frac{dr}{r\psi(r)} < \infty, \quad \text{and} \quad \int_e^\infty \frac{dr}{\phi(r)} = \infty. \tag{1.1}$$

However, for simplicity, we always regard $\phi(r) \leqq r$ in the sequal.

In order to avoid nonessential complications, we have restricted ourselves to discuss a meromorphic map $f : \mathbb{C}^n \to \mathbb{P}^m$ and hyperplanes (As a matter of fact, Lang, Cherry, and Wong and Stoll have more general settings in spaces and divisors). Suppose $H_1, H_2, \ldots, H_q$ are arbitrary hyperplanes in $\mathbb{P}^m$ in general position. We denote the error term of f in the second main theorem by

$$S(f, \{H_j\}_{j=1}^q, r) = (q-m-1)T_f(r) - \sum_{j=1}^q N_f(H_j, r) + N(R_f, r), \tag{1.2}$$

where R_f is a ramification divisor of f to be defined in Sect. 6 and 7 respectively. It is possible that $S(f, \{H_j\}_{j=1}^q, r)$ is negative. Thus the classical second main theorem (e.g. see [14]) states that there is a constant $C > 1$ such that

$$S(f, \{H_j\}_{j=1}^q, r) \leqq C \log(rT_f(r)) \tag{1.3}$$

for all large r outside a set of finite Lebesgue measure. If f is of finite order, then, for all large r,

$$S(f, \{H_j\}_{j=1}^q, r) \leqq C \log r\,. \tag{1.4}$$

In 1992, under the assumption of f being non-degenerate and $n \geqq m$, Cherry [3] extended Lang's result [10] and showed

$$S(f, \{H_j\}_{j=1}^q, r) \leqq m\,(\log T_f(r) + \log \psi(T_f^2(r)) + \log \psi(C\,r\,T_f^2(r)\psi(T_f^2(r)))) + O(1) \tag{1.5}$$

for all large r outside a set of finite Lebesgue measure, where ψ is defined as in (1.1).

Recently, under the assumption of f being linearly non-degenerate, and any n and m, Wong and Stoll [19] showed (1.3) can be improved to be, for any $\varepsilon > 0$,

$$S(f, \{H_j\}_{j=1}^q, r) \leqq \frac{m(m+1)}{2}(\log T_f(r) + (2+\varepsilon)\log\log T_f(r)) + O(\log r) \tag{1.6}$$

for all large r outside a set of finite Lebesgue measure.

The main purpose of this paper is to derive a precise form of the second main theorem of meromorphic maps under the above setting. In fact, we show the inequality (1.5), hence (1.3), can be sharpened to be

$$S(f, \{H_j\}_{j=1}^q, r) \leqq m(\log T_f(r) + \log \psi(T_f(r))) + O(1) \tag{1.7}$$

for all large r outside a set of finite Lebesgue measure, where ψ is defined as in (1.1); and the inequality (1.6), and hence (1.3), can be improved to

$$S(f, \{H_j\}_{j=1}^q, r) \leqq \frac{m(m+1)}{2}(\log T_f(r) + \log \psi(T_f(r))) + O(1) \tag{1.8}$$

for all large r outside a set of finite Lebesgue measure, where ψ is defined as in (1.1). Recall that there are many $\psi's$ such that $\psi(r) < \log^{2+\varepsilon}(r)$ for any $\varepsilon > 0$ and both (1.7) and (1.8) sharpen (1.3) in different cases. To accomplish these, we need Lemma 6 which concerns the higher order logarithmic partial derivative by means of reducing it to the one variable by fiber integration, instead of the negative curvature methods used by Vitter in [16]. This lemma has a different format than the usual logarithmic derivative lemma. The second technical ingredient is Lemma 8, which gives sharp estimates on the order of partial derivatives which come from the ramification divisor in the second main theorem. With these in hand, we are able to show inequalities (1.7) and (1.8).

Furthermore, the method in this paper brings out another interesting estimate on the error term of meromorphic mappings of finite order, which cannot be obtained from either Lang and Cherry's method or Wong and Stoll's. This is the first investigation of the error term of meromorphic mappings of finite order as far as the author knows. In order to match up (1.4), we have that if f is of finite order p, then, for any $\varepsilon > 0$,

$$S(f, \{H_j\}_{j=1}^q, r) \leqq m(p - 1 + \varepsilon) \log r \tag{1.9}$$

for all large r, where we assume that f is non-degenerate. When f is linearly non-degenerate and is of finite order p, then, for any $\varepsilon > 0$,

$$S(f, \{H_j\}_{j=1}^q, r) \leqq \frac{m(m+1)}{2}(p - 1 + \varepsilon) \log r \tag{1.10}$$

for any large r. It follows that $S(f, \{H_j\}_{j=1}^q, r)$ is always negative if the order of f is less than 1.

More interesting, there is a big difference between coefficients in (1.7) and (1.8) when $m > 1$; one is m, the other is $m(m+1)/2$. In the beginning, the author was skeptical about the sharpness of the coefficient $m(m+1)/2$ in (1.6), which was proved by Wong and Stoll in [19], although their results are very good considering their weaker assumption. Later, Wong convinced the author the coefficient $m(m+1)/2$ could be sharp and encouraged the author to prove the sharpness. Therefore, another important part of this paper is to verify the leading coefficients m in (1.7) and (1.9), and $m(m+1)/2$ in (1.8) and (1.10) are sharp. Thus one can claim that the leading coefficient in the error term for a linearly non-degenerate mapping should be $m(m+1)/2$. Since the maps constructed by the author are not linear degenerate and algebraically degenerate, it is an interesting problem to determine the best possible coefficient in the error term for an algebraically non-degenerate map. The author thanks the referee for suggesting to post this interesting question here.

There is no overlap between inequalities (1.7) and (1.8), also (1.9) and (1.10), when $m > 1$ since we trade off the upper bound with the non-degeneracy. Moreover, the method used in this paper is straightforward compared with the method used by Vitter [16], Lang [10] and Cherry [3], Wong and Stoll [19]. All their proofs are based on the complex differential geometry method. Additionally, Wong and Stoll's proof contains some very difficult estimates on the associated maps. However, our proofs are based on the Cartan's method by means of a new logarithmic partial derivative lemma. Furthermore, our results are better than theirs as we have discussed above.

2 Notations and preliminaries

For $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$, we define, for any $r \in \mathbb{R}^+$,

$$\|z\| = (|z_1|^2 + \cdots + |z_n|^2)^{1/2} \quad \text{and} \quad B_n(r) = \{z \in \mathbb{C}^n; \|z\| < r\},$$

$$S_n(r) = \{z \in \mathbb{C}^n; \|z\| = r\} \quad \text{and} \quad B_n[r] = \{z \in \mathbb{C}^n; \|z\| \leqq r\}.$$

Let $d = \partial + \bar{\partial}$ and $d^c = (\partial - \bar{\partial})/4\pi i$; we write,

$$\omega_n(z) = dd^c \log\|z\|^2 \quad \text{and} \quad \sigma_n(z) = d^c \log\|z\|^2 \wedge \omega_n^{n-1}(z), \quad \text{for } z \in \mathbb{C}^n \setminus \{0\}\,;$$

$$v_n(z) = dd^c \|z\|^2 \quad \text{and} \quad \rho_n(z) = v_n^n(z), \quad \text{for } z \in \mathbb{C}^n\,.$$

Thus $\sigma_n(z)$ defines a positive measure on $S_n(r)$ with total measure one and ρ_n is Lebesgue measure on $\mathbb{C}^n$ normalized such that $B_n(r)$ has measure r^{2n}. Moreover, when we restrict v_n to $S_n(r)$, we obtain that

$$v_n(z) = r^2 \omega_n(z) \quad \text{and} \quad \int_{B_n(r)} \omega_n^n = 1\,.$$

Let N and M be connected, complex manifolds of dimensions n and m respectively. Let U be a non-empty open subset of N such that $N \setminus U$ is an analytic set. Let $F : N \to M$ be a holomorphic map on U. Set $\Gamma = \{(x, F(x));\ x \in U\}$. The map F is said to be meromorphic on N if

(i) The closure of Γ in $N \times M$ is an analytic subvariety of $N \times M$.

(ii) The projection $\rho : \Gamma \to N$ is proper, i.e. $\Gamma \cap (K \times M)$ is compact for any compact subset K in N.

We call $I_f = \{x \in N;\ \#\rho^{-1}(x) > 1\}$ the set of indeterminacy. It is clear that I_f is an analytic subvariety of N of codimension at least 2. Therefore the U can be chosen so that $I_f = N \setminus U$. Furthermore if $F : \mathbb{C}^n \to \mathbb{P}^m$ is meromorphic, then F can be represented by a holomorphic mapping $f : \mathbb{C}^n \to \mathbb{C}^{m+1}$ such that $f = (f_0, f_1, \ldots, f_m)$, and

$$I_f = \{z \in \mathbb{C}^n; f_0(z) = f_1(z) = \cdots = f_m(z) = 0\}, \quad \text{and} \quad F = \pi \circ f \text{ on } U$$

where $\pi : \mathbb{C}^{m+1} \setminus \{0\} \to \mathbb{P}^m$ is $\pi(w) \equiv [w]$ = complex line through 0 and w. We call f a reduced representative of F (the only factors common to $f_0, \ldots, f_m$ are units). F will often be identified with its reduced representative f.

Let $I = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ be a multi-index with $\alpha_j \in \mathbb{Z}^+ \cup \{0\}$ with $1 \leqq j \leqq n$. We denote the length of I by $|I| = \sum_{j=1}^n \alpha_j$, and define

$$\partial^I f = \left(\frac{\partial^{|I|} f_1}{\partial z_1^{\alpha_1} \cdots \partial z_n^{\alpha_n}}, \ldots, \frac{\partial^{|I|} f_m}{\partial z_1^{\alpha_1} \cdots \partial z_n^{\alpha_n}} \right)$$

and $f_{z_j^k} = \partial^k f / \partial z_j^k = (\partial^k f_1 / \partial z_j^k, \ldots, \partial^k f_m / \partial z_j^k)$, for any holomorphic map

$$f = (f_1, \ldots, f_m) : \mathbb{C}^n \to \mathbb{C}^m\,.$$

Let N be a complex manifold of complex dimension n. We denote by $\mathcal{D}^{p,q}(N)$ the space of compactly supported complex valued $C^\infty(p,q)$-forms on N. This is endowed with the structure of a linear space and the usual topology. The currents of bidegree (p,q) are the linear functionals T on the space $\mathcal{D}^{p,q}(N)$, i.e. the elements of the dual space of $\mathcal{D}^{n-p,n-q}(N)$. The set of these currents will be denoted by $\mathcal{D}_{p,q}(N)$. This space is provided with the topology of weak convergence: a sequence $T_\nu \to T$ if $T_\nu(g) \to T(g)$ for every $g \in \mathcal{D}^{p,q}(N)$. Similarly, we define the set of currents of degree k $\mathcal{D}_k(N)$

to be the set of the elements of the dual space of $\mathscr{D}^{2n-k}(N)$. The currents of degree $2n$, which are defined on the space $\mathscr{D}^0(N)$, coincide with the generalized functions. One can interpret in a similar way the currents of degree 0 defined on the forms of maximal degree.

The concept of a current includes in particular both the concept of a form and the concept of an analytic set. Let α be a form of bidegree $(n-p, n-q)$ with locally integrable coefficients. Then α determines a current $[\alpha]$ of the same bidegree which acts on $\mathscr{D}^{p,q}(N)$ by the rule

$$[\alpha](g) = \int_N \alpha \wedge g, \quad \text{for} \quad g \in \mathscr{D}^{p,q}(N).$$

In exactly the same way, a k-dimensional analytic set $A \subset N$ defines a current $[A]$ of dimension (k,k) by

$$[A](g) = \int_{A_{\mathrm{reg}}} g \quad \text{for} \quad g \in \mathscr{D}^{2k}(N),$$

where A_{reg} is the set of regular points of A. Clearly, the current $[A] \in \mathscr{D}^{n-k,n-k}(N)$ and $d[A] = 0$.

Let f be a meromorphic function in N, the divisor of f is the current of bidegree $(1,1)$ defined by

$$D_f = \sum_j a_j[A_j] - \sum_j b_j[B_j],$$

where f has zeroes of multiplicity a_j on A_j and poles of multiplicity b_j on B_j, and A_j and B_j are irreducible components of complex analytic hypersurfaces in N. One of our basic tools is the Poincaré-Lelong formula

$$D_f = dd^c[\log|f|^2]. \tag{2.1}$$

For all $0 < s < r$, the growth of a meromorphic mapping $f: C^n \to \mathbb{P}^m$ is measured by its characteristic function

$$T_f(r,s) = \int_s^r \frac{1}{t^{2n-1}} \int_{B_n[t]} f^*(\omega_0) \wedge v_n^{n-1}\, dt = \int_s^r \frac{1}{t^{2n-1}} \int_{B_n[t]} dd^c \log\|f\|^2 \wedge v_n^{n-1}\, dt,$$

where ω_0 is the Fubini-Study metric on $\mathbb{P}^m$. Sometimes, for simplicity, we write $T_f(r)$ instead of $T_f(r,s)$ if no confusion occurs.

We say that a meromorphic map $f : \mathbb{C}^n \to \mathbb{P}^m$ is of finite order p if

$$\limsup_{r\to\infty} \frac{\log T_f(r)}{\log r} = p < \infty.$$

For $A \in \mathbb{C}^{m+1}$, a hyperplane in $\mathbb{P}^m$ is defined as $H_A = \{[w] \in \mathbb{P}^m;\ (w,A) = 0\}$. In the sequel we always write $H = H_A$ with $\|A\| = 1$. If the image of f is not contained in H, their intersection is measured by the counting function

$N_f(H,r)$ defined as follows:

$$N_f(H,r) = \int_s^r \frac{dt}{t^{2n-1}} \int_{H_f[t]} v_n^{n-1}$$

for $H_f = f^{-1}(H)$ and $H_f[t] = H_f \cap B_n[t]$ and the integral over $H_f[t]$ understood to mean "counting multiplicity". More generally, we define

Definition *Let $\Phi \in \mathscr{D}_{1,1}(\mathbb{C}^n)$ such that $d\Phi = 0$, and Φ is representable by integration. The order function $N(\Phi,r)$ for the divisor Φ is given by*

$$N(\Phi,r) = \int_s^r \Phi(\chi_{\mathbb{C}^n[t]} v^{n-1}) \frac{dt}{t^{2n-1}}\,.$$

It is easily verified that $N_f(H,r) = N(D_{(f,A)},r)$ and $N(f^*(\omega_0),r) = T_f(r)$. Moreover, Lemma (3.3) in [16] asserts that if u is pluripotential on $\mathbb{C}^n$ then

$$N(dd^c[u],r) = \frac{1}{2} \int_{S_n(r)} u\sigma_n + O(1)\,. \tag{2.2}$$

It follows when $H = H_A$ and $u = \log|(f,A)|$ that Jensen's theorem states

$$N_f(H,r) = \int_{S_n(r)} \log|(f,A)|\sigma_n + O(1)\,. \tag{2.3}$$

Let $H = H_A$ with $\|A\| = 1$ be a hyperplane. The "closeness" of the image of f to H is measured by the proximity function

$$m_f(H,r) = \int_{S_n(r)} \log \frac{\|f\|}{|(f,A)|}\sigma_n\,,$$

where $|(f,A)|/\|f\|$ is the norm of the holomorphic section of hyperplane bundle defining H pulled back to $\mathbb{C}^n$ via f. Applying $u = \log(\|f\|^2/|(f,A)|)$ to (2.1), we obtain the first main theorem

$$T_f(r) = N_f(H,r) + m_f(H,r) + O(1)\,. \tag{2.4}$$

Furthermore,

$$T_f(r) = \int_{S_n(r)} \log\|f\|\sigma_n + O(1)\,. \tag{2.5}$$

Let n and m be any positive integers. A meromorphic map $f : \mathbb{C}^n \to \mathbb{P}^m$ is called linearly non-degenerate if the image of f is not contained in a hyperplane. If $n \geqq m$, we also use the usual definition of non-degenerate: a meromorphic map $f : \mathbb{C}^n \to \mathbb{P}^m$ is called non-degenerate if the image of f contains an open set in $\mathbb{P}^m$.

3 Results

Theorem 1 (Logarithmic Derivative Lemma) *Let ψ and ϕ be defined as in* (1.1) *and assume f is a non-constant meromorphic function in $\mathbb{C}^n$ and*

$I = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ *is a multi-index of length* $l = \sum_{j=1}^{n} \alpha_j$. *Then*

$$\int_{S_n(r)} \log^+ \left| \frac{\partial^I f}{f} \right| \sigma_n \leqq l \log^+ \frac{T_f(r)\psi(T_f(r))}{\phi(r)} + O(1),$$

for all large r *outside a set* E *with* $\int_E dr/\phi(r) < \infty$.

Since the logarithmic derivative lemma plays an important role in Nevanlinna theory, it has been studied by many people for many years. Theorem 1 here, when $\phi(r) \equiv 1$ and $\psi(r) = (\log r)^{1+\varepsilon}$ (for any $\varepsilon > 0$) and $l = 1$, is stronger than the logarithmic derivative lemma obtained by Vitter [16], or Biancofiore and Stoll [1] since we use precise estimates of Gol'dberg and Grinshtein [4], and Kolokolnikov [7]. Moreover, a logarithmic derivative lemma recently showed by Miles [12], and Hinkkanen [6] is a special case of our Theorem 1 when $n = 1$ and $l = 1$. Furthermore, the proof of Theorem 1 is derived from our Lemma 6 which is another setting of a logarithmic derivative lemma.

Theorem 2 (Second Main Theorem) *Let* ψ *and* ϕ *be defined as in* (1.1), n *and* m *any positive integers and assume* $f : \mathbb{C}^n \to \mathbb{P}^m$ *is a meromorphic map which is non-rational and linearly non-degenerate. Suppose* $H_1, H_2, \ldots, H_q$ *be arbitrary hyperplanes in* $\mathbb{P}^m$ *in general position. Then*

$$S(f, \{H_j\}, r) \leqq \frac{m(m+1)}{2} \log \frac{T_f(r)\psi(T_f(r))}{\phi(r)} + O(1) \tag{3.1}$$

for all large r *outside a set* E *with* $\int_E dr/\phi(r) < \infty$;

In addition, if f *is of finite order* p, *then, for any* $\varepsilon > 0$,

$$S(f, \{H_j\}, r) \leqq \frac{m(m+1)}{2}(p + \varepsilon - 1)\log r + O(1) \tag{3.2}$$

for all large r, *where* $S(f, \{H_j\}_{j=1}^{q}, r)$ *is defined as in* (1.2).

Wong and Stoll in [18] and [19] introduced the concept of secondary defect for any hyperplanes $\{H_j\}$ in general position. They showed

$$\delta_2(f, \{H_j\}) \equiv \liminf_{r\to\infty} \frac{S(f, \{H_j\}, r)}{\log T_f(r)} \leqq \frac{m(m+1)}{2},$$

where $S(f, \{H_j\}_{j=1}^{q}, r)$ is defined as in (1.2). However, Theorem 2 gives

Corollary *Under the assumptions of Theorem* 2, *we have*

$$\begin{aligned} \delta_2(f, \{H_j\}) &\leqq m(m+1)/2, \quad \text{if } f \text{ is of infinite order;} \\ &\leqq \frac{m(m+1)}{2}\left(1 - \frac{1}{p}\right), \quad \text{if } f \text{ is of finite order } p; \\ &= -\infty, \quad \text{if } f \text{ is of order zero.} \end{aligned}$$

Proof of Corollary. Taking $\psi(r) = \log^{3/2} r$ and $\phi(r) = 1$ in (3.1), we have the first inequality since $T_f(r)$ goes to infinity as $r \to \infty$.

When f is of finite order p, then we have from (3.2) that, for any $\varepsilon > 0$,

$$\liminf_{r\to\infty} \frac{S(f,\{H_j\},r)}{\log T_f(r)} \leqq \frac{m(m+1)}{2}(p+\varepsilon-1)\liminf_{r\to\infty} \frac{\log r}{\log T_f(r)}$$

$$\leqq \frac{m(m+1)}{2}(p+\varepsilon-1)\frac{1}{p}.$$

By arbitrariness of ε, the corollary is proved completely. □

Wong and Stoll in [18] and [19] also brought up the concept of third defect for any hyperplanes $\{H_j\}$ in general position since they did not get the best second term in the left side of (1.6). Now we know from Theorem 2 that the third defect is always zero. We no longer see any usefulness for the third defect.

Theorem 3 (Second Main Theorem) *Let n and m be positive integers with $n \geqq m$ and suppose $f : \mathbb{C}^n \to \mathbb{P}^m$ is a non-rational and non-degenerate meromorphic map. Suppose $H_1, H_2, \ldots, H_q$ are arbitrary hyperplanes in $\mathbb{P}^m$ in general position. Then*

$$S(f,\{H_j\},r) \leqq m\log\frac{T_f(r)\psi(T_f(r))}{\phi(r)} + O(1) \tag{3.3}$$

for all large r outside a set E with $\int_E dr/\phi(r) < \infty$;

In addition, if f is of finite order p, then, for any $\varepsilon > 0$,

$$S(f,\{H_j\},r) \leqq m(p+\varepsilon-1)\log r \tag{3.4}$$

for all large r; where $S(f,\{H_j\}_{j=1}^q,r)$ is defined as in (1.2).

Corollary *Under the assumptions of Theorem 3, we have*

$$\delta_2(f,\{H_j\}) \leqq m, \quad \text{if } f \text{ is of infinite order;}$$

$$\leqq m(1-1/p), \quad \text{if } f \text{ is of finite order } p;$$

$$= -\infty, \quad \text{if } f \text{ is of order zero.}$$

When $m = 1$, Theorem 2 coincides with Theorem 3. However, Theorem 3 is not better than Theorem 2 even if we assume $n \geqq m$ in Theorem 2 since linear non-degeneracy is much weaker than non-degeneracy. When $n = m = 1$, our theorem 3 is still better than Hinkkanen's result [6] because [6] had $\log^+$ in the left side of (3.1) instead of log. So (3.2) cannot be obtained by using Hinkkanen's result. Moreover, if f has a regular growth, e.g. $T_f(r) \approx Cr^p$, then (3.2) and (3.4) can be improved to be

$$S(f,\{H_j\},r) \leqq \frac{m(m+1)}{2}\left(1-\frac{1}{p}\right)\log T_f(r) + O(1)$$

and

$$S(f, \{H_j\}, r) \leqq m\left(1 - \frac{1}{p}\right) \log T_f(r) + O(1),$$

respectively. Obviously, these inequalities are much better than (3.1) and (3.3). Generally speaking, (3.1) and (3.3) can be always improved if we only consider a subclass of meromorphic mappings.

Theorem 4 *There are a meromorphic map $f : \mathbb{C}^n \to \mathbb{P}^m$ (for some n and m) and hyperplanes $H_1, H_2, \ldots, H_q$ for some q in general position in $\mathbb{P}^m$ such that*

$$S(f, \{H_j\}, r) = \left(\frac{m(m+1)}{2} + o(1)\right) \log T_f(r)$$

for all large r, where $S(f, \{H_j\}_{j=1}^q, r)$ is defined as in (1.2).

Theorem 5 *There are a meromorphic map $f : \mathbb{C}^n \to \mathbb{P}^m$ (for some n and m) and hyperplanes $H_1, H_2, \ldots, H_q$ (for some q) in general position in $\mathbb{P}^m$ such that f is of finite order p (for some p) and*

$$S(f, \{H_j\}, r) = \frac{m(m+1)}{2}(p - 1 + o(1)) \log r$$

for all large r, where $S(f, \{H_j\}_{j=1}^q, r)$ is defined as in (1.2).

If we take $\psi(r) \equiv \log^{1+\varepsilon} r$ for any $\varepsilon > 0$ and $\phi(r) \equiv 1$, then clearly (3.1) can be written as,

$$\begin{aligned} S(f, \{H_j\}, r) &\leqq \frac{m(m+1)}{2} \log T_f(r) + (1+\varepsilon) \log\log T_f(r) + O(1) \\ &= \left(\frac{m(m+1)}{2} + o(1)\right) \log T_f(r) \end{aligned}$$

for all large r outside a set of finite Lebesgue measure. It follows that Theorems 4 and 5 show the sharpness of Theorem 2 in a certain sense. Moreover, the mappings in Theorems 4 and 5 are algebraically degenerate and not linearly degenerate. This observation leads an interesting question to determine what the best possible coefficient is if the map is algebraically non-degenerate.

The map f in the proofs of Theorems 4 and 5 is from $\mathbb{C}^2$ to $\mathbb{P}^3$, i.e. $n < m$ (it is not difficult to find a map from $\mathbb{C}$ to $\mathbb{P}^2$ which makes Theorems 4 and 5 be true by using the method in the proofs of Theorems 4 and 5). Clearly, the map f can be regarded as a map from $\mathbb{C}^n$ to $\mathbb{P}^3$ for any $n \geqq 2$. Thus we see from a little changes in the proofs of Theorems 4 and 5 that Theorem 2 is sharp whenever $n \geqq m$ (compare with Theorem 3) and $n < m$.

Naturally, the following theorems show the sharpness of Theorem 3. In fact, the sharpness of (3.3) was shown by the author in [20] in a general setting and the best example can be found in [23] in one complex variable case. However, examples here are different from examples in [20]. Moreover, Theorems 6 and 7 can be proved by utilizing the maps in the proofs of Theorems 4 and 5 without much effort.

Theorem 6 *There are a meromorphic map $f : \mathbb{C}^n \to \mathbb{P}^m$ (for some n and m with $n \geqq m$) and hyperplanes $H_1, H_2, \ldots, H_q$ (for some q) in $\mathbb{P}^m$ in general position such that*

$$S(f, \{H_j\}, r) = (m + o(1)) \log T_f(r)$$

for all large r, where $S(f, \{H_j\}_{j=1}^q, r)$ is defined as in (1.2).

Theorem 7 *There are a meromorphic map $f : \mathbb{C}^n \to \mathbb{P}^m$ (for some n and m) and hyperplanes $H_1, H_2, \ldots, H_q$ (for some q) in $\mathbb{P}^m$ in general position such that f is of finite order p (for some p) and*

$$S(f, \{H_j\}, r) = m(p - 1 + o(1)) \log r$$

for all large r, where $S(f, \{H_j\}_{j=1}^q, r)$ is defined as in (1.2).

4 Lemmas

Lemma 1 *Let f be a non-constant meromorphic function in $\mathbb{C}$. For arbitrary α with $0 < \alpha < 1$, there exists a constant C such that for arbitrary r and R with $r < R$, we have*

$$\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{f'(re^{i\theta})}{f(re^{i\theta})}\right|^\alpha d\theta \leqq \frac{C}{\cos \alpha\pi/2}\left\{\left(\frac{R}{r(R-r)}\right)^\alpha [m_f(R,0) + m_f(R,\infty)]^\alpha + \frac{1}{r^\alpha}[n_f(R,0) + n_f(R,\infty)]^\alpha\right\}$$

Proof. Without loss of generality, we assume that $f(0) = 1$, otherwise we consider $z^k f(z)$ for some integer k. Hence, we have from [4],

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{f'(re^{i\theta})}{f(re^{i\theta})}\right|^\alpha d\theta &\leqq \frac{1}{2\pi}\int_0^{2\pi} d\theta\left\{\frac{1}{2\pi}\int_0^{2\pi}|\log|f(Re^{i\phi})||\frac{2R}{|Re^{i\phi} - z|^2}d\phi\right\}^\alpha \\ &\quad + \frac{1}{2\pi}\int_0^{2\pi}\left|\sum_{|c_k|<R}\frac{\delta_k \bar{c}_k}{R^2 - \bar{c}_k z}\right|^\alpha d\theta + \frac{1}{2\pi}\int_0^{2\pi}\left|\sum_{|c_k|<R}\frac{\delta_k}{z - c_k}\right|^\alpha d\theta \\ &= I_1 + I_2 + I_3\,,\end{aligned}$$

where $z = re^{i\theta}$, $\delta_k^2 = 1$, and

$$I_1 \leqq \left\{\frac{R}{r(R-r)}(m_f(R,0) + m_f(R,\infty))\right\}^\alpha,$$

$$I_2 \leqq 2^{2-\alpha}\sec\frac{\alpha\pi}{2}\left\{\frac{1}{R}(n_f(R,0) + n_f(R,\infty))\right\}^\alpha.$$

By [7], we obtain

$$I_3 \leqq 2^{2+\alpha} \sec \frac{\alpha\pi}{2} \left\{ \frac{1}{r}(n_f(R,0) + n_f(R,\infty)) \right\}^{\alpha} .$$

It follows that the lemma is proved. □

Lemma 2 *Let $r \in \mathbb{R}^+$ and h be a function on $S_n(r)$ such that $h\sigma_n$ is integrable over $S_n(r)$, then, for $p(w) = \sqrt{r^2 - |w|^2}$,*

$$\int_{S_n(r)} h\sigma_n = \frac{1}{r^{2n-2}} \int_{\bar{B}_{n-1}(r)} \left(\int_{S_1(p(w))} h(w,\zeta)\sigma_1(\zeta) \right) \rho_{n-1}(w) .$$

Lemma 3 *Let f be a non-constant meromorphic function in $\mathbb{C}^n$, and $c \in \mathbb{P}^1$ then, if $r > 0$,*

$$\frac{1}{r^{2n-2}} \int_{\bar{B}_{n-1}(r)} n_{f_{[w]}}(p(w),c)\rho_{n-1}(w) \leqq n_f(r,c) ,$$

where $f_{[w]}(z) = f(w,z)$ for $w \in \mathbb{C}^{n-1}$ and $z \in \mathbb{C}$; and $p(w) = \sqrt{r^2 - |w|^2}$, for $w \in \bar{B}_{n-1}(r)$.

Complete proofs of the two lemmas above can be found in [1]. Lemma 4 is from [22]. Here, for the sake of completeness, we give the proof of Lemma 4.

In the sequel, C is always thought as to be an absolute positive constant, although the value may vary in each appearance and we write

$$n_f(t,0,\infty) = n_f(t,0) + n_f(t,\infty) \quad \text{and} \quad m_f(t,0,\infty) = m_f(t,0) + m_f(t,\infty) .$$

Lemma 4 *Let f be a meromorphic function in $\mathbb{C}^n$. Then for any $0 < \alpha < 1/2$, there is a constant $C > 1$ such that for any $r < R$, and any $j \in \{1,2,\ldots,n\}$, we have*

$$\int_{S_n(r)} \left| \frac{f_{z_j}}{f} \right|^{\alpha} \sigma_n \leqq C \left(\frac{R}{r} \right)^{\alpha(2n-2)} \left(\frac{R}{r(R-r)} \right)^{\alpha} T_f^{\alpha}(R) .$$

Proof. Without loss of generality, we take $j = n$ and set $s = (R+r)/2$. Applying Lemma 2 and Lemma 1 with $p(w) = \sqrt{r^2 - |w|^2}$ and $P(w) = \sqrt{s^2 - |w|^2}$, we obtain

$$\begin{aligned} \int_{S^n(r)} \left| \frac{f_{z_n}}{f} \right|^{\alpha} \sigma_n &= r^{2-2n} \int_{\bar{B}_{n-1}(r)} \int_{S_1(p(w))} \left| \frac{f'_{[w]}(z)}{f_{[w]}(z)} \right|^{\alpha} \sigma_1(z)\rho_{n-1}(w) \\ &\leqq r^{2-2n} \int_{\bar{B}_{n-1}(r)} \frac{C}{\cos \alpha\pi/2} \left\{ \left(\frac{P(w)}{p(w)(P(w) - p(w))} \right)^{\alpha} \right. \\ &\quad \left. \times\, m^{\alpha}_{f_{[w]}}(P(w),0,\infty) + \frac{1}{p^{\alpha}(w)} n^{\alpha}_{f_{[w]}}(P(w),0,\infty) \right\} \rho_{n-1}(w) . \end{aligned} \tag{4.1}$$

Clearly, for any $w \in \bar{B}_{n-1}(r)$, since $p(w) \leqq P(w)r/s$,

$$\frac{P(w)}{P(w) - p(w)} \leqq \frac{s}{s-r} \,. \tag{4.2}$$

Moreover, for any $0 < \beta < 1$, set $C = \int_{\bar{B}_{n-1}(1)} \frac{1}{(1-\tau^2)^{\beta/2}} \rho_{n-1}(\tau)$, then,

$$\int\limits_{\bar{B}_{n-1}(r)} \frac{1}{p^{\beta}(w)} \rho_{n-1}(w) = r^{2n-2} \int\limits_{\bar{B}_{n-1}(1)} \frac{1}{r^{\beta}(1-\tau^2)^{\beta/2}} \rho_{n-1}(\tau) = \frac{Cr^{2n-2}}{r^{\beta}} \,. \tag{4.3}$$

For any $c \in \mathbb{P}^1$, applying Lemma 2 gives

$$\begin{aligned}
s^{2n-2} m_f(s,c) &= s^{2n-2} \int\limits_{S_n(s)} \log^+ |1/(f(z) - c)| \sigma_n(z) \\
&= \int\limits_{\bar{B}_{n-1}(s)} \left(\int\limits_{S_1(P(w))} \log^+ |1/(f_{[w]} - c)| \sigma_1 \right) \rho_{n-1}(w) \\
&= \int\limits_{\bar{B}_{n-1}(s)} m_{f_{[w]}}(P(w), c) \rho_{n-1}(w) \\
&\geqq \int\limits_{\bar{B}_{n-1}(r)} m_{f_{[w]}}(P(w), c) \rho_{n-1}(w) \,.
\end{aligned} \tag{4.4}$$

Therefore, using (4.2), the Hölder inequality, and (4.3) for $\beta = \alpha/(1-\alpha)$ and (4.4), we get that

$$\begin{aligned}
r^{2-2n} &\int\limits_{\bar{B}_{n-1}(r)} \left(\frac{P(w)}{p(w)(P(w) - p(w))} \right)^{\alpha} m^{\alpha}_{f_{[w]}}(P(w), 0, \infty) \rho_{n-1}(w) \\
&\leqq \left(\frac{s}{s-r} \right)^{\alpha} r^{2-2n} \int\limits_{\bar{B}_{n-1}(r)} \frac{1}{p^{\alpha}(w)} m^{\alpha}_{f_{[w]}}(P(w), 0, \infty) \rho_{n-1}(w) \\
&\leqq \left(\frac{s}{s-r} \right)^{\alpha} r^{2-2n} \left(\int\limits_{\bar{B}_{n-1}(r)} m_{f_{[w]}}(P(w), 0, \infty) \rho_{n-1}(w) \right)^{\alpha} \\
&\quad \times \left(\int\limits_{\bar{B}_{n-1}(r)} p^{-\alpha/(1-\alpha)}(w) \rho_{n-1}(w) \right)^{1-\alpha} \\
&\leqq C \left(\frac{s}{r(s-r)} \right)^{\alpha} \left(\frac{s}{r} \right)^{\alpha(2n-2)} m^{\alpha}_f(s, 0, \infty) \,.
\end{aligned} \tag{4.5}$$

Similarly, using the Hölder inequality, (4.3) and Lemma 3, we have

$$\begin{aligned} & r^{2-2n} \int_{\bar{B}_{n-1}(r)} \frac{1}{p^{\alpha}(w)} n^{\alpha}_{f_{[w]}}(P(w),0,\infty)\rho_{n-1}(w) \\ & \leqq r^{2-2n} \left(\int_{\bar{B}_{n-1}(r)} n_{f_{[w]}}(P(w),0,\infty)\rho_{n-1}(w) \right)^{\alpha} \\ & \quad \times \left(\int_{\bar{B}_{n-1}(r)} p^{-\alpha/(1-\alpha)}(w)\rho_{n-1}(w) \right)^{1-\alpha} \\ & \leqq r^{2-2n}(s^{2n-2} n_f(s,0,\infty))^{\alpha} \left(\frac{Cr^{2n-2}}{r^{\alpha/(1-\alpha)}} \right)^{1-\alpha} \\ & \leqq \frac{C}{r^{\alpha}} \left(\frac{s}{r} \right)^{\alpha(2n-2)} n^{\alpha}_f(s,0,\infty) . \end{aligned} \tag{4.6}$$

Noting $s = (R + r)/2$, we have

$$m_f(s,\infty) \leqq T_f(R) + O(1) \quad \text{and} \quad m_f(s,0) \leqq T_f(R) + O(1);$$

$$\frac{s}{r} \leqq \frac{R}{r} \quad \text{and} \quad \frac{s}{r(s-r)} \leqq \frac{2R}{r(R-r)};$$

$$n_f(s,\infty) \leqq \frac{R}{R-s} N_f(s,\infty) \leqq \frac{R}{R-s} T_f(R) = \frac{2R}{R-r} T_f(R);$$

$$n_f(s,0) \leqq \frac{2R}{R-r}(T_f(R) + O(1)) .$$

Combining these estimations with (4.1), (4.5) and (4.6), we obtain the Lemma. □

Lemma 5 *Let $f : \mathbb{C}^n \to \mathbb{P}^m$ be a non-constant meromorphic mapping such that $f_0 \not\equiv 0$ and set*

$$G = (f_1/f_0, f_2/f_0, \ldots, f_m/f_0) \quad \textit{and}$$

$$[1, G_{z_j}] = [1, (f_1/f_0)_{z_j}, (f_2/f_0)_{z_j}, \ldots, (f_m/f_0)_{z_j}] .$$

Then, for any $j \in \{1, 2, \ldots, n\}$ and for any $r < R$,

$$T_{[1,G_{z_j}]}(r) \leqq 2T_f(r) + m \log^+ \left\{ \left(\frac{R}{r} \right)^{2n-2} \frac{R}{r(R-r)} T_f(R) \right\} + O(1) .$$

Proof. Set $G_k = f_k/f_0$, then $(G_k)_{z_j} = (f_k)_{z_j}/f_0 - (f_0)_{z_j} f_k/f_0^2$. Then the reduced representative of the meromorphic map $[1, G_{z_j}]$ is of the form $\beta^{-1}(f_0^2, f_0^2 G_{z_j})$ where β is a holomorphic function on $\mathbb{C}^n$. Thus we have from (2.5) and Jensen's theorem (2.3) with $e_0 = (1, 0, \ldots, 0) \in \mathbb{C}^{m+1}$

$$T_{[1,G_{z_j}]}(r) = \frac{1}{2}\int_{S_n(r)} \log(|\beta|^{-2}|f_0|^4(1 + \|G_{z_j}\|^2))\sigma_n + O(1)$$

$$\leqq \frac{1}{2}\int_{S_n(r)} \log(1 + \|G_{z_j}\|^2)\sigma_n + 2N_f(H_{e_0}, r) - N(D_\beta, r) + O(1)\,. \quad (4.7)$$

Since

$$\log(1 + \|G_{z_j}\|^2) \leqq \log^+\|G\|^2 + \log^+(\|G_{z_j}\|/\|G\|)^2 + \log 2$$

$$\leqq \log(1 + \|G\|^2) + \sum_{k=1}^{m} \log^+|(G_k)_{z_j}/G_k|^2 + O(1)\,,$$

we have from (4.7) and Lemma 4 that

$$T_{[1,G_{z_j}]}(r) \leqq \frac{1}{2}\int_{S_n(r)} \log(1 + \|G\|^2)\sigma_n + 2N_f(H_{e_0}, r)$$

$$+ \sum_{k=1}^{m}\int_{S_n(r)} \frac{1}{\alpha}\log^+\left|\frac{(G_k)_{z_j}}{G_k}\right|^\alpha \sigma_n + O(1)$$

$$\leqq \frac{1}{2}\int_{S_n(r)} \log(1 + \|G\|^2)\sigma_n + 2N_f(H_{e_0}, r)$$

$$+ \sum_{k=1}^{m} \log^+\left\{C\left(\frac{R}{r}\right)^{(2n-2)}\left(\frac{R}{r(R-r)}\right) T_{G_k}(R)\right\} + O(1)\,. \quad (4.8)$$

Applying the first main theorem (2.4) to f and H_{e_0} and noting $\|f\|/(f, e_0) = (1 + \|G\|^2)^{1/2}$, we have

$$T_f(r) = \frac{1}{2}\int_{S_n(r)} \log(1 + \|G\|^2)\sigma_n + N_f(H_{e_0}, r) + O(1)\,. \quad (4.9)$$

Consequently,

$$N_f(H_{e_0}, r) \leqq T_f(r) + O(1)\,. \quad (4.10)$$

Furthermore, we have from the definition of T_f that for all k,

$$T_{G_k}(r) = T_{[f_0, f_k]}(r) \leqq T_f(r)\,. \quad (4.11)$$

It follows from (4.8)–(4.11) that the Lemma is proved. □

Lemma 6 *Let f be a non-constant meromorphic function in $\mathbb{C}^n$ and $I = (v_1, v_2, \ldots, v_n)$ a multi-index of length $l = \sum_{j=1}^{n} v_j$. For any α with $0 < l\alpha < 1/2$, there is a constant $C > 1$ such that for any $r < \rho < R$, we* *have*

$$\int_{S_n(r)} \left|\frac{\partial^I f}{f}\right|^\alpha \sigma_n \leqq C\left(\frac{\rho}{r}\right)^{l\alpha(2n-2)}\left(\frac{\rho}{r(\rho-r)}\right)^{l\alpha} T_f^{l\alpha}(R)\,.$$

Proof. We prove the lemma by utilizing induction on the number of non-zero elements in I. First we assume that there is only one non-zero element in I, say, $I = (l, 0, \ldots, 0)$. Then

$$\partial^I f = f_{z_1^l} = \frac{f_{z_1^l}\, f_{z_1^{l-1}}}{f_{z_1^{l-1}}\, f_{z_1^{l-2}}} \cdots \frac{f_{z_1}}{f}\,.$$

It turns out from the Hölder inequality and Lemma 4 that, for $r < \rho$,

$$\begin{aligned}
\int_{S_n(r)} \left|\frac{f_{z_1^l}}{f}\right|^\alpha \sigma_n &= \int_{S_n(r)} \left|\frac{f_{z_1^l}}{f_{z_1^{l-1}}}\right|^\alpha \left|\frac{f_{z_1^{l-1}}}{f_{z_1^{l-2}}}\right|^\alpha \cdots \left|\frac{f_{z_1}}{f}\right|^\alpha \sigma_n \\
&\leqq \left(\int_{S_n(r)} \left|\frac{f_{z_1^l}}{f_{z_1^{l-1}}}\right|^{l\alpha} \sigma_n\right)^{1/l} \cdots \left(\int_{S_n(r)} \left|\frac{f_{z_1}}{f}\right|^{l\alpha} \sigma_n\right)^{1/l} \\
&\leqq C\left(\frac{\rho}{r}\right)^{l\alpha(2n-2)}\left(\frac{\rho}{r(\rho-r)}\right)^{l\alpha} T^\alpha_{f_{z_1^{l-1}}}(\rho) \\
&\qquad \cdots T^\alpha_{f_{z_1}}(\rho) T^\alpha_f(\rho)\,.
\end{aligned} \tag{4.12}$$

Applying Lemma 5 to the meromorphic function f when $m = 1$, we have that, for any $j \in \{1, 2, \ldots, n\}$ and for any $r < R$,

$$T_{f_{z_j}}(r) \leqq 2T_f(r) + \log\left\{\frac{R}{r}\frac{R}{r(R-r)}T_f(r)\right\} + O(1)\,. \tag{4.13}$$

Using (4.13) for $f_{z_1^{l-1}}, f_{z_1^{l-1}}, \ldots, f_{z_1}$ consecutively, we obtain a constant C such that, for any k with $1 \leqq k \leqq l-1$ and any $\rho < R$,

$$T_{f_{z_1^k}}(\rho) \leqq CT_f(R)\,. \tag{4.14}$$

It follows from (4.12) and (4.14) that the lemma is proved in this case.

Now suppose the lemma is true when the number of non-zero elements in I is $n-1$. The lemma will be proved if we can show that the lemma holds if $\alpha_j > 0$ for any $1 \leqq j \leqq n$. Set

$$I_{n-1} = (v_1, \ldots, v_{n-1}, 0)\,,$$

then

$$\frac{\partial^I f}{f} = \frac{(\partial^{I_{n-1}} f)_{z_n^{v_n}}}{\partial^{I_{n-1}} f}\frac{\partial^{I_{n-1}} f}{f} \quad \text{and} \quad l = v_n + |I_{n-1}|\,.$$

Therefore, applying the Hölder inequality and the induction hypothesis, we have

$$\int_{S_n(r)} \left|\frac{\partial^l f}{f}\right|^{\alpha} \sigma_n \leqq \left(\int_{S_n(r)} \left|\frac{(\partial^{l_{n-1}} f)_{z_n^{\nu_n}}}{\partial^{l_{n-1}} f}\right|^{2\alpha} \sigma_n\right)^{1/2} \left(\int_{S_n(r)} \left|\frac{\partial^{l_{n-1}} f}{f}\right|^{2\alpha} \sigma_n\right)^{1/2}$$

$$\leqq C\left(\frac{\rho}{r}\right)^{(\nu_n+|l_{n-1}|)\alpha(2n-2)} \left(\frac{\rho}{r(\rho-r)}\right)^{(\nu_n+|l_{n-1}|)\alpha}$$

$$\times\, T^{\nu_n\alpha}_{\partial^{l_{n-1}} f}(\rho) T_f^{|l_{n-1}|\alpha}(\rho)\,. \tag{4.15}$$

Again making use of (4.13), we get a constant $C > 1$ such that for any $\rho < R$,

$$T_{\partial^{l_{n-1}} f}(\rho) \leqq C T_f(R)\,.$$

It turns out from (4.15) that

$$\int_{S_n(r)} \left|\frac{\partial^l f}{f}\right|^{\alpha} \sigma_n \leqq C\left(\frac{\rho}{r}\right)^{l\alpha(2n-2)} \left(\frac{\rho}{r(\rho-r)}\right)^{l\alpha} T_f^{\nu_n\alpha}(R) T_f^{|l_{n-1}|\alpha}(R)\,.$$

Thus Lemma 6 is proved completely. □

Lemma 7 *Let ψ and ϕ be defined as in* (1.1) *and T a nondecreasing function in $\mathbb{R}^+$, then there is a constant $C > 1$ such that*

$$T\left(r + \frac{\phi(r)}{\psi(T(r))}\right) \leqq CT(r)$$

for all large r outside a set E with $\int_E dr/\phi(r) < \infty$.

This is a sort of standard growth lemma. The interesting readers can find a proof of the lemma from [6] or [5].

Lemma 8 *Let $F : \mathbb{C}^n \to \mathbb{P}^m$ be a linearly non-degenerate meromorphic map and*

$$f = (f_0, f_1, \ldots, f_m)$$

a reduced representative of the meromorphic map F, then there are the multi-indices $\beta_1, \ldots, \beta_m$ such that $|\beta_j| \leqq j$ for any $1 \leqq j \leqq m$ and f, $\partial^{\beta_1} f, \ldots, \partial^{\beta_m} f$ are linearly independent over $\mathbb{C}$.

Proof. Since F is linearly non-degenerate, $f_0, f_1, \ldots, f_m$ are linearly independent functions in $\mathbb{C}^n$. For any positive integer k, set

$$\mathscr{F}_k = \{\partial^\alpha f : |\alpha| = k\} \quad \text{and} \quad \mathscr{F}_0 = \{f\}\,.$$

We first claim that there is at least one element in $\mathscr{F}_1$, say f_{z_1}, which is independent with f. In fact, if the claim is not so, then for any $f_{z_j} \in \mathscr{F}_1$, f_{z_j} and f are linearly dependent; i.e. for any j with $0 \leqq j \leqq m$, there is a c_j such that

$$f_{z_j}(z) = c_j f(z)\,. \tag{4.16}$$

Taking partial derivatives of (4.16), we obtain that for any $h \in \cup_{k=2}^{\infty}\mathscr{F}_k$, h and f are linearly dependent over $\mathbb{C}$. Moreover, f has a global expansion at any point. Hence,

$$\begin{aligned} f(z) &= f(z_0) + \sum_{j=1}^{n} f_{z_j}(z_0)(z_j - (z_0)_j) + \cdots \\ &= f(z_0) + f(z_0)\sum_{j=1}^{n} c_j(z_j - (z_0)_j) + f(z_0)\sum_{j,l}^{n} c_{jl}(z_j - (z_0)_j)(z_l - (z_0)_l)\cdots \\ &\equiv f(z_0)p(z) \end{aligned}$$

where $z = (z_1, \ldots, z_n)$, $z_0 = ((z_0)_1, \ldots, (z_0)_n)$ and $p(z)$ is an entire function in $\mathbb{C}^n$. Consequently,

$$f_j(z) = f_j(z_0)p(z), \quad \text{for} \quad j = 0,\ 1, \ldots, m\,.$$

It turns out that f_0, $f_1, \ldots, f_m$ are linearly dependent functions. This is a contradiction.

Now suppose that

$$f, \partial^{\beta_1} f, \ldots, \partial^{\beta_{m-1}} f \quad (|\beta_j| \leqq j)$$

are linearly independent and we claim that there exists β_m such that $\partial^{\beta_m} f \in \mathscr{F}_1 \cup \mathscr{F}_2 \cup \cdots \cup \mathscr{F}_m$ and

$$f, \partial^{\beta_1} f, \ldots, \partial^{\beta_{m-1}} f, \partial^{\beta_m} f$$

are linearly independent. If the claim is false, then f, $\partial^{\beta_1} f, \ldots, \partial^{\beta_{m-1}} f$ consists of a maximal linearly independent subset in $\mathscr{F}_0 \cup \mathscr{F}_1 \cup \cdots \cup \mathscr{F}_m$. It turns out from taking partial derivatives that f, $\partial^{\beta_1} f, \ldots, \partial^{\beta_{m-1}} f$ consist of a maximal linearly independent subset in $\cup_0^{\infty}\mathscr{F}_j$. Thus, the global expansion at point $z_0 \in \mathbb{C}^n$ gives

$$f(z) = f(z_0)p_0(z) + \partial^{\beta_1} f(z_0)p_1(z) + \cdots + \partial^{\beta_{m-1}} f(z_0)p_{m-1}(z)\,,$$

where $p_j (j = 0, \ldots, m-1)$ is an entire function in $\mathbb{C}^n$ and $f(z_0)$, $\partial^{\beta_1} f(z_0)$, $\ldots, \partial^{\beta_{m-1}} f(z_0)$ are vectors in $\mathbb{C}^{m+1}$. Therefore, for any $j = 0, 1, \ldots, m$,

$$f_j(z) = (f_j(z_0),\ \partial^{\beta_1} f_j(z_0), \ldots, \partial^{\beta_{m-1}} f_j(z_0)) \begin{pmatrix} p_0(z) \\ p_1(z) \\ \vdots \\ p_{m-1}(z) \end{pmatrix}.$$

Consequently, f_0, $f_1, \ldots, f_m$ are linearly dependent functions which contradicts our assumption that f is linear non-degenerate. □

Lemma 9 *Let f be an entire function in $\mathbb{C}$, then, for $z = (z_1, \ldots, z_n)$ and $n \geqq 2$,*

$$\int_{S_n(r)} f(z_1)\sigma_n(z) = \frac{n-1}{\pi} \int_0^r \int_0^{2\pi} \frac{(r^2 - t^2)^{n-2}}{r^{2n-2}} t f(te^{i\theta}) d\theta dt .$$

Proof. First let $r = 1$, $B_n(1) = B_n$ and $S_n(1) = S_n$, and set, for any $0 < s < \infty$,

$$h(s) = \int_{sB_n} f(z_1)\omega_n^n(z) .$$

Integration in polar coordinates gives $h(s) = 2n \int_0^s t^{2n-1} \int_{S_n} f(tz_1)\sigma_n(z)dt$. Consequently,

$$h'(1) = 2n \int_{S_n} f(z_1)\sigma_n(z) . \tag{4.17}$$

On the other hand, noting Fubini's theorem with an orthogonal projection of $\mathbb{C}^n$ to $\mathbb{C}$, h may be written as

$$h(s) = c(n) \int_{B_1} (s^2 - |w|^2)^{n-1} f(w)\omega_1(w) ,$$

where $c(n)$ is a constant depending on the normalization of ω_1. Thus

$$h'(1) = c(n) \int_{B_1} 2(n-1)(1 - |w|^2)^{n-2} f(w)\omega_1(w) . \tag{4.18}$$

By comparing (4.17) to (4.18), we obtain

$$\int_{S_n} f(z_1)\sigma_n(z) = 2(n-1)c(n) \int_{B_1} (1 - |w|^2)^{n-2} f(w)\omega_1(w) . \tag{4.19}$$

Let $f = 1$, then

$$1 = 2(n-1)c(n) \int_{B_1} (1 - |w|^2)^{n-2} f(w)\omega_1(w)$$

$$= 2(n-1)c(n) \int_0^1 \int_0^{2\pi} (1 - t^2)^{n-2} t \, dt d\theta = 2\pi c(n) ,$$

so that $c(n) = 1/(2\pi)$. Substituting $1/(2\pi)$ for $c(n)$ in (4.19) gives the lemma when $r = 1$. If $r \neq 1$, set $z = r\xi$, then

$$\int_{S_n(r)} f(z_1)\sigma_n(z) = \int_{S_n} f(r\xi_1)\sigma_n(r\xi)$$

$$= \frac{n-1}{\pi} \int_0^1 (1 - t^2)^{n-2} \int_0^{2\pi} t f(rte^{i\theta}) d\theta dt$$

$$= \frac{n-1}{\pi} \int_0^r \left(1 - \left(\frac{s}{r}\right)^2\right)^{n-2} \int_0^{2\pi} \frac{s}{r^2} f(se^{i\theta}) d\theta ds ,$$

which implies the lemma. □

5 Proof of Theorem 1

Using the concavity of logarithmic function and Lemma 6, we obtain that, for any small positive α,

$$\begin{aligned}\int_{S_n(r)} \log^+ \left|\frac{\partial^l f}{f}\right| \sigma_n &\leqq \frac{1}{\alpha} \log^+ \left(\int_{S_n(r)} \left|\frac{\partial^l f}{f}\right|^\alpha \sigma_n \right) + O(1) \\ &\leqq \frac{1}{\alpha} \log^+ \left\{ C \left(\frac{\rho}{r}\right)^{l\alpha(2n-2)} \left(\frac{\rho}{r(\rho-r)}\right)^{l\alpha} T_f^{l\alpha}(R) \right\} \\ &\quad + O(1) \\ &\leqq l \log^+ \left\{ \left(\frac{\rho}{r}\right)^{2n-2} \frac{\rho}{r(\rho-r)} T_f(R) \right\} + O(1) \end{aligned} \tag{5.1}$$

for any $r < \rho < R$. Put

$$R = r + \frac{\phi(r)}{\psi(T_f(r))} \quad \text{and} \quad \rho = \frac{R+r}{2} = r + \frac{\phi(r)}{2\psi(T_f(r))} . \tag{5.2}$$

Applying Lemma 7 to the increasing function T_f, we get

$$T_f(R) = T_f\left(r + \frac{\phi(r)}{\psi(T_f(r))}\right) \leqq CT_f(r) \tag{5.3}$$

for all large r outside a set E with $\int_E dr/\phi(r) < \infty$. In addition, for all large r,

$$\rho/r \leqq 2 \quad \text{and} \quad 1/(\rho - r) \leqq 2\psi(T_f(r))/\phi(r) . \tag{5.4}$$

Therefore substituting (5.2)–(5.4) into (5.1) shows that the theorem is proved. □

6 Proof of Theorem 2

Let $f : \mathbb{C}^n \to \mathbb{C}^{m+1}$ also denote a reduced representative of the meromorphic map. Set

$$f = (f_0, f_1, \ldots, f_m) .$$

f being linearly non-degenerate is equivalent to the fact f_0, $f_1, \ldots, f_m$ are linearly independent in $\mathbb{C}^n$. By Lemma 8, there are multi-indices β_j with $|\beta_j| = l_j \leqq j$ for $0 \leqq j \leqq m$ such that f, $\partial^{\beta_1} f$, $\partial^{\beta_2} f, \ldots, \partial^{\beta_m} f$ are linearly independent. Set

$$J(f) \equiv J(f_0,\dots,f_m) \equiv f \wedge \partial^{\beta_1} f \wedge \cdots \wedge \partial^{\beta_m} f$$
$$= \begin{vmatrix} f_0 & f_1 & \cdots & f_m \\ \partial^{\beta_1} f_0 & \partial^{\beta_1} f_1 & \cdots & \partial^{\beta_1} f_m \\ \cdots & \cdots & \cdots & \cdots \\ \partial^{\beta_m} f_0 & \partial^{\beta_m} f_1 & \cdots & \partial^{\beta_m} f_m \end{vmatrix}$$
$$= \sum_{0 \leqq i_0, i_1, \dots, i_m \leqq m} (-1)^{sign(i_0, i_1, \dots, i_m)} (\partial^{\beta_{i_0}} f_0)(\partial^{\beta_{i_1}} f_1) \cdots (\partial^{\beta_{i_m}} f_m)$$

where we always regard the index under the $\sum$ as $i_j \neq i_k$ for all $j \neq k$.

Clearly, $J(f)$ is a holomorphic function in $\mathbb{C}^n$. Define the ramification divisor $R_f \equiv D_J$.

Without loss of generality we may assume that the number of hyperplanes $q \geqq m+1$ and $H_j = \{[w] \in \mathbb{P}^m;\ (w, A_j) = 0\}$ where the A_j are unit vectors in $\mathbb{C}^{m+1}$. Otherwise, we can add more hyperplanes such that the new set of hyperplanes consists of $m+1$ hyperplanes in general position. In addition, $S(f, \{H_j\}_1^q, r)$ is nondecreasing with respect to q since (1.2) can be written as

$$S(f, \{H_j\}_1^q, r) = \sum_{j=1}^{q} m_f(H_j, r) - (m+1)T_f(r) + N(R_f, r) + O(1)\,,$$

and $m_f(H_j, r) \geqq 0$. Set

$$g_j(z) \equiv (f(z), A_j) \quad j = 1, \dots, q\,.$$

Thus, for any $\gamma = (\gamma_0, \dots, \gamma_m)$, where $\gamma_j \in \{1, \dots, q\} (j = 0, \dots, m)$ and $\gamma_j \neq \gamma_k$ if $j \neq k$, we get from general position of the hyperplanes that there exists a non-singular matrix B_γ such that

$$\begin{pmatrix} f_0(z) \\ \vdots \\ f_m(z) \end{pmatrix} = B_\gamma \begin{pmatrix} g_{\gamma_0}(z) \\ \vdots \\ g_{\gamma_m}(z) \end{pmatrix},$$

i.e. for each $j \in \{0, \dots, m\}, f_j(z) = \sum_{k=0}^{m} b_{jk}(\gamma) g_{\gamma_k}(z)$. Set

$$G_\gamma \equiv (g_{\gamma_0},\ g_{\gamma_1} \dots, g_{\gamma_m}) : \mathbb{C}^n \to \mathbb{C}^{m+1}\,,$$

then, for any large $r > 0$,

$$T_{G_\gamma}(r) = \int_{S_n(r)} \log \|G_\gamma\| \sigma_n + O(1) \leqq T_f(r) + O(1) \leqq 2T_f(r)\,, \tag{6.1}$$

where $O(1)$ depends on γ. Since there are only finitely many γ's, we always regard $O(1)$ as an absolute constant in the sequel. Furthermore, there is a non-zero constant c_γ such that

$$J(f) = c_\gamma J(G_\gamma) \equiv c_\gamma G_\gamma \wedge \partial^{\beta_1} G_\gamma \wedge \cdots \wedge \partial^{\beta_m} G_\gamma\,, \tag{6.2}$$

For any $\alpha \in \mathbb{R}^+$ and any $\{a_{ij}\} \subset \mathbb{R}^+$, there exists a constant $C = C(\alpha) > 1$ such that

$$\left(\sum_{i,j} a_{ij}\right)^{\alpha} \leqq C \sum_{i,j} a_{ij}^{\alpha} . \tag{6.3}$$

For $r < \rho < R$ and sufficiently small α, set $s \equiv \alpha(m+1)$ and $t \equiv 1/(m+1)$, we have from the concavity of logarithmic function, (6.3), the Hölder inequality and Lemma 6, and noting $T_{g_{\gamma_j}}(r) \leqq CT_{G_i}(r)$ and $|\beta_j| = l_j \leqq j$ for all j, that

$$\begin{aligned}
&\int_{S_n(r)} \log \sum_{\gamma} \frac{|J(g_{\gamma_0}, \dots, g_{\gamma_m})|}{|g_{\gamma_0} g_{\gamma_1} \cdots g_{\gamma_m}|} \sigma_n = \frac{1}{\alpha} \int_{S_n(r)} \log \left(\sum_{\gamma} \frac{|J(G_\gamma)|}{|g_{\gamma_0} \cdots g_{\gamma_m}|} \right)^{\alpha} \sigma_n \\
&\leqq \int_{S_n(r)} \frac{1}{\alpha} \log \left(\sum_{\gamma} \sum_{0 \leqq i_0, i_1, \dots, i_m \leqq m} \left| \frac{\partial^{\beta_{i_0}} g_{\gamma_0}}{g_{\gamma_0}} \frac{\partial^{\beta_{i_1}} g_{\gamma_1}}{g_{\gamma_1}} \cdots \frac{\partial^{\beta_{i_m}} g_{\gamma_m}}{g_{\gamma_m}} \right| \right)^{\alpha} \sigma_n + O(1) \\
&\leqq \frac{1}{\alpha} \log \left(\int_{S_n(r)} \sum_{\gamma} \sum_{0 \leqq i_0, i_1, \dots, i_m \leqq m} \left| \frac{\partial^{\beta_{i_0}} g_{\gamma_0}}{g_{\gamma_0}} \frac{\partial^{\beta_{i_1}} g_{\gamma_1}}{g_{\gamma_1}} \cdots \frac{\partial^{\beta_{i_m}} g_{\gamma_m}}{g_{\gamma_m}} \right|^{\alpha} \sigma_n \right) + O(1) \\
&\leqq \frac{1}{\alpha} \log \left\{ \sum_{\gamma} \sum_{0 \leqq i_0 < i_1, \dots, i_m \leqq m} \left(\int_{S_n(r)} \left| \frac{\partial^{\beta_{i_0}} g_{\gamma_0}}{g_{\gamma_0}} \right|^{s} \sigma_n \right)^{t} \right. \\
&\qquad \left. \cdots \left(\int_{S_n(r)} \left| \frac{\partial^{\beta_{i_m}} g_{\gamma_m}}{g_{\gamma_m}} \right|^{s} \sigma_n \right)^{t} \right\} + O(1) \\
&\leqq \frac{1}{\alpha} \log \left\{ \sum_{\gamma} C \left(\frac{\rho}{r}\right)^{l_1 \alpha (2n-2)} \left(\frac{\rho}{r(\rho - r)} \right)^{l_1 \alpha} T_{G_\gamma}^{l_1 \alpha}(R) \right. \\
&\qquad \left. \cdots C \left(\frac{\rho}{r}\right)^{l_m \alpha (2n-2)} \left(\frac{\rho}{r(\rho - r)} \right)^{l_m \alpha} T_{G_\gamma}^{l_m \alpha}(R) \right\} + O(1) \\
&\leqq \frac{m(m+1)}{2} \log \left\{ \left(\frac{\rho}{r}\right)^{2n-2} \frac{\rho}{r(\rho - r)} \left(\sum_{\gamma} T_{G_\gamma}^{\frac{sm}{2}}(R) \right)^{\frac{2}{sm}} \right\} + O(1) ,
\end{aligned} \tag{6.4}$$

where the indexes γ's are combinations of $\{1, \dots, q\}$ taken $m+1$ numbers at a time, and the last inequality holds because $l_1 + l_2 + \cdots + l_m \leqq m(m+1)/2$. It follows from (6.4) and (6.1) that, for any $r < \rho < R$,

$$\int_{S_n(r)} \log \sum_{\gamma} \frac{|J(G_\gamma)|}{|g_{\gamma_0} g_{\gamma_1} \cdots g_{\gamma_m}|} \sigma_n \leqq \frac{m(m+1)}{2} \log \left\{ \left(\frac{\rho}{r}\right)^{2n-2} \frac{\rho}{r(\rho - r)} T_f(R) \right\} + O(1) . \tag{6.5}$$

At any $z \in \mathbf{C}^n$ the g_j's may be ordered so that $|g_1(z)| \leqq |g_2(z)| \leqq \cdots \leqq |g_q(z)|$, then there is a constant C such that

$$\log \|f(z)\| = \log \left(\sum_{j=0}^{m} \left| \sum_{k=1}^{m+1} b_{jk}(\gamma) g_k(z) \right|^2 \right)^{1/2}$$

$$\leqq \log |g_j(z)| + C, \quad j = m+2, \dots, q, \tag{6.6}$$

where $\gamma = (1, 2, \dots, m+1)$ and C can be chosen independently of z (at different points z one orders the g_j's differently and gets different constants but there are only finitely many possibilities) and also, noting (6.2),

$$|(g_{m+2} \cdots g_q)(z)| = \frac{|(g_1 \cdots g_q)(z)|}{|J(f_0, \dots, f_m)(z)|} \frac{|J(f_0, \dots, f_m)(z)|}{|(g_1 g_2 \cdots g_{m+1})(z)|}$$

$$= \frac{|(g_1 \cdots g_q)(z)|}{|J(f_0, \dots, f_m)(z)|} \frac{|J(G_\gamma)(z)|}{|(g_{\gamma_1} \cdots g_{\gamma_{m+1}})(z)|} |c_\gamma| .$$

Therefore, there is an absolute constant $C > 1$ such that

$$|g_{m+2} \cdots g_q| \leqq C \frac{|g_1 \cdots g_q|}{|J(f_0, \dots, f_m)|} \left(\sum_\gamma \frac{|J(G_\gamma)|}{|g_{\gamma_1} \cdots g_{\gamma_{m+1}}|} \right), \tag{6.7}$$

noting the right side of this inequality is independent of the order of g_j's.

On the other hand, we have from Jensen's formula that

$$N_f(H_j, r) = \int_{S_n(r)} \log |g_j| \sigma_n + O(1), \tag{6.8}$$

and from the Poincaré-Lelong formula (2.1) and (2.2) that

$$N(R_f, r) = N(dd^c[\log J^2(f)],\ r) = \int_{S_n(r)} \log |J(f)| \sigma_n + O(1). \tag{6.9}$$

It follows from (6.6)–(6.9) that

$$(q - m - 1) T_f(r) = (q - m - 1) \int_{S_n(r)} \log \|f\| \sigma_n + O(1)$$

$$\leqq \int_{S_n(r)} \log |g_{m+2} \cdots g_q| \sigma_n + O(1)$$

$$\leqq \int_{S_n(r)} \log \frac{|g_1 g_2 \cdots g_q|}{|J(f_0, \dots, f_m)|} \sigma_n$$

$$+ \int_{S_n(r)} \log \sum_\gamma \frac{|J(G_\gamma)|}{|g_{\gamma_1} \cdots g_{\gamma_{m+1}}|} \sigma_n + O(1)$$

$$= \sum_{j=1}^{q} N_f(H_j, r) - N(R_f, r)$$

$$+ \int_{S_n(r)} \log \sum_\gamma \frac{|J(G_\gamma)|}{|g_{\gamma_1} \cdots g_{\gamma_{m+1}}|} \sigma_n + O(1). \tag{6.10}$$

Therefore, combining (6.5) and (6.10) gives

$$S(f, \{H_j\}, r) \leqq \frac{m(m+1)}{2} \log\left\{\left(\frac{\rho}{r}\right)^{2n-2} \frac{\rho}{r(\rho - r)} T_f(R)\right\} + O(1). \quad (6.11)$$

Again taking ρ and R as in (5.2) and using Lemma 7, we get (3.1).

If f is of finite order p, then, for any $\varepsilon > 0$, we have, for all large r,

$$T_f(r) \leqq r^{p+\varepsilon}.$$

We take $R = 2r$ and $\rho = 3r/2$. Hence (6.11) gives

$$S(f, \{H_j\}, r) \leqq \frac{m(m+1)}{2} \log\left(\frac{C}{r} r^{p+\varepsilon}\right) + O(1)$$

$$\leqq \frac{m(m+1)}{2}(p + \varepsilon - 1)\log r + O(1).$$

which is the second part of the theorem. □

7 Proof of Theorem 3

Let $f : \mathbb{C}^n \to \mathbb{C}^{m+1}$ also denote a reduced representative of the meromorphic map. Set

$$f = (f_0, f_1, \dots, f_m).$$

Since the map f is non-degenerate and $n \geqq m$, we can, without loss of generality, choose coordinates on $\mathbb{C}^n$ such that the determinant

$$\begin{vmatrix} f_0 & f_1 & \cdots & f_m \\ (f_0)_{z_1} & (f_1)_{z_1} & \cdots & (f_m)_{z_1} \\ \cdots & \cdots & \cdots & \cdots \\ (f_0)_{z_m} & (f_1)_{z_m} & \cdots & (f_m)_{z_m} \end{vmatrix}$$

is not identically equal to zero. It follows that $f, f_{z_1}, f_{z_2}, \dots, f_{z_m}$ are linearly independent. Define the ramification divisor $R_f = D_{f \wedge f_{z_1} \wedge \cdots \wedge f_{z_m}}$. Thus the proof of Theorem 2 can be carried over here sentence by sentence if we define

$$J(f) \equiv f \wedge f_{z_1} \wedge \cdots \wedge f_{z_m}.$$

Therefore we have

$$S(f, \{H_j\}, r) \leqq \int_{S_n(r)} \log \sum_{\gamma} \frac{|J(G_\gamma)|}{|g_{\gamma_1} \cdots g_{\gamma_{m+1}}|} \sigma_n + O(1)$$

$$\leqq \int_{S_n(r)} \frac{1}{\alpha} \log\left(\sum_{\gamma} \sum_{0 \leqq i_0, i_1, \dots, i_m \leqq m} \left|\frac{(g_{\gamma_0})_{z_1}}{g_{\gamma_0}} \frac{(g_{\gamma_1})_{z_2}}{g_{\gamma_1}} \cdots \frac{(g_{\gamma_m})_{z_m}}{g_{\gamma_m}}\right|\right)^{\alpha} \sigma_n$$

$$+ O(1)$$

$$\leqq m \log \left\{ C \left(\frac{\rho}{r}\right)^{(2n-2)} \frac{\rho}{r(\rho - r)} T_f(R) \right\} + O(1) .$$

The remainder of the proof is similar to the proof of Theorem 2. □

8 Proof of Theorem 4

Set $\alpha(r) = e^{(\log r)^2} \log r$ and let $\{r_j\}$ be the sequence defined by

$$\alpha(r_j) = 2^{j+1}, \quad j = 1, 2, 3, \ldots . \tag{8.1}$$

This sequence $\{r_j\}$ is uniquely determined, strictly increasing and unbounded. Let

$$g(z) = \prod_1^{\infty} \left(1 + \left(\frac{z}{r_j}\right)^{n_j}\right) , \tag{8.2}$$

where

$$n_j = 2^j \tag{8.3}$$

We prove that g is entire and (8.23) and (8.24) hold. In these estimates $N_g(r, 0)$ and $n_g(r, 0)$ will be replaced by $N(r)$ and $n(r)$.

Let $r > 0$, with $r \in [r_{k-1}, r_k)$. Then $n(r) = 2^k - 2 \leqq \alpha(r) - 2$ and

$$n(r) = \frac{1}{2}\alpha(r_k) - 2 \geqq \frac{1}{2}\alpha(r) - 2 . \tag{8.4}$$

The upper bound for $n(r)$ and the definition of $N(r)$ yield that

$$N(r) \leqq \frac{1}{2} e^{(\log r)^2} - 2 \log r + C , \quad (r > r_0) . \tag{8.5}$$

Choose $a \in (\frac{1}{2}, 1)$, and set

$$s_j = r_j - a^j, \quad S_j = r_j + a^j , \tag{8.6}$$

$$E_j = [s_j, S_j], \quad E = \cup E_j , \tag{8.7}$$

and observe that $\mu(E) \leqq \sum_{j=1}^{\infty} 2a^j = A$, where A depends only on a.

If $r > S_k$, we have from (8.6) and $r_j < j + 1$ that for all large k,

$$\log \frac{r}{r_k} \geqq \log \left(1 + \frac{a^k}{r_k}\right) \geqq \frac{a^k}{2r_k} \geqq \frac{a^k}{2(1+k)} ,$$

so that, by (8.3),

$$\left(\frac{r}{r_k}\right)^{n_k} \geqq \exp\left(\frac{(2a)^k}{2(k+1)}\right) . \tag{8.8}$$

If $r < s_k$, we see from (8.7) and $r_j < j+1$ that for all large k

$$\log\frac{r}{r_k} \leqq \log\left(1-\frac{a^k}{r_k}\right) \leqq -\frac{a^k}{r_k} \leqq -\frac{a^k}{(1+k)}$$

and so

$$\left(\frac{r}{r_k}\right)^{n_k} \leqq \exp\left(-\frac{(2a)^k}{1+k}\right) . \tag{8.9}$$

Note that if $r \in (r_k, r_{k+1})\backslash E$, then $r > S_j$ ($j = 1,\ldots,k$) and $r < s_j$ ($j = k+1,\ldots$). If $z = re^{i\theta}$ for any such r then

$$\begin{aligned}\log|g(z)| &= \sum_{j=1}^{k} n_j \log\left|\frac{z}{r_j}\right| + \sum_{j=1}^{k}\log\left|\left(\frac{r_j}{z}\right)^{n_j}+1\right| + \sum_{j=k+1}^{\infty}\log\left|1+\left(\frac{z}{r_j}\right)^{n_j}\right| \\ &= N(r) + \sum_{j=1}^{k}\log\left|\left(\frac{r_j}{z}\right)^{n_j}+1\right| + \sum_{j=k+1}^{\infty}\log\left|1+\left(\frac{z}{r_j}\right)^{n_j}\right| \\ &= N(r) + I_1 + I_2 .\end{aligned} \tag{8.10}$$

Since $2a > 1$, we see from (8.8) and (8.9) that if r is large enough, then

$$|I_1| \leqq \sum_{j=1}^{k}\left(\frac{r_j}{r}\right)^{n_j} \leqq \sum_{j=1}^{\infty}\exp\left(-\frac{(2a)^j}{2(j+1)}\right) + O(1) = O(1),$$

$$|I_2| \leqq \sum_{j=k+1}^{\infty}\left(\frac{r}{r_j}\right)^{n_j} \leqq \sum_{j=1}^{\infty}\exp\left(-\frac{(2a)^j}{1+j}\right) + O(1) = O(1).$$

Thus g is entire; indeed

$$\log|g(z)| = N(r) + O(1) \tag{8.11}$$

for all large $r \notin E$ and $z = re^{i\theta}$. It follows that

$$N(r) \leqq T_g(r) \leqq \log M_g(r) = N(r) + O(1), \quad (r > r_*, r \notin E) \tag{8.12}$$

where $M_g(r) = \max_{|z|=r}|g(z)|$. Therefore, we have from (8.5) and (8.12) that

$$T_g(r) = N(r) + O(1) \leqq \alpha(r), \quad (r > r_*, r \notin E) . \tag{8.13}$$

Furthermore, if $r \in E$, then there exists j such that $r \in E_j$ and

$$\begin{aligned}\log^+|f(re^{i\theta})| &\leqq \log M_g(r) \leqq \log M_g(S_j) \leqq \alpha(S_j) = \exp((\log(r_j + a^j))^2) \\ &\leqq \exp((1+o(1))(\log(r_j - a^j))^2) \leqq \exp((1+o(1))(\log r)^2) .\end{aligned}$$

Consequently, for all large r,

$$\log^+|f(re^{i\theta})| \leqq \alpha(r) \leqq e^{(1+o(1))(\log r)^2} . \tag{8.14}$$

As above, we assume that $r \in (r_k, r_{k+1})\backslash E$, with k large. Thus

$$\frac{zg'(z)}{g(z)} = \sum_{j=1}^{k} n_j + \sum_{j=1}^{k} \frac{-n_j}{1+(z/r_j)^{n_j}} + \sum_{j=k+1}^{\infty} \frac{n_j(z/r_j)^{n_j}}{1+(z/r_j)^{n_j}}$$

$$= n(r) + J_1 + J_2 .$$

Since $2a > 1$, we see from (8.2), (8.8) and (8.9) that

$$|J_1| \leqq 2\sum_{j=1}^{k} \frac{n_j}{(r/r_j)^{n_j}} \leqq 2\sum_{j=1}^{\infty} 2^j \exp\left(-\frac{(2a)^j}{2(1+j)}\right) + O(1) = O(1) ,$$

$$|J_2| \leqq \sum_{j=k+1}^{\infty} \frac{n_j(r/r_j)^{n_j}}{1-(r/r_j)^{n_j}} \leqq 3\sum_{j=1}^{\infty} 2^j \exp\left(-\frac{(2a)^j}{(1+j)}\right) + O(1) = O(1) .$$

Consequently, if $z = re^{i\theta}$, then

$$\frac{zg'(z)}{g(z)} = n(r) + O(1) , \quad (r > r_*, r \notin E) , \tag{8.15}$$

and

$$\left|\frac{g''(z)}{g(z)}\right| = \left|\left(\frac{g'(z)}{g(z)}\right)^2 - \frac{n(r)+O(1)}{r^2}\right|$$

$$\geqq (1+o(1))n^2(r)/r^2 \quad (r > r_*, r \notin E) , \tag{8.16}$$

It follows from (8.15) and (8.4) that

$$\frac{1}{2\pi}\int_0^{2\pi} \log\left|\frac{g'(z)}{g(z)}\right| d\theta \geqq \log n(r) - \log r - o(1)$$

$$\geqq (\log r)^2 + \log\log r - \log r - o(1)$$

$$= (1+o(1))(\log r)^2 \tag{8.17}$$

for all large $r \notin E$ and $z = re^{i\theta}$. Similarly, (8.16) and (8.4) give

$$\frac{1}{2\pi}\int_0^{2\pi} \log\left|\frac{g''(z)}{g(z)}\right| d\theta \geqq 2(\log r)^2 + \log\log r - 2\log r - o(1)$$

$$= (2+o(1))(\log r)^2 \tag{8.18}$$

for all large $r \notin E$ and $z = re^{i\theta}$.

Now we show that inequalities (8.17) and (8.18) hold for all large r. In fact, for $r \in [r_k, S_k]$, we have from (8.15) that

$$\frac{zg'(z)}{g(z)} = n(r) - \frac{n_k}{1+(z/r_k)^{n_k}} + O(1) , \tag{8.19}$$

Moreover, since if $|a| \geqq 1$ and $a \neq -1$, then $\mathrm{Re}\{1/(1+a)\} \leqq 1/2$; so, for $r \in [r_k, S_k]$, (8.19) gives

$$\frac{zg'(z)}{g(z)} \geqq \mathrm{Re}\frac{zg'(z)}{g(z)} = n(r) - \frac{n_k}{2} + O(1) \geqq \frac{3}{4}n(r) + O(1)\,. \tag{8.20}$$

Thus we yield from (8.20) and (8.17) that

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi}\log\left|\frac{g'(z)}{g(z)}\right| d\theta &\geqq \log n(r) - \log r - O(1)\\ &\geqq (\log r)^2 + \log\log r - \log r - O(1)\end{aligned} \tag{8.21}$$

for all large $r \in [r_k, S_k]$.

If $r \in [s_{k+1}, r_{k+1})$, then applying Jensen's formula to g'/g, and noting the fact

$$N_g(0,r) = N_g(0, s_{k+1}) \quad \text{and} \quad N_{g'}(0,r) \geqq N_{g'}(0, s_{k+1})\,,$$

we get from (8.17) and (8.6) that

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi}\log\left|\frac{g'}{g}(re^{i\theta})\right| d\theta &= N_{g'/g}(0,r) - N_{g'/g}(\infty,r) + O(1)\\ &= N_{g'}(0,r) - N_g(0,r) + O(1)\\ &\geqq N_{g'}(0,s_{k+1}) - N_g(0,s_{k+1}) + O(1)\\ &\geqq N_{g'/g}(0,s_{k+1}) - N_{g'/g}(\infty,s_{k+1}) + O(1)\\ &= \frac{1}{2\pi}\int_0^{2\pi}\log\left|\frac{g'}{g}(s_{k+1}e^{i\theta})\right| d\theta + O(1)\\ &\geqq (1+o(1))(\log s_{k+1})^2 = (1+o(1))(\log(r_k - a^k)^2\\ &\geqq (1+o(1))(\log r)^2, \quad (r \to \infty)\,.\end{aligned} \tag{8.22}$$

Combining (8.17), (8.21) and (8.22), we see

$$\frac{1}{2\pi}\int_0^{2\pi}\log\left|\frac{g'}{g}(re^{i\theta})\right| d\theta \geqq (1+o(1))(\log r)^2 \tag{8.23}$$

for all large r. Similarly, we have, for all large r,

$$\frac{1}{2\pi}\int_0^{2\pi}\log\left|\frac{g''}{g}(re^{i\theta})\right| d\theta \geqq (2+o(1))(\log r)^2\,. \tag{8.24}$$

We now are ready to prove Theorem 4. Let g be as in (8.1) and set $f : \mathbb{C}^2 \to \mathbb{P}^3$ be defined by

$$f(z_1,z_2) = [1 : g(z_1) : g(z_2) : g(z_1)g(z_2)]\,.$$

Obviously, $f(\mathbb{C}^n)$ is not contained in a hyperplane and is contained in a hypersurface in $\mathbb{P}^3$. We take the hyperplanes

$$H_j = \{[w] = [w_0 : \cdots : w_3] \in \mathbb{P}^3; w_{j-1} = 0\}, \quad j = 1,2,3,4$$

and set the multi-indices

$$I_1 = (1,0), \quad I_2 = (0,2) \quad \text{and} \quad I_3 = (1,2).$$

Moreover, let

$$J = f \wedge f_{z^{I_1}} \wedge f_{z^{I_2}} \wedge f_{z^{I_3}}$$

$$= \begin{vmatrix} 1 & g(z_1) & g(z_2) & g(z_1)g(z_2) \\ 0 & g'(z_1) & 0 & * \\ 0 & 0 & g''(z_2) & * \\ 0 & 0 & 0 & g'(z_1)g''(z_2) \end{vmatrix} = (g'(z_1)g''(z_2))^2,$$

and noting $R_f = D_J$, we have from the proof of Theorem 2 that

$$S(f,\{H_j\},r) = (q-m-1)T_f(r) - \sum_{j=1}^{4} N_f(H_j,r) + N(R_f,r)$$

$$= \sum_{j=1}^{2} \int_{S_n(r)} \log \left| \frac{1}{g(z_j)} \right| \sigma_n - \int_{S_n(r)} \log |g(z_1)g(z_2)| \sigma_n + \int_{S_n(r)} \log |J| \sigma_n$$

$$= 2 \int_{S_n(r)} \log \left| \frac{g'(z_1)}{g(z_1)} \frac{g''(z_2)}{g(z_2)} \right| \sigma_n .$$

It follows from (8.23), (8.24), and Lemma 9 for $n = 2$ that

$$S(f,\{H_j\},r)$$

$$= \frac{2}{\pi} \int_0^r \int_0^{2\pi} \frac{1}{r^2} t \log \left| \frac{g'}{g}(te^{i\theta}) \right| d\theta dt + \frac{2}{\pi} \int_0^r \int_0^{2\pi} \frac{1}{r^2} t \log \left| \frac{g''}{g}(te^{i\theta}) \right| d\theta dt$$

$$\geqq \frac{4}{r^2}(1 + o(1)) \int_0^r t(\log t)^2 dt + \frac{4}{r^2}(2 + o(1)) \int_0^r t(\log t)^2 dt$$

$$\geqq 6(1 + o(1))(\log r)^2 . \tag{8.25}$$

Now estimate $T_f(r)$. Indeed, we have from (8.14) that, for all large r,

$$T_f(r) = \int_{S_n(r)} \log\|f\|\sigma_n + O(1) \leqq C \int_{S_n(r)} \log^+|g(z_1)|\sigma_n + O(1)$$

$$= C \int_0^r \int_0^{2\pi} \frac{1}{r^2} t \log |g(te^{i\theta})| d\theta dt + O(1)$$

$$\leqq \frac{C}{r^2} \int_0^r t \exp((1 + o(1))(\log t)^2) dt \leqq C \exp((1 + o(1))(\log r)^2) .$$

Consequently,

$$\log T_f(r) \leqq (1+o(1))(\log r)^2 + O(1)\,. \tag{8.26}$$

It follows from (8.25) and (8.26) that Theorem 4 is proved. □

9 Proof of Theorem 5

Let $f : \mathbb{C}^2 \to \mathbb{P}^3$ be defined by

$$f(z_1,z_2) = [1 : \exp(z_1^p) : \exp(z_2^p) : \exp(z_1^p + z_2^p)]\,,$$

where $p > 1$ is any positive integer. We take the hyperplanes

$$H_j = \{[w] = [w_0,\dots,w_3] \in \mathbb{P}^3; w_{j-1} = 0\} \quad j = 1, 2, 3, 4$$

and set the multi-indexes

$$I_1 = (1,0),\quad I_2 = (0,2) \quad \text{and} \quad I_3 = (1,2)\,.$$

Moreover let

$$J = f \wedge f_{z_{I_1}} \wedge f_{z_{I_2}} \wedge f_{z_{I_3}}$$

$$= \begin{vmatrix} 1 & e^{z_1^p} & e^{z_2^p} & * \\ 0 & pz_1^{p-1}e^{z_1^p} & 0 & * \\ 0 & 0 & pz_2^{p-2}(p-1+pz_2^p)e^{z_2^p} & * \\ 0 & 0 & 0 & p^2z_1^{p-1}z_2^{p-2}(p-1+pz_2^p)e^{z_1^p+z_2^p} \end{vmatrix}$$

$$= p^4 z_1^{2p-2} z_2^{2p-4}(p-1+pz_2^p)^2\exp(2z_1^p+2z_2^p) \equiv P(z_1,z_2)\exp(2z_1^p+2z_2^p)\,,$$

and noting $R_f = D_J$, we get from Lemma 9 that

$$\begin{aligned} S(f,\{H_j\},r) &= (q-m-1)T_f(r) - \sum_{j=1}^{4} N_f(H_j,r) + N(R_f,r) \\ &= -\sum_{j=1}^{2} \int_{S_n(r)} \log|e^{z_j^p}|\sigma_n - \int_{S_n(r)} \log|e^{z_1^p+z_2^p}|\sigma_n + \int_{S_n(r)} \log|J|\sigma_n \\ &= \int_{S_n(r)} \log|P(z_1,z_2)|\sigma_n \\ &= (4p-6)\int_{S_n(r)} \log|z_1|\sigma_n + 2\int_{S_n(r)} \log|p-1+pz_2^p|\sigma_n + O(1) \\ &\geqq 6(p-1)\log r + O(1)\,. \end{aligned}$$

Now compute the order of f. In fact,

$$\begin{aligned} T_f(r) &= \int_{S_n(r)} \log\|f\|\sigma_n + O(1) \geqq \int_{S_n(r)} \log(1+|\exp(z_1^p)|)\sigma_n + O(1) \\ &\geqq \frac{1}{\pi}\int_0^r\int_0^{2\pi} \frac{t}{r^2}\log(1+|\exp(t^p e^{ip\theta})|)\,d\theta\,dt + O(1) \geqq Cr^p + O(1)\,, \end{aligned}$$

and

$$T_f(r) = \int_{S_n(r)} \log \|f\| \sigma_n + O(1) \leqq C \int_{S_n(r)} \log(1 + |\exp(z_1^p)|) \sigma_n + O(1)$$

$$\leqq Cr^p + O(1) .$$

It follows that f has order p. □

10 Proof of Theorem 6

Let g be defined as in (8.2) and set

$$f(z_1, z_2, z_3) = [1 : g(z_1) : g(z_2)] : \mathbb{C}^3 \to \mathbb{P}^2 ,$$

and the hyperplanes

$$H_j = \{[w] = [w_0 : w_1 : w_2] \in \mathbb{P}^2; w_{j-1} = 0\}, \quad j = 1, 2, 3.$$

In a manner similar to the proof of Theorem 4, we get from Lemma 9 and (8.23) that

$$S(f, \{H_j\}, r) = \int_{S_n(r)} \log \frac{|g'(z_1)g'(z_2)|}{|g(z_1)g(z_2)|} \sigma_n = 2 \int_{S_n(r)} \log \left| \frac{g'(z_1)}{g(z_1)} \right| \sigma_n$$

$$= 2 \frac{2}{\pi} \int_0^r \int_0^{2\pi} \frac{r^2 - t^2}{r^4} t \log \left| \frac{g'}{g}(te^{i\theta}) \right| d\theta dt$$

$$\geqq 8(1 + o(1)) \int_0^r \frac{r^2 - t^2}{r^4} t (\log t)^2 dt + O(1)$$

$$= (2 + o(1))(\log r)^2 . \tag{10.1}$$

Furthermore, we obtain from (8.26) that $\log T_f(r) \leqq (1 + o(1))(\log r)^2 + O(1)$. It follows from (10.1) that the theorem is proved. □

11 Proof of Theorem 7

The proof of the theorem is quite similar to the proof of Theorem 6. For any positive integer $p > 1$, set the map

$$f(z_1, z_2, z_3) = [1 : e^{z_1^p} : e^{z_2^p}] : \mathbb{C}^3 \to \mathbb{P}^2 ,$$

and the hyperplanes

$$H_j = \{[w] = [w_0 : w_1 : w_2] \in \mathbb{P}^2; w_{j-1} = 0\}, \quad j = 1, 2, 3 .$$

Thus we have

$$\begin{aligned} S(f,\{H_j\},r) &= \int_{S_n(r)} \log|pz_1^{p-1}pz_2^{p-1}|\sigma_n \\ &= 2\frac{2}{\pi}\int_0^r\int_0^{2\pi}\frac{r^2-t^2}{r^4}t\log t^{p-1}d\theta dt + O(1) \\ &= 2(p-1+o(1))\log r\,. \end{aligned}$$

Clearly, the function f is of order p. Hence Theorem 7 is proved. □

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