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An arithmetic Riemann–Roch theorem

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Introduction

We prove in this paper an arithmetic analog of the Riemann–Roch–Grothendieck theorem for the determinant of the cohomology of an Hermitian vector bundle of arbitrary rank on a family of arithmetic varieties of arbitrary dimension. We also show that high powers of ample line bundles on arithmetic varieties have small sections.

Let X and Y be regular quasi-projective flat schemes over $\mathbb{Z}$. Consider an Hermitian vector bundle $\bar{E} = (E, h)$ on $X: E$ is an algebraic vector bundle on X and h is an Hermitian metric on the associated holomorphic vector bundle on $X(\mathbb{C})$, which is invariant under complex conjugation. In [GS2] we defined arithmetic Chow groups $\widehat{CH}^p(X)$, $p \geq 0$, and in [GS3] we attached to (E, h) arithmetic characteristic classes such as the Chern character $\widehat{ch}(E, h) \in \widehat{CH}^{\cdot}(X)_{\mathbb{Q}} = \bigoplus_{p \geq 0} \widehat{CH}^p(X) \otimes_{\mathbb{Z}} \mathbb{Q}$, and the Todd class $\widehat{Td}(E, h)$. Assume now that $f: X \to Y$ is a smooth projective morphism from X to Y. The determinant of cohomology $\lambda(E) = \det Rf_*(E)$ is an algebraic (graded) line bundle on Y. Choose an Hermitian metric h_f, invariant by conjugation, on the relative tangent space Tf, whose restriction to each fiber of f over $Y(\mathbb{C})$ is Kähler. The line bundle $\lambda(E)$ can then be equipped with the Quillen metric h_Q ([Q2], [BGS1] or 4.1.1 below).

Our main result (Theorem 7) computes the first arithmetic Chern class of $(\lambda(E), h_Q)$ in the $\mathbb{Q}$-vector space $\widehat{CH}^1(Y) \otimes_{\mathbb{Z}} \mathbb{Q}$. It reads

$$(1) \quad \hat{c}_1(\lambda(E), h_Q) = f_*(\widehat{ch}(E, h)\widehat{Td}(Tf, h_f) - a(ch(E_{\mathbb{C}})Td(Tf_{\mathbb{C}})R(Tf_{\mathbb{C}})))^{(1)} .$$

Here $\alpha^{(1)}$ is the component of degree one of $\alpha \in \widehat{CH}^{\cdot}(Y)_{\mathbb{Q}}$, a is the map from the real cohomology of $Y(\mathbb{C})$ to $\widehat{CH}^{\cdot}(Y)_{\mathbb{Q}}$ defined in [GS2, 3.3.4] and in 2.2.1 below, and R is the additive characteristic class (in real cohomology) attached to the power series

$$R(x) = \sum_{\substack{m \text{ odd} \\ m \geq 1}} \left(2\zeta'(-m) + \left(1 + \frac{1}{2} + \dots + \frac{1}{m} \right) \zeta(-m) \right) \frac{x^m}{m!}$$

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which we introduced in [GS4] ($\zeta(s)$ is the Riemann zeta function, and $\zeta'(s)$ its derivative).

Formula (1) was conjectured in [GS4, Conjecture 1]. The main step in the proof of this formula consists in factoring the map f as the composition $f = g \circ i$, where $i: X \to P$ is a closed (regular) immersion and $g: P = \mathbb{P}_Y^N \to Y$ is the N-dimensional projective space over Y. Choose a resolution

$$0 \to E_m \to E_{m-1} \to \ldots \to E_0 \to i_* E \to 0$$

of the coherent sheaf $i_* E$ on P, and (arbitrary) Hermitian metrics on $E_j, j \geqq 0$, as well as a Kähler metric on Tg. We show that (1) for f and $\bar{E}$ follows from the same identity for g and $\bar{E}_j, j \geqq 0$. Indeed, the difference

$$\hat{c}_1(\lambda(E), h_Q) - \sum_{j \geqq 0} (-1)^j \hat{c}_1(\lambda(E_j), h_Q)$$

was computed by Bismut and Lebeau [BL], while the corresponding alternating sum of the right-hand sides in (1) was computed in [BGS3, Theorem 4.13].

We are thus reduced to the case of the projection $g: \mathbb{P}_Y^N \to Y$. When E is the trivial bundle and $Y = \operatorname{Spec}(\mathbb{Z})$, formula (1) was shown in [GS4, Theorem 1]. The general case follows by simple reductions, using the closed immersions $\mathbb{P}^N \to \mathbb{P}^{N+1}$ and the main step above.

This proof of (1) was described in [GS7]. The details are given in paragraphs 4.2.3 and 4.2.4 which, when f is smooth, can be read independently from the rest of the paper.

We also generalize (1) in several ways, in order to allow singularities on the special fibers of X or Y over $\mathbb{Z}$ (this might be of some use, since resolution of singularities is not currently available for schemes of finite type over $\mathbb{Z}$). More specifically, we consider two cases. Case(i): Y is regular, the generic fiber $X_{\mathbb{Q}}$ is smooth, f is projective, and smooth over $X_{\mathbb{Q}}$, and $\mathscr{F}$ is a coherent sheaf on X, which is locally free on $X_{\mathbb{Q}}$ and equipped with an Hermitian metric on $X(\mathbb{C})$. Case(ii): $X_{\mathbb{Q}}$ and $Y_{\mathbb{Q}}$ are smooth, f is l.c.i., and $\bar{E}$ is an Hermitian vector bundle on X.

To make sense of a Riemann–Roch–Grothendieck theorem for $\lambda(\mathscr{F})$ in case (i), or $\lambda(E)$ in case (ii), we need to extend our previous constructions in [GS2] and [GS3] to the singular case. So we introduce "homological Chow groups" $\widehat{CH}_\cdot(X)$, cap-products between $\widehat{CH}^\cdot$ and $\widehat{CH}_\cdot$, and more generally some kind of "operational formalism" in the sense of Fulton [Fu2]. In case (ii), the statement (1) becomes an identity in $\widehat{CH}_\cdot(Y)_{\mathbb{Q}}$, and $\widehat{Td}\,(Tf)$ has to be replaced by the arithmetic Todd class of the relative tangent complex to f (see 2.6.2). In Case (i), we define a notion of Chern character with supports, and then a characteristic class $\tau(\bar{\mathscr{F}}) \in \widehat{CH}_\cdot(X)_{\mathbb{Q}}$ which takes the place of $\widehat{ch}\,(E, h)\,\widehat{Td}\,(Tf, h_f)$ in formula (1). Our theorem (Theorem 7) is then in the style of the singular Riemann Roch theorem of [BFM]. This requires us to combine the Grassmannian graph construction of [BFM] with the study of complex immersions in [BGS2] and [BGS3].

The plan of this paper is as follows. In Sect. 1 we study the Grassmannian graph construction from the algebraic geometric point of view. In particular we show an interesting rigidity property of this construction (Theorem 2), and deduce from it a technical lemma, to be used in Sect. 3. In Sect. 2 we introduce $\widehat{CH}_\cdot$, show some functorial properties of these groups, and define cap products and characteristic classes of Hermitian vector bundles; we also replace $\mathbb{Z}$ by a more general base. In Sect. 3 we define the Chern character with supports and the transformation τ. The proof that τ is independent of choices (Theorem 5) uses the technical lemma of the first section. We can then proceed with the proof of the main theorem (Theorem 7) in Sect. 4. Several reformulations of this Riemann–Roch–Grothendieck theorem are also given, including one involving a notion of "arithmetic Betti numbers" (4.1.6). Finally, in Sect. 5, we use a (weak) version of this result to find bounded (nontrivial) sections of $\mathscr{F} \otimes S^n E$ where $\bar{\mathscr{F}}$ is an Hermitian coherent sheaf and $\bar{E}$ an Hermitian ample vector bundle on a projective arithmetic variety X, n is a large integer, and $S^n E$ the n-th symmetric power of E (Theorem 9). The proof uses a result of Bismut and Vasserot about the asymptotic behaviour of the analytic torsion of $\mathscr{F}_{\mathbb{C}} \otimes S^n E_{\mathbb{C}}$ [B–V], and a lemma of Gromov to compare the sup and L^2 norms on these.

Special cases of our results were announced in [GS6, GS7, S3, G3]. Theorem 7 was first shown by Deligne for smooth families of curves, up to universal constants [De]. In [F3], Faltings extends our result to higher degrees, when X, Y are regular and f is smooth. In [Vo], Vojta used a variant of Theorem 9 in his new proof of the Mordell conjecture.

1 The Grassmannian graph construction

1.1 Definition and basic properties

1.1.1 Let X be an integral (i.e. reduced and irreducible) scheme, and suppose that $E.$ is a chain complex of bundles (i.e. locally free coherent sheaves) on X. Denote by $C. = C(E.)$ the split acyclic complex with $C_i = E_i \oplus E_{i-1}$ and differential $d_i: C_i \to C_{i-1}$, $d_i(x, y) = (y, 0)$. Notice that $C(E.)$ is an additive functor of the graded bundle $E.$ (it does not depend on the differential on $E.$). Furthermore there is a natural map of complexes

$$\gamma: E. \to C(E.)$$
$$x \mapsto (x, d(x))$$

which is the inclusion of a sub-bundle in each degree (i.e. γ is locally split).

If $\phi: E. \to F.$ is a map of complexes, then $C(\phi)\cdot\gamma_E = \gamma_F \cdot \phi$. If furthermore ϕ is null-homotopic, i.e. if there exists h such that $\phi = d\cdot h + h\cdot d$, then $C(\phi)$ is also null homotopic. Namely, if we define

$$C(h): E_i \oplus E_{i-1} \to F_{i+1} \oplus F_i$$
$$(x, y) \mapsto (h(x), -h(y) + \phi(x))$$

we get, on $C(E.)$, $d\cdot C(h) + C(h)\cdot d = C(\phi)$. This homotopy is compatible with the natural transformation γ, i.e. that $C(h)\cdot\gamma_E = \gamma_F \cdot h$.

We suppose now that $E_i = 0$ for $i < 0$. Let $\mathbf{P}^1$ be the projective line over $\mathbf{Z}$ and $\mathcal{O}_{\mathbf{P}^1}(i\infty)$ the invertible sheaf of meromorphic functions on $\mathbf{P}^1$ which have poles of order at most order i along the divisor ∞ and are regular on the affine line $\mathbf{A}^1 = \mathbf{P}^1 - \{\infty\}$. Notice that $\mathcal{O}_{\mathbf{P}^1}(i\infty)$ is contained in $\mathcal{O}_{\mathbf{P}^1}((i+1)\infty)$. By pulling back along the projection $X \times \mathbf{P}^1 \to X$ (where $X \times \mathbf{P}^1$ is the product over $\mathbf{Z}$), we can view $E.$ as defining a complex of sheaves on $X \times \mathbf{P}^1$. Let $\tilde{C}. = \tilde{C}(E.)$ be the C-construction applied to the graded bundle $\bigoplus_i E_i(i)$ where $E_i(i)$ is E_i twisted by $\mathcal{O}_{\mathbf{P}^1}(i\infty)$. The sheaf E_i is a subsheaf of $E_i(i)$ and they are equal on $X \times \mathbf{A}^1$. Hence, via the map γ_E, $E_i|_{X\times\mathbf{A}^1}$ is a sub-bundle of $\tilde{C}_i\,|_{X\times\mathbf{A}^1}$. Let $\pi: \mathbf{G} \to X \times \mathbf{P}^1$ be the product of the Grassmann bundles $\mathbf{G}(n_i, \tilde{C}_i)$ parameterizing rank $n_i = \operatorname{rank}(E_i)$ sub-bundles of $\tilde{C}_i$ over $X \times \mathbf{P}^1$. Over$X \times \mathbf{A}^1$, the map γ_E defines a section s of π. The following definition appears in [BFM, II.1].

Definition 1 The Zariski closure $W = W(E.)$ of $s(X \times \mathbf{A}^1)$ in $\mathbf{G}$ is called the *Grassmannian graph of E..*

Since π is proper so is its restriction to W (which we shall also denote π), and since $X \times \mathbf{A}^1$ is integral so is W. By construction π is an isomorphism over $X \times \mathbf{A}^1$; however the (effective Cartier) divisor $W_\infty = \pi^{-1}(X \times \{\infty\})$ cut out by W at infinity will in general not be isomorphic to X. By construction, there is a sub-bundle $\tilde{E}_i \subset \pi^*(\tilde{C}_i)$ which coincides with E_i over $X \times \mathbf{A}^1$. Notice that, since W is integral, this property *characterizes* $\tilde{E}_i$ as a sub-bundle of $\pi^*(\tilde{C}_i)$.

1.1.2 Let us now summarize, in the following proposition, some properties of the Grassmannian graph construction, whose proof can be found in [BFM, II.1 and II.2], and [BGS3, Sect. 4].

Theorem 1 (i) *Assume that the restriction of E. to a nonempty open subset $U \subset X$ is acyclic. Then there is a canonical splitting of π over $U \times \mathbf{P}^1$. Denote by $\tilde{X}$ the closure in W_∞ of the image of $U \times \{\infty\}$ by this section. Then the cycle $Z = [W_\infty] - [\tilde{X}]$ is supported in the inverse image by π of $X - U$, and the restriction of $\tilde{E}.$* to $\tilde{X}$ *is split acyclic.*

(ii) *Suppose that $i: X \to P$ is a regular immersion of a closed subscheme, $\mathcal{F}$ is a locally free sheaf on X, and $E. \to i_*\mathcal{F}$ is a finite locally free resolution. Then $W(E.)$ is isomorphic to the total space of the deformation to the normal cone construction of* [BFM, 1.5]. *Hence the immersion $X \times \mathbf{P}^1 \to P \times \mathbf{P}^1$ induces a closed immersion $j: X \times \mathbf{P}^1 \to W$, such that $\tilde{E}.$ is a resolution of $j_*\mathcal{F}$.*

Furthermore $\tilde{P}$ is the blow up of P along X, and $W_\infty \simeq \mathbf{P}(N_{X/P} \oplus 1) \cup P$. In particular $|Z|$ is the projective completion $\mathbf{P}(N_{X/P} \oplus 1)$ of the normal bundle of X in P.

Finally, on $|Z|$ there is an exact sequence

$$0 \to G. \to \tilde{E}.|_{\mathbf{P}(N_{X/P}\oplus 1)} \to K.(H) \otimes \pi^*(\mathcal{F}) \to 0\,,$$

where $K.(H)$ is the tautological Koszul complex on $\mathbf{P}(N_{X/P} \oplus 1)$, which is a resolution of $\mathcal{O}_X$ when X is imbedded into $\mathbf{P}(N_{X/P} \oplus 1)$ by the zero section, and G. is acyclic.

1.1.3 We shall now prove a few additional properties of the Grassmannian graph. For instance, we need to know how $W(E.)$ depends upon $E.$. First we state a general lemma.

Lemma 1 *Let V be an integral scheme, and suppose that $A \subset E$ and $B \subset F$ are locally free sub-sheaves of locally free sheaves.*

1. *If $\phi: A \to B$ is a homomorphism which vanishes on a Zariski dense open subset of V, then ϕ vanishes on the whole of V.*
2. *If $B \subset F$ is a sub-bundle, and $\phi: E \to F$ is homomorphism such that, over a Zariski dense open subset of V, $\phi(A) \subset B$, then $\phi(A) \subset B$ over the whole of V.*

Proof. The first statement is true because B is a torsion free module, while the second statement follows from the first by considering the induced map $A \to F/B$ (since, by assumption, F/B is locally free). □

Lemma 2 *$\tilde{E}.$ is a sub-complex of $\pi^*(\tilde{C}.)$*

Proof. It suffices to show that $d_{\tilde{C}}(\tilde{E}_i) \subset \tilde{E}_{i-1}$, since $\tilde{C}$ is a complex. This is true on the dense open subset $X \times \mathbf{A}^1 \subset W$. But W is integral and, by definition of the Grassmannian graph, $\tilde{E}_{i-1}$ is a sub-bundle in $\pi^*(\tilde{C}_{i-1})$. Therefore, by Lemma 1, $d_{\tilde{C}}(\tilde{E}_i) \subset \tilde{E}_{i-1}$ on the whole of W. □

Lemma 3 *If $E.$ has locally free homology sheaves then $W(E.) \simeq X \times \mathbf{P}^1$.*

Proof. The complex $E.$ breaks up into short exact sequences:

$$0 \to Z_i \to E_i \xrightarrow{d} B_{i-1} \to 0$$

(2)
$$0 \to B_i \to Z_i \to H_i(E.) \to 0 ,$$

where Z_i and B_i denote the subsheaves of E_i consisting of cycles and boundaries respectively. Since we assume that the H_i's are locally free, it follows by induction on i that all the sheaves in the above exact sequences are locally free too. Consider the map

$$\eta_i : E_i(i) \oplus E_{i-1}(i-1) \to \frac{B_{i-1}(i) \oplus E_{i-1}(i-1)}{B_{i-1}(i-1)}$$

which maps (u, v) to the class of (du, v); here $B_{i-1}(i-1)$ is mapped diagonally into $B_{i-1}(i) \oplus E_{i-1}(i-1)$ by the inclusions $B_{i-1}(i-1) \subset B_{i-1}(i)$ and
$B_{i-1}(i-1) \subset Z_{i-1}(i-1) \subset E_{i-1}(i-1)$. Since the sheaves in the sequences (2) are all locally free, the target of η_i is locally free, and hence its kernel is too. Over $X \times \mathbf{A}^1$, the homomorphism η_i is equivalent to the map

$$\eta_i : E_i(i) \oplus E_{i-1}(i-1) \to E_{i-1}(i-1)$$

sending (u, v) to $v - du$, and hence the restriction of $\mathrm{Ker}(\eta_i)$ to $X \times \mathbf{A}^1$ is isomorphic to the inclusion of E_i into $E_i \oplus E_{i-1}$ via $x \mapsto (x, dx)$. So the sub-bundle $\mathrm{Ker}(\eta_i) \subset \tilde{C}_i$ determines an extension over $X \times \mathbf{P}^1$ of the section $s : X \times \mathbf{A}^1 \to \mathbf{G}(n_i, \tilde{C}_i)$ defined in Section 1.1.1, and hence $W(E.) = X \times \mathbf{P}^1$, as desired. □

Lemma 4 *Let $E.$ be a complex of locally free sheaves on X, and let $f : P \to X$ be a flat map. Then $W(f^*E.) = P \times_X W(E.)$.*

Proof. First observe that $\tilde{C}(f^*E.) = (f \times 1_{\mathbf{P}^1})^*(\tilde{C}(E.))$. Hence $W(f^*E.)$ is the Zariski closure of $P \times \mathbf{A}^1$ in $P \times \mathbf{G}$, which is equal to to the Zariski closure of $f^{-1}(s(X \times \mathbf{A}^1))$. Since f is flat, it is open and f^{-1} preserves the operation of Zariski closure. Thus $W(f^*E.) = f^{-1}(W(E.)) = P \times_X W(E.)$. □

Lemma 5 *Let $\phi : E. \to F.$ be a map of complexes of locally free sheaves on X. Assume ϕ is a monomorphism and* Coker(ϕ) *is an acyclic complex of locally free sheaves.*

Then $W(E.) = W(F.)$ *and the pull-back of* ϕ *to* $X \times \mathbf{A}^1$ *extends to a map of complexes* $\tilde{\phi}: \tilde{E}. \to \tilde{F}.$ *on* $W(E.)$. *Furthermore* $\mathrm{Coker}(\tilde{\phi})$ *is acyclic, and split acyclic over* $\{\infty\}$.

Proof. Since $X \times \mathbf{A}^1$ is dense in W there can exist at most one isomorphism $W(E.) \simeq W(F.)$ which is the identity on $X \times \mathbf{A}^1$. To define it we may work locally on X, hence we can assume that X is affine. By induction on the degree i of F_i, we may therefore assume that the complex $F.$ is the direct sum of $E.$ with an acyclic complex $G.$, and hence that $\tilde{C}(F.) \simeq \tilde{C}(E.) \oplus \tilde{C}(G.)$. Let m_i, n_i and p_i be the ranks of E_i, F_i, and G_i, respectively. The direct sum decomposition of $F.$ gives a closed embedding

$$\mathbf{G}(m_i, \tilde{C}_i(E.)) \times_{X \times \mathbf{P}^1} \mathbf{G}(p_i, \tilde{C}_i(G.)) \to \mathbf{G}(n_i, \tilde{C}_i(F.))$$

which is compatible with the sections of these Grassmannians over $X \times \mathbf{A}^1$. By Lemma 3, $W(G.) = X \times \mathbf{P}^1$, hence, via the embedding above, $W(E.) \simeq W(F.)$.

Observe that, on $W(E.) = W(F.)$, we have an exact sequence

$$0 \to \tilde{C}(E.) \to \tilde{C}(F.) \to \tilde{C}(G.) \to 0 .$$

The induced sequence

$$0 \to \tilde{E}. \to \tilde{F}. \to \tilde{G}. \to 0$$

is exact since this is true locally.

The fact that $\tilde{G}.$ is acyclic on W and split acyclic over $\{\infty\}$ is shown as in [BGS3, Lemma 4.5]. □

Corollary 1 *Let* $\phi: E. \to F.$ *be a quasi-isomorphism between bounded complexes of locally free sheaves on* X. *Then* $W(E.) = W(F.)$, *and the complexes* $\tilde{E}.$ *and* $\tilde{F}.$ *are quasi-isomorphic as complexes on* $W(E.)$.

Proof. Apply the previous lemma to the inclusion of $E.$ and $F.$ into the mapping cylinder of ϕ. □

Corollary 2 *Let* M *be an integral regular scheme, and* $i: X \subset M$ *a closed subscheme. Let* $W(X/M)$ *denote the Grassman-graph construction for any resolution of* $i_* \mathcal{O}_X$ *by locally free sheaves on* M. *Then, given a locally free coherent sheaf* $\mathcal{F}$ *on* X, *any resolution* $E. \simeq i_* \mathcal{F}$ *by locally free coherent sheaves on* M *extends to a complex of locally free coherent sheaves on* $W(X/M)$ *which is a complex of sub-bundles of* $\tilde{C}(E.)$.

Proof. Notice that, by Corollary 1, $W(X/M)$ is independent of the choice of resolution $F.$ of $i_* \mathcal{O}_X$ by locally free coherent sheaves.

Locally on X, $E.$ is quasi-isomorphic to a direct sum of copies of such a resolution $F.$. By the previous lemma $F.$ and hence any finite direct sum $F.^{\oplus n}$ extends to a subcomplex of $\tilde{C}(F.^{\oplus n}) \simeq (\tilde{C}(F.)^{\oplus n})$ on $W(X/M)$. Now $W(X/M) = \bigcup W(X \cap U/U)$ as U runs through any open cover of M. If we choose the open cover so that, on each U, $E.$ is quasi-isomorphic to a sum of copies of $F.$, then $E.$ extends as a subcomplex of $\tilde{C}(E.)$ on each $W(X \cap U/U)$, and hence, by the uniqueness of such extensions, it extends as a subcomplex on the whole of W. □

1.1.4

Lemma 6 *If $E.$ and $F.$ are chain complexes of locally free coherent sheaves on X, the identity map on $X \times \mathbf{A}^1$ extends uniquely to a map from the Zariski closure $W(E., F.)$ of $X \times \mathbf{A}^1$ in $W(E.) \times_{X \times \mathbf{P}^1} W(F.)$ to $W(E. \oplus F.)$.*

Proof. On the variety $W(E.) \times_{X \times \mathbf{P}^1} W(F.)$ by pulling back from the two factors we obtain sub-bundles $\tilde{E}.$ of $\tilde{C}(E.)$ and $\tilde{F}.$ of $\tilde{C}(F.)$ extending $E.$ and $F.$ from $X \times \mathbf{A}^1$. The direct sum $\tilde{E}. \oplus \tilde{F}.$ is a sub-bundle of $\tilde{C}(E.) \oplus \tilde{C}(F.) \simeq \tilde{C}(E. \oplus F.)$ and hence is classified by a map from the fiber product $W(E.) \times_{X \times \mathbf{P}^1} W(F.)$ to the Grassmannian of sub-bundles of $\tilde{C}(E. \oplus F.)$. This map agrees with the standard section s over $X \times \mathbf{A}^1$, and hence maps $W(E., F.)$ to $W(E. \oplus F.)$. □

Lemma 7 (i) *Any map $\phi : E. \to F.$ of chain complexes of locally free sheaves on X extends to a map $\tilde{\phi} : \tilde{E} \to \tilde{F}.$ on $W(E., F.)$. The map $\phi \mapsto \tilde{\phi}$ is additive.*

(ii) *If $\phi : E. \to F.$ and $\psi : F. \to G.$ are maps of complexes of locally free coherent sheaves on X, then on $W(E., F., G.)$ (which is defined analogously to $W(E., F.)$), we have $\widetilde{(\psi \circ \phi)} = \tilde{\psi} \circ \tilde{\phi}$.*

Proof. (i) By pulling back from $X \times \mathbf{P}^1$ we may view $\tilde{C}.(\phi)$ as a map of complexes on $W(E., F.)$. On $X \times \mathbf{A}^1 \subset W(E., F.)$ we know that $\tilde{C}.(\phi)(E.) \subset F.$. Hence, by Lemma 1, $\tilde{C}.(\phi)(\tilde{E}.) \subset \tilde{F}.$. Since $\tilde{C}.(\phi)$ is additive in ϕ, it follows (by Lemma 1 applied to the map $(\widetilde{\phi_1 + \phi_2}) - (\tilde{\phi}_1 + \tilde{\phi}_2)$), that $\tilde{\phi}$ is also additive.

We omit the proof of (ii). □

Let us temporarily simplify our notation, and write W for $W(E., F.)$. Let W_∞ be the inverse image of $X \times \{\infty\}$ under the projection $\pi : W \to X \times \mathbf{P}^1$. It follows from Lemma 3 that, as in Theorem 1 (which is the special case where $F. = 0$ and $E.$ is generically acyclic) $[W_\infty] = Z + [\tilde{X}]$, in which $\pi|_{\tilde{X}} : \tilde{X} \to X$ is birational, and the support $|Z|$ of Z is contained in the inverse image of the proper closed subset of X where the homology sheaves $H.(E. \oplus F.)$ are not locally free. By definition, if U is the complement of this closed subset in X, $\tilde{X}$ is the closure in W_∞ of the image of $U \times \{\infty\}$ by the section of π over $U \times \mathbf{P}^1$ obtained in Lemma 3.

Theorem 2 *Let $\phi : E. \to F.$ be a map of complexes on X, and let h be a null-homotopy of ϕ. Then h extends uniquely from $X \times \mathbf{A}^1$ to a nullhomotopy $\tilde{h}$ of $\tilde{\phi}$ on W.*

The restriction of $\tilde{h}$ to W_∞ depends only on ϕ and not on the choice of h. It is additive in ϕ.

The restriction of $\tilde{h}$ to $|Z|$ depends only (and additively) on the restriction of ϕ to $\pi(|Z|)$.

Proof. On $X \times \mathbf{P}^1$ we can define a map

$$\tilde{C}_i(h) : E_i(i) \oplus E_{i-1}((i-1)) \to F_{i+1}((i+1)) \oplus F_i(i)$$

by $\tilde{C}_i(h)(x, y) = (h(x), \; -h(y) + \phi(x))$, where we remember that we have an embedding of $\mathcal{O}_{\mathbf{P}^1}(i\infty)$ into $\mathcal{O}_{\mathbf{P}^1}((i+1)\infty)$. In order to show the existence and uniqueness of $\tilde{h}$, we now use Lemma 1, and we observe that $\tilde{C}(h)$ restricted to the dense open subset $X \times \mathbf{A}^1 \subset W(E., F.)$ is a nullhomotopy of $C(\phi)$.

On W we have a commutative diagram:

$$\begin{array}{ccc} \tilde{E}_i & \rightarrow & E_i(i) \oplus E_{i-1}(i-1) \\ \tilde{h} \downarrow & & \downarrow h_C = \begin{bmatrix} h(1) & 0 \\ \phi & -h(1) \end{bmatrix} \\ \tilde{F}_{i+1} & \rightarrow & F_{i+1}(i+1) \oplus F_i(i) \end{array}$$

Here $h(1): E_i(i) \rightarrow F_{i+1}(i+1)$ is the composition of h with the inclusion $F_{i+1}(i) \rightarrow F_{i+1}(i+1)$. Since the restriction of this inclusion to $X \times \{\infty\}$ vanishes, the restriction of $h(1)$ vanishes too. Hence the restriction of $\tilde{h}$ to infinity does not depend on the choice of h at all, but rather is the restriction to $\tilde{E}.$ of the map

$$\begin{bmatrix} 0 & 0 \\ \phi & 0 \end{bmatrix}$$

from $\tilde{C}(E.)$ to $\tilde{C}(F.)$. Therefore at a point $w \in W$ it depends linearly on the map ϕ at the image of w in X. The assertions of the theorem follow from this remark. □

1.2 A technical lemma

1.2.1 We need some facts from homological algebra.

Lemma 8 *Let $E.$ and $F.$ be bounded complexes of sheaves of abelian groups on a topological space X. Suppose that there is a finite open cover $\{U_\alpha\}$ of X and quasi-isomorphisms*

$$\psi_\alpha : E.|_{U_\alpha} \rightarrow F.|_{U_\alpha}$$

such that on each intersection $U_\alpha \cap U_\beta$, ψ_α and ψ_β are homotopic, i.e. there exists a map

$$\psi_{\beta\alpha} : E.|_{U_\alpha \cap U_\beta} \rightarrow F.|_{U_\alpha \cap U_\beta}$$

such that

$$\psi_\beta - \psi_\alpha = d\psi_{\beta\alpha} + \psi_{\beta\alpha} d ,$$

and such that, on each triple intersection $U_\alpha \cap U_\beta \cap U_\gamma$,

$$\psi_{\beta\alpha} - \psi_{\gamma\alpha} + \psi_{\gamma\beta} = 0 .$$

Then $E.$ and $F.$ are isomorphic in the derived category of bounded complexes of sheaves.

Proof. Consider the complex $\check{\mathscr{C}}^*(\{U_\alpha\}, F.)$ of sheaves on X with sections over each open set U consisting of the total complex of the Čech bi-complex $\check{C}^*(\{U_\alpha \cap U\}, F.)$. The natural augmentation $\eta: F. \rightarrow \check{\mathscr{C}}^*(\{U_\alpha\}, F.)$ is a quasi-isomorphism. It suffices to show therefore that there is a quasi-isomorphism $E. \rightarrow \check{\mathscr{C}}^*(\{U_\alpha\}, F.)$. We define such a map $\psi^\cdot$ as follows. Writing $\psi^\cdot = \bigoplus_k \psi^k$ with $\psi^k : E_i \rightarrow \bigoplus_{l-k=i} \check{\mathscr{C}}^k(\{U_\alpha\}, F_l)$ we set

$$\psi^0 = \bigoplus_\alpha \psi_\alpha ,$$

$$\psi^1 = \bigoplus_{\alpha,\beta} \psi_{\beta\alpha} ,$$

$$\psi^k = 0 \quad \text{if } k > 1 .$$

Then it is straightforward to check that $\psi^{\cdot}$ is a map of complexes. To see that $\psi^{\cdot}$ is a quasi-isomorphism, we can work locally, and assume that $X = U_\alpha$ for some α. Then $\eta\psi_\alpha$ and $\psi^{\cdot}$ are homotopic, and since ψ_α and η are quasi-isomorphisms, it follows that $\psi^{\cdot}$ is too. □

Lemma 9 *Suppose that $E_{..}$ and $G_{..}$ are bounded double complexes in an abelian category, with d' (resp. d'') the first (resp. second) differential. Assume that $\psi_{..}$ is a map of bigraded objects such that:*

1. *For each k,*

$$\psi_{k,\cdot}: E_{k,\cdot} \to G_{k,\cdot}$$

is a quasi-isomorphism of complexes.

2. *For each k, l there exists a map $\chi_{k,l}: E_{k,l} \to G_{k-1,l+1}$ such that*

$$d'\psi_{k,\cdot} - \psi_{k-1,\cdot}.d' = (-1)^{k-1} d''\chi_{k,\cdot} + (-1)^k \chi_{k,\cdot}.d'' .$$

3. *$d'\chi + \chi d'$ vanishes.*

Then the map $\psi + (-1)^k\chi$ on $E_{k,\cdot}$ induces a quasi-isomorphism between the total complexes of $E_{..}$ and $G_{..}$

Proof. First we check that $\phi = \psi + (-1)^k\chi$ on $E_{k,\cdot}$ induces a chain map on $\mathrm{Tot}(E_{..})$. Let $D = d' + d''$ be the total differential. We want

$$D\cdot\phi - \phi\cdot D = (d'\cdot\psi - \psi\cdot d') + (d''\cdot\psi - \psi\cdot d'') + (-1)^k(d'\cdot\chi + \chi\cdot d') + (-1)^k(d''\cdot\chi - \chi\cdot d'')$$

to be zero. But the second and third terms in the right hand side of this equation vanish by 1. and 3., while the other two terms have sum zero by 2..

Now given a double complex $X_{..}$ consider the filtration

$$F_i X_{..} = \bigoplus_{k \leq i} X_{k,\cdot} .$$

then ϕ preserves this filtration on $E_{..}$ and $G_{..}$, and the induced map on the associated graded objects is the sum of the quasi-isomorphisms $\psi_{k,\cdot}$. Hence ϕ is itself a quasi-isomorphism. □

1.2.2 Let X be an integral scheme, quasi-projective over a regular noetherian integral domain A. Suppose that $j: X \to M$ and $k: X \to P$ are two embeddings of X into regular quasi-projective varieties over A, and that there is a smooth map $q: P \to M$, such that $qk = j$.

Consider the following diagram, in which the square is Cartesian, g is the section of p induced by k, and $f = j \times_M 1_P$.

$$\begin{array}{ccccc} X & \xrightarrow{g} & X\times_M P & \xrightarrow{f} & P \\ & & \downarrow p & & \downarrow q \\ & & X & \xrightarrow{j} & M . \end{array}$$

Since it is a section of a smooth map, g is a regular embedding. This implies that the direct image by g_* of any bundle on X has a finite global resolution by bundles on $X \times_M P$. This is a standard fact that we shall use several times: being regular, g is perfect and, since $X \times_M P$ is quasi-projective we can apply [BGI, II Proposition 2.2.9.b] (see loc.cit. Definition 2.2.4 and 2.2.5).

So let $V_\cdot \to g_* \mathcal{O}_X$ be a resolution by locally free coherent sheaves on $X \times_M P$ of the direct image of the trivial bundle on X. If $\mathscr{F}$ is a coherent sheaf on X, $g_* \mathcal{O}_X$ and $p^* \mathscr{F}$ are Tor-independent (p is smooth), and so $V_\cdot \otimes p^* \mathscr{F}$ is a resolution (though not by locally free sheaves) of $g_* \mathscr{F}$. Now let $E_\cdot \to j_* \mathscr{F}$ be a locally free resolution of the direct image of $\mathscr{F}$ on M, and let $W(E_\cdot) = W$ be the corresponding Grassmann-graph construction.

On P we may choose a resolution of $f_*(V_\cdot \otimes p^* \mathscr{F})$ by a double complex $G_{\cdot\cdot}$ of locally free coherent sheaves so that, for each i, $G_{i,\cdot} \to f_*(V_i \otimes p^* \mathscr{F})$ is a resolution (proceed by induction on i, using the fact that this is true when $V_\cdot$ has length one). By Corollary 2 we know that, since each $G_{i,\cdot}$ is a resolution of a locally free coherent sheaf on X, it extends as a complex $\tilde{G}_{i,\cdot}$ to $W((X \times_M P)/P)$. Now observe that $W((X \times_M P)/P) = W \times_M P$, by Lemma 3. The horizontal component d' of the differential on $G_{\cdot\cdot}$ can be viewed as determining for each i a map of complexes $d' : G_{i,\cdot} \to G_{i-1,\cdot}$, where the differential on $G_{i,\cdot}$ is $(-1)^i d''$, and hence by Lemma 6 we get a map $\tilde{d}' : \tilde{G}_{i,\cdot} \to \tilde{G}_{i-1,\cdot}$, such that $\tilde{d}'^2 = 0$. Thus we have a double complex $\tilde{G}_{\cdot\cdot}$ on $W(X/M) \times_M P$.

From the results in the first section, we see that this double complex $\tilde{G}_{\cdot\cdot}$ has the following properties. If $\pi : W \times_M P \to \mathbf{P}^1 \times P$ is the projection, then for each l, $\tilde{G}_{l,\cdot}$ is acyclic on $\pi^{-1}[\mathbf{P}^1 \times (P - q^{-1}(X))]$. Over $\{0\} \in \mathbf{P}^1$, $\tilde{G}_{\cdot\cdot} \simeq G_{\cdot\cdot}$. Over $\{\infty\} \in \mathbf{P}^1$, if we define $\tilde{M} \subset W_\infty$ as above, $W_\infty \times_M P \simeq (Z \cup \tilde{M}) \times_M P$, with $\pi(Z) \subset X \times_M P \subset P$. The restriction of the double complex $\tilde{G}$ to $\tilde{M} \times_M P$ has split acyclic columns $\tilde{G}_{l,\cdot}$ and $\operatorname{Tot} \tilde{G}|_{\tilde{M} \times_M P}$ is therefore acyclic.

Let $|Z|$ be the support in W of the cycle $Z = [W_\infty] - [\tilde{M}]$. We have $\pi(|Z|) = X$. We denote by $v : |Z| \times_M P \to X \times_M P$ and $q_\infty : |Z| \times_M P \to Z$ the projections induced by π and q respectively, and by $E_\cdot^Z$ the restriction to $|Z|$ of the canonical extension of $E_\cdot$ to W.

Lemma 10 *There is an isomorphism in the derived category of bounded complexes of locally free coherent sheaves on* $|Z| \times_M P$:

$$\operatorname{Tot} \tilde{G}|_{|Z| \times_M P} \simeq v^*(V^\cdot) \otimes q_\infty^*(E_\cdot^Z) .$$

Proof. For every integer k, since each V_k is locally free coherent on $X \times_M P$, in a small open set where V_k is trivial of rank r_k, the sheaf $f_*(V_k \otimes p^* \mathscr{F}) = f_*(p^* \mathscr{F})^{r_k} = q^*(j_* \mathscr{F})^{r_k}$ has a resolution by $q^* E_\cdot^{r_k}$. It also has a (global) resolution by $G_{k,\cdot}$. Hence there exists a locally finite covering $\{U_\alpha\}$ of P by affine opens, and, for each U_α, an isomorphism

$$\bar{\theta}_\alpha : V_k \to \mathcal{O}^{r_k}_{X \times_M P}$$

on $U_\alpha \cap (X \times_M P)$, and a chain equivalence

$$\phi_\alpha : E_\cdot^{r_k} \to G_{k,\cdot} ,$$

on U_α, resolving the isomorphism

$$f_*(\bar{\theta}_\alpha \otimes 1) : f_*(V_k \otimes p^* \mathscr{F}) \to f_*(p^* \mathscr{F})^{r_k} .$$

On $U_\alpha \cap U_\beta \cap (X \times_M P)$ there is a transition matrix $\bar{\theta}_{\beta\alpha} = \bar{\theta}_\beta \bar{\theta}_\alpha^{-1}$ for V_k, which we may lift to an $r_k \times r_k$ matrix of functions $\theta_{\beta\alpha}$ on $U_\alpha \cap U_\beta$.

Now consider the two maps ϕ_α and $\phi_\beta(\theta_{\beta\alpha} \otimes 1)$ from $E_\cdot^{r_k}$ to $G_{k,\cdot}$. The compositions of these maps with the quasi-isomorphism from $G_{k,\cdot}$ to $f_*(V_k \otimes p^* \mathscr{F})$ are the same, hence, since $E^{r_k}_{\cdot}$ is a complex of locally free coherent sheaves, these maps are homotopic over the affine open $U_\alpha \cap U_\beta$. Let us make a choice of homotopy $\phi_{\beta\alpha}$, satisfying

$$d(\phi_{\beta\alpha}) = \phi_\beta(\theta_{\beta\alpha} \otimes 1) - \phi_\alpha \,.$$

By Lemma 6 the maps $\phi_\alpha, \phi_\beta, \theta_{\beta\alpha}$, and $\phi_{\beta\alpha}$ extend to the inverse images of the sets U_α, U_β, and $U_\alpha \cap U_\beta$ in $W \times_M P$. We denote these extensions by $\tilde{\phi}_\alpha, \tilde{\phi}_\beta, \tilde{\theta}_{\beta\alpha}$, and $\tilde{\phi}_{\beta\alpha}$. Notice that $\tilde{\theta}_{\beta\alpha}$ is the inverse image by π of $\theta_{\beta\alpha}$. In particular, its restriction to $|Z| \times_M P$ coincides with $v^*(\bar{\theta}_{\beta\alpha}) = v^*(\bar{\theta}_\beta)v^*(\bar{\theta}_\alpha)^{-1}$. On the covering $\{|Z| \times_M U_\alpha\}$ we consider the trivializations $\bar{\theta}_\alpha = v^*(\bar{\theta}_\alpha)$ of $v^*(V_k)$ and the maps

$$\psi_\alpha = \tilde{\phi}_\alpha(\tilde{\theta}_\alpha \otimes 1) : v^*(V_k) \otimes q_\infty^*(E_{\cdot}^{Z})|_{|Z| \times_M P \cap U_\alpha} \to \tilde{G}_{k,\cdot}|_{|Z| \times P \cap U_\alpha} \,.$$

On the intersection $|Z| \times_M (U_\alpha \cap U_\beta)$ the map

$$\psi_{\beta\alpha} = \tilde{\phi}_{\beta\alpha}(\tilde{\theta}_\alpha \otimes 1)$$

is a homotopy between ψ_β and ψ_α, since

$$\begin{aligned} d\psi_{\beta\alpha} &= d(\tilde{\phi}_{\beta\alpha})|_{|Z| \times P}(\tilde{\theta}_\alpha \otimes 1) = (\tilde{\phi}_\beta \tilde{\theta}_{\beta\alpha} - \tilde{\phi}_\alpha)|_{|Z| \times P}(\tilde{\theta}_\alpha \otimes 1) \\ &= \tilde{\phi}_\beta|_{|Z| \times P}((v^*(\tilde{\theta}_{\beta\alpha})v^*(\bar{\theta}_\alpha)) \otimes 1) - \tilde{\phi}_\alpha|_{|Z| \times P}(v^*(\bar{\theta}_\alpha) \otimes 1) \\ &= \tilde{\phi}_\beta(\tilde{\theta}_\beta \otimes 1) - \tilde{\phi}_\alpha(\tilde{\theta}_\alpha \otimes 1) = \psi_\beta - \psi_\alpha \,. \end{aligned}$$

Furthermore, by Theorem 2, $\tilde{\phi}_{\beta\alpha}|_{|Z| \times_M P}$ depends only on the restriction of $\phi_\beta \theta_{\beta\alpha} - \phi_\alpha$ to $X \times_M P$. Therefore we deduce from Theorem 2 that, on the triple intersections $|Z| \times_M (U_\alpha \cap U_\beta \cap U_\gamma)$,

$$\psi_{\beta\alpha} - \psi_{\gamma\alpha} + \psi_{\gamma\beta} = 0 \,.$$

Indeed, $(\psi_{\beta\alpha} - \psi_{\gamma\alpha} + \psi_{\gamma\beta})(\tilde{\theta}_\alpha \otimes 1)^{-1}$ is the restriction to $|Z|$ of the extension to W of a null-homotopy of $(\phi_\beta\theta_{\beta\alpha} - \phi_\alpha) - (\phi_\gamma\theta_{\gamma\alpha} - \phi_\alpha) + (\phi_\gamma\theta_{\gamma\beta} - \phi_\beta)\theta_{\beta\alpha}$, whose restriction to $X \times_M P$ is zero since

$$\bar{\theta}_{\gamma\beta}\bar{\theta}_{\beta\alpha} = \bar{\theta}_\gamma \bar{\theta}_\beta^{-1} \bar{\theta}_\beta \bar{\theta}_\alpha^{-1} = \bar{\theta}_{\gamma\alpha} \,.$$

Hence, by Lemma 8, the family $\{\psi_\alpha, \psi_{\beta\alpha}\}$ defines a quasi-isomorphism of complexes of sheaves on $|Z| \times_M P$:

$$v^*(V_k) \otimes q_\infty^*(E_{\cdot}^{Z}) \to \operatorname{Tot} \check{\mathscr{C}}^{\cdot}(\{U_\alpha\}, \tilde{G}_{k,\cdot}) \,.$$

The differentials $d_k : V_k \to V_{k-1}$ can also be lifted locally to maps of complexes $d_{k,\alpha} : E_k^{r_k} \to E_{k-1}^{r_{k-1}}$, over each U_α. After composition with the augmentation from $G_{k-1,\cdot}$ to $f_*(V_{k-1} \otimes p^*\mathscr{F})$, the two maps $d_k^G \circ \phi_\alpha$ and $\phi_\alpha \circ d_{k-1}$ coincide. Hence these two maps are homotopic. Let ϕ'_α be the homotopy and χ_α the restriction to $|Z| \times_M P$ of the canonical extension of ϕ'_α to $W \times_M P$. Restricting to $|Z| \times_M P$ we get, over each U_α, a diagram

$$\begin{array}{ccc} V_k \otimes \tilde{E}_{\cdot} & \xrightarrow{\psi_\alpha} & \tilde{G}_{k,\cdot} \\ {\scriptstyle d' = d_k^V \otimes 1} \downarrow & & \downarrow {\scriptstyle d'} \\ V_{k-1} \otimes \tilde{E}_{\cdot} & \xrightarrow{\psi_\alpha} & \tilde{G}_{k-1,\cdot} \end{array}$$

with $d'\psi_\alpha - \psi_\alpha d' = (-1)^{k-1} d'' \chi_\alpha + (-1)^k \chi_\alpha (1 \otimes d_{\tilde{E}})$. From the equality

$$d'(d'\phi_\alpha - \phi_\alpha d') + (d'\phi_\alpha - \phi_\alpha d')d' = 0$$

on $X \times_M P$ we get, by Theorem 2, $d'\chi_\alpha + \chi_\alpha d' = 0$. Hence, by Lemma 9, $\psi_\alpha + (-1)^k \chi_\alpha$ defines a quasi-isomorphism

$$\operatorname{Tot}(v^*(V) \otimes q_\infty^*(\tilde{E}.)) \to \operatorname{Tot}(\tilde{G}.,.)$$

on U_α. Now consider

$$\Psi = \{\psi_\alpha + (-1)^k \chi_\alpha, \psi_{\beta\alpha}\} :$$

$$\operatorname{Tot}(v^*(V.) \otimes q_\infty^*(E.^Z)) \to \operatorname{Tot}(\check{\mathscr{C}}^\cdot(\{U_\alpha\}, \tilde{G}.,.)) .$$

To show that Ψ is a quasi-isomorphism, which will conclude the proof of Lemma 10, we shall apply once more Lemma 8. For that the only identity left to be shown in $d'\psi_{\beta\alpha} - \psi_{\beta\alpha} d' = (-1)^k (\chi_\beta - \chi_\alpha)$. This follows from Theorem 2 by an argument as above, where we compose both sides of this equality with $(\tilde{\theta}_\alpha \otimes 1)^{-1}$ and notice that d' commutes with $\tilde{\theta}_{\beta\alpha} \otimes 1$ on $X \times_M P$. Details are left to the reader. □

2 Arithmetic Chow homology

2.1 The construction

In [GS2] we defined an *arithmetic ring* to be an excellent regular noetherian integral domain A, together with a finite non-empty set Σ of monomorphisms from A into the complex numbers, and a conjugate linear involution F_∞ of the product $\mathbf{C}^\Sigma$ which commutes with the diagonal embedding of A into $\mathbf{C}^\Sigma$.

When dealing with dimensions of cycles, it will be convenient to restrict our choice of ground ring. We say that A is "good" if it is equicodimensional (i.e., all maximal ideals have the same height), and Jacobson (i.e., any prime ideal is the intersection of the maximal ideals containing it). We shall write $e = e(A)$ for the dimension of A, and F for the fraction field of A. Examples of good arithmetic rings are the ring of S-integers in an algebraic number field, or the algebraic number field itself (where in both cases we take Σ to be the set of all embeddings of the ring into the complex numbers and F_∞ to be complex conjugation), or the complex numbers. Notice that a more general case is considered in [Fu2], Chapter 20, and [G2].

An *arithmetic variety* over the arithmetic ring A is a scheme X which is flat and quasi-projective over Spec(A), and has smooth generic fiber X_F. Till the end of Section 3 we make the assumption that the ground ring A is good. This implies that X is Jacobson, and that each irreducible component of X is equicodimensional (see [EGA4] 10.4). It follows that each Zariski open set in X has the same dimension as X, and that if $Z \subset X$ is an integral subscheme of X for which Z_F is non-empty, then $\dim Z_F$ equals $\dim Z - e$. Note that the set $X(\mathbf{C})$ of complex points of $X \otimes_A \mathbf{C}^\Sigma$ is a quasi-projective complex manifold. The conjugate linear involution F_∞ of $\mathbf{C}^\Sigma$ induces an anti-holomorphic involution, which we also denote F_∞, of $X(\mathbf{C})$. For each natural number p, we write $A^{p,p}(X_\mathbf{R})$ [resp. $\mathscr{D}^{p,p}(X_\mathbf{R})$] for the vector space of real differential forms [resp. real differential forms with distribution coefficients] α on $X(\mathbf{C})$ such that $F_\infty^*(\alpha) = (-1)^p \alpha$. Let $\mathscr{D}_{p,p}(X_\mathbf{R})$ be the space of real currents on $X(\mathbf{C})$ which is the topological dual (with respect to the Schwartz topology) of the space $A_c^{p,p}(X_{\mathbf{R}})$ of forms with compact support, and let $A_{p,p}(X_{\mathbf{R}}) \in \mathscr{D}_{p,q}(X_{\mathbf{R}})$ be the subspace consisting of currents which are smooth, i.e. represented by integration against smooth forms.

If X is a scheme and p is a natural number, following [EGA4] and [Fu2], we define $Z_p(X)$ to be the free abelian group on the set of integral subschemes of X of dimension p; elements of this group are called *dimension p cycles*. Following [GS2] 1.2.3, if X is an arithmetic variety, we write $\tilde{\mathscr{D}}_{q,q}(X_{\mathbf{R}})$ for the quotient of $\mathscr{D}_{q,q}(X_{\mathbf{R}})$ by the subgroup $(\operatorname{Im}\partial + \operatorname{Im}\bar{\partial})$. If $Z \in Z_p(X)$, then its restriction to the generic fibre X_F is $p - e$ dimensional. (Recall that e is the dimension of the ground ring A.) Hence the current δ_Z given by integration over the restriction of the cycle Z to $X(\mathbf{C})$ lies in $\mathscr{D}_{p-e,p-e}(X_{\mathbf{R}})$. (See [GS2] 1.1.2 for more details; the only difference here is that we are grading by dimension rather than by codimension.) A *Green current* for Z is an element $g_Z \in \tilde{\mathscr{D}}_{p-e+1,p-e+1}(X_{\mathbf{R}})$, such that the current $dd^c(g_Z) + \delta_Z$ is smooth. Let $\hat{Z}_p(X)$ be the subgroup of $Z_p(X) \oplus \tilde{\mathscr{D}}_{p-e+1,p-e+1}(X_{\mathbf{R}})$ consisting of pairs $(Z = \sum n_i[Z_i], g_Z)$ such that g_Z is a Green current for Z.

If $Y \subset X$ is an integral subscheme of dimension $p + 1$, and $f \in k(Y)^*$ is a rational function on Y, the divisor of f, $\operatorname{div}(f)$, is a cycle of dimension p; see [Fu1] for details. The complex points $Y(\mathbf{C})$ of Y form an analytic subspace of $X(\mathbf{C})$; we shall write $i\colon Y(\mathbf{C}) \to X(\mathbf{C})$ for the inclusion. As in [GS2, 3.3.3], the function $-\log|f|^2$ on the nonsingular locus of $Y(\mathbf{C})$ defines a current $i_*[-\log|f|^2] \in \mathscr{D}_{p-e+1,p-e+1}(X_{\mathbf{R}})$, (for which we shall often write just $-\log|f|^2$). Of course this current is zero if $Y \cap X_F$ is empty. By the Poincaré–Lelong formula [GH], we know that $i_*[-\log|f|^2]$ is a Green current for $\operatorname{div}(f)$:

$$dd^c i_*[-\log|f|^2] + \delta_{\operatorname{div}(f)} = 0\,.$$

Hence the pair $(\operatorname{div}(f),\ i_*[-\log|f|^2])$ is an element of $\hat{Z}_p(X)$; we write $\hat{R}_p(X) \subset \hat{Z}_p(X)$ for the subgroup generated by these classes.

Definition 2 If X is an arithmetic variety over a good arithmetic ring A and p a natural number, the p-th *arithmetic Chow homology group* $\widehat{CH}_p(X)$ is the quotient of $\hat{Z}_p(X)$ by the subgroup $\hat{R}_p(X)$.

2.2 Elementary properties

The following are, except where noted, direct translations of the corresponding results for the arithmetic Chow groups (graded by codimension) defined in [GS2]. We have therefore omitted any duplicate proofs.

2.2.1 We start by recalling the notation and definitions (with some minor modifications to account for the grading by dimension) of [GS2] for the various natural maps we can construct involving $\widehat{CH}_.(X)$. First, there are maps

$$\zeta\colon \widehat{CH}_p(X) \to CH_p(X)$$

$$(Z, g_Z) \mapsto Z\,,$$

$$a\colon \tilde{A}_{p-e+1,p-e+1}(X_{\mathbf{R}}) \to \widehat{CH}_p(X)$$

$$\alpha \mapsto (0, \alpha)\,,$$

and

$$\omega : \widehat{CH}_p(X) \to A_{p-e,p-e}(X_{\mathbf{R}})$$
$$(Z, g_Z) \mapsto \omega(Z, g_Z) = \delta_Z + dd^c g_Z \,.$$

These are well defined since $\zeta(\widehat{\mathrm{div}}(f)) = \mathrm{div}(f)$ and $\omega(\widehat{\mathrm{div}} f)) = 0$.

Let $CH_{p,p+1}(X)$ be the group (see [G1])

$$CH_{p,p+1}(X) = \frac{\mathrm{Ker}\{\mathrm{div} : \bigoplus_{y \in X_{p+1}} k(y)^* \to Z_p(X)\}}{\mathrm{Im}\{t : \bigoplus_{z \in X_{p+2}} K_2 k(z) \to \bigoplus_{y \in X_{p+1}} k(y)^*\}} \,,$$

where the map t is the differential in the Quillen spectral sequence, [Q1, Sect. 7]. It follows from the arguments of [GS2, 3.3.5], that the map

$$\bigoplus_{y \in X_{p+1}} k(y)^* \to \tilde{\mathscr{D}}_{p-e+1,p-e+1}(X_{\mathbf{R}})$$
$$f \in k(y)^* \mapsto -\log|f|^2$$

factors through a map.

$$\rho : CH_{p,p+1}(X) \to H^{\mathrm{I}}_{p-e+1,p-e+1}(X_{\mathbf{R}}) \subset \tilde{\mathscr{D}}_{p-e+1,p-e+1}(X_{\mathbf{R}}) \,.$$

Here $H^{\mathrm{I}}_{p-e+1,p-e+1}(X_{\mathbf{R}})$ is the kernel of dd^c. We shall also write $Z_{p-e,p-e}(X_{\mathbf{R}}) \subset A_{p-e,p-e}(X_{\mathbf{R}})$ for the kernel of d, and $H^{\mathrm{II}}_{p-e,p-e}(X_{\mathbf{R}})$ for its quotient by the image of dd^c.

As in [GS2, 3.3.5], we have exact sequences:

$$CH_{p,p+1}(X) \xrightarrow{\rho} \tilde{A}_{p-e+1,p-e+1}(X_{\mathbf{R}}) \xrightarrow{a} \widehat{CH}_p(X) \xrightarrow{\zeta} CH_p(X) \to 0$$

and

$$CH_{p,p+1}(X) \xrightarrow{\rho} H^{\mathrm{I}}_{p-e+1,p-e+1}(X_{\mathbf{R}}) \xrightarrow{a} \widehat{CH}_p(X) \xrightarrow{(\zeta,-\omega)}$$

$$CH_p(X) \oplus Z_{p-e,p-e}(X_{\mathbf{R}}) \xrightarrow{c-h} H^{\mathrm{II}}_{p-e,p-e}(X_{\mathbf{R}}) \to 0 \,.$$

2.2.2 Let $f : X \to Y$ be a proper map between arithmetic varieties, which restricts to a smooth map $X_F \to Y_F$. Then the proof of [GS2, 3.6.1] applies, after replacing the grading by codimension by the grading by dimension, to give a *push forward map* $f_* : \widehat{CH}.(X) \to \widehat{CH}.(Y)$, which is given explicitly on arithmetic cycles by $f_*(Z, g_Z) = (f_* Z, f_* g_Z)$. Here $f_* Z$ is the usual push forward of algebraic cycles, as in [Fu1], while $f_* g_Z$ is the push forward on currents induced by pullback on forms with compact support; note that $f_* g_Z$ is a Green current for Z because $f : X(\mathbf{C}) \to Y(\mathbf{C})$ is smooth.

The map f_* is part of a morphism from the exact sequences of 2.2.1 for X to the corresponding exact sequences for Y. In particular, for any arithmetic variety X, the canonical map $X^{\mathrm{red}} \to X$, where X^{red} is the reduced scheme attached to X, induces an isomorphism on $\widehat{CH}_p$ for all $p \geq 0$, as can be checked from the corresponding statement for CH_p by applying the five Lemma to the map of exact sequences in 2.2.1.

2.2.3 Now let $f: X \to Y$ be a flat map between arithmetic varieties which is smooth over F. Then, again as in [GS2, 3.6.1], we can construct a pull back map $f^*: \widehat{CH}_p(Y) \to \widehat{CH}_{p+d}(X)$ where d is the dimension of the fibres of f (we assume for simplicity that d is the same for all connected components of X).

Lemma 11 *Let $f: X \to Y$ be a flat map between arithmetic varieties, which is smooth over Y_F, and let $g: P \to Y$ be a proper map, again smooth over Y_F. Let Z be the fibre product of X and P over Y, with $p: Z \to P$ and $q: Z \to X$ the two projections. Then given $x \in \widehat{CH}.(P)$, we have $q_* p^*(x) = f^* g_*(x) \in \widehat{CH}.(X)$.*

Proof. First we observe that the two functors $q_* p^*$ and $f^* g_*$ agree at the level of algebraic cycles, not just modulo rational equivalence. This is because the corresponding two (derived) functors from the category of $\mathcal{O}_P$-modules to the category of $\mathcal{O}_X$-modules agree. It then suffices to know that the maps from currents on P to currents on X agree, since then the two maps will have the same effect on Green currents. By duality, this follows directly from the fact that integration of compactly supported differential forms over the fibres of a smooth map commutes with base change.

2.2.4 If X is equidimensional and regular of dimension n, then $\widehat{CH}_p(X) \simeq \widehat{CH}^{n-p}(X)$. This is an elementary property of the gradings by codimension and dimension.

2.2.5 If ϕ is a K_1-chain on X which is supported in the special fibres of X (i.e. the support of ϕ lies in $X - X_F$), then $\widehat{\operatorname{div}}(\phi) = (\operatorname{div}(\phi), 0)$. Therefore, in defining $\widehat{CH}.(X)$, we can first divide $\hat{Z}.(X)$ by arithmetic cycles which are divisors of K_1-chains of this type. More precisely:

Definition 3 *Let X be an arithmetic variety; then $\tilde{Z}_p(X)$ is the quotient of $Z_p(X)$ by the subgroup consisting of all* div(f) *for which f is a rational function on a $(p+1)$-dimensional subvariety $W \subset X$ such that $W \cap X_F$ is empty.*

Notice that a “cycle” V in $\tilde{Z}_p(X)$ restricts to a well defined cycle on the generic fibre X_F, and hence that it makes sense to talk of a Green current g_V for V.

The group $\widehat{CH}_p(X)$ is generated by pairs $(\tilde{Z}, g) \in \tilde{Z}_p(X) \oplus \tilde{\mathcal{D}}_{p-e+1,p-e+1}(X_{\mathbf{R}})$, where g is a Green current for $\tilde{Z}$. The relations are generated by of all classes $\widehat{\operatorname{div}}(f)$ with $f \in k(y)^*$, for $y \in X_F$ a point of dimension $p - e + 1$.

2.2.6 Let X be an arithmetic variety and $D \subset X$ a principal effective Cartier divisor on X such that D_F is smooth, and let $i: D \to X$ be the inclusion. For any map $f: Y \to X$ of arithmetic varieties whose restriction $f_F: Y_F \to X_F$ to the generic fibre is transverse to D_F, we may define as follows a pull-back map

$$i^* = i_Y^*: \widehat{CH}.(Y) \to \widehat{CH}.(f^{-1}(|D|)) .$$

Let (Z, g) be an arithmetic cycle on Y. By the Moving Lemma on Y_F, we may assume that Z_F meets $f^{-1}(D_F)$ properly. We then set

$$i^*(Z, g) = (Z.f^*D, i^*g) ,$$

where $Z.f^*D$ is defined by the method of [Fu2, Remark 2.3]. Namely, by linearity, we may assume that Z is irreducible. If $Z \subset f^{-1}(|D|)$, we put $Z.f^*D = 0$, and otherwise $Z.f^*D$ is the Weil divisor associated to the Cartier divisor j^*f^*D where $j: Z \to Y$ is the inclusion. Notice that Zf^*D depends only on the restriction of Z to $Y - |f^*D|$.

We must check that i^* respects linear equivalence. If (Z, g) and (W, h) represent the same class in $\widehat{CH}.(Y)$, then, as in [GS2, 4.2.6], we can find a K_1-chain ϕ on Y such that $\widehat{\mathrm{div}}(\phi) = (Z - W, g - h)$, and ϕ meets $|f^*(D_F)|$ almost properly on Y_F (with the terminology of loc.cit.). It follows that

$$i^*(-\log(|\phi|^2)) = i^*(h - g) = i^*h - i^*g\ .$$

We need to check that $i^*(\mathrm{div}(\phi))$ is rationally equivalent to zero. By linearity, we may assume that ϕ is a rational function r on an integral subscheme $T \subset Y$. Suppose that T is not contained in $f^{-1}(|D|)$ (otherwise there is nothing to check), and let $i^*T = \sum_S n_S[S]$. Define $i^*(\phi) = \sum n_S\{r|_S\}$, where the sum is taken over all S's which are not contained in $|\mathrm{div}(\phi)|$. On the other hand, if t is an equation for $f^*(D)$ on Y and $\mathrm{div}(\phi) = \sum_V m_V[V]$, we get

$$i^*(\widehat{\mathrm{div}}(\phi)) = \sum m_V\, \widehat{\mathrm{div}}(t \mid V)\ ,$$

where the sum runs over all irreducible subvarieties V of codimension one in the support of ϕ which are not contained in the divisor of t. It follows that $i^*(\mathrm{div}(\phi)) - \mathrm{div}(i^*(\phi))$ is the divisor of a K_1-chain ψ supported on the components of excess intersection of $f^{-1}(|D|)$ and $\mathrm{div}(\phi)$ (see the proof of Lemma 23 below). In particular, the support of ψ does not meet Y_F. It follows that

$$i^*(\widehat{\mathrm{div}}(\phi)) = \widehat{\mathrm{div}}(i^*(\phi) + \psi)\ ,$$

i.e. i^* preserves linear equivalence.

When f is proper, for any $x \in \widehat{CH}.(Y)$,

$$i^*f_*(x) = f_*^D i^*(x)$$

in $\widehat{CH}.(|D|)$, where f^D is the restriction of f to the inverse image of $|D|$. This may be checked directly on cycles as in [Ful, Proposition 2.3.c].

2.2.7 Another (and more general) case when we can define a pull back map is when $f: X \to Y$ is a local complete intersection (l.c.i.) morphism between arithmetic varieties, i.e. there is a factorization $f = g \circ i$, with $i: X \to P$ a regular embedding, and $g: P \to Y$ a smooth morphism. Then there is a pull back map

$$f^*: \widehat{CH}_p(Y) \to \widehat{CH}_{p+d}(X)$$

where $d = \dim(X) - \dim(Y)$ (for simplicity, we assume here that X and Y are irreducible).

This is constructed by the method of 4.4 in [GS2]. Specifically, we have a map $g^*: \widehat{CH}_p(Y) \to \widehat{CH}_{p+n}(P)$, $n = \dim(P) - \dim(Y)$, since g is smooth and in particular flat. It therefore suffices to define a map

$$i^*: \widehat{CH}_{p+n}(P) \to \widehat{CH}_{p+d}(X)\ .$$

This is done for cycles in Theorem 4.4.1, (i) to (v), in [GS2], where the only use of regularity in the construction of the map i^* is to ensure that i is a regular embedding (there is a typographical error in the statement of Theorem 4.4.1 in loc. cit., in that the closed embedding is called f, but is twice written i). Indeed, the construction in Fulton's book [Fu2] that is appealed to ibidem is the Gysin map attached to a regular embedding in homological Chow groups, which is valid when varieties are singular. The proof in [GS2, Theorem 4.4.2] that the pull-back on cycles respects rational equivalence applies equally well to our current situation, the only change being to replace the grading by codimension with grading by dimension. Theorem 4.4.3 (1) to (4) also applies, with the same changes, showing that f^* is independent of the factorization chosen.

The pull back by l.c.i. maps has the following properties. In the situation of 2.2.6, the inclusion $i: D \to X$ is l.c.i., and both definitions of i^* on $\widehat{CH}_.(X)$ agree. When $f: X \to Y$ is both l.c.i. flat, and smooth on X_F, the map f^* agrees with the map in 2.2.3 above. If $f: X \to Y$ and $g: Y \to Z$ are two l.c.i. maps, then $(gf)^* = g^* f^*$ from $\widehat{CH}_.(Z)$ to $\widehat{CH}_.(X)$.

Furthermore, given an l.c.i. map $f: X \to Y$ and a proper map $g: P \to Y$ (smooth on P_F), such that f and g are Tor-independent, then

$$f^* g_* = g_{X*} f_P^* : \widehat{CH}_.(P) \to \widehat{CH}_.(X) ,$$

where $g_X: X \times_Y P \to X$ and $f_P: X \times_Y P \to P$ are the two projections (notice f_P is still l.c.i. under our assumption). By Lemma 11 applied to smooth maps, to prove this we need only consider the case where f is a regular immersion. We may check this identity on a generator (Z, g_Z) of $\widehat{CH}_.(P)$ with Z_F transverse to f_P and g_Z of logarithmic type. The identity is true for g_Z because integration of forms along fibres is compatible with pull-backs (compare (3) in Lemma 12 below), and it is also true in the algebraic Chow groups of $f^{-1}(g(|Z|))$ by [Fu2, Theorem 6.2.a)], which remains valid over A as explained in [GS2, 4.4.1].

2.3 *Cap products*

2.3.1 We shall now show that the construction in [GS2] of cup products on the arithmetic Chow cohomology groups can be extended to give a cap product between arithmetic Chow cohomology and homology. These cap products are described somewhat in the style of Fulton's operational theory [Fu2].

First notice that the real vector space $\tilde{A}(X_{\mathbf{R}}) = \bigoplus_{p \geq 0} \tilde{A}^{p,p}(X_{\mathbf{R}})$ is a contravariant functor from arithmetic varieties to rings without unit, where on $\tilde{A}(X_{\mathbf{R}})$ we consider the $*$-product: $\phi * \psi = \phi \wedge dd^c(\psi)$. Given a class $x \in \widehat{CH}_q(X)$ and $\phi \in \tilde{A}(X_{\mathbf{R}})$, we define their cap product

$$\phi \cap x = \phi \cap (Z, g_Z) = (0, \phi * g_Z) = a(\phi \cdot \omega(x)) .$$

Here we have written $x = (Z, g_Z)$, and "$\cdot$" for the product

$$A^{p,p}(X_{\mathbf{R}}) \otimes \mathscr{D}_{q-e+1,q-e+1}(X_{\mathbf{R}}) \to \mathscr{D}_{q-p-e+1,q-p-e+1}(X_{\mathbf{R}})$$

which is induced by the wedge product of forms with distribution coefficients.

Theorem 3 *Given a map $f: X \to Y$ of arithmetic varieties, with Y regular, there is a cap product:*

$$\widehat{CH}^p(Y) \otimes \widehat{CH}_q(X) \to \widehat{CH}_{q-p}(X)_{\mathbf{Q}}$$

$$y \otimes x \mapsto y ._f x$$

which we also denote $y \cap x$ if $X = Y$. This product has the following properties.

1. *$\omega(y._f x) = f^*\omega(y) \wedge \omega(x)$, and, for any $\eta \in \tilde{A}(Y_{\mathbf{R}})$, $a(\phi)._f x = a(f^*\phi) \cap x$.*

2. *$\widehat{CH}.(X)_{\mathbf{Q}}$ is a graded $\widehat{CH}^{\cdot}(Y)_{\mathbf{Q}}$ module*; i.e. *if $x \in \widehat{CH}_q(X)_{\mathbf{Q}}$, $y \in \widehat{CH}^p(Y)_{\mathbf{Q}}$, and $y' \in \widehat{CH}^{p'}(Y)_{\mathbf{Q}}$, then $y._f(y'._f x) = (yy')._f x$, where yy' is the cup product defined in* [GS2, 4.2.3].

3. *If $g: Y \to Y'$ is a map of arithmetic varieties with Y' regular, $y' \in \widehat{CH}^p(Y')$ and $x \in \widehat{CH}_q(X)$, then $y'._{gf} x = (g^*(y'))._f x$. Here g^* is the pull back map defined in* [GS2, 4.4.3].

4. *If $h: X' \to X$ is projective, and smooth over X_F, $x' \in \widehat{CH}_q(X')$ and $y \in \widehat{CH}^p(Y)$, then $y._f(h_*(x')) = h_*(y._{fh} x')$. That is, the push forward h_* is (after tensoring with $\mathbf{Q}$) a map of $\widehat{CH}^{\cdot}(Y)_{\mathbf{Q}}$ modules.*

5. *If $h: X' \to X$ is smooth over F, and either flat or l.c.i., $x \in \widehat{CH}_q(X)$ and $y \in \widehat{CH}^p(Y)$, then $h^*(y._f x') = y._{fh} h^*(x')$. That is, the pull back map h^* is (after tensoring with $\mathbf{Q}$) a map of $\widehat{CH}^{\cdot}(Y)_{\mathbf{Q}}$ modules.*

6. *Let $i: D \to X'$ be the inclusion of a principal effective Cartier divisor, $h: X \to X'$ a morphism which meets D_F properly, and $i_X: h^{-1}(D) \to X$ the inclusion of the inverse image of D. Then, for any $x \in \widehat{CH}.(X)$ and $y \in \widehat{CH}^{\cdot}(Y)$, the following holds in $\widehat{CH}.(|h^{-1}(D)|)$:*

$$y._{fi_X} i^*(x) = i^*(y._f x) .$$

Proof. We shall first define $y._f x$. Without loss of generality we may suppose that Y is equidimensional of dimension n, since being regular it will be a disjoint union of such varieties, and we may then consider the cap product for each component of Y separately.

Let $x = (V, g_V) \in \widehat{CH}_q(X)$, with V an algebraic cycle on X. We may assume that V is a prime cycle, i.e. that $V \subset X$ is a q-dimensional subvariety. By the Moving Lemma, we may assume that $y = (W = \sum_i n_i W_i, g_W) \in \widehat{CH}^p(Y)$, with each $f^{-1}(W_i)$ meeting V properly on the generic fibre X_F. Since Y is regular, $[\mathcal{O}_{W_i}] \in K_0^{W_i}(Y)$, and hence $f^*[\mathcal{O}_{W_i}] \in K_0^{V \cap f^{-1}(W_i)}(X)$. So we have $f^*[\mathcal{O}_{W_i}] \cap [\mathcal{O}_V] \in K'_0(V \cap f^{-1}(W_i))$. Now by [S1 Theorem 8 v)], and [GS1, Theorem 8.2], $K'_0(V \cap f^{-1}(W_i))_{\mathbf{Q}} \simeq \bigoplus_{r \geq 0} CH_r(V \cap f^{-1}(W_i))_{\mathbf{Q}}$, with $CH_r(V \cap f^{-1}(W_i))_{\mathbf{Q}} \simeq \mathrm{Gr}_r K'_0(V \cap f^{-1}(W_i))_{\mathbf{Q}}$, where Gr. is the graded associated to the filtration Fil. by dimension of support. By 2.2.5, to define the cap product arithmetic cycle it will suffice to define an algebraic cycle $[V]._f[W] \in \tilde{Z}_{q-p}(X)$, together with a Green current for $[V]._f[W]$. We shall produce the cycle $[V]._f[W]$ in the group $CH_{q-p}(V \cap f^{-1}(|W|))_{\mathbf{Q}}$, which maps naturally to $\tilde{Z}_{q-p}(X)$ (since each $f^{-1}(W_i)$ meets V properly on the generic fibre X_F). To do this it suffices to show that

$$f^*[\mathcal{O}_{W_i}] \cap [\mathcal{O}_V] \in \mathrm{Fil}_{q-p} K'_0(V \cap f^{-1}(W_i))_{\mathbf{Q}} .$$

Since X is quasi-projective, we can factor $f = \pi \circ i$, with $\pi: U \to Y$ the smooth projection from a Zariski open subset U of $\mathbf{P}^n_Y$, and i a closed immersion. By the associativity of the tensor product, $f^*[\mathcal{O}_{W_i}] \cap [\mathcal{O}_V]$ may be calculated in the K-theory with supports of U, i.e. via the isomorphism $\mathrm{Fil}^{N-\cdot} K_0^V(U)_{\mathbf{Q}} \simeq \mathrm{Fil}.K_0(V)_{\mathbf{Q}}$; here N is the dimension of U, and $\mathrm{Fil}^{\cdot}$ is the filtration by codimension of support. The needed assertion about the cap product cycle now follows from the multiplicativity of the filtration by codimension of supports on K-theory with rational coefficients for a regular scheme, see [S1, Theorem 7 iv)] and [GS1, Theorem 8.3].

Turning to the construction of the associated Green current, let $\widetilde{V(\mathbf{C})}$ be a resolution of singularities of $V(\mathbf{C})$, and $k: \widetilde{V(\mathbf{C})} \to X(\mathbf{C})$ the map induced by the inclusion $V \to X$. Then as in [GS2, 2.1], since W_i meets f properly over F, the current

$$\delta_V \wedge f^* g_{W_i} \stackrel{\text{def}}{=\!=} k_*(fk)^* g_{W_i}$$

is well defined if we choose for g_{W_i} a Green form of log type along $W_i(\mathbf{C})$ in the sense of [GS2, 1.3.5]. Such a choice always exists after adding an element of the form $\partial(u) + \bar{\partial}(v)$ (loc. cit.). We now set

$$(V, g_V)._f(W_i, g_{W_i}) = ([V]._f[W], \delta_V \wedge f^* g_{W_i} + g_V \wedge \omega_W),$$

which is an arithmetic cycle by [GS2, 2.1.4].

We must show that the cap product is compatible with rational equivalence. Suppose that $x = (V, g_V)$ with V a subvariety of X, as above, and that (W, g_W), $(W', g_{W'})$ are two arithmetic cycles on X, representing the same class $y \in \widehat{CH}^{\cdot}(Y)$, both meeting V_F and f_F properly. By [GS2, 4.2.6], we may assume that $(W, g_W) - (W', g_{W'}) = \widehat{\operatorname{div}}(\phi)$ with ϕ a K_1-chain on Y which meets V_F and f_F almost properly. Following op. cit. we may assume that $\phi = [Z].\{\tilde{\phi}\}$, with $\tilde{\phi}$ a rational function on Y, the divisor of which meets f_F and V_F properly, and which is a unit on any component of $f(V_F) \cap W_F$ for which $f^{-1}(V_F) \cap W_F$ has excess dimension. Then by the method of Lemma 4.2.5 of [GS2], $(V, g_V)._f(\widehat{\operatorname{div}}(\phi)) = \operatorname{div}(\psi)$, where ψ is the K_1-chain on X equal to $([V]._f[Z]).f^*(\tilde{\phi})$.

Similarly, if (V, g_V) and $(V', g_{V'})$ are two representatives of x, we can write $(V, g_V) - (V', g_V') = \widehat{\operatorname{div}}(\phi)$, with ϕ a K_1-chain on X. By the Moving Lemma, and the fact that the cap product is independent of the choice of representative for y, we can choose a representative $y = (W, g_W)$, with W_F meeting f_F and ϕ_F properly on the fibre over $\operatorname{Spec}(F)$. As before,

$$((V, g_V) - (V', g_{V'}))._f(W, g_W) = \widehat{\operatorname{div}}(\phi._f[W]),$$

where the K_1-chain $\phi._f[W]$ is defined by the cap product

$$K'_1(|\phi| - |\operatorname{div}(\phi)|) \otimes K_0^W(Y) \to K'_1((|\phi| - |\operatorname{div}(\phi)|) \cap f^{-1}(|W|)).$$

This proves that the cap product is well defined.

Property 1 follows immediately from the definition.

Turning to the proof of Property 2, we may assume that X_F and Y_F are projective (compare [GS2, 4.2.7. (ii)]), and that x, y, and y' are represented by arithmetic cycles (V, g_V), (W, g_W), $(W', g_{W'})$, respectively, such that W and W' meet properly on Y_F, meet f_F properly, and $f^{-1}W$ and $f^{-1}W'$ meet V properly on X_F. Then the required associativity is the conjunction of two facts. First, by [GS2, Lemma 4.4.3], the following identity holds in $\tilde{\mathscr{D}}^{\cdot}(X_{\mathbf{R}})$

$$g_V * f^*(g_W * g_{W'}) = g_V * (f^*(g_W) * f^*(g_{W'})),$$

and by op. cit. 2.2,

$$g_V * (f^*(g_W) * f^*(g_{W'})) = (g_V * f^*(g_W)) * f^*(g_{W'}).$$

Second, the product on K-theory with supports is associative, as follows from the associativity of the tensor product, see [GS1].

The proof of Property 3 on the Green current side is just the fact that if we represent the classes x and y by arithmetic cycles (V, g_V) and (W, g_W) for which the associated algebraic cycles intersect properly, the pull back $(f'f)^* g_W$, and the product $g_V * (f'f)^* g_W$ were both defined using pull-backs and wedge products of smooth forms of log type, and hence are functorial and associative. On the cycle side, we just appeal again to the associativity of the tensor product.

On the cycle side the proof of Property 4 reflects the projection formula for K-theory, while for Green currents, assuming proper intersections and representing the Green currents by forms of log type, we are reduced to the projection formula for integration of smooth forms over the fibres of a proper smooth map.

To prove Property 5, first consider the case where h is flat. Let $x = (V, g_V) \in \widehat{CH}_q(X)$ with V a prime cycle, and let $y = (W, g_W)$ with W a prime cycle meeting V_F properly; it follows that W also meets $h^{-1}(V_F)$ properly. The equality $h^*(g_V * f^*(g_W)) = h^*(g_V) * (fh)^* g_W$ follows from the Lemma of section 4.4.3 of [GS2] and the fact that Green currents of log type pull back to Green currents of log type, see 2.1.3 and 2.1.4 of op. cit., so that $(fh)^* g_W = h^* f^* g_W$ at the level of forms. Next we check that we have an equality of cycles

$$h^*([V] \cdot_f [W]) = [h^{-1}(V)] \cdot_{fh} [W]$$

in $CH_{q-p+d}(h^{-1}(V) \cap (fh)^{-1}(W))_{\mathbb{Q}}$. It suffices to observe that

$$h^*([\mathcal{O}_V] \cap f^*[\mathcal{O}_W]) = [\mathcal{O}_{h^*(V)}] \cap (fh)^*[\mathcal{O}_W] \in \mathrm{Fil}_{q-p+d} K'_0(h^{-1}(V \cap f^{-1}(W)))_{\mathbb{Q}}$$

by the associativity of the tensor product and the flatness of h. Turning to the case when h is l.c.i., since a smooth map is flat, we need only consider the case of a regular immersion $h: X' \to X$. Again the equality of Green currents follows from the regular case, since X'_F and X_F are smooth. On the cycle side, we use the compatibility of the pull back of cycles via deformation to the normal cone with products on K-theory, which may be verified by embedding the whole deformation to the normal cone family in a regular variety.

The proof of 6 follows from the definitions in 2.2.6. q.e.d.

2.3.2 We shall also need the following projection formula:

Lemma 12 *Let $f: X \to M$ be a map of arithmetic varieties with M regular, and suppose $p: P \to M$ is a proper smooth map of arithmetic varieties of relative dimension d. Then if we write $g: X \times_M P \to P$ and $q: X \times_M P \to X$ for the projections, we have, for all $\alpha \in \widehat{CH}_p(X)$ and $\gamma \in \widehat{CH}^q(P)$,*

$$q_*(q^*(\alpha) \cdot_g \gamma) = \alpha \cdot_f p_* \gamma \, .$$

Proof. Suppose that α (resp. γ) is the class of the arithmetic cycle (Z, g_Z) (resp. (W, g_W)), where Z and W are irreducible cycles which are flat over the base (if not the statement is purely algebraic, see below). We may assume that g_Z (resp. g_W) is of logarithmic type along $Z(\mathbb{C})$ (resp. $W(\mathbb{C})$), and, by the Moving Lemma, that the closed sets $g^{-1}(W_F)$ and $q^{-1}(Z_F)$ meet properly in $(X \times_M P)_F$ (i.e., each irreducible component of $g^{-1}(W_F)$ meets properly the irreducible set $q^{-1}(Z_F)$). When $\dim p(W_F) \neq \dim(W_F)$, the cycle $p_*(W)$ is zero by definition of p_* on cycles [Fu1].

But then any fibre of the map $p: W \to p(W)$ has positive dimension, hence the same is true for any fibre of the map $q: q^{-1}(Z) \cap g^{-1}(W) \to Z \cap f^{-1}(W)$. It follows that in that case the cycle components of both $q_*(q^*(\alpha) \cdot_g \gamma)$ and $\alpha \cdot_f p_*(\gamma)$ vanish (for our choice of representatives of α and γ). When $\dim p(W_F) = \dim(W_F)$, it follows from our transversality assumptions that any component of $q^{-1}(Z_F) \cap g^{-1}(W_F)$ is generically finite over its image, and that the components of

$$q(q^{-1}(Z_F) \cap g^{-1}(W_F)) = Z_F \cap f^{-1}(p(W_F))$$

have the same dimension. Their multiplicities are equal by the Tor formula and the projection formula, since p and q are smooth.

On the other hand, the equality of currents

$$(3) \quad q_*(q^*(g_Z) * g^*(g_W)) = q_*(q^*(\delta_Z) g^*(g_W) + q^*(g_Z) g^*(\omega_W))$$
$$= \delta_Z q_* g^*(g_W) + g_Z q_* g^*(\omega_W) = g_Z * q_* g^*(g_W),$$

when tested on (compactly supported) form of appropriate degree, is an equality of indefinite integrals on the open set $X(\mathbb{C}) - (Z(\mathbb{C}) \cap f^{-1} p(W(\mathbb{C})))$ (unless $p(W(\mathbb{C})) = M(\mathbb{C})$, in which case (3) can be checked directly). The identity (3) then follows from the fact that integration of forms along fibres of p and q commutes with base change by the map $Z(\mathbb{C}) \to M(\mathbb{C})$.

Finally, to check the identity of the lemma for our choice of representatives of α and β, by 2.2.5 we need only to check that the cycle classes $q_*(q^*[Z] \cdot_g [W])$ and $[Z] \cdot_f p_*[W]$ are equal in the algebraic Chow group $CH_\cdot(Z \cap f^{-1} p(W))_{\mathbb{Q}}$. Since the cap product on Chow homology is defined using algebraic K-theory [GS1, S1], this follows from the identity of derived functors $Lf^* Rp_* = Rq_* Lg^*$, i.e. base change for direct images in K-theory with supports (see [BGI IV 3.1.0] and [Q1, Proposition 2.11]).

2.4 Characteristic classes for Hermitian algebraic vector bundles

2.4.1 In this section we recall some results in complex geometry concerning Bott–Chern secondary characteristic classes and their singular analogs.

Let X be a complex manifold and $\bar{E} = (E, h)$ an holomorphic vector bundle E on X equipped with a smooth Hermitian metric h. Denote by

$$ch(\bar{E}) = \operatorname{tr} \exp\left(\frac{i}{2\pi} \nabla^2\right)$$

the usual form representing the Chern character of E, where ∇ is the Hermitian holomorphic connection on E attached to h [GH]. Given an exact sequence

$$\mathcal{E}: 0 \to S \to E \to Q \to 0$$

of holomorphic vector bundles on X, and (arbitrary) Hermitian metrics h', h, h'' on S, E, Q respectively, the Bott–Chern secondary characteristic class

$$\widetilde{ch}(\mathcal{E}) \in \tilde{A}(X) = \bigoplus_{p \geq 0} A^{pp}(X)/(\operatorname{Im} \partial + \operatorname{Im} \bar{\partial})$$

solves the equation

$$dd^c \widetilde{ch}(\mathscr{E}) = ch(\bar{S}) - ch(\bar{E}) + ch(\bar{Q}),$$

(with $dd^c = \bar{\partial}\partial/(2\pi i)$). We refer the reader to [BC, Do, BGS1, GS3] for its construction. To simplify notations we write $\widetilde{ch}(\mathscr{E})$ instead of $\widetilde{ch}(\mathscr{E}, h', h, h'')$. This class $\widetilde{ch}(\mathscr{E})$ is functorial and is characterized by the fact that it vanishes when $\mathscr{E}$ is (metrically) split, i.e. (E, h) is the orthogonal direct sum of (S, h') with (Q, h'').

In special cases we shall use a different notation for $\widetilde{ch}(\mathscr{E})$. When $Q = 0$, i.e. if we are given an isomorphism $\theta: S \to E$, we shall write $\widetilde{ch}(\bar{S}, \bar{E}, \theta)$ instead of $\widetilde{ch}(\mathscr{E})$. If, in addition, θ is the identity on $S = E$, we shall write $\widetilde{ch}(E, h', h)$ instead of $\widetilde{ch}(\mathscr{E})$.

Notice that these secondary characteristic classes exist for other characteristic classes and not only for the Chern character [GS3]. For instance, given a metrized exact sequence $\mathscr{E}$ as above, there exist classes $\widetilde{Td}(\mathscr{E})$ and $\widetilde{Td^{-1}}(\mathscr{E})$ in $\tilde{A}(X)$, attached to the Todd class and its inverse. They solve the equations

$$dd^c \widetilde{Td}(\mathscr{E}) = Td(\bar{S})Td(\bar{Q}) - Td(\bar{E})$$

and

$$dd^c \widetilde{Td^{-1}}(\mathscr{E}) = Td^{-1}(\bar{S})Td^{-1}(\bar{Q}) - Td^{-1}(\bar{E}).$$

More generally, given a finite acyclic complex of vector bundles

$$E.: 0 \to E_m \to E_{m-1} \to \dots \to E_0 \to 0,$$

and arbitrary metrics on each E_j, there exists a class $\widetilde{ch}(E.) \in \tilde{A}(X)$ such that

$$dd^c \widetilde{ch}(E.) = -ch(\bar{E}.)$$

where, by definition, $ch(\bar{E}.) = \sum_{j \geqq 0} (-1)^j ch(\bar{E}_j)$ [BGS1, GS3].

This definition is again generalized in [BGS2], where Bott–Chern singular currents are introduced. Given a closed immersion $i: Y \to X$ of (smooth) complex manifolds and an Hermitian vector bundle $\bar{F}$ on Y, consider a resolution

$$0 \to E_m \to E_{m-1} \to \dots \to E_0 \to i_* F \to 0$$

of the coherent sheaf $i_* F$ by holomorphic vector bundles on X. The homology groups $H_k(i^* E.)$, for $k \geqq 0$, of the restriction to Y of the complex of sheaves

$$E.: E_m \to E_{m-1} \to \dots \to E_0$$

are canonically isomorphic to the bundles $\bigwedge^k(N^*) \otimes F$, where N^* is the dual of the normal bundle of Y in X (see [BGS2, 1.a] for references). We then have the following definition [B]:

Definition 4 Given a metric on N, we say that a choice of metrics on E_j, $j \geqq 0$, satisfies condition (A) when the isomorphisms

$$H_k(i^* E.) \simeq \bigwedge^k (N^*) \otimes F$$

are isometries for the metric induced by $\bar{E}.$ on homology, and the metric induced by $\bar{N}$ and $\bar{F}$ on $\bigwedge^k (N^*) \otimes F$.

Such a choice of metrics always exists [B, Proposition 1.6]. Using such metrics, we defined in [BGS2, (2.4)] a class of currents

$$\widetilde{ch}(E.) \in \bigoplus_{p \geqq 0} \mathscr{D}^{pp}(X)/(\operatorname{Im} \partial + \operatorname{Im} \bar{\partial})$$

solving the equation of currents

$$(4) \qquad dd^c \widetilde{ch}(E.) = i_*(ch(\bar{F}) Td^{-1}(\bar{N})) - ch(\bar{E}.) .$$

When $F = 0$ we recover the previous definition. This Bott–Chern singular current $\widetilde{ch}(E.)$ was denoted $T(h^\xi)$ in [BGS2] and [BGS3, 1.g], where ξ is $E.$ and h^ξ is the metric on $E.$. We refer to loc. cit. for several properties of $\widetilde{ch}(E.)$. Again, by abuse of notation, we omit to mention in our notation $\widetilde{ch}(E.)$ the dependence of this class upon our choices of metrics (on F, N and $E.$).

2.4.2 If X is an arithmetic variety, an Hermitian algebraic bundle $\bar{E}$ on X is a pair (E, h) consisting of a vector bundle E on the scheme X, and a C^∞ metric h on the induced holomorphic bundle over $X(\mathbb{C})$ which is invariant under F_∞. If $f: Y \to X$ is a morphism of arithmetic varieties, then $f^*(\bar{E}) = (f^*(E), f^*(h))$ is an Hermitian bundle on Y.

As in [GS3, 6.1], we can form the arithmetic Grothendieck ring of X, $\hat{K}_0(X)$. This is the quotient of the free abelian group on the set of pairs $(\bar{E}, \eta)$, with $\eta \in \sum_{p \geqq 0} \tilde{A}^{p,p}(X_{\mathbf{R}})$, by the subgroup generated by all expressions of the form $(\bar{E}', \eta') - (\bar{E}, \eta) + (\bar{E}'', \eta'') - (0, \beta)$, where $\mathscr{E}$ is an exact sequence of bundles

$$0 \to E' \to E \to E'' \to 0$$

over X, equipped with arbitrary metrics, and $\beta = \widetilde{ch}(\mathscr{E}) + \eta' - \eta + \eta''$, where $\widetilde{ch}(\mathscr{E})$ is the Bott–Chern secondary Chern character class. We get in this way a contravariant functor $X \to \hat{K}_0(X)$ from arithmetic varieties to rings [GS3, loc. cit.].

Theorem 4 *There is a biadditive pairing*

$$\hat{K}_0(X) \otimes \widehat{CH}.(X) \to \widehat{CH}.(X)_{\mathbf{Q}} ,$$

which we write $\alpha \otimes x \mapsto \widehat{ch}(\alpha) \cap x$, *with the following properties.*

1. *If* $f: X \to Y$ *is a morphism of arithmetic varieties, with* Y *regular,* $\alpha \in \hat{K}_0(Y)$ *and* $x \in \widehat{CH}.(X)$, *then*

$$\widehat{ch}(f^*\alpha) \cap x = \widehat{ch}(\alpha)._f x .$$

2. *If* $(0, \eta) \in \hat{K}_0(X)$ *and* $x \in \widehat{CH}.(X)$, *then*

$$\widehat{ch}((0, \eta)) \cap x = a(\eta\omega(x)) .$$

3. *If* $\alpha \in \hat{K}_0(X)$ *and* $x \in \widehat{CH}.(X)$, *then*

$$\omega(\widehat{ch}(\alpha) \cap x) = ch(\alpha) \wedge \omega(x) .$$

4. *The pairing makes* $\widehat{CH}.(X)_{\mathbf{Q}}$ *into a* $\hat{K}_0(X)$ *module; i.e. for all* $\alpha, \beta \in \hat{K}_0(X)$, *and* $x \in \widehat{CH}.(X)_{\mathbf{Q}}$, *we have*

$$\widehat{ch}(\alpha) \cap (\widehat{ch}(\beta) \cap x) = (\widehat{ch}(\alpha\beta)) \cap x .$$

Here $\alpha\beta$ *is the product in* $\hat{K}_0(X)$ *(see* [GS3]).

5. *If* $f: Y \to X$ *is a flat or* l.c.i. *morphism of arithmetic varieties, let* $\alpha \in \hat{K}_0(X)$ *and* $x \in \widehat{CH}.(X)$. *Then*

$$\widehat{ch}(f^*\alpha) \cap f^*(x) = f^*(\widehat{ch}(\alpha) \cap x) .$$

6. *if* $f: X \to Y$ *is a proper morphism of arithmetic varieties, smooth over* Y_F, *let* $\alpha \in \hat{K}_0(Y)$ *and* $x \in \widehat{CH}.(X)$. *Then*

$$f_*(\widehat{ch}(f^*\alpha) \cap x) = \widehat{ch}(\alpha) \cap f_*(x) .$$

7. *If* $i: D \to X$ *is the inclusion of a principal effective Cartier divisor,* $f: Y \to X$ *as in* 2.2.6., $i_Y: f^{-1}(|D|) \to Y$ *the inclusion, and* $\bar{E}$ *an Hermitian vector bundle on* Y, *for any* $x \in \widehat{CH}.(X)$ *we have*

$$\widehat{ch}(i_Y^*(\bar{E})) \cap i^*(x) = i^*(\widehat{ch}(\bar{E}) \cap x) .$$

Proof. We start by considering a generator $\alpha = (E, h, \eta)$ of $\hat{K}_0(X)$. Since we are assuming that all our varieties are quasi-projective over A, we know that there exists a vector bundle U over an arithmetic variety M, with M smooth over A, a map $u: X \to M$, and an isomorphism $\theta: E \to u^*U$. Fix an arbitrary metric on U; then $\bar{U}$ has a Chern character $\widehat{ch}(\bar{U}) \in \widehat{CH}^{\cdot}(M)$. Given $x \in \widehat{CH}.(X)$ consider the class

$$\widehat{ch}(\bar{U})._u x + a \circ \widetilde{ch}(\bar{E}, p^*\bar{U}, \theta) \cap x + a(\eta) \cap x .$$

(Here, and in the discussion following, we have written $a \circ \widetilde{ch}(.)$ instead of $a(\widetilde{ch}(.))$.) We claim that this is independent of the choice of the triple $(u, \bar{U}, \theta)$. For if $(u': X \to M', \bar{U}', \theta')$ is a second such triple, let $\mathrm{Iso}(U', U)$ be the variety, smooth over A, which parameterizes isomorphisms $U \to U'$. There are projections $p: \mathrm{Iso}(U', U) \to M$, and $p': \mathrm{Iso}(U', U) \to M'$, and an isomorphism $\psi: p^*U \to p'^*U'$. By definition of $\mathrm{Iso}(U', U)$ there is a unique map $\delta: X \to \mathrm{Iso}(U', U)$

such that $p \circ \delta = u, p' \circ \delta = u'$, and $\delta^*(\psi)$ is the isomorphism $\theta'(\theta)^{-1}$. Then given $x \in \widehat{CH}.(X)$, we get

$$\begin{aligned} x._u\widehat{ch}(\bar{U}) - x._{u'}\widehat{ch}(\bar{U}') &= x._{p\delta}\widehat{ch}(\bar{U}) - x._{p'\delta}\widehat{ch}(\bar{U}') \\ &= x._\delta(p^*\widehat{ch}(\bar{U}) - p'^*\widehat{ch}(\bar{U})) \\ &= x._\delta(a \circ \widetilde{ch}(p^*\bar{U}, p'^*\bar{U}', \psi)) \\ &= x \cap \delta^*(a \circ \widetilde{ch}(p^*\bar{U}, p'^*\bar{U}', \psi)) \\ &= x \cap (a \circ \widetilde{ch}(u^*(\bar{U}), u'^*(\bar{U}'), \theta'(\theta)^{-1})) \\ &= x \cap (a \circ \widetilde{ch}(\bar{E}, p'^*\bar{U}', \theta) - a \circ \widetilde{ch}(\bar{E}, p^*\bar{U}, \theta')) \end{aligned}$$

as desired.

Since the cap product is biadditive, this pairing is additive in x. In order to show that we get a map $\hat{K}_0(X) \otimes \widehat{CH}.(X) \to \widehat{CH}.(X)_{\mathbf{Q}}$, it suffices to show that given an exact sequence

$$\mathscr{E}: 0 \to \bar{E}' \to \bar{E} \to \bar{E}'' \to 0$$

of bundles on X, we have, for any choice of metrics on the bundles,

$$(\widehat{ch}(\bar{E}') - \widehat{ch}(\bar{E}) + \widehat{ch}(\bar{E}''))\cap x = a \circ \widetilde{ch}(\mathscr{E}) \cap x\ .$$

Choose triples $(u: X \to M, \bar{U}, \theta)$, $(u': X \to M', \bar{U}', \theta')$, and $(u'': X \to M'', \bar{U}'', \theta'')$ representing the terms in the exact sequence. There is a variety P, smooth over A, which parameterizes exact sequences $0 \to U' \to U \to U'' \to 0$. That is, there are projections $q: P \to M, q'\ : P \to M', q'': P \to M''$, and a universal exact sequence

$$\mathscr{U}: 0 \to q'^*U' \to q^*U \to q''^*U'' \to 0\ ,$$

with the obvious universal property. In particular there is a map $f: X \to P$, such that $qf = u$, $q'f = u'$, $q''f = u''$, and an isomorphism of exact sequences $f^*(\mathscr{U}) \simeq \mathscr{E}$. Since P is smooth over A, we know from [GS3, 4.8(ii)] that

$$q'^*(\widehat{ch}(\bar{U}') - q^*(\widehat{ch}(\bar{U}) + q''^*(\widehat{ch}(\bar{U}''))) = a \circ \widetilde{ch}(\mathscr{U})\ .$$

From the biadditivity of the cap product $\widehat{CH}^{\cdot}(M) \otimes \widehat{CH}.(X) \to \widehat{CH}.(X)_{\mathbf{Q}}$ we see that

$$\begin{aligned} &(\widehat{ch}(\bar{E}') - \widehat{ch}(\bar{E}) + \widehat{ch}(\bar{E}''))\cap x \\ &\quad = \widehat{ch}(\bar{U}')._{u'}x - \widehat{ch}(\bar{U})._u x + \widehat{ch}(\bar{U}'')._{u''}x + a \circ \widetilde{ch}(\bar{E}', u'^*\bar{U}', \theta') \cap x \\ &\qquad - a \circ \widetilde{ch}(\bar{E}, u^*\bar{U}, \theta) \cap x + a \circ \widetilde{ch}(\bar{E}'', u''^*\bar{U}'', \theta'') \cap x \end{aligned}$$

$$= \widehat{ch}(q'^*\bar{U}')._f x - \widehat{ch}(q^* \bar{U})._f x + \widehat{ch}(q''^*\bar{U}'')._f x + a \circ \widetilde{ch}(\bar{E}', u'^*\bar{U}', \theta') \cap x$$

$$- a \circ \widetilde{ch}(\bar{E}, u^*\bar{U}, \theta) \cap x + a \circ \widetilde{ch}(\bar{E}'', u''^*\bar{U}'', \theta'') \cap x$$

$$= f^*(a \circ \widetilde{ch}(\mathscr{U})) \cap x + a \circ \widetilde{ch}(\bar{E}', u'^*\bar{U}', \theta') \cap x - a \circ \widetilde{ch}(\bar{E}, u^*\bar{U}, \theta) \cap x$$

$$+ a \circ \widetilde{ch}(\bar{E}'', u''^*\bar{U}'', \theta'') \cap x$$

$$= a \circ \widetilde{ch}(\mathscr{E}) \cap x$$

by [GS3, Proposition 1.3.4], as desired.

We now verify Properties 1–7 in turn.

Properties 1 and 2 are part of the construction of the pairing, while Property 3 is a straightforward computation.

For Property 4, since the product is biadditive we can consider the four cases where $\alpha = (0, \eta)$, or $\alpha = f^*(\bar{E})$ for $f: X \to Y$ a map to an arithmetic variety smooth over A, and $\bar{E}$ an Hermitian bundle on Y, and $\beta = (0, \eta')$, or $\beta = g^*(\bar{E}')$ for $g: X \to Y'$ a map to a regular arithmetic variety, and $\bar{E}'$ a Hermitian bundle on Y'. The case where $\alpha = (0, \eta)$, and $\beta = (0, \eta')$ reduces immediately to Theorem 3, Property 2. If $\alpha = (f^*\bar{E}, 0)$, and $\beta = (g^*, \bar{E}', 0)$, then $\alpha \cup \beta = ((f, g)^*(p^*\bar{E} \otimes p'^*\bar{E}'), 0)$, where $(f, g): X \to Y \times Y'$ is the induced map, and p and p' are the projections from $Y \times Y'$. Hence, by Theorem 3, Property 2,

$$\begin{aligned}
\widehat{ch}(\alpha \cup \beta) \cap x &= \widehat{ch}(p^*\bar{E} \otimes p'^*\bar{E}')._{(f,g)} x \\
&= (\widehat{ch}(p^*\bar{E})\widehat{ch}(p'^*\bar{E}'))._{(f,g)} x \\
&= \widehat{ch}(p^*\bar{E})._{(f,g)}(\widehat{ch}(p'^*\bar{E}')._{(f,g)} x) \\
&= \widehat{ch}(\alpha) \cap (\widehat{ch}(\beta) \cap x) .
\end{aligned}$$

Now consider

$$\begin{aligned}
\widehat{ch}(0, \eta) \cap (\widehat{ch}(\bar{E}, 0) \cap x) &= a(\eta \cdot \omega(\widehat{ch}(\bar{E}, 0))) \cap x \\
&= a(\eta \wedge (ch(\bar{E}) \wedge \omega(x))) \\
&= a((\eta \wedge ch(\bar{E})) \wedge \omega(x)) \\
&= \widehat{ch}(0, \eta \wedge ch(\bar{E})) \cap x \\
&= \widehat{ch}((0, \eta)(\bar{E}, 0)) \cap x .
\end{aligned}$$

The proof of the fourth case is similar, and we therefore omit it.

Property 5 (resp. 6, resp. 7) follow from Property 5 (resp. 4, resp. 6) in Theorem 3. □

Remarks. The same method allows us to define other characteristic classes like the Todd class $\widehat{Td}(\bar{E}) \cap x$ for any Hermitian vector bundle $\bar{E}$ (and more generally $\widehat{Td}(\alpha) \cap x$ for any $\alpha \in \hat{K}_0(X)$) from the regular case [GS3]. These satisfy properties similar to those in Theorem 4 (of course $\widehat{Td}$ is multiplicative rather than additive). These other classes are also given by standard universal polynomials in the components of $\widehat{ch}$. Notice also that, by [GS3, Theorem 7.2.1], for any $\alpha \in \hat{K}_0(X)$ there is a unique $\beta \in \hat{K}_0(X)$ such that $\widehat{Td}(\alpha) = \widehat{ch}(\beta)$.

As mentioned in 2.3.1, the formalism of cap products is inspired by the operational theory of Chow groups. It is probably the case that, for any X (and more generally for any map $Y \to X$), one could define operational (resp. bivariant) arithmetic Chow groups in the style of [Fu2]. The Chern character $\widehat{ch}(\alpha)$ would lie in these operational Chow groups.

2.5 Hermitian Coherent Sheaves

2.5.1

Definition 25 Let X be an arithmetic variety. An *Hermitian coherent sheaf* $\bar{\mathscr{F}}$ on X consists of a pair $(\mathscr{F}, h)$, with $\mathscr{F}$ a coherent sheaf of $\mathcal{O}_X$-modules on X which has locally free restriction to the generic fibre X_F, and h a smooth metric on the associated holomorphic vector bundle over $X(\mathbf{C})$, invariant under complex conjugation.

Given an arithmetic variety X, let $\hat{P}(X)$ be the free abelian group on pairs $(\bar{\mathscr{F}}, \eta)$, with $\bar{\mathscr{F}}$ an Hermitian coherent sheaf on X, and $\eta \in \tilde{A}^*(X_{\mathbf{R}})$. Let $\hat{E}(X) \subset \hat{P}(X)$ be the subgroup generated by elements of the form

$$(\bar{\mathscr{F}}', \eta') - (\bar{\mathscr{F}}, \eta) + (\bar{\mathscr{F}}'', \eta'') - (0, \beta)$$

where

$$\mathscr{E}: 0 \to \bar{\mathscr{F}}' \to \bar{\mathscr{F}} \to \bar{\mathscr{F}}'' \to 0$$

is an exact sequence of coherent sheaves on X, locally free on X_F, equipped with arbitrary Hermitian metrics, and

$$\beta = \widetilde{ch}(\mathscr{E}) + \eta' - \eta + \eta'' .$$

The Grothendieck group $\hat{K}'_0(X)$ is, by definition the quotient of $\hat{P}(X)$ by $\hat{E}(X)$.

2.5.2 For any X there is a homomorphism of abelian groups $\kappa: \hat{K}_0(X) \to \hat{K}'_0(X)$ which comes from viewing any vector bundle as a coherent sheaf.

Lemma 13 *If X is regular, the map $\kappa: \hat{K}_0(X) \to \hat{K}'_0(X)$ is an isomorphism.*

Proof. It suffices to show that κ has an inverse. Let $(\bar{\mathscr{F}}, \eta)$ be a generator of $\hat{K}'_0(X)$. We define $\kappa(0, \eta) = (0, \eta)$. To define $\kappa(\bar{\mathscr{F}}, 0)$, we procede as follows. Since X is regular, the coherent sheaf underlying $\bar{\mathscr{F}}$ has a finite resolution $0 \to E_n \to \ldots \to E_0 \to \mathscr{F} \to 0$ by locally free sheaves on X. We may choose arbitrary metrics h_i on the $E_.$'s, to get Hermitian bundles $\bar{E}_i$. Since $\mathscr{F}$ is locally free over X_F, the resolution $\varepsilon: \bar{E}_. \to \bar{\mathscr{F}}$ restricts to an acyclic complex of Hermitian vector bundles on $X(\mathbf{C})$. Let $\widetilde{ch}(\varepsilon)$ be the corresponding Bott–Chern class, so that $dd^c(\widetilde{ch}(\varepsilon)) = ch(\bar{E}.) - ch(\bar{\mathscr{F}})$. Now consider the class $\sigma(\bar{\mathscr{F}}, \eta)$ of $\sum_{i=0}^{n}(-1)^i(\bar{E}_i, 0) - (0, \widetilde{ch}(\varepsilon))$ in $\hat{K}_0(X)$. We claim that it is independent of the choices made, so that we can define σ to be the inverse of κ. Suppose that $\varepsilon' : \bar{E}'. \to \bar{\mathscr{F}}$ is a second resolution of $\bar{\mathscr{F}}$ by Hermitian bundles on X. Then, by a standard argument, we may assume there exists a quasi-isomorphism $\alpha : E. \to E'.$ which is a morphism of resolutions from ε to ε'. From [GS3, Proposition 1.3.4] (whose proof extends, by induction, to complexes of arbitrary length) we get that

$$\widetilde{ch}(\varepsilon) - \widetilde{ch}(\varepsilon') = \sum_{i=0}^{n} (-1)^i \widetilde{ch}(E_i, E'_i, \alpha)\,,$$

which implies that σ is independent of the choices made. One can check that σ is an inverse to κ. □

The homomorphism $\kappa : \hat{K}_0(X) \to \hat{K}'_0(X)$ is a special case of a cap product

$$\cap : \hat{K}_0(X) \otimes \hat{K}'_0(X) \to \hat{K}'_0(X)$$

defined by

$$(\bar{\mathscr{E}}, \eta) \cap (\bar{\mathscr{F}}, \theta) = (\bar{\mathscr{E}} \otimes \bar{\mathscr{F}}, ch(\bar{\mathscr{E}})\theta + \eta\, ch(\bar{\mathscr{F}}) + \eta * \theta)\,.$$

One can check that this cap product makes $\hat{K}'_0(X)$ into a $\hat{K}_0(X)$-module (compare [GS3, Theorem 7.3.2]). The map κ is the cap product by the class in $\hat{K}'_0(X)$ of the structural sheaf with trivial metric.

2.6 Todd classes

2.6.1 In this section we study properties of the Bott–Chern secondary characteristic classes (see 2.4.1 above) in the case of the Todd class and its inverse.

First notice that, given an exact sequence

$$\mathscr{E} : 0 \to S \to E \to Q \to 0$$

of holomorphic vector bundles on a complex manifold X, endowed with arbitrary metrics, the following equality holds in $\tilde{A}(X)$:

$$\widetilde{Td^{-1}}(\mathscr{E}) = -\widetilde{Td}(\mathscr{E})\,Td^{-1}(\bar{S})\,Td^{-1}(\bar{E})\,Td^{-1}(\bar{Q})\,. \tag{5}$$

To check this, just notice that both sides of (5) have the same image by dd^c, depend functorially on $\mathscr{E}$, and vanish when $\mathscr{E}$ is split. Therefore they coincide by [GS3] Theorem 1.2.2.

Lemma 14

(i) *Let*

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & S & = & S & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & A & \to & B & \to & Q & \to & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \to & C & \to & D & \to & Q & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

be a commutative diagram of bundles on X, with exact lines and columns. Call these respectively C_1, C_2, C_3 and *L_1, L_2, L_3 (from left to right and top to bottom). Choose arbitrary metrics on the bundles S, Q, A, B, C, D. Then the following identity holds in $\tilde{A}(X)$:*

$$\widetilde{Td}(L_3)\,Td(\bar{A})\,Td^{-1}(\bar{C})\,Td^{-1}(\bar{D}) - \widetilde{Td}(L_2)\,Td^{-1}(\bar{D})$$

$$+ \widetilde{Td}(C_2)\,Td^{-1}(\bar{D}) - \widetilde{Td}(C_1)\,Td^{-1}(\bar{C}) = 0\,.$$

(i′) Under the hypotheses of (i), the following holds

$$- \widetilde{Td^{-1}}(L_3)\,Td(\bar{B}) - \widetilde{Td}(L_2)\,Td^{-1}(\bar{C})\,Td^{-1}(\bar{Q})$$

$$+ \widetilde{Td}(C_2)\,Td^{-1}(\bar{D}) - \widetilde{Td}(C_1)\,Td^{-1}(\bar{C}) = 0\,.$$

(ii) Let

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & S & = & S & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & S' & \to & A & \to & B & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & S' & \to & C & \to & D & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

be a commutative diagram with exact lines L_1, L_2, L_3 and exact columns C_1, C_2, C_3, with arbitrary metrics on S, S', A, B, C, D. Then, in $\tilde{A}(X)$,

$$\widetilde{Td}(C_2)\,Td^{-1}(\bar{C}) - \widetilde{Td}(C_3)\,Td^{-1}(\bar{D})$$

$$+ \widetilde{Td}(L_3)\,Td(\bar{A})\,Td^{-1}(\bar{C})\,Td^{-1}(\bar{D})\,Td^{-1}(\bar{S}') - \widetilde{Td}(L_2)\,Td^{-1}(\bar{D})\,Td^{-1}(\bar{S}') = 0\,.$$

(iii) Let (D)

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & S_1 & \to & E_1 & \to & Q_1 & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & S_2 & \to & E_2 & \to & Q_2 & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & S_3 & \to & E_3 & \to & Q_3 & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

be a commutative diagram with exact lines L_1, L_2, L_3 and exact columns C_1, C_2, C_3, with arbitrary metrics on all the bundles. Then, in $\tilde{A}(X)$,

$$\widetilde{Td}(L_1)Td^{-1}(\bar{Q}_1)Td^{-1}(\bar{Q}_3)Td(\bar{E}_3) - \widetilde{Td}(L_2)Td^{-1}(\bar{Q}_2)$$

$$+ \widetilde{Td}(L_3)Td^{-1}(\bar{Q}_1)Td^{-1}(\bar{Q}_3)Td(\bar{E}_1) + (\widetilde{Td}(L_1) * \widetilde{Td}(L_3))Td^{-1}(\bar{Q}_1)Td^{-1}(\bar{Q}_3)$$

$$= \widetilde{Td}(C_1) - \widetilde{Td}(C_2)Td^{-1}(\bar{Q}_1)Td^{-1}(\bar{Q}_3) - \widetilde{Td^{-1}}(C_3)Td(\bar{E}_1)Td(\bar{E}_3)$$

$$+ (\widetilde{Td}(C_2) * \widetilde{Td^{-1}}(C_3)),$$

where the $*$-product is the one defined in 2.3.1.

Proof. To prove the first identity in (i), after multiplication by $Td(\bar{C})Td(\bar{D})$, we compute

$$\widetilde{Td}(L_3)Td(\bar{A}) - \widetilde{Td}(L_2)Td(\bar{C}) + \widetilde{Td}(C_2)Td(\bar{C}) - \widetilde{Td}(C_1)Td(\bar{D})$$

$$= \widetilde{Td}(L_3)(Td(\bar{S})Td(\bar{C}) - dd^c\widetilde{Td}(C_1)) - \widetilde{Td}(L_2)Td(\bar{C}) + \widetilde{Td}(C_2)Td(\bar{C})$$

$$+ \widetilde{Td}(C_1)(-Td(\bar{Q})Td(\bar{C}) + dd^c\widetilde{Td}(L_3))$$

$$= [\widetilde{Td}(L_3)Td(\bar{S}) - \widetilde{Td}(L_2) + \widetilde{Td}(C_2) - \widetilde{Td}(C_1)Td(\bar{Q})]Td(\bar{C}).$$

But, from [GS3, Pro-position 1.3.2], we get

$$\widetilde{Td}(L_3)Td(\bar{S}) = \widetilde{Td}(L_3 \oplus L_1)$$

and

$$\widetilde{Td}(C_1)Td(\bar{Q}) = \widetilde{Td}(C_3 \oplus C_1).$$

From [GS3, Proposition 1.3.4] we conclude that

$$\widetilde{Td}(L_3)Td(\bar{S}) - \widetilde{Td}(L_2) + \widetilde{Td}(C_2) - \widetilde{Td}(C_1)Td(\bar{Q}) = 0\,,$$

which proves (i).

The proof of (i′) is similar; one approach is to substract from (i).

The proof of (ii) is similar to (i) (first multiply by $Td(\bar{S}')Td(\bar{C})Td(\bar{D})$). It follows also from (i) by looking at the dual diagram, since $\widetilde{Td}(\mathscr{E}^*) = \widetilde{Td}(\mathscr{E})^*$, where $\alpha^* = (-1)^p\alpha$ when $\alpha \in \tilde{A}^{p,p}(X)$ (this follows from the axiomatic characterization of $\widetilde{Td}(\mathscr{E})$).

To prove (iii) one may also reduce this equation to Proposition 1.3.4. in [GS3] or use the following argument (which also works for (i), (i′) and (ii)). Let η be the difference of the left-hand side and the right-hand side of this equation. Notice that η is functorial in the diagram (D), and vanishes when both the lines and columns are (metrically) split. Furthermore $dd^c(\eta) = 0$. As in the proof of the Proposition 1.3.4 in [GS3] we may define a diagram $(\tilde{D})$ of bundles on $X \times \mathbb{P}^1$ with exact lines and columns, whose restriction to 0 (resp. ∞)) coincides with (D) (resp. has split lines). Let $\tilde{\eta}$ be the corresponding form, and $p: X \times \mathbb{P}^1 \to X$ the projection. We get

$$0 = p_*(dd^c(\tilde{\eta})(\log|z|^2)) = p_*(\tilde{\eta}\, dd^c(\log|z|^2)) = \tilde{\eta}_{|X\times 0} - \tilde{\eta}_{|X\times\infty}\,.$$

So, to prove the vanishing of η we may assume that the lines in (D) are split. By repeating the argument, we may also assume that the columns are split. q.e.d.

2.6.2 Let $f: X \to Y$ be a morphism of arithmetic varieties. We assume that f is a local complete intersection morphism (l.c.i. morphism) which is smooth on the generic fiber $Y_{\mathbb{Q}}$. Choose a Hermitian metric on the complex relative tangent space $Tf_{\mathbb{C}}$. We shall attach to these data a Todd class $\widehat{Td}(f)$.

Since X is quasi-projective we may imbed X in a projective space $\mathbb{P}^N$ and let $i: X \to Y \times \mathbb{P}^N = P$ be the product of this imbedding with the map f. We get a factorization $f = gi$, where $g: P \to Y$ is the first projection:

$$\begin{array}{ccc} X & \xrightarrow{\;i\;} & P \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle g} & \\ & Y\,. & \end{array}$$

Since f is l.c.i. and g is smooth, the immersion i is regular. Denote by $N = N_{X/P}$ the normal bundle of X in P and Tg the relative tangent-bundle of g. Choose Hermitian metrics on N and Tg. Recall from [GS3] and 2.4.2 that we can attach to these Hermitian bundles (operational) Todd classes $\widehat{Td}(\bar{N})$ acting on $\widehat{CH}^{\cdot}(X)_{\mathbb{Q}}$ and $\widehat{Td}(\overline{Tg})$ acting on $\widehat{CH}^{\cdot}(P)_{\mathbb{Q}}$.

On $X(\mathbb{C})$ there is an exact sequence of vector bundles

$$\mathscr{E}: 0 \to Tf_{\mathbb{C}} \to i^* Tg_{\mathbb{C}} \to N_{\mathbb{C}} \to 0\,.$$

We denote by $\widetilde{Td}(\mathscr{E}) \in \tilde{A}^{\cdot}(X)$ the Bott–Chern secondary characteristic class attached to this sequence [BC, GS3]. In particular

$$dd^c\widetilde{Td}(\mathscr{E}) = Td(\overline{Tf}_{\mathbb{C}} \oplus \bar{N}_{\mathbb{C}}) - Td(i^*\overline{Tg}_{\mathbb{C}})\,.$$

Let

$$(6) \qquad \widetilde{Td}(f/g) = \widetilde{Td}(\mathscr{E})Td(\bar{N}_{\mathbb{C}})^{-1} \in \tilde{A}^{\cdot}(X_{\mathbb{R}}) ,$$

so that

$$dd^c \widetilde{Td}(f/g) = Td(\overline{Tf_{\mathbb{C}}}) - Td(i^* \overline{Tg_{\mathbb{C}}})Td(\bar{N}_{\mathbb{C}})^{-1} .$$

For any $\alpha \in \widehat{CH}_{\cdot}(X)$ we define

$$(7) \qquad \widehat{Td}(f) \cap \alpha = \widehat{Td}(i^* \overline{Tg}) \cap (\widehat{Td}^{-1}(\bar{N}) \cap \alpha) + \widetilde{Td}(f/g) \cap \alpha \in \widehat{CH}_{\cdot}(X)_{\mathbb{Q}} .$$

When $Y = \mathrm{Spec}(A)$ is the ground ring, we also write $\widehat{Td}(X)$ instead of $\widehat{Td}(f)$, and $\widetilde{Td}(X/P)$ instead of $\widetilde{Td}(f/g)$.

If we assume that X and Y are regular and the ground ring is not necessarily good, we can also define

$$\widehat{Td}(f) = \widehat{Td}(i^* \overline{Tg})\widehat{Td}^{-1}(\bar{N}) + \widetilde{Td}(f/g) \in \widehat{CH}^{\cdot}(X)_{\mathbb{Q}} .$$

Proposition 1 (i) *The class $\widehat{Td}(f)$ depends only on the choice of metrics on $Tf_{\mathbb{C}}$, and not on the choice of i, g, nor on the metrics on N and Tg.*

(ii) *Let $f: X \to Y$ and $g: Y \to Z$ be two maps between regular arithmetic varieties. Assume that f and g are smooth over F and choose a metric on $Tf_{\mathbb{C}}$, $Tg_{\mathbb{C}}$ and $T(gf)_{\mathbb{C}}$. Then the following identity holds in $\widehat{CH}^{\cdot}(X)$*

$$\widehat{Td}(gf) = \widehat{Td}(f) f^*(\widehat{Td}(g)) - a(\widetilde{Td}(\mathscr{E}')) ,$$

where $\mathscr{E}'$ is the exact sequence

$$0 \to Tf_{\mathbb{C}} \to T(gf)_{\mathbb{C}} \to f^* Tg_{\mathbb{C}} \to 0 .$$

Proof. To prove (i), given two factorizations $f = g_1 i_1 = g_2 i_2$ as above, we may consider the fiber product $P_1 \times_Y P_2$ and the diagonal imbedding, so we are led to consider a diagram

$$\begin{array}{ccccc} & & X & & \\ & \overset{j}{\swarrow} & i \downarrow & \overset{f}{\searrow} & \\ P' & \xrightarrow{p} & P & \xrightarrow{g} & Y \end{array}$$

where p, g and $h = g \circ p$ are smooth. We want to show that, for arbitrary choices of metrics,

$$\widehat{Td}(i^* \overline{Tg}) \cap (\widehat{Td}^{-1}(\bar{N}_{X/P}) \cap \alpha) + \widetilde{Td}(f/g) \cap \alpha$$
$$= \widehat{Td}(j^* \overline{Th}) \cap (\widehat{Td}^{-1}(\bar{N}_{X/P'}) \cap \alpha) + \widetilde{Td}(f/h) \cap \alpha .$$

(The proof in the regular case is similar).

On $X(\mathbb{C})$ we have a commutative diagram with exact rows and columns

$$
\begin{array}{ccccccc}
 & & & 0 & & 0 & \\
 & & & \downarrow & & \downarrow & \\
 & & & Tf_{\mathbb{C}} & \overset{\mathrm{id}}{\simeq} & Tf_{\mathbb{C}} & \\
 & & & \downarrow & & \downarrow & \\
0 & \to j^*Tp_{\mathbb{C}} & \to & j^*Th_{\mathbb{C}} & \to & i^*Tg_{\mathbb{C}} & \to 0 \\
 & \wr\,\mathrm{id} & & \downarrow & & \downarrow & \\
0 & \to j^*Tp_{\mathbb{C}} & \to & N_{X(\mathbb{C})/P'(\mathbb{C})} & \to & N_{X(\mathbb{C})/P(\mathbb{C})} & \to 0 \\
 & & & \downarrow & & \downarrow & \\
 & & & 0 & & 0 & .
\end{array}
$$

Let L_1, L_2, L_3 (resp. C_1, C_2, C_3) be the three lines (resp. columns) in this diagram. By definition

$$\widetilde{Td}(f/h) = \widetilde{Td}(C_2)/Td(\bar{N}_{X/P'}) ,$$

and

$$\widetilde{Td}(g/h) = \widetilde{Td}(C_3)/Td(\bar{N}_{X/P}) .$$

So we compute

$$
\begin{aligned}
&\widehat{Td}(i^*\overline{Tg}) \cap (\widehat{Td}^{-1}(\bar{N}_{X/P}) \cap \alpha) + \widetilde{Td}(f/g) \cap \alpha - \widehat{Td}(j^*\overline{Th}) \\
&\quad \cap (\widehat{Td}^{-1}(\bar{N}_{X/P'}) \cap \alpha) - \widetilde{Td}(f/h) \cap \alpha \\
&= [\widehat{Td}(j^*\overline{Tp})\widehat{Td}(i^*\overline{Tg}) - \widehat{Td}(j^*\overline{Th})]\widehat{Td}^{-1}(\bar{N}_{X/P})\widehat{Td}^{-1}(j^*\overline{Tp}) \cap \alpha + [\widehat{Td}(\bar{N}_{X/P'}) \\
&\quad - \widehat{Td}(j^*\overline{Tp})\widehat{Td}(\bar{N}_{X/P})]\widehat{Td}(j^*\overline{Th})\widehat{Td}^{-1}(\bar{N}_{X/P'})\widehat{Td}^{-1}(\bar{N}_{X/P})\widehat{Td}^{-1}(j^*\overline{Tp}) \cap \alpha \\
&\quad + \widetilde{Td}(f/g) \cap \alpha - \widetilde{Td}(f/h) \cap \alpha \\
&= a(x) \cap \alpha
\end{aligned}
$$

with

$$
\begin{aligned}
x = {} & \widetilde{Td}(L_3)Td(j^*\overline{Th})Td^{-1}(\bar{N}_{X/P'})Td^{-1}(\bar{N}_{X/P})Td^{-1}(j^*\overline{Tp}) \\
& - \widetilde{Td}(L_2)Td^{-1}(\bar{N}_{X/P})Td^{-1}(j^*\overline{Tp}) \\
& + \widetilde{Td}(C_3)Td^{-1}(\bar{N}_{X/P'}) - \widetilde{Td}(C_2)Td^{-1}(\bar{N}_{X/P}) .
\end{aligned}
$$

By Lemma 14(ii) applied to the diagram above, we know that $x = 0$. This proves (i).

To prove (ii), since f and g are l.c.i., by a standard argument there exists a commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{i} & M' & \xrightarrow{j} & M \\ & {}_f\searrow & \downarrow p & & \downarrow q \\ & & Y & \xrightarrow{k} & M'' \\ & & & {}_g\searrow & \downarrow r \\ & & & & Z\,. \end{array}$$

in which p, q and r are smooth, i, j, and k are regular immersions, and the square is cartesian. Let us write T_p, T_q, T_r, T_{rq}, N_i, N_j, N_k, and N_{ji} for the relative tangent bundles and the normal bundles respectively of the maps p, q, r, rq, i, j, k, and ji. Note that $N_j \simeq p^* N_k$, while $T_p \simeq j^* T_q$. We choose arbitrary metrics on these bundles, except for requiring that the two isomorphisms we just mentioned are isometries. On $X(\mathbb{C})$ we get the following commutative diagram with exact lines L_1, L_2, L_3 and exact columns C_1, C_2, C_3 (we omit to write the subscripts $\mathbb{C}$):

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & T_f & \to & i^* T_p & \to & N_i & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & T_{gf} & \to & i^* j^* T_{rq} & \to & N_{ji} & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & f^* T_g & \to & f^* k^* T_r & \to & f^* N_k & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0\,. & & \end{array}$$

By definition of $\widehat{Td}$ we have

$$\widehat{Td}(f) = \widehat{Td}(i^* \bar{T}_p) \widehat{Td}^{-1}(\bar{N}_i) + \widetilde{Td}(L_1) Td^{-1}(\bar{N}_i)\,,$$

$$f^* \widehat{Td}(g) = \widehat{Td}(f^* k^* \bar{T}_r) \widehat{Td}^{-1}(f^* \bar{N}_k) + \widetilde{Td}(L_3) Td^{-1}(f^* \bar{N}_k)\,,$$

and

$$\widehat{Td}(gf) = \widehat{Td}(i^* j^*\ \bar{T}_{rq}) \widehat{Td}^{-1}(\bar{N}_{ji}) + \widetilde{Td}(L_2) Td^{-1}(\bar{N}_{ji})\,.$$

Using the first two formulas we find

$$\begin{aligned} \widehat{Td}(f) f^* \widehat{Td}(g) - \widetilde{Td}(C_1) = {} & \widehat{Td}(i^* \bar{T}_p) \widehat{Td}^{-1}(\bar{N}_i) \widehat{Td}(f^* k^* \bar{T}_r) \widehat{Td}^{-1}(f^* \bar{N}_k) \\ & + \widetilde{Td}(L_1) Td^{-1}(\bar{N}_i) Td(f^* k^* \bar{T}_r) Td^{-1}(f^* \bar{N}_k) \\ & + \widetilde{Td}(L_3) Td^{-1}(f^* \bar{N}_k) Td(i^* \bar{T}_p) Td^{-1}(\bar{N}_i) \\ & + (\widetilde{Td}(L_1) * \widetilde{Td}(L_3)) Td^{-1}(\bar{N}_i) Td^{-1}(f^* \bar{N}_k) - \widetilde{Td}(C_1)\,. \end{aligned}$$

Using the exact sequences C_2 and C_3 we may rewrite as follows the formula for $\widehat{Td}(gf)$:

$$\widehat{Td}(gf) = \widehat{Td}(i^*\bar{T}_p)\widehat{Td}^{-1}(\bar{N}_i)\widehat{Td}(f^*k^*\bar{T}_r)\widehat{Td}^{-1}(f^*\bar{N}_k) + (\widetilde{Td}(C_2) * \widetilde{Td^{-1}}(C_3))$$

$$- \widetilde{Td}(C_2)Td^{-1}(\bar{N}_i)Td^{-1}(f^*\bar{N}_k) - \widetilde{Td^{-1}}(C_3)Td(i^*\bar{T}_p)Td(f^*k^*\bar{T}_r)$$

$$+ \widetilde{Td}(L_2)Td^{-1}(\bar{N}_{ji}) .$$

Comparing the right hand sides in these two formulas follows from Lemma 14 (iii) applied to the diagram above. q.e.d.

2.6.3 Concerning the classes $\widetilde{Td}(f/g)$ we shall need the following results. We assume that the ground ring is $\mathbb{C}$ and first consider the case where Y is a point. Then we have the following

Lemma 15 (i) *If we choose two different metrics h' and h'' on the tangent bundle to P, then*

$$\widetilde{Td}(X/P, h') - \widetilde{Td}(X/P, h'') = \widetilde{Td}(T_P, h', h'')Td^{-1}(\bar{N}_{X/P}) .$$

(ii) *If h' and h'' are two different metrics on the normal bundle $N_{X/P}$, then*

$$\widetilde{Td}(X/P, h') - \widetilde{Td}(X/P, h'') = -k^*(\widetilde{Td^{-1}}(N, h', h'')Td(\bar{T}_P)).$$

Proof. (i) follows from Lemma 14 (i) with $S = 0$, $\bar{B} = (T_P, h')$, $\bar{D} = (T_P, h'')$, $\bar{A} = \bar{C} = i^*\bar{T}_P$ and $\bar{Q} = \bar{N}_{X/P}$, when (ii) is the special case $M = P$ of the next lemma. q.e.d.

Now consider two closed immersions $j: X \to M$ and $k: X \to P$ of smooth complex manifolds, and $q: P \to M$ a smooth map such that $qk = j$.

Lemma 16 *The following identity holds in $\tilde{A}(X)$:*

$$\widetilde{Td}(X/P) - \widetilde{Td}(X/M) = \widetilde{Td^{-1}}(0 \to k^*\bar{T}_q \to \bar{N}_{X/P} \to \bar{N}_{X/M} \to 0)k^*Td(\bar{T}_P)$$

$$+ \widetilde{Td}(0 \to k^*\bar{T}_q \to k^*\bar{T}_P \to j^*\bar{T}_M \to 0)Td^{-1}(\bar{N}_{X/M})Td^{-1}(k^*\bar{T}_q) .$$

Proof. On X we have a commutative diagram with exact lines and columns:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & T_X & = & T_X & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & k^*T_q & \to & k^*T_P & \to & j^*T_M & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & k^*T_q & \to & N_{X/P} & \to & N_{X/M} & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

By applying formula (5) to $\widetilde{Td}^{-1}(L_3)$ and then Lemma 14 ii) the proof is complete. q.e.d.

2.7 Segre classes

Let X be an arithmetic variety and $\bar{E} = (E, h)$ a Hermitian bundle of rank r on X. Denote by $P = \mathbb{P}(E)$ the projective space of E in the sense of Grothendieck, representing rank one quotients of E, and $f: P \to X$ the projection. On P we have the canonical exact sequence

$$\mathscr{E}: 0 \to S \to f^*E \to \mathcal{O}(1) \to 0 ,$$

where $\mathcal{O}(1)$ is the tautological line bundle. We equip f^*E with the metric f^*h, and S (resp. $\mathcal{O}(1)$) with the induced (resp. quotient) metric. For every integer $k \geqq 0$ define

$$\hat{s}'_k(\bar{E}) = f_*(\hat{c}_1(\overline{\mathcal{O}(1)})^{k+r-1}) \in \widehat{CH}^k(X) .$$

Following [E] we define an element R_k in $\tilde{A}^{k-1,k-1}(X)$, $k > 0$, as follows. Let $\mathscr{E}^\vee$ be the dual of $\mathscr{E}$ and $\mathscr{E}^\vee(1) = \mathscr{E}^\vee \otimes \mathcal{O}(1)$. For the obvious choice of metrics, let $\tilde{c}_r(\mathscr{E}^\vee(1)) \in \tilde{A}^{r-1,r-1}(P)$ be the r-th Bott–Chern class of this exact sequence. Then R_k is th k-th coefficient of the formal power series

$$\sum_{k>0} R_k t^k = \left(\sum_{k>0} f_*(c_1(\overline{\mathcal{O}(1)})^{k-1} \tilde{c}_r(\mathscr{E}^\vee(1)))t^k \right) \left(\sum_{j>0} c_j(\bar{E})(-t)^j \right)^{-1} , \tag{8}$$

where we have used the module structure of $\tilde{A}^{\cdot}(X)$ on closed forms in $A^{\cdot}(X)$.

The arithmetic *Segre class* $\hat{s}_k(\bar{E}) \in \widehat{CH}^k(X)$, $k \geqq 0$, is defined by $\hat{s}_0(\bar{E}) = 1$ and

$$\hat{s}_k(\bar{E}) = \hat{s}'_k(\bar{E}) + a(R_k) \quad \text{when} \quad k > 0 . \tag{9}$$

Lemma 17 *The following identity of formal power series relates Segre classes and Chern classes:*

$$\sum_{j \geqq 0} \hat{c}_j(\bar{E})(-t)^j = \left(\sum_{k \geqq 0} \hat{s}_k(\bar{E}) t^k \right)^{-1} .$$

Proof. From the exact sequence

$$\mathscr{E}^\vee(1): 0 \to \mathcal{O}_P \to f^*(E^\vee)(1) \to S^\vee(1) \to 0\,,$$

the additivity of Chern classes, and their behaviour under tensoring by a line bundle (see [GS3]), we get

$$\begin{aligned} a(\tilde{c}_r(\mathscr{E}^\vee(1))) &= \hat{c}_r(\overline{\mathcal{O}_P} \oplus \bar{S}^\vee(1)) - \hat{c}_r(f^*(\bar{E}^\vee)(1)) \\ &= -\,\hat{c}_r(f^*(\bar{E}^\vee)(1)) = -\sum_{j \geqq 0} f^*(\hat{c}_j(\bar{E}^\vee))\hat{c}_1(\overline{\mathcal{O}(1)})^{r-j}\,. \end{aligned}$$

Furthermore $\hat{c}_j(E^\vee) = (-1)^j \hat{c}_j(\bar{E})$ (loc. cit.). If we apply a to (8) we get, by the projection formula,

$$\begin{aligned} \sum_{k>0} a(R_k)t^k &= -\left(\sum_{\substack{k>0 \\ j \geqq 0}} f_*(\hat{c}_1(\overline{\mathcal{O}(1)})^{k-1} f^*(\hat{c}_j(\bar{E}^\vee))\hat{c}_1(\overline{\mathcal{O}(1)})^{r-j})t^k \right) \\ &\quad \cdot \left(\sum_{j \geqq 0} \hat{c}_j(\bar{E})^\vee (-t)^j \right)^{-1} \\ &= -\sum_{j \geqq 0} \hat{c}_j(\bar{E})(-t)^j \left(\sum_{k>0} f_*(\hat{c}_1(\overline{\mathcal{O}(1)})^{k-j+r-1})t^{k-j} \right) \\ &\quad \cdot \left(\sum_{j \geqq 0} \hat{c}_j(\bar{E})(-t)^j \right)^{-1} \end{aligned}$$

since $f_*(\hat{c}_1(\overline{\mathcal{O}(1)})^n) = 0$ unless $n \geqq r-1$. Using $\hat{s}_0(\bar{E}) = 1$, the definition of $\hat{s}'_k(\bar{E})$ and the equalities we just proved, we conclude that

$$\sum_{k>0} a(R_k)t^k = -\sum_{k \geqq 0} \hat{s}'_k(\bar{E})t^k + \left(\sum_{j \geqq 0} \hat{c}_j(\bar{E})(-t)^j \right)^{-1}$$

and the lemma follows from (9). q.e.d.

It follows from this lemma that Segre classes provide an alternative way to define Chern classes, hence all characteristic classes of Hermitian vector bundles, rather than the splitting principle used in [GS3] (compare [E] and [F3]).

3 A characteristic class for Hermitian coherent sheaves

3.1 The Chern character with supports

Let P be an integral arithmetic variety (over a good arithmetic ring), and $i: X \to P$ a closed arithmetic proper subvariety, with a choice of metric on the normal bundle to $X(\mathbf{C})$ in $P(\mathbf{C})$. Suppose that we are given a bounded complex $E.$ of locally free sheaves on P, acyclic off X, which over the generic fibre P_F is a resolution of a Hermitian bundle $\bar{\mathscr{F}}$ on X_F. Then we can define a *Chern character with supports*, $\widehat{ch}_P^X(E.) \in \widehat{CH}.(X)_{\mathbf{Q}}$ as follows.

Let $\pi: W \to P \times \mathbf{P}^1$ be the Grassmann graph construction associated to the complex $E.$. As in Sect. 1.1, we consider the cycle $Z = [W_\infty] - [\tilde{P}]$ and write $|Z|$

for its support. Let $\tilde{E}.$ be the extension of $E.$ to W, and $E.^Z$ its restriction to $|Z|$. Since $E.^Z$ is a resolution over $|Z|_F$ of the direct image of $\mathscr{F}$, we may equip the bundles E_i^Z with Hermitian inner products satisfying condition (A) (see 2.4.1, Definition 4), and consider, as in loc. cit., the Bott–Chern singular current $\widetilde{ch}(E.^Z)$ of this complex. Let us write $\pi^Z : |Z| \to X$ for the projection induced by the map $\pi : W \to P$. Observe that, because of Eq. (4), the current $dd^c(\pi_*^Z(\widetilde{ch}(E.^Z)))$ is smooth on X, and hence, by [GS2, Theorem 1.2.2(i)], that $a(\pi_*^Z(\widetilde{ch}(E.^Z)))$ makes sense as an element in $\widehat{CH}.(X)$. Furthermore, the cycle Z may be viewed as giving a class in $\widehat{CH}_{\dim(Z)}(|Z|) = CH_{\dim(Z)}(|Z|)$.

Definition 6 The Chern character of $E.$ with supports in X is:

$$\widehat{ch}_P^X(E.) = \pi_*^Z(\widehat{ch}(\bar{E}.^Z) \cap Z) + a(\pi_* \widetilde{ch}(E.^Z)) .$$

We shall write sometimes $\widehat{ch}^X(E.)$ rather than $\widehat{ch}_P^X(E.)$. This class depends on the choice of metrics on $\mathscr{F}$ and on the normal bundle to $X(\mathbf{C})$ in $P(\mathbf{C})$, but it is independent of the choice of metrics on $E.^Z$. This follows from the case $E''. = E'. = E.$ of the following lemma.

Lemma 18 *If $k : E'. \to E''.$ is a quasi-isomorphism which, over F, induces a morphism of resolutions of $\mathscr{F}$, then*

$$\widehat{ch}^X(E'.) = \widehat{ch}^X(E''.) .$$

Proof. Replacing the quasi-isomorphism by its mapping cone, if necessary, we may suppose that k is injective in each degree. By Lemma 5 we know that $W(E'.) = W(E''.)$ and that the map k induces a map $k^Z : E'.^Z \to E''.^Z$ which is a monomorphism with cokernel a split acyclic complex of locally free coherent sheaves. Let us now choose arbitrary metrics on the complexes $E'.^Z$, $E''.^Z$, which satisfy condition (A), and let us also make a choice of metrics on the quotient complex compatible with the splittings. By Theorem 4

$$\begin{aligned}&(\widehat{ch}(\bar{E}''.^Z) - \widehat{ch}(\bar{E}'.^Z) + \widehat{ch}(\overline{E''.^Z/E'.^Z})) \cap Z \\ &\quad = a\left[\sum_m (-1)^{m+1} \widetilde{ch}(0 \to (E''^Z_m) \to (E'^Z_m) \to (E''^Z_m/E'^Z_m) \to 0)\right].\end{aligned}$$

However by [BGS3, Theorem 2.9], the right hand side of this equation is equal to

$$a[\widetilde{ch}(E'.^Z) - \widetilde{ch}(E''.^Z) - \widetilde{ch}(E''.^Z/E''.^Z)] .$$

Since $E''.^Z/E'.^Z$ is split acyclic both $\widehat{ch}(\overline{E''.^Z/E'.^Z})$ and $\widetilde{ch}(E''.^Z/E'.^Z)$ vanish. So we conclude that

$$(\widehat{ch}(\bar{E}''.^Z) - \widehat{ch}(\bar{E}'.^Z)) \cap Z = a[\widetilde{ch}(E'.^Z) - \widetilde{ch}(E''.^Z)] . \qquad \square$$

Corollary 3 *Let $\bar{\mathscr{F}}$ be an Hermitian coherent sheaf on X and suppose that P is regular. For any resolution $E.$ of $\bar{\mathscr{F}}$ by a complex of locally free coherent sheaves on P, the* *class* $\widehat{ch}^X(E.) \in \widehat{CH}.(X)$ *is independent of the choice of resolution. We shall denote it* $\widehat{ch}^X(\bar{\mathscr{F}})$.

Proof. Given any two resolutions $E'.$ and $E''.$ of $\bar{\mathscr{F}}$, there exists a third resolution $E.$ which maps quasi-isomorphically to each of the first two resolutions. By the previous lemma, it then follows that the class $\widehat{ch}^X(E.)$ in $\widehat{CH}.(X)$ is independent of the choice of resolution. □

Lemma 19

(i) $\omega(\widehat{ch}^X(E.)) = ch(\bar{\mathscr{F}})Td^{-1}(\bar{N})$.

(ii) *If we are given two metrics* h' *and* h'' *on the normal bundle of* $X(\mathbf{C})$ *in* $P(\mathbf{C})$, *then the difference of the associated Chern characters with supports is given by the formula*

$$\widehat{ch}^X(E.)' - \widehat{ch}^X(E.)'' = a(ch(\bar{\mathscr{F}})\widetilde{Td}^{-1}(N, h', h'')) .$$

(iii) *If we are given a short exact sequence* $\mathscr{A}$ *of complexes of locally free coherent sheaves:*

$$0 \to E.' \to E. \to E.'' \to 0$$

which on the generic fibre is a resolution of the exact sequence of Hermitian coherent sheaves:

$$0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0 ,$$

then

$$\widehat{ch}^X(E'.) - \widehat{ch}^X(E.) + \widehat{ch}^X(E''.) = a(\widetilde{ch}(\mathscr{A})Td^{-1}(\bar{N}_{X/P}))$$

Proof. Recall from (4) that

$$dd^c\widetilde{ch}(\bar{E}.^Z) = i_*(ch(\bar{\mathscr{F}})Td^{-1}(\bar{N}_{X/P})) - ch(\bar{E}.^Z) .$$

Since $\pi_* i_*$ is the identity, we get part (i) of the lemma. To prove part (ii), consider the embedding $i \times 1_{\mathbf{P}^1}: X \times \mathbf{P}^1 \to P \times \mathbf{P}^1$, and give the normal bundle to this embedding a metric which restricts to h' at 0 and to h'' at ∞. Given part (i) of the lemma, the proof now proceeds by the same method as [GS3], section 1.2. The proof of part (iii) is similar, except that in the family of embeddings $i \times 1_{\mathbf{P}^1}$ we consider a Hermitian coherent sheaf $\tilde{\mathscr{F}}$ which is flat over $\mathbf{P}^1$, and which restricts to $\mathscr{F}$ at 0 and to $\mathscr{F}' \oplus \mathscr{F}''$ at ∞. This sheaf $\tilde{\mathscr{F}}$ may be constructed as in [GS3, 1.2.3], and we give it an Hermitian metric for which the restriction isomorphisms at 0 and at ∞ are isometries. □

Given the definition of $\hat{K}'_0(X)$, one immediately obtains:

Corollary 4 *Assume* P *is regular. Then the map*

$$(\bar{\mathscr{F}}, \gamma) \mapsto \widehat{ch}^X(\bar{\mathscr{F}}) + a(\gamma)$$

defines a homomorphism of abelian groups

$$\hat{K}'_0(X) \to \widehat{CH}.(X)_{\mathbf{Q}}$$

Lemma 20 *We have the following module property of the Chern character with supports. If G is a locally free coherent sheaf with Hermitian metric on P, then:*

$$\widehat{ch}_P^X(E.\otimes i^*\bar{G}) = \widehat{ch}_P^X(E.)._i\widehat{ch}(\bar{G}) .$$

Proof. Note that $W(E.\otimes G) = W(E.)$, and that on this scheme we have $\widetilde{E.\otimes G} \simeq \tilde{E}.\otimes G$. On restricting to $|Z|$, we find that

$$(E.\otimes G)^Z = E.^Z \otimes \pi^{Z*}i^*G .$$

Now observe that

$$\widetilde{ch}(E.^Z \otimes \pi^{Z*}i^*G) = \widetilde{ch}(E.^Z)\pi^{Z*}i^*ch(\bar{G}) ,$$

[BGS3, 1.3.3] and the result follows from the projection formula for currents. □

Lemma 21 *Let $X \to P$ be a closed embedding of arithmetic varieties, and suppose that $g: P \to Y$ is a proper map of arithmetic varieties, such that the induced maps from $P(\mathbf{C})$ and $X(\mathbf{C})$ to $Y(\mathbf{C})$ are both smooth. If $\bar{E}.$ is a complex of Hermitian locally free coherent sheaves on P, acyclic off X, which on the generic fibre $P_{\mathbf{Q}}$ is a resolution, satisfying condition* (A), *of a Hermitian coherent sheaf $\bar{\mathscr{F}}$ on $X_{\mathbf{Q}}$, then, writing f for the restriction of g to X, we have the following equality in $\widehat{CH}.(Y)_{\mathbf{Q}}$;*

$$f_*(\widehat{ch}_P^X(E.)._i\alpha) = g_*(\widehat{ch}(E.)\cap\alpha) + a(g_*\widetilde{ch}(E.)\omega(\alpha)) ,$$

where $\alpha = \widehat{ch}(x)$, for $x \in \hat{K}_0(P)_{\mathbf{Q}}$, or P is regular and $\alpha \in \widehat{CH}^{\cdot}(P)$.

Proof. First we fix the metric on the normal bundle of $X \times \mathbf{P}^1$ in the Grassman-graph construction $W = W(E.)$, as in [BGS3, Sect. 4(d)], via the isomorphism

$$N_{X\times\mathbf{P}^1/W} \simeq p^*N_{X/P}(-\infty) ,$$

where p is the projection from $Y \times \mathbf{P}^1$ to Y.

On $\mathbf{C}$-valued points, the map $\tilde{\pi} = g \circ \pi$ from W to $Y \times \mathbf{P}^1$ induces a proper map of complex manifolds. As in loc. cit., choose a metric on $\tilde{E}.$ such that its restriction to 0 coincides with the one on $E.$, its restriction to $\tilde{P}$ is split acyclic, and its restriction to $|Z|$ satisfies condition (A).

Consider the class $\tilde{\pi}_*(\widehat{ch}(\tilde{E}.)._q\alpha) + a(\tilde{\pi}_*\widetilde{ch}(\tilde{E}.)q^*\omega(\alpha))$ in $\widehat{CH}.(Y\times\mathbf{P}^1)_{\mathbf{Q}}$, where $q: W \to P$ is the projection. The restriction of this class to $\{\infty\}$ is the left-hand side of the equation in the lemma, while the restriction to $\{0\}$ is the right-hand side. By Theorem 4.4.6 of [GS2] (which remains valid when X has singularities away from the generic fibre X_F), the difference of these two is

$$\int_{\mathbf{P}^1} \tilde{\pi}_*(\omega(\widehat{ch}(\tilde{E}.) + a(\widetilde{ch}(\tilde{E}.)))q^*\omega(\alpha))dd^c(-\log|z|^2)$$

where z is the parameter on $\mathbf{P}^1$. However, by (4), $\omega(\widehat{ch}(\tilde{E}.) + a(\widetilde{ch}(\tilde{E}.)))$ is equal to $j_*(ch(\bar{\mathscr{F}})Td^{-1}(N_{X\times\mathbf{P}^1/W}))$, where $j: X \times \mathbf{P}^1 \to W$ is the inclusion. Applying the projection formula for integration over the fiber, it remains to observe as in [BGS3, loc. cit.], that

$$\int_{\mathbf{P}^1} ch(p^* N_{Y/P}(-\infty)) \log|z|^2 = 0 . \qquad \square$$

Lemma 22 (*i*) *Suppose that* $i: X \to P$ *is a closed regular embedding and* $\bar{\mathscr{F}}$ *is an Hermitian vector bundle on* X; *then for any resolution* $E. \to i_* \mathscr{F}$ *by locally free sheaves on* P,

$$\widehat{ch}^X(E.) = \widehat{ch}(\bar{\mathscr{F}}) \cap (\widehat{Td}^{-1}(\bar{N}_{X/P}) \cap [X]) .$$

(*ii*) *Suppose that* X *and* P *are both regular. Then for any Hermitian coherent sheaf* $\bar{\mathscr{F}}$ *on* X

$$\widehat{ch}^X(\bar{\mathscr{F}}) = \widehat{ch}(\bar{\mathscr{F}}) \cap \widehat{Td}^{-1}(\bar{N}_{X/P}) .$$

Proof. (i) Since the inclusion $i: X \to P$ is a regular embedding, the cycle Z at infinity is irreducible and equal to $\mathbf{P}(N \oplus 1)$, with $N = N_{X/P}$ (Theorem 1(ii)). Let H be the tautological codimension one sub-bundle of $N \oplus 1$ on $\mathbf{P}(N \oplus 1)$. Then the Koszul complex $K.(H)$ is a resolution of $s_* \mathscr{O}_X$, where s is the zero section, and hence $K.(H) \otimes \pi^*(\mathscr{F})$ is a resolution of $s_* \mathscr{F}$. Furthermore, we know from Theorem 1 (and Theorem 4.8 of [BGS3]) that there is a quasi-isomorphism:

$$\phi: E.^Z \simeq K.(H) \otimes \pi^*(\mathscr{F})$$

in which ϕ is an epimorphism with split acyclic kernel. If we give $K.(H)$ the metric obtained by viewing H as a sub-bundle of $N \oplus 1$, then $K.(H) \otimes \pi^*(\mathscr{F})$ automatically satisfies condition (A) as a resolution of $s_* \mathscr{F}$ (see [BGS3] loc. cit.). Hence, as in Lemma 18,

$$a(\widetilde{ch}(E.^Z) - \widetilde{ch}(K.(H) \otimes \pi^*(\mathscr{F}))) = (\widehat{ch}(K.(H) \otimes \pi^*(\mathscr{F})) - \widehat{ch}(E.^Z)) \cap [Z].$$

The result now follows from Theorem 4.13 of [BGS3]. Part (ii) follows immediately from part (i). $\square$

Lemma 23 *Let* $E.$ *be a complex of locally free coherent sheaves on the arithmetic variety* P, *and let* $X \subset P$ *be the support of the homology of* $E.$. *Suppose that* X_F, *viewed as a reduced subscheme of* P_F, *is smooth over* F, *and that, on* P_F, $E.$ *is a resolution of an Hermitian vector bundle on* X_F. *Let* $i: D \to P$ *be an arithmetic subvariety which is a principal (i.e. effective Cartier) divisor and meets* X *transversally over* F. *Let us write* $\{D_\alpha\}$ *for the irreducible components of* D, n_α *for the multiplicity of* D_α *in the Weil divisor* $[D] = \sum_\alpha n_\alpha [D_\alpha]$ *associated to* D, *and* $\eta^\alpha: X \cap D_\alpha \to X \cap D$ *for the inclusion. Then*:

$$i_X^* \widehat{ch}_P^X(E.) = \sum_\alpha n_\alpha \eta_*^\alpha \widehat{ch}_{D_\alpha}^{X \cap D_\alpha}(E.|_{D_\alpha}) \in \widehat{CH}.(X \cap D) ,$$

where $i_X: X \cap D \to X$ *is the inclusion.*

This is also true when, more generally, D *is a divisor on* P, *which is contained in a Zariski open subset* $U \subset P$, *and is principal as a divisor on* U.

Proof. Let us fix some notation. Write G_P for the product of Grassmannian bundles $\prod_m G(n_m, \tilde{C}_m(E.))$ over $P \times \mathbf{P}^1$ as in 1.1.1, G_D for the restriction of this to $D \times \mathbf{P}^1$ (which may be identified with the corresponding product of Grassmannians for the restriction of $E.$ to D). Let $i_G: G_D \to G_P$ be the inclusion of the divisor G_D into G_P. Let $G_{P\infty}$ be the fiber of G_P over $P \times \{\infty\}$, which is a divisor in G_P, and let $i_\infty: G_{D\infty} \to G_{P\infty}$ be the inclusion of the corresponding divisor in G_D. Write $j: G_{P\infty} \to G_P$ for the inclusion, and for each α, j_{D_α} for the corresponding inclusion over $D_\alpha \times \mathbf{P}^1$. Observe that $G_{P\infty} \cap G_D = G_{D\infty}$.

Now $G_{D\infty}$ is a principal divisor in the pull-back G_D^0 of G_D over $D \times (\mathbf{A}^1 - \{0\})$ (we give G_P^0 a similar meaning); let us write $t = 0$ for the equation of this divisor. Following 2.2.6, we have a pull back map $j_D^*: Z_k(G_D) \to Z_{k-1}(G_{D\infty})$ on cycles as well as on $\widehat{CH}.$, together with similar pull back maps j^*, i_∞^*, and $i_{G_D}^*$.

Let $W \subset G_P$ be the Grassmannian graph of $E.$ and $Z = j^*([W]) - [\tilde{P}]$ on $G_{P\infty}$. Since on the complement of the divisor $G_{P\infty}$ the variety W is the image of the section of G_P over $P \times \mathbf{A}^1$ corresponding to the graphs of the differentials in $E.$, we have an equality of cycles

$$i^*([W]) = \sum_\alpha n_\alpha [W_{D_\alpha}]$$

on the open set $G_P - G_{P\infty}$. The cycle $j_{D_\alpha}^*[W_{D_\alpha}] = Z_{D_\alpha} + [\tilde{D}_\alpha]$ is used to compute the Chern character with supports of $E.|_{D_\alpha}$. Since $j^* i^*([W])$ depends only on the restriction of $[W]$ to $G_P - G_{P\infty}$,

$$j^* i^*([W]) = \sum_\alpha n_\alpha (Z_{D_\alpha} + [\tilde{D}_\alpha]) .$$

However, we know, following [Fu2, Corollary 2.4.2], that the maps $j^* i^*$ and $i^* j^*$ agree up to rational equivalence. Thus there is a K_1-chain ϕ on $G_{D\infty}$ such that

$$\operatorname{div}(\phi) = j^* i^*[W] - i^* j^*[W] .$$

We claim that this K_1-chain can be chosen so that its support does not intersect the generic fibre. For the varieties G_D, W, and $G_{P\infty}$ all meet transversally over F, hence the cycle $j^* i^*[W] - i^* j^*[W]$ is supported over the special fibres. Examining the proof of Theorem 2.4 in [Fu2], we see that the K_1-chain ϕ is constructed by blowing up the components of the intersection which have excess intersection, which in this case are all supported over special fibres. Another argument, using K-theory and the Gersten complex ([Q1], [G1]) may be given as follows. Let f be an equation for D; then the symbol $\{f, t\}$ defines an element of $K_2(W \cap (G_P - (G_{|D|} \cup G_{P\infty})))$, and hence in K_2 of the function field of W. The differential of this element in the Gersten complex is a K_1 chain ψ supported on $\operatorname{div}(f) \cup \operatorname{div}(t)$; see [GS2, 4.2.5 and 3.3.5], and [Q1, Sect. 7]. Since the differential in the Gersten complex is compatible with products [G1, Sect. 8], one knows that, on the components of $\operatorname{div}(f)$ where t does not vanish, ψ is equal to $\psi_t = \operatorname{div}(f) * \{t\}$, while, on the components where t vanishes but f does not, ψ is equal to $-\psi_f = -\operatorname{div}(t) * \{f\}$. Now observe that $\operatorname{div}(\psi_t) = j_D^* i_G^*[W]$, while $\operatorname{div}(\psi_f) = i_{G_D}^* j^*[W]$. Since the composition of two differentials in the Gersten complex is zero, $\operatorname{div}(\psi) = 0$, and hence

$$j^* i^*[W] - i^* j^*[W] = \operatorname{div}(\phi) ,$$

i.e.

$$\sum_\alpha n_\alpha (Z_{D_\alpha} + [\tilde{D}_\alpha]) - i^*(Z + [\tilde{P}]) = \operatorname{div}(\phi) ,$$

where $\phi = \psi - \psi_t + \psi_f$ is a K_1-chain supported on $\operatorname{div}(f) \cap \operatorname{div}(t) \cap W$. Since D and X are smooth and meet transversally over F, it follows from the identification of the generic fibre of the Grassmannian-graph construction with deformation to the normal cone (see Theorem 1 ii)), that over F the two cycles $Z_{D_F} + [\tilde{D}_F]$ and $i^*(Z_F + [\tilde{P}_F])$ coincide, and hence that the cycle ϕ constructed above is supported over the special fibers. Furthermore, since away from X, W and W_{D_α} are isomorphic to $P \times \mathbf{P}^1$ and $D_\alpha \times \mathbf{P}^1$ respectively, we know that the support of ϕ lies over $X \cap |D|$.

Now choose metrics on $\tilde{E}.$ as in the proof of Lemma 21, and consider the class:

$$i^* \widehat{ch}^X(E.) = i^*(\pi^Z_*[\widehat{ch}(E.^Z) \cap Z] + a[\pi^Z_* \widetilde{ch}(E^Z)]) .$$

Since $X(\mathbf{C})$ and $D(\mathbf{C})$ are smooth and intersect transversely in $P(\mathbf{C})$,

$$i^*(a[\pi^Z_* \widetilde{ch}(E.^Z)]) = a[\pi^{Z_D}_* \widetilde{ch}(E.^{Z_D})] .$$

Therefore we must show that

$$i^* \pi_* (\widehat{ch}(E.^Z) \cap Z) = \sum_\alpha n_\alpha \eta^\alpha_* (\pi^{Z_{D_\alpha}}_* (\widehat{ch}(E.^{Z_{D_\alpha}}) \cap Z_{D\alpha}))$$

in $\widehat{CH}.(X \cap |D|)$ where $\eta^\alpha : X \cap D_\alpha \to X \cap |D|$ is the inclusion. Since $\eta^\alpha \circ \pi^{Z_{D_\alpha}}$ factors through the inclusion of Z_{D_α} into $|Z| \cap |D| := (\pi^Z)^{-1}(|D|)$ followed by the projection from $|Z| \cap |D|$ to $X \cap |D|$, we know, using Theorem 4 Property 6 for this inclusion, that the right-hand side of this formula is equal to $\pi^{|Z| \cap |D|}_* (\widehat{ch}(E.^Z|_{|Z| \cap |D|}) \cap (\sum_\alpha n_\alpha Z_{D_\alpha}))$. By the statement at the end of 2.2.6. and Theorem 4 Property 2 applied to i, we know that the left-hand side of the equation is equal to $\pi^{|Z| \cap |D|}_* (\widehat{ch}(E.^Z|_{|Z| \cap |D|}) \cap i^* Z)$. As we saw above

$$i^* Z = \sum_\alpha n_\alpha Z_{D_\alpha} - \operatorname{div}(\phi) + \tau$$

in $\widehat{CH}.(|Z| \cap |D|)$, where $\tau = \sum_\alpha n_\alpha [\tilde{D}_\alpha] - i^*[\tilde{P}]$. Since the support of τ is contained in $\tilde{P} \cap |Z|$, $E.^Z|_{|\tau|}$ is (metrically) split, and hence $\widehat{ch}(E.^Z|_{|\tau|}) \cap \tau = 0$ in $\widehat{CH}.(|\tau|)$. Because the support of the K_1-chain ϕ does not meet X_F, $\widehat{\operatorname{div}}(\phi) = (\operatorname{div}(\phi), 0)$, and therefore

$$\widehat{ch}(E.^Z|_{|\phi|}) \cap \widehat{\operatorname{div}}(\phi) = 0$$

in $\widehat{CH}.(|\phi|)$. Hence

$$\pi^{|Z| \cap |D|}_* (\widehat{ch}(E.^Z|_{|Z| \cap |D|}) \cap i^* Z) = \pi^{|Z| \cap |D|}_* \left(\widehat{ch}(E.^Z|_{|Z| \cap |D|}) \cap \left(\sum_\alpha n_\alpha Z_{D_\alpha} \right) \right),$$

and we are done.

This completes the proof of the main assertion of the lemma. Since the construction of the Chern character with supports is compatible with pull-back by flat maps, we get the second assertion. □

3.2 The construction of τ

3.2.1 We wish to define an analog in our situation of the Riemann–Roch transformation used by Baum et al. in proving the singular Riemann–Roch Theorem, [BFM]. This will be a map

$$\tau : \hat{K}'_0(X) \to \widehat{CH}.(X)_{\mathbf{Q}} ,$$

depending only on the choice of an Hermitian metric on the tangent bundle to $X(\mathbf{C})$. So let X be an arithmetic variety, $\bar{\mathscr{F}}$ an Hermitian coherent sheaf on X, and $i: X \to P$ a closed immersion of X into a regular irreducible arithmetic variety P. Let us fix a metric on the normal bundle of $X(\mathbf{C}) \subset P(\mathbf{C})$. From Corollary 3, we get from these data a class $\widehat{ch}_P^X(\bar{\mathscr{F}}) \in \widehat{CH}_\cdot(X)_\mathbf{Q}$. Let us now, in addition, fix metrics on the tangent bundles of $X(\mathbf{C})$ and $P(\mathbf{C})$. Recall from 2.6.2 that there is a secondary characteristic class $\widetilde{Td}(X/P) \in \tilde{A}(X_\mathbf{R})$, such that we have an equation of forms on $X(\mathbf{C})$:

$$dd^c \widetilde{Td}(X/P) = Td(\bar{T}_{X(\mathbf{C})}) - i^* Td(\bar{T}_{P(\mathbf{C})}) Td^{-1}(\bar{N}_{X(\mathbf{C})/P(\mathbf{C})}) \,.$$

Recall also that we defined in 2.6.2 the arithmetic Todd class $\widehat{Td}(P)$ in the arithmetic Chow group of P.

Definition 7 We set

$$\tau_P(\bar{\mathscr{F}}) = \widehat{ch}_P^X(\bar{\mathscr{F}})._i\widehat{Td}(P) + a(ch(\bar{\mathscr{F}})\widetilde{Td}(X/P)) \,.$$

The following lemma follows directly from the equations above and Lemma 19.1.

Lemma 24 *With the notation above,* $\omega(\tau_P(\bar{\mathscr{F}})) = ch(\bar{\mathscr{F}}) Td(\bar{T}_{X(\mathbf{C})})$.

Ultimately, we wish to show that, given $\bar{\mathscr{F}}$, this class only depends on the choice of metric on the tangent bundle to X. Hence we must show independence of all other choices made. First we have:

Lemma 25 *The class* $\tau_P(\bar{\mathscr{F}})$ *does not depend on the choice of metrics on* $T_{P(\mathbf{C})}$ *and* $N_{X(\mathbf{C})/P(\mathbf{C})}$.

Proof. If we have two different metrics h' and h'' on $T_{P(\mathbf{C})}$, then, from Lemma 15(i), we get

$$\begin{aligned}
\tau_P(\bar{\mathscr{F}}, h') - \tau_P(\bar{\mathscr{F}}, h'') &= \widehat{ch}_P^X(\bar{\mathscr{F}})._i\widehat{Td}(P, h') - \widehat{ch}_P^X(\bar{\mathscr{F}})._i\widehat{Td}(P, h'') \\
&\quad + a(ch(\bar{\mathscr{F}})\widetilde{Td}(X/P, h') - ch(\bar{\mathscr{F}})\widetilde{Td}(X/P, h'')) \\
&= \widehat{ch}_P^X(\bar{\mathscr{F}})._i(a(\widetilde{Td}(T_P, h', h'')) \\
&\quad + a(ch(\bar{\mathscr{F}})[\widetilde{Td}(X/P, h') - \widetilde{Td}(X/P, h'')]) \\
&= a(ch(\bar{\mathscr{F}})[\widetilde{Td}(T_P, h', h'') Td^{-1}(\bar{N}_{X/P}) + \widetilde{Td}(X/P, h') \\
&\quad - \widetilde{Td}(X/P, h'')]) = 0 \,.
\end{aligned}$$

Now if we put two metrics on the normal bundle we get

$$\begin{aligned}
\tau_P(\bar{\mathscr{F}}, h') - \tau_P(\bar{\mathscr{F}}, h'') &= [\widehat{ch}_P^X(\bar{\mathscr{F}}, h') - \widehat{ch}_P^X(\bar{\mathscr{F}}, h'')]._i\widehat{Td}(P) \\
&\quad + a(ch(\bar{\mathscr{F}})(\widetilde{Td}(X/P, h') - \widetilde{Td}(X/P, h''))) \,,
\end{aligned}$$

which is zero by Lemma 19(2) and Lemma 15(ii). □

Having eliminated the dependence on metrics, it only remains to show:

Theorem 5 *The class $\tau_P(\bar{\mathscr{F}})$ is independent of the choice of the embedding of X into a regular integral variety P.*

We shall proceed in several steps.

Lemma 26 *Let $j: X \to M$ and $k: X \to P$ be two embeddings of X into regular integral varieties, and suppose that there is a smooth map $q: P \to M$, such that $qk = j$. Then $\tau_P(\bar{\mathscr{F}}) = \tau_M(\bar{\mathscr{F}})$.*

Proof. We are in the situation of Sect. 1.2.2, and we shall use the notations given there. In particular, there is a complex $\tilde{G}_. = \mathrm{Tot}(\tilde{G}_{..})$ over $W \times_M P$. Its restriction $G_.^Z$ to $|Z| \times_M P$ is quasi-isomorphic to $v^*(V_.) \otimes q_\infty^*(E_.^Z)$ by Lemma 10.

Choose Hermitian metrics on the normal bundles $N_{X(\mathbf{C})/M(\mathbf{C})}$ and $N_{X(\mathbf{C})/P(\mathbf{C})}$ and on the relative tangent bundle $T_{P(\mathbf{C})/M(\mathbf{C})}$. The normal bundle of $X(\mathbf{C})$ in $X(\mathbf{C}) \times_M P(\mathbf{C})$ coincides with $k^* T_{P(\mathbf{C})/M(\mathbf{C})}$, and $|Z|(\mathbf{C}) = \mathbf{P}(N_{X(\mathbf{C})/M(\mathbf{C})} \oplus \mathbf{1})$, therefore the normal bundle of $X(\mathbf{C})$ in $|Z|(\mathbf{C}) \times_M P(\mathbf{C})$ is isomorphic to $N_{X(\mathbf{C})/M(\mathbf{C})} \oplus k^* T_{P(\mathbf{C})/M(\mathbf{C})}$. We endow it with the orthogonal direct sum of the two chosen metrics. Let us write $\widetilde{Td}^{-1}(X, P, M)$ for the Bott–Chern secondary class associated to the characteristic class Td^{-1} and the exact sequence of bundles on $X(\mathbf{C})$:

$$0 \to k^* T_{P(\mathbf{C})/M(\mathbf{C})} \to N_{X(\mathbf{C})/P(\mathbf{C})} \to N_{X(\mathbf{C})/M(\mathbf{C})} \to 0 .$$

Lemma 27 *If $Y \subset |Z| \times_M P$ is the support of the homology of $G_.^Z$, and $h: Y \to X$ is the projection map, then, for the choices we made of metrics on normal bundles, we have*

$$\widehat{ch}_P^X(\bar{\mathscr{F}}) = \sum_\beta n_\beta h_*^\beta(\widehat{ch}_{Z_\beta \times_M P}^{Y^\beta}(G_.^Z|_{Z_\beta})) - a(\widetilde{Td}^{-1}(X, P, M) ch(\bar{\mathscr{F}}))$$

Here the Z_β are the irreducible components of $|Z|$, and $Z = \sum_\beta n_\beta Z_\beta$, $Y^\beta = Z_\beta \times_M P \subset Y$, and $h^\beta: Y^\beta \to X$ is the induced projection.

Proof. Let T be the support on $W \times_M P$ of the homology of $\tilde{G}_.$. There is a natural projection from T to $X \times \mathbf{P}^1$, which is an isomorphism over $X \times \mathbf{A}^1$ and such that the inverse image of $X \times \{\infty\}$ is Y. Given any $t \in \mathbf{P}^1$, we write T_t for the inverse image in T of $X \times \{t\}$. Notice that the generic fibre of T is isomorphic to the one of $X \times \mathbf{P}^1$. Let $j_0: P \to W \times_M P$ and $j_\infty: W_\infty \times_M P \to W \times_M P$ be the inclusions corresponding to $\{0\}$ and $\{\infty\}$ in $\mathbf{P}^1$.

The normal bundle of $X(\mathbf{C}) \times \mathbf{P}^1(\mathbf{C})$ in $W(\mathbf{C})$ is isomorphic to $h^*(N_{X(\mathbf{C})/M(\mathbf{C})}(-1))$, where $h: X \times \mathbf{P}^1 \to X$ is the projection; see [BGS3, Sect. 4(d)]. We metrize it by the chosen metric on $h^*(N_{X(\mathbf{C})/M(\mathbf{C})})$ tensored with the standard metric on the tautological line bundle over $\mathbf{P}^1(\mathbf{C})$. By Lemma 23 we know that

$$j_0^*(\widehat{ch}_{W \times_M P}^T(\tilde{G}_.)) = \widehat{ch}_P^X(G_.) = \widehat{ch}_P^X(\bar{\mathscr{F}}) ,$$

and that

$$j_\infty^*(\widehat{ch}_{W \times_M P}^T(\tilde{G})) = \sum_\beta n_\beta h_*^\beta(\widehat{ch}_{Z_\beta \times_M P}^{Y^\beta}(G_.^Z|_{Z_\beta}))$$

(notice that $T \cap (W_\infty \times_M P) = Y$, and $Y \cap (\tilde{M} \times_M P) = \emptyset$). By the method of [BGS3, Sect. 4], we see that

$$\widehat{ch}_P^X(\bar{\mathscr{F}}) - \sum_\beta n_\beta h_*^\beta(\widehat{ch}_{Z_\beta \times_M P}^{Y^\beta}(G.^Z|_\beta))$$

is equal to the integral over $\mathbf{P}^1$ of $ch(\bar{\mathscr{F}}) Td^{-1}(\bar{N}_{T(\mathbf{C})/W(\mathbf{C})}) \log|z|^2$, and this implies that

$$\widehat{ch}_P^X(\bar{\mathscr{F}}) - \sum_\beta n_\beta h_*^\beta(\widehat{ch}_{Z_\beta \times_M P}^{Y^\beta}(G.^Z|_{Z_\beta})) = -a(\widetilde{Td}^{-1}(X, P, M) ch(\bar{\mathscr{F}}))$$

(see also below the proof of Lemma 28). □

Now let us prove Lemma 26. Noting that, for each irreducible component Z_β of Z, $Z_\beta \times_M P = Z_\beta \times_X (X \times_M P)$, we have a Cartesian diagram:

$$\begin{array}{ccc} Z_\beta \times_M P & \xrightarrow{q_\beta} & Z_\beta \\ \downarrow v_\beta & & \downarrow b_\beta \\ X \times_M P & \xrightarrow{p} & X \end{array}$$

On $G.^Z$, $V.$, and $E.^Z$ we choose metrics satisfying condition (A) with respect to our choices of metrics on $\mathscr{F}$ and, before Lemma 27 on the normal bundles involved. For all β the inverse image by v_β of $g(X(\mathbf{C}))$ is transverse in the complex points of $Z_\beta \times_M P$ to the inverse image by q of the zero section $j^Z: X(\mathbf{C}) \to \mathbf{P}(N_\mathbf{C} \oplus 1)$. Therefore, as in [BGS3, Sect. 2.b], the complex $v_\beta^* \bar{V}. \otimes q_\beta^* \bar{E}.^Z$ is a resolution of $(j^Z \times g)_* \mathscr{F}$ satisfying condition (A) for the one component $Z_\beta(\mathbb{C})$ (with multiplicity one) of $Z(\mathbb{C})$ which is non-empty.

Since both of the complexes $G.^Z|_{Z_\beta(\mathbf{C}) \times_M P}$ and $v_\beta^* V \otimes q_\beta^* E.^Z$ are resolutions of the Hermitian vector bundle $\mathscr{F}(\mathbf{C})$, if $Z_\beta(\mathbf{C}) \neq \emptyset$, we have that

$$h_*^\beta \widehat{ch}_{Z_\beta \times_M P}^{Y_\beta}(G.^Z|_{Z_\beta \times_M P}) = h_*^\beta \widehat{ch}_{Z_\beta \times_M P}^{Y_\beta}(v_\beta^* V. \otimes q_\beta^* E.^Z|_{Z_\beta}) .$$

If $Z_\beta(\mathbf{C}) = \emptyset$, the same formula remains true as an identity in $\widehat{CH}.(b_\beta(Z_\beta)) = CH.(b_\beta(Z_\beta))$, simply because the complexes are quasi-isomorphic.

The projection map $\pi = p \circ v = b \circ q_\infty$ from $|Z| \times_M P$ to X is smooth on complex points, and maps $Y(\mathbf{C})$ isomorphically onto $X(\mathbf{C})$ via h. Hence, applying Lemma 21 for each β to the maps $\pi^\beta: Z_\beta \times_M P \to X$ and $h^\beta: Y^\beta = Y \cap (Z_\beta \times_M P) \to X$, and noting that $Z_\beta(F)$ is empty for all but one β, we find that:

$$\sum_\beta n_\beta h_*^\beta(\widehat{ch}_{Z_\beta \times_M P}^{Y_\beta}(v_\beta^* V. \otimes q_\beta^* \bar{E}.^Z|_{Z_\beta}))$$

$$= \sum_\beta n_\beta \pi_*^\beta(\widehat{ch}(v_\beta^* V. \otimes q_\beta^* E.^Z|_{Z_\beta}) \cap [Z_\beta \times_M P]) + a(\pi_*(\widetilde{ch}(v^* \bar{V}. \otimes q_\infty^* \bar{E}.^Z))) .$$

Let us now compute the first term in the above expression, treating each term in the summation individually; nothing that $\pi^\beta = b_\beta \circ q_\beta$, we start with the direct image by q_β. In $\widehat{CH}.(Z_\beta)$, we have, by Theorem 4, Properties 4 and 6, that:

$$\begin{aligned} q_{\beta *}(\widehat{ch}(v_\beta^* V. \otimes q_\beta^* E.^Z|_{Z_\beta}) \cap [Z_\beta \times_M P]) \\ &= q_{\beta *}(\widehat{ch}(q_\beta^* E.^Z|_{Z_\beta}) \cap (\widehat{ch}(v_\beta^* V.) \cap [Z_\beta \times_M P])) \\ &= \widehat{ch}(E.^Z|_{Z_\beta}) \cap q_{\beta *}(\widehat{ch}(v_\beta^* V.) \cap [Z_\beta \times_M P]) . \end{aligned}$$

Now $v_\beta^*(\bar{V}.)$ is a resolution of $s_*(\mathcal{O}_{Z_\beta})$, where $s: Z_\beta \to Z_\beta \times_M P$ is the section of the smooth map $q_\beta: Z_\beta \times_M P \to Z_\beta$ induced by the section $g: X \to X \times_M P$; over the manifold of complex points $Z_\beta \times_M P(\mathbf{C})$, $v_\beta^*(\bar{V}.)$ satisfies condition (A), since it is the pull-back by a submersion between complex manifolds of a complex which satisfies condition (A). Applying Lemma 21 (with $\alpha = 1$) to the diagram

$$\begin{array}{ccc} Z_\beta & \xrightarrow{s} & Z_\beta \times_M P \\ & & \downarrow q_\beta \\ & & Z_\beta \ , \end{array}$$

we find that

$$q_{\beta *}(\widehat{ch}(v_\beta^* \bar{V}.) \cap [Z_\beta \times_M P]) + a(q_{Z_\beta *}\widetilde{ch}(\bar{V}.)) = \widehat{ch}{}^{Z_\beta}_{Z_\beta \times_M P}(\mathcal{O}_{Z_\beta}) \cap [Z_\beta] \ ,$$

which by Lemma 22(i) is equal to

$$\widehat{Td}^{-1}(\bar{N}_{Z_\beta/Z_\beta \times_M P})^{-1} \cap [Z_\beta] = \widehat{Td}^{-1}(b_\beta^* \bar{N}_{X/X \times_M P})^{-1} \cap [Z_\beta] \ .$$

Now recalling that

$$\widehat{ch}{}^X_M(\bar{\mathscr{F}}) = \sum_\beta n_\beta b_*^\beta \widehat{ch}(\bar{E}^Z|_{Z_\beta}) + a(b_* \widetilde{ch}(\bar{E}.^Z)) \ ,$$

together with Lemma 27 and all the equalities we proved after it, we find that:

$$\begin{aligned} \widehat{ch}{}^X_P(\bar{\mathscr{F}}) &= (\widehat{ch}{}^X_M(\bar{\mathscr{F}}) - a(b_* \widetilde{ch}(\bar{E}.^Z)))\widehat{Td}^{-1}(\bar{N}_{X/X \times_M P}) \\ &\quad - a(b_*(ch(\bar{E}.^Z) q_{\infty *} \widetilde{ch}(v^* \bar{V}.))) + a(\pi_* \widetilde{ch}(v^* \bar{V}. \otimes q_\infty^* E.^Z)) \\ &\quad - a(\widetilde{Td}^{-1}(X, P, M) ch(\bar{\mathscr{F}})) \\ &= \widehat{ch}{}^X_M(\bar{\mathscr{F}}) \widehat{Td}^{-1}(\bar{N}_{X/X \times_M P}) - a(b_* \widetilde{ch}(\bar{E}.^Z) \widehat{Td}^{-1}(\bar{N}_{X(\mathbf{C})/X(\mathbf{C}) \times_M P(\mathbf{C})})) \\ &\quad - a(b_*(ch(\bar{E}.^Z) q_{\infty *} \widetilde{ch}(v^* \bar{V}.))) + a(\pi_* \widetilde{ch}(v^* \bar{V}. \otimes q_\infty^* \bar{E}.^Z)) \\ &\quad - a(\widetilde{Td}^{-1}(X, P, M) ch(\bar{\mathscr{F}})) \ . \end{aligned}$$

We may simplify this expression using the following identity, which is a consequence of Theorem 2.7 of [BGS3] together with the projection formula for direct image of currents:

$$\pi_*(\widetilde{ch}(v^* \bar{V}. \otimes q_\infty^* \bar{E}.^Z)) = b_* ch(\bar{E}.^Z) p_* \widetilde{ch}(\bar{V}.) + Td^{-1}(\bar{N}_{X(\mathbf{C})/X(\mathbf{C}) \times_M P(\mathbf{C})}) b_* \widetilde{ch}(\bar{E}.^Z) \ ,$$

to obtain

$$\widehat{ch}{}^X_P(\bar{\mathscr{F}}) = \widehat{ch}{}^X_M(\bar{\mathscr{F}}) \widehat{Td}^{-1}(\bar{N}_{X/X \times_M P}) - a(\widetilde{Td}^{-1}(X, P, M) ch(\bar{\mathscr{F}})) \ .$$

From this equality and the definition of τ, we deduce that

$$\tau_P(\mathscr{F}) = \widehat{ch}_P^X(\mathscr{F})._k\widehat{Td}(P) + a(ch(\bar{\mathscr{F}})\widetilde{Td}(X/P))$$

$$= (\widehat{ch}_M^X(\mathscr{F}) \cap \widehat{Td}^{-1}(\bar{N}_{X/X\times P}))._k\widehat{Td}(P)$$

$$- a(ch(\bar{\mathscr{F}})\widetilde{Td}^{-1}(X, P, M)k^*Td(\bar{T}_{P(\mathbf{C})}) + ch(\bar{\mathscr{F}})\widetilde{Td}(X/P)) .$$

But the normal bundle of X in $X \times_M P$ coincides with $k^* T_{P/M} = k^* T_q$. Therefore, by applying Proposition 1 ii) to the map q and the defining map of Y, and Theorem 3, Property 3, for $j = qk$ we get, from the equation above,

$$\tau_P(\mathscr{F}) = \widehat{ch}_M^X(\mathscr{F})._j\widehat{Td}(M) + a(ch(\bar{\mathscr{F}})x) ,$$

where

$$x = -\widetilde{Td}(0 \to k^* T_{P(\mathbf{C})/M(\mathbf{C})} \to k^* T_{P(\mathbf{C})} \to j^* T_{M(\mathbf{C})} \to 0)$$

$$\cdot Td^{-1}(k^*\bar{T}_{P(\mathbf{C})/M(\mathbf{C})})Td^{-1}(\bar{N}_{X(\mathbf{C})/M(\mathbf{C})})$$

$$+ \widetilde{Td}(X/P) - \widetilde{Td}^{-1}(X, P, M) \cap k^* Td(\bar{T}_{P(\mathbf{C})}) .$$

From Lemma 16 we know that $x = \widetilde{Td}(X/M)$, and therefore

$$\tau_P(\mathscr{F}) = \tau_M(\mathscr{F}) . \qquad \square$$

From Lemma 26 we may now prove Theorem 5. First we see that $\tau_P(\bar{\mathscr{F}})$ is independent of the embedding $k: X \to P$ for P smooth (and integral). Indeed, given $k: X \to P$ and $j: X \to M$ two embeddings of X into smooth varieties, we consider the product embedding $i: X \to P \times M$ and apply the lemma to the two projections from the product.

In general, given a closed embedding $j: X \to M$ with M regular and integral, we can choose a closed embedding $f: M \to P$ with P smooth and integral. Let $N = N_{M/P}$ be the normal bundle of M in P and $s: M \to \mathbf{P}(N \oplus 1)$ the zero section. Note that $\mathbf{P}(N \oplus 1)$ is regular, and that the projection $q: \mathbf{P}(N \oplus 1) \to M$ is smooth. Hence by Lemma 26 $\tau_M(\bar{\mathscr{F}}) = \tau_{\mathbf{P}(N\oplus 1)}(\bar{\mathscr{F}})$, where we embed X into $\mathbf{P}(N \oplus 1)$ via $s \circ j$. Hence to end the proof of Theorem 5, it suffices to prove:

Lemma 28 *With the notation above,*

$$\tau_P(\bar{\mathscr{F}}) = \tau_{\mathbf{P}(N\oplus 1)}(\bar{\mathscr{F}}) .$$

Proof. Choose metrics on the normal bundles of $X(\mathbf{C})$ in $P(\mathbf{C})$ and $M(\mathbf{C})$, and on N. The normal bundle of $X(\mathbf{C})$ in $\mathbf{P}(N \oplus 1)(\mathbf{C})$ is the direct sum of $j^*(N_\mathbf{C})$ with $N_{X(\mathbf{C})/M(\mathbf{C})}$; we endow it with the orthogonal sum of their metrics. Given the formula for τ_P in terms of the Chern character with supports it then suffices to compare $\widehat{ch}_P^X(\bar{\mathscr{F}})$ with $\widehat{ch}_{\mathbf{P}(N\oplus 1)}^X(\bar{\mathscr{F}})$. By Lemma 23 the Chern character with supports is compatible with restriction to principal divisors, so, by an argument similar to [GS2, 4.4.6] (see also [BGS3 4.12]), we know that the difference of these two classes is the integral over $\mathbf{P}^1$ of $ch(\bar{\mathscr{F}})Td^{-1}(\bar{N}_{(X\times\mathbf{P}^1)(\mathbf{C})/W(\mathbf{C})})\log|z|^2$, where W is the deformation to the normal cone for the inclusion of M into P, and $X\times\mathbf{P}^1\to W$ is the natural inclusion. The bundle $N_{(X\times\mathbf{P}^1)(\mathbf{C})/W(\mathbf{C})}$ is an extension of $N_{(M\times\mathbf{P}^1)(\mathbf{C})/W(\mathbf{C})}$ by $N_{X(\mathbf{C})/M(\mathbf{C})}$ which coincides with $N_{X(\mathbf{C}),P(\mathbf{C})}$ over $X(\mathbf{C})\times 0$, and with the normal bundle of $X(\mathbf{C})$ in $\mathbf{P}(N\oplus 1)(\mathbf{C})$ over $X(\mathbf{C})\times\{\infty\}$. Therefore, by [GS3, 1.2.3], we get

$$\widehat{ch}_P^X(\bar{\mathscr{F}})-\widehat{ch}_{\mathbf{P}(N\oplus 1)}^X(\bar{\mathscr{F}})$$
$$=-a(ch(\bar{\mathscr{F}})\widetilde{Td^{-1}}(0\to N_{X(\mathbf{C})/M(\mathbf{C})}\to N_{X(\mathbf{C})/P(\mathbf{C})}\to j^*(N_{\mathbf{C}})\to 0)).$$

Similarly, since $\bar{T}_{\mathbf{P}(N\oplus 1)(\mathbf{C})}$ is the orthogonal direct sum of $\bar{T}_{M(\mathbf{C})}$ with $\bar{N}_{\mathbf{C}}$, we get, on M,

$$f^*(\widehat{Td}(P))=s^*(\widehat{Td}(\mathbf{P}(N\oplus 1))-a(\widetilde{Td}(0\to\bar{T}_{M(\mathbf{C})}\to f^*(\bar{T}_{P(\mathbf{C})})\to\bar{N}_{\mathbf{C}}\to 0)).$$

From these two equalities and Definition 7 we get, with $i=f\circ j$,

$$\tau_P(\bar{\mathscr{F}})=\widehat{ch}_P^X(\bar{\mathscr{F}})_i\widehat{Td}(P)+a(ch(\bar{\mathscr{F}})\widetilde{Td}(X/P))=\tau_{\mathbf{P}(N\oplus 1)}(\bar{\mathscr{F}})+a(ch(\bar{\mathscr{F}})y)$$

where

$$y=\widetilde{Td}(X/P)-\widetilde{Td^{-1}}(0\to N_{X(\mathbf{C})/M(\mathbf{C})}\to N_{X(\mathbf{C})/P(\mathbf{C})}\to j^*N_{\mathbf{C}}\to 0)k^*Td(\bar{T}_{P(\mathbf{C})})$$
$$-\widetilde{Td}(X/\mathbf{P}(N\oplus 1))$$
$$-j^*(\widetilde{Td}(0\to\bar{T}_{M(\mathbf{C})}\to f^*(\bar{T}_{P(\mathbf{C})})\to\bar{N}_{\mathbf{C}}\to 0))Td^{-1}(\bar{N}_{X(\mathbf{C})/\mathbf{P}(N\oplus 1)(\mathbf{C})}).$$

The fact that $y=0$ follows from the remark that $\widetilde{Td}(X/\mathbf{P}(N\oplus 1))=\widetilde{Td}(X/M)$ and from Lemma 14 (i'). □

3.2.2 Having shown in Theorem 5 that $\tau(\bar{\mathscr{F}})$ is independent of choices, we may now give a few properties of this class.

Theorem 6 (i) *There exists a canonical isomorphism of* $\mathbf{Q}$*-vector spaces*

$$\tau:\hat{K}'_0(X)_{\mathbf{Q}}\to\widehat{CH}_{\cdot}(X)_{\mathbf{Q}}$$

mapping the class of $(\bar{\mathscr{F}},\eta)$ *to* $\tau(\bar{\mathscr{F}})+a(\eta)$.

(ii) *For any* $x\in\hat{K}'_0(X)$ *and* $y\in\hat{K}_0(X)$, *one has*

$$\tau(x\cap y)=\tau(x)\cap\widehat{ch}(y).$$

(iii) *When* X *is regular, for any* $x\in\hat{K}_0(X)=\hat{K}'_0(X)$,

$$\tau(x)=\widehat{ch}(x)\widehat{Td}(X)$$

in $\widehat{CH}^{\cdot}(X)_{\mathbf{Q}}=\widehat{CH}_{\cdot}(X)_{\mathbf{Q}}$.

Proof. The fact that τ is well defined on $\hat{K}'_0(X)$ follows from Corollary 4. To show (i), consider the diagram

$$\begin{array}{ccccccccc} K'_1(X)_{\mathbb{Q}} & \to & \bigoplus_{p\geqq 0}\tilde{A}^{p,p}(X) & \to & \hat{K}'_0(X)_{\mathbb{Q}} & \to & K'_0(X)_{\mathbb{Q}} & \to & 0 \\ \downarrow\tau & & \downarrow \mathrm{id} & & \downarrow\tau & & \downarrow\tau & & \\ \bigoplus_{p\geqq 0} CH_{p,p+1}(X)_{\mathbb{Q}} & \to & \bigoplus_{p\geqq 0}\tilde{A}^{p,p}(X) & \to & \widehat{CH}.(X)_{\mathbb{Q}} & \to & CH.(X)_{\mathbb{Q}} & \to & 0\,. \end{array}$$

The algebraic maps τ on $K'_1(X)_{\mathbb{Q}}$ and $K'_0(X)_{\mathbb{Q}}$ can be defined using the Chern character with supports from K-theory to the graded quotients of its γ-filtration [S1, Theorem 4, 7.1., and Theorem 8], mimicking [G1, Theorem 4.1]; they are isomorphisms. The rows in the above diagram are exact (for the top row, proceed as in [GS3, Theorem 6.2(i)]; notice that any coherent sheaf on X has finite resolution by coherent sheaves which are locally free on X_F). Furthermore, the diagram commutes: this follows from the definitions and, for the left hand square, where it can be checked on the generic fibre, from [GS3, 7.2.1]. By the five Lemma we conclude that

$$\tau: \hat{K}'_0(X)_{\mathbb{Q}} \to \widehat{CH}.(X)_{\mathbb{Q}}$$

is an isomorphism.

The module propery (iii) follows from Lemma 20, Property 1 in Theorem 4 and Property 2 in Theorem 3.

To prove (iii), we apply Definition 7 to the case of the identity map $X \to X$ and Lemma 22(ii). □

4 Riemann–Roch

4.1 The statement

4.1.1 Let X be a smooth projective complex variety of complex dimension d and h_X an Hermitian metric on the holomorphic tangent bundle TX over X, satisfying the Kähler condition $d\omega_0 = 0$, where ω_0 is the normalized Kähler form attached to h_X. In any local holomorphic coordinate chart (z_α) on X we have

$$\omega_0 = \sum_{\alpha,\beta} \frac{i}{2\pi} h_X\left(\frac{\partial}{\partial z_\alpha}, \frac{\partial}{\partial z_\beta}\right) dz_\alpha d\bar{z}_\beta\,. \tag{10}$$

Let $\bar{E}$ be an Hermitian vector bundle on X, $q \geqq 0$ an integer, and

$$A^{0q}(X,E) = A^{0q}(X) \underset{C^\infty(X)}{\otimes} C^\infty(X,E)$$

the vector space of forms of type $(0,q)$ on X with values in E. Since

$$A^{0q}(X) = C^\infty(X, \textstyle\bigwedge^q(\overline{TX}^*))\,,$$

where $\overline{TX}^*$ is the dual of the complex conjugate of the complex vector bundle TX, it inherits from h_X a pointwise scalar product with values in $C^\infty(X)$. By tensoring with the metric on E, we get a pointwise scalar product on $A^{0q}(X,E)$. The L^2 scalar product of two sections s and t in $A^{0q}(X,E)$ is defined by the formula

$$\langle s,t\rangle_{L^2} = \int_X \langle s(x), t(x)\rangle \frac{\omega_0^d}{d!}$$

where d is the complex dimension of X and $\langle s(x), t(x)\rangle$ is the pointwise scalar product. The Cauchy–Riemann operator

$$\bar{\partial}: A^{0q}(X, E) \to A^{0,q+1}(X, E)$$

has a (formal) adjoint $\bar{\partial}^*$:

$$\langle s, \bar{\partial} t\rangle_{L^2} = \langle \bar{\partial}^* s, t\rangle_{L^2}$$

when $s \in A^{0,q+1}(X, E)$ and $t \in A^{0,q}(X, E)$.

Consider the Laplace operator $\Delta_q = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$ on $A^{0q}(X, E)$. The cohomology $H^q(X, E)$ may be computed using the Dolbeault resolution on X, and, by Hodge theory, it is isomorphic to the subspace $\operatorname{Ker}(\Delta_q)$ of harmonic forms in $A^{0q}(X, E)$. We may therefore endow it with the L^2 scalar product. Let h_{L^2} denote the induced scalar product on the complex line

$$\lambda(E) = \bigoplus_{q \geqq 0} \det H^q(X, E)^{(-1)^q}.$$

(If V is a complex vector space, $\det(V)$ is its maximal exterior power, and if L is a complex line, L^{-1} is its dual.)

The zeta function of Δ_q

$$\zeta_q(s) = \operatorname{Tr}(\Delta_q^{-s} | \operatorname{Ker}(\Delta_q)^\perp)$$

is convergent when $\operatorname{Re}(s) > d$. It can be analytically continued to the whole complex plane and is regular at the origin. Following Ray and Singer [RS] we define the analytic torsion

$$T(X, E) = \sum_{q \geqq 0} (-1)^{q+1} q \zeta_q'(0),$$

where $\zeta_q'(0)$ is the derivative of $\zeta_q(s)$ at the origin. On $\lambda(E)$, Quillen considered the metric

$$h_Q = h_{L^2} \exp(T(X, E)).$$

Notice that these definitions are those in [GS4] and [BL], whose results will be used below.

4.1.2 Let $f: X \to Y$ be a projective morphism of arithmetic varieties over an arithmetic ring A. Assume that the restriction of f to the generic fibre X_F is smooth. On the relative tangent bundle $Tf_{\mathbb{C}}$ choose an Hermitian metric h_f whose restriction to every fibre $X_y = f^{-1}(y)$, $y \in Y(\mathbb{C})$, is Kähler.

Let $\bar{\mathscr{F}}$ be an Hermitian coherent sheaf on X. Assume that $Rf_*\mathscr{F}$ is a perfect complex on Y. According to Grothendieck and Knudsen–Mumford [KM], one may then define a (graded) line bundle

$$\lambda(\mathscr{F}) = \det Rf_*(\mathscr{F})$$

on Y, called the determinant of cohomology. For every $y \in Y(\mathbb{C})$, the fiber $\lambda(\mathscr{F})_y$ is canonically isomorphic to

$$\bigotimes_{q \geqq 0} \det H^q(X_y(\mathbb{C}), \mathscr{F}_{\mathbb{C}})^{(-1)^q}$$

and can be given the Quillen metric h_Q. It was shown in [BGS1, Corollary 3.9], that this metric h_Q is smooth on the line bundle $\lambda(\mathscr{F})_{\mathbb{C}}$.

We shall consider two cases where $Rf_* \mathscr{F}$ is perfect:
(i) Y is regular;
(ii) f is l.c.i. and $\mathscr{F} = F$ is locally free.
In case (i), any coherent sheaf on Y has a finite resolution by locally free coherent sheaves, and $R^q f_* \mathscr{F}$ is coherent for all $q \geqq 0$. Therefore (see [KM]) $Rf_* \mathscr{F}$ is perfect and

$$\lambda(\mathscr{F}) = \bigotimes_{q \geqq 0} \det R^q f_*(\mathscr{F})^{(-1)^q} .$$

In Case (ii) we may view F as a perfect complex and we notice that f is a perfect morphism [GBI, Exposé 3, Exemple 4.1.1]).
We shall omit the sign questions in the definition of $\lambda(\mathscr{F})$ (see [KM]) since they play no role in what follows.

4.1.3 Let $H^{\cdot}(X(\mathbb{C}), \mathbb{R})$ be the (singular) real cohomology of $X(\mathbb{C})$, and $ch(\mathscr{F}_{\mathbb{C}})$, $ch(E_{\mathbb{C}})$, $Td(f_{\mathbb{C}})$ the usual Chern character class of $\mathscr{F}_{\mathbb{C}}$ (resp. $E_{\mathbb{C}}$, resp. the Todd class of $Tf_{\mathbb{C}}$) in $H^{\cdot}(X(\mathbb{C}), \mathbb{R})$. We shall also consider the characteristic class $R(f_{\mathbb{C}}) = R(Tf_{\mathbb{C}}) \in H^*(X(\mathbb{C}), \mathbb{R})$ introduced in [GS4]. Namely, the class R is contravariant, additive on exact sequences, and, when $L_{\mathbb{C}}$ is a holomorphic line bundle with first Chern class $c_1(L_{\mathbb{C}}) \in H^2(X(\mathbb{C}), \mathbb{R})$, the following identity holds in the real cohomology of $X(\mathbb{C})$:

$$R(L_{\mathbb{C}}) = \sum_{\substack{m \text{ odd} \\ m \geqq 1}} \left(2\zeta'(-m) + \left(1 + \frac{1}{2} + \ldots + \frac{1}{m} \right) \zeta(-m) \right) \frac{c_1(L_{\mathbb{C}})^m}{m!} .$$

Here $\zeta(s)$ is the Riemann zeta function, and $\zeta'(s)$ its derivative. We shall consider the image of $R(f_{\mathbb{C}})$ by the map

$$a : H^{\cdot}(X(\mathbb{C}), \mathbb{R}) \to \widehat{CH}_{\cdot}(X)$$

defined as in 2.2.1 (real cohomology maps to $\tilde{A}^{\cdot}(X(\mathbb{C}))$).

Given $x \in \widehat{CH}^{\cdot}(X)$ (resp. $x \in CH_{\cdot}(X)$) we denote by $x^{(p)} \in \widehat{CH}^p(X)$ (resp. $x_{(p)} \in \widehat{CH}_p(X)$) its component of degree p.

Finally, given any Hermitian metrics on $T_{X(\mathbb{C})}$ and $T_{Y(\mathbb{C})}$, we let $\widetilde{Td}(f) \in \tilde{A}^{\cdot}(X_{\mathbb{R}})$ be the Bott–Chern secondary Todd class (2.6.1) of the exact sequence of Hermitian vector bundles on $X(\mathbb{C})$

$$0 \to Tf_{\mathbb{C}} \to T_{X(\mathbb{C})} \to f^* T_{Y(\mathbb{C})} \to 0 .$$

4.1.4

Theorem 7 Let $f : X \to Y$ be a morphism of arithmetic varieties whose restriction to X_F is smooth. On the relative tangent bundle $Tf_{\mathbb{C}}$ choose an Hermitian metric whose restriction to any fibre of f is Kähler.

(i) Assume that the ground ring is good and that Y is regular. Let $\bar{\mathscr{F}}$ be an Hermitian coherent sheaf on X. Then, for any choice of Hermitian metrics on $T_{X(\mathbb{C})}$ and $T_{Y(\mathbb{C})}$, the following identity holds in $\widehat{CH}^1(Y)_{\mathbb{Q}}$

$$(11) \quad \tau(\lambda(\mathscr{F}), h_Q)^{(1)} = f_*(\tau(\bar{\mathscr{F}}) + a(ch(\bar{\mathscr{F}}_{\mathbb{C}})\widetilde{Td}(f)) - a(ch(\mathscr{F}_{\mathbb{C}})Td(f_{\mathbb{C}})R(f_{\mathbb{C}})))^{(1)} .$$

(ii) Assume f is l.c.i. and let $\bar{F}$ be an Hermitian vector bundle on X. When the ground ring is good, given any element $\alpha \in \widehat{CH}_p(Y)_{\mathbb{Q}}$, $p \geqq 1$, the following identity holds in $\widehat{CH}_{p-1}(Y)_{\mathbb{Q}}$:

(12)
$$\hat{c}_1(\lambda(F), h_Q) \cap \alpha = f_*((\widehat{ch}(\bar{F})\widehat{Td}(f) - a(ch(F_{\mathbb{C}})Td(f_{\mathbb{C}})R(f_{\mathbb{C}}))) \cap f^*(\alpha))_{(p-1)}.$$

Similarly, when X and Y are regular (and the ground ring is arbitrary), the identity

(13)
$$\hat{c}_1(\lambda(F), h_Q) = f_*(\widehat{ch}(\bar{F})\widehat{Td}(f) - a(ch(F_{\mathbb{C}})Td(f_{\mathbb{C}})R(f_{\mathbb{C}})))^{(1)}$$

holds in $\widehat{CH}^1(Y)_{\mathbb{Q}}$.

For the definitions of $\widehat{ch}(\bar{F})$ and $\widehat{Td}(f)$ see [GS3], 2.4.2 and 2.6.2.

4.1.5 Notice that when X and Y are regular, the ground ring is good, f is l.c.i. and $\mathscr{F} = F$, the three statements in Theorem 7 are equivalent (by 2.2.4. and Theorem 6(iii)).

It is interesting to consider Theorem 7(i) when $Y = \operatorname{Spec}(\mathbb{Z})$, in which case one gets an arithmetic analog of the Riemann–Roch theorem of Hirzebruch [Hz], which is an equality of real numbers rather than integers. So let $\bar{\mathscr{F}}$ be an Hermitian coherent sheaf on an arithmetic variety X. Denote by $\sharp S$ the cardinal of a finite set S, and by $H^q(X, \mathscr{F})_{\text{tors}}$ the torsion subgroup of $H^q(X, \mathscr{F})$. Define

(14)
$$\chi_Q(\bar{\mathscr{F}}) = \sum_{q \geqq 0} (-1)^q (\log \sharp H^q(X, \mathscr{F})_{\text{tors}} - \log \operatorname{vol}(H^q(X(\mathbb{C}) \mathscr{F}_{\mathbb{C}})^+ / H^q(X, \mathscr{F})) + \tfrac{1}{2} q \zeta_q'(0)),$$

where $\zeta_q'(0)$ is defined as in 4.1.1, $H^q(X(\mathbb{C}), \mathscr{F}_{\mathbb{C}})^+$ is the subspace of $H^q(X(\mathbb{C}), \mathscr{F}_{\mathbb{C}})$ fixed by F_∞, and $\operatorname{vol}(H^q(X(\mathbb{C}), \mathscr{F}_{\mathbb{C}})^+ / H^q(X, \mathscr{F}))$ the volume of its quotient by the lattice $H^q(X, \mathscr{F}) / H^q(X, \mathscr{F})_{\text{tors}}$ for the volume form attached to the L^2-metric.

Then Theorem 7(i) reads

(15)
$$\chi_Q(\bar{\mathscr{F}}) = f_*(\tau(\bar{\mathscr{F}}) - a(ch(\mathscr{F}_{\mathbb{C}})Td(f_{\mathbb{C}})R(f_{\mathbb{C}})))^{(1)}$$

in $\widehat{CH}^1(\operatorname{Spec}(\mathbb{Z})) = \mathbb{R}$ (in this identification, the codimension one cycle $(\sum n_p[p], \lambda)$ on $\operatorname{Spec}(\mathbb{Z})$ is mapped to $\sum_p n_p \log(p) + \lambda/2$ for any finite set of integers n_p and any real number λ).

Indeed $\tau(\lambda(\mathscr{F}), h_Q)^{(1)} = \chi_Q(\bar{\mathscr{F}})$ since the tangent space to Y is trivial. Furthermore $\chi_Q(\bar{\mathscr{F}})$ coincides with $\hat{c}_1(\lambda(\mathscr{F}), h_Q)$ by the definition of Quillen's metric and the fact that for any Hermitian coherent sheaf $\bar{\mathscr{F}}$ on $Y = \operatorname{Spec}(\mathbb{Z})$ (i.e. a finitely generated $\mathbb{Z}$-module with metric), the class of $\det(\bar{\mathscr{F}})$ in $\widehat{CH}^1(\operatorname{Spec}(\mathbb{Z})) = \mathbb{R}$ is equal to

$$\log \sharp(\mathscr{F}_{\text{tors}}) - \log \operatorname{vol}((\mathscr{F}_{\mathbb{C}})^+ / \mathscr{F})$$

(to prove this one just needs to check the case where $\mathscr{F}$ is torsion-free and the case where $\mathscr{F}$ is a finite cyclic group).

4.1.6. In order to make the analogy between $\chi_Q(\bar{\mathscr{F}})$ and the Euler characteristic of a vector bundle more explicit, one may proceed as follows. Let

$B^q = B^q(\mathscr{F}_{\mathbb{C}}) \subset A^{0q}(X(\mathbb{C}), \mathscr{F}_{\mathbb{C}})$ be the image of $\bar{\partial}$, and $\zeta'_{B^q}(0)$ the derivative at zero of the zeta function

$$\zeta_{B^q}(s) = \mathrm{Tr}(\Delta_q^{-s} \mid B^q)$$

of the restriction of Δ_q to B^q. Since B^q is isomorphic to $\bar{\partial}^*(A^{0,q+1}(X(\mathbb{C}), \mathscr{F}_{\mathbb{C}}))$, one has [RS]

(16)
$$\sum_{q \geqq 0} (-1)^q q \zeta'_q(0) = \sum_{q \geqq 0} (-1)^q \zeta'_{B^q}(0) .$$

On the other hand, following [GS5], for any finitely generated lattice M, equipped with a norm $\|\cdot\|$ on $M \otimes_{\mathbb{Z}} \mathbb{R}$, we define

$$h^0(M, \|\cdot\|) = \log \sharp \{m \in M, \|m\| \leqq 1\} ,$$

and

$$h^1(M, \|\cdot\|) = h^0(M^*, \|\cdot\|^*) ,$$

where $M^* = \mathrm{Hom}(M, \mathbb{Z})$ is equipped with the dual norm $\|\cdot\|^*$. If $n = rk(M)$ is the dimension of $M \otimes_{\mathbb{Z}} \mathbb{Q}$, it was proved in [GS5, Theorem 2], that

$$|h^0(M, \|\cdot\|) - h^1(M, \|\cdot\|) - \log \sharp M_{\mathrm{tors}} + \log \mathrm{vol}(M \otimes \mathbb{R}/M)|$$

is bounded above by an explicit constant $C(n)$. It follows from this and (16) that, if we define the "arithmetic Betti numbers" of $\bar{\mathscr{F}}$ by the formula

(17)
$$b_q(\bar{\mathscr{F}}) = h^0(H^q(X, \mathscr{F}), \|\cdot\|_{L^2}) + h^1(H^{q-1}(X, \mathscr{F}), \|\cdot\|_{L^2}) + \tfrac{1}{2}\zeta'_{B^q}(0)$$

we have

$$\left| \chi_{\mathrm{Q}}(\bar{\mathscr{F}}) - \sum_{q \geqq 0} (-1)^q b_q(\bar{\mathscr{F}}) \right| \leqq \sum_{q \geqq 0} C(n_q)$$

where $n_q = \dim_{\mathbb{C}} H^q(X(\mathbb{C}), \mathscr{F}_{\mathbb{C}})$ is purely topological.

Furthermore, these numbers $b_q(\bar{\mathscr{F}})$ behave well under Serre duality. Namely, if ω_X is the relative dualizing sheaf of X over $\mathbb{Z}$ (recall that $X \to \mathrm{Spec}(\mathbb{Z})$ is l.c.i), equipped with the metric coming from the isomorphism $\omega_{X,\mathbb{C}} \simeq \bigwedge^d(TX(\mathbb{C})^*)$, and if we let $\bar{\mathscr{F}}^\vee = \bar{\omega}_X \otimes \bar{\mathscr{F}}^*$, then, if $\mathscr{F}$ is locally free,

(18)
$$b_{d+1-q}(\bar{\mathscr{F}}^\vee) = b_q(\bar{\mathscr{F}}) .$$

To prove (18), we first notice that algebraic relative Serre–Grothendieck duality applied to f implies that

(19)
$$\sharp H^q(X, \mathscr{F})_{\mathrm{tors}} = \sharp H^{d+1-q}(\mathscr{F}^\vee)_{\mathrm{tors}} ,$$

and that furthermore the lattice $H^q(X, \mathscr{F})/H^q(X, \mathscr{F})_{\mathrm{tors}}$ is dual to $H^{d-q}(\mathscr{F}^\vee)/H^{d-q}(\mathscr{F}^\vee)_{\mathrm{tors}}$. On the other hand, the analytic Serre duality on $X(\mathbb{C})$ between the Dolbeault resolutions of $\mathscr{F}_{\mathbb{C}}$ and $\mathscr{F}^\vee_{\mathbb{C}}$ (see also [GS4, 1.4]) respects the L^2-metrics and induces isomorphisms

$$B^q(\mathscr{F}_{\mathbb{C}}) \simeq \bar{\partial}^*(A^{0,d+1-q}(\mathscr{F}^\vee_{\mathbb{C}})) \simeq \bar{\partial}(A^{0,d-q}(\mathscr{F}^\vee_{\mathbb{C}}))$$

compatible with the action of Laplace operators. Therefore

$$\zeta'_{B^q(\mathscr{F}_{\mathbb{C}})}(0) = \zeta'_{B^{d+1-q}}(\mathscr{F}^\vee_{\mathbb{C}}(0) ,$$

and (18) follows from this, (16), (17) and (19).

Notice that $h^0(M, \|\cdot\|)$ and $h^1(M, \|\cdot\|)$ are nonnegative real numbers. It would be of interest to find lower bounds for $\zeta'_{B^q}(0)$, $q \geqq 0$. We conjecture that such lower bounds exist, which do not depend on the metric on E.

4.1.7 Let us check, following [GS4, Theorem 2.1.1], an example of Theorem 7. We first note that if X is a Riemann surface and h is a Hermitian metric on the holomorphic tangent bundle T_X of X, then with the definition of 4.1.1 above, the Laplacian on functions is given by

$$\Delta_0(f) = -\left(h\left(\frac{\partial}{\partial z}, \frac{\partial}{\partial z}\right)\right)^{-1} \frac{\partial^2 f}{\partial z\, \partial \bar{z}} \tag{20}$$

in any holomorphic coordinate chart with coordinate z. We now consider the case of $Y = \mathrm{Spec}(\mathbb{Z})$, $X = \mathbb{P}^1_{\mathbb{Z}}$, and $\bar{F}$ equal to the trivial Hermitian line bundle $\bar{\mathcal{O}}_{\mathbb{P}^1}$ on $\mathbb{P}^1_{\mathbb{Z}} = \mathrm{Proj}(\mathbb{Z}[X, Y])$. We fix the Hermitian metric on $T\mathbb{P}^1(\mathbb{C})$ by requiring it to be invariant under the unitary group $U(2)$, and requiring that on the affine line $\mathbb{A}^1_{\mathbb{Z}} = \mathrm{Spec}(\mathbb{Z}[z]) \subset \mathbb{P}^1_{\mathbb{Z}}$, where $z = X/Y$, $\dfrac{\partial}{\partial z}$ have norm 1 at the origin. Then, by (10), the restriction of the Kähler form to the z-plane is

$$\omega_0 = \frac{i}{2\pi} \frac{dz\, d\bar{z}}{(1 + |z|^2)^2}\,.$$

Notice that this gives $\mathbb{P}^1(\mathbb{C})$ volume one. By (20) the Laplacian on $A^{0,0}(\mathbb{P}^1(\mathbb{C}))$ is

$$\Delta_0 = -(1 + |z|^2)^2 \frac{\partial^2}{\partial z\, \partial \bar{z}}$$

on $\mathbb{A}^1_{\mathbb{Z}}(\mathbb{C}) \simeq \mathbb{C}$. The eigenvalues of Δ are $k(k+1)$, for $k = 0, 1, 2, \ldots$, with multiplicity $2k+1$. For instance $\dfrac{1 - z\bar{z}}{1 + z\bar{z}}, \dfrac{z}{1 + z\bar{z}}$ and $\dfrac{\bar{z}}{1 + z\bar{z}}$ are independent eigenvectors with eigenvalue 2. From [V, Proposition 3.1], it follows that

$$\zeta'_{\Delta_0}(0) = 4\zeta'(-1) - 1/2\,.$$

The L^2-norm of the trivial section of $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1})$ is $\int_{\mathbb{P}^1(\mathbb{C})} \omega_0 = 1$. Hence

$$\chi_Q(\bar{\mathcal{O}}_{\mathbb{P}^1}) = 1/4 - 2\zeta'(-1)\,. \tag{21}$$

Now the right-hand side of (15) is equal to

$$f_*(\hat{c}_1(\mathbb{P}^1_{\mathbb{Z}})^2)/12 - (2\zeta'(-1) + \zeta(-1)) = (f_*(\hat{c}_1(\mathbb{P}^1_{\mathbb{Z}})^2) + 1)/12 - 2\zeta'(-1)\,. \tag{22}$$

We know that there is an isomorphism of invertible sheaves on $\mathbb{P}^1_{\mathbb{Z}}$:

$$\phi : T\mathbb{P}^1_{\mathbb{Z}} \simeq \mathcal{O}_{\mathbb{P}^1}(2)\,.$$

Since the Hermitian metrics on these sheaves are $U(2)$ invariant, and the isomorphism ϕ is $\mathrm{SL}_2(\mathbb{Z})$-invariant, the norm of ϕ is constant, equal to its value at the origin. But at the origin ϕ induces an isomorphism of the free rank one $\mathbb{Z}$-modules generated by $\frac{\partial}{\partial z}$ and by Y^2, both of which have norm 1, hence ϕ is an isometry. It follows that

$$\hat{c}_1(\mathbb{P}^1_{\mathbb{Z}}) = 2\hat{c}_1(\overline{\mathcal{O}(1)}) .$$

As shown in [GS3, 5.4], if $f: \mathbb{P}^1_{\mathbb{Z}} \to \mathrm{Spec}(\mathbb{Z})$ is the projection, then

$$f_*(\hat{c}_1(\mathcal{O}(1))^2) = 1/2 \in \widehat{CH}^1(\mathrm{Spec}(\mathbb{Z})) \simeq \mathbb{R} , \tag{23}$$

hence $f_*(\hat{c}_1(\mathbb{P}^1_{\mathbb{Z}})^2) = 2$. From (21), (22), (23) we see that Theorem 7 holds for $\mathbb{P}^1_{\mathbb{Z}}$ and the trivial line bundle.

4.2 The proof

4.2.1 Being quasi-projective, X is contained in a projective space $\mathbb{P}^N$, and we let $i: X \to Y \times \mathbb{P}^N = P$ be the product of this embedding with the map f. We get a factorization $f = g \circ i$, where g is the first projection:

$$\begin{array}{ccc} X & \xrightarrow{i} & P \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle g} & \\ & Y . & \end{array}$$

The map i is a closed immersion (since f is proper) and g is smooth.

In case (i), since Y is regular, so is P, and we may choose a resolution

$$0 \to E_k \to E_{k-1} \to \ldots \to E_0 \to i_*\mathcal{F} \to 0 \tag{24}$$

of $i_*\mathcal{F}$ by vector bundles on P. In case (ii) such a resolution also exists for i_*F, since i is l.c.i. and P is quasi-projective (see 1.2.2).

Let us choose a Kähler metric on $P(\mathbb{C})$. We may restrict this metric to $X(\mathbb{C})$, $Tf_{\mathbb{C}}$ and the normal bundle $N_{\mathbb{C}} = N_{X(\mathbb{C})/P(\mathbb{C})}$. Now choose a metric on each bundle E_j, $j \geqq 0$, in such a way that hypothesis (A) is satisfied (see 2.4.1). It will be enough to prove that the theorem holds with this choice of metric on $Tf_{\mathbb{C}}$, since, as shown in [GS4, Theorem 1.4(i)], based on [BGS1], this will imply that Theorem 7 is true for any choice of metric on $Tf_{\mathbb{C}}$.

We shall deduce Theorem 7 in two steps:

1) we prove that Theorem 7 for g and $\bar{E}_j, j \geqq 0$, implies Theorem 7 for $\bar{\mathcal{F}}$ (or $\bar{F}$) and f;
2) we prove that Theorem 7 holds for g and any Hermitian vector bundle on P.

4.2.2 Let us prove 1) in case (i). According to [KM] the resolution (24) induces a canonical isomorphism

$$\sigma: \lambda(\mathcal{F}) \xrightarrow{\sim} \bigoplus_{j \geqq 0} \lambda(E_j)^{(-1)^j} .$$

The norm of this isomorphism for the Quillen metric was computed by Bismut and Lebeau [BL]. Let $ch(E.) = \sum_{j \geqq 0} (-1)^j ch(E_j)$ be the Chern character of $E.$ in the real cohomology of $P(\mathbb{C})$, $\widetilde{ch}(E.) \in \tilde{A}^{\cdot}(P(\mathbb{C}))$ the torsion of the resolution

$\underset{\sim}{E}$ as defined in 2.4.1, $Td(\bar{g}_{\mathbb{C}})$ the Todd form of $\overline{Tg_{\mathbb{C}}}$ in $A^{\cdot}(P(\mathbb{C}))$, and $\widetilde{Td}(f/g)) \in \tilde{A}^{\cdot}(X(\mathbb{C}))$ as in 2.6.2. Then, according to [BL, Theorem 0.1],

$$\log \| \sigma \|_{\mathrm{Q}}^{2} = [f_*(ch(\mathscr{F}_{\mathbb{C}})R(N_{\mathbb{C}})Td(f_{\mathbb{C}})) + g_*(\widetilde{ch}(E.)Td(\bar{g}_{\mathbb{C}})) + f_*(ch(\bar{\mathscr{F}}_{\mathbb{C}})\widetilde{Td}(f/g))]^{(0)}.$$

Here f_* (resp. g_*) denotes integration of forms along the fibres of a smooth map, and direct image of cohomology classes.

The line bundle $\lambda(E.) = \bigotimes_{j \geq 0} \lambda(E_j)^{(-1)^j}$ may be equipped with the tensor product of Quillen metrics. We also define $\widehat{ch}(\bar{E}.) = \sum_{j=0}^{k} (-1)^j \widehat{ch}(\bar{E}_j)$, $\tau(\bar{E}.) = \sum_{j=0}^{k} (-1)^j \tau(\bar{E}_j)$, etc. Since

$$\hat{c}_1(\lambda(\mathscr{F}), h_{\mathrm{Q}}) - \hat{c}_1(\lambda(E.), h_{\mathrm{Q}}) = a(\log \| \sigma \|_{\mathrm{Q}}^{2})$$

(see [GS3, 1.2.5 and (4.8.2)]), we get, for the difference of the left hand sides of equation (11) for $\mathscr{F}$ and $\bar{E}.$:

$$\begin{aligned} \tau(\lambda(\mathscr{F}), h_{\mathrm{Q}})^{(1)} - \tau(\lambda(E.), h_{\mathrm{Q}})^{(1)} &= ((\widehat{ch}(\lambda(\bar{\mathscr{F}})) - \widehat{ch}(\lambda(\bar{E}.)))\widehat{Td}(Y))^{(1)} \\ &= ((\hat{c}_1(\lambda(\bar{\mathscr{F}})) - \hat{c}_1(\lambda(\bar{E}.)))\widehat{Td}(Y))^{(1)} \\ &= a[f_*(ch(\mathscr{F}_{\mathbb{C}})R(N_{\mathbb{C}})Td(f_{\mathbb{C}}))Td(T_{Y(\mathbb{C})}) \\ &\quad + g_*(\widetilde{ch}(E.)Td(\bar{g}_{\mathbb{C}}))Td(\bar{T}_{Y(\mathbb{C})}) \\ &\quad + f_*(ch(\bar{\mathscr{F}}_{\mathbb{C}})\widetilde{Td}(f/g))Td(\bar{T}_{Y(\mathbb{C})})]^{(1)}. \end{aligned} \tag{25}$$

We wish to compare this with the difference of the right-hand sides in (11). Since R is additive and Td is multiplicative, we may use the Riemann–Roch theorem for i in ordinary cohomology to get

$$\begin{aligned} f_*(ch(\mathscr{F}_{\mathbb{C}})R(N_{\mathbb{C}})Td(f_{\mathbb{C}}))Td(T_{Y(\mathbb{C})}) &= g_*(ch(E._{,\mathbb{C}})R(g_{\mathbb{C}})Td(P(\mathbb{C}))) \\ &\quad - f_*(ch(\mathscr{F}_{\mathbb{C}})R(f_{\mathbb{C}})Td(X(\mathbb{C}))). \end{aligned} \tag{26}$$

Now let

$$x := f_*(\tau(\bar{\mathscr{F}}) + a(ch(\bar{\mathscr{F}}_{\mathbb{C}})\widetilde{Td}(f))) - g_*(\tau(\bar{E}.) + a(ch(\bar{E}._{,\mathbb{C}})\widetilde{Td}(g))). \tag{27}$$

We need to show that this is equal to

$$a((f_*(ch(\bar{\mathscr{F}}_{\mathbb{C}})\widetilde{Td}(f/g)) + g_*(\widetilde{ch}(E.)Td(\bar{g}_{\mathbb{C}})))Td(\overline{T_{Y(\mathbb{C})}}).$$

Since P is regular, by Theorem 6,

$$\tau(\bar{E}.) = \widehat{ch}(\bar{E}.)\widehat{Td}(P),$$

while

$$\tau(\bar{\mathscr{F}}) = \widehat{ch}_X^P(\bar{\mathscr{F}})_i\widehat{Td}(P) + a(ch(\bar{\mathscr{F}}_{\mathbb{C}})\widetilde{Td}(X/P)).$$

By Lemma 21,

$$f_*(\widehat{ch}_X^P(\bar{\mathscr{F}})_{\cdot i}\widehat{Td}(P)) = g_*(\widehat{ch}(\bar{E}.)\widehat{Td}(P)) + a(g_*(\widetilde{ch}(E.)\,Td(\bar{T}_{P(\mathbb{C})}))) .$$

Therefore

$$\text{(28)} \qquad x = a(f_*(ch(\bar{\mathscr{F}}_{\mathbb{C}})\widetilde{Td}(X/P) + ch(\bar{\mathscr{F}}_{\mathbb{C}})\widetilde{Td}(f)) + g_*(\widetilde{ch}(E.)\,Td(\bar{T}_{P(\mathbb{C})})) - g_*(ch(\overline{E._{,\mathbb{C}}})\,\widetilde{Td}(g)) .$$

By definition

$$dd^c(\widetilde{Td}(g)) = Td(\bar{g}_{\mathbb{C}}).g^*\,Td(\bar{T}_{Y(\mathbb{C})}) - Td(\bar{T}_{P(\mathbb{C})}) ,$$

and

$$dd^c(\widetilde{ch}(E.)) = i_*(ch(\bar{\mathscr{F}}_{\mathbb{C}})\,Td^{-1}(\bar{N}_{\mathbb{C}})) - ch(\bar{E}._{\mathbb{C}}) .$$

Therefore, in $\tilde{A}^{\cdot}(Y_{\mathbb{R}})$,

$$\begin{aligned} g_*(\widetilde{ch}(E.)\,Td(\bar{T}_{P(\mathbb{C})})) &= g_*(\widetilde{ch}(E.)\,Td(\bar{g}_{\mathbb{C}}))\,Td(\bar{T}_{Y(\mathbb{C})}) - g_*(\widetilde{ch}(E.)dd^c(\widetilde{Td}(g))) \\ &= g_*(\widetilde{ch}(E.)\,Td(\bar{g}_{\mathbb{C}}))\,Td(\bar{T}_{Y(\mathbb{C})}) - g_*(dd^c(\widetilde{ch}(E.))\widetilde{Td}(\mathrm{g})) \\ &= g_*(\widetilde{ch}(E.)\,Td(\bar{g}_{\mathbb{C}}))\,Td(\bar{T}_{Y(\mathbb{C})}) + g_*(ch(\bar{E}._{\mathbb{C}})\widetilde{Td}(g)) \\ &\quad - f_*(ch(\bar{\mathscr{F}}_{\mathbb{C}})\,Td^{-1}(\bar{N}_{\mathbb{C}})i^*\widetilde{Td}(g)) . \end{aligned}$$

Combining this with (28) we get

$$\text{(29)} \quad x = a(f_*(ch(\bar{\mathscr{F}}_{\mathbb{C}})\widetilde{Td}(f)) + f_*(ch(\bar{\mathscr{F}}_{\mathbb{C}})\widetilde{Td}(X/P)) - f_*(ch(\bar{\mathscr{F}}_{\mathbb{C}})\,Td^{-1}(\bar{N}_{\mathbb{C}})i^*\widetilde{Td}(g)) + g_*(\widetilde{ch}(E.)\,Td(\bar{g}_{\mathbb{C}})g^*\,Td(\bar{T}_{Y(\mathbb{C})})) .$$

Now consider the following diagram with exact rows and columns:

$$\begin{array}{ccccccccc} & & 0 & & 0 & & & & \\ & & \downarrow & & \downarrow & & & & \\ 0 & \to & Tf_{\mathbb{C}} & \to & i^*Tg_{\mathbb{C}} & \to & N_{\mathbb{C}} & \to & 0 \\ & & \downarrow & & \downarrow & & \wr\| & & \\ 0 & \to & i^*T_{X(\mathbb{C})} & \to & i^*T_{P(\mathbb{C})} & \to & N_{\mathbb{C}} & \to & 0 \\ & & \downarrow & & \downarrow & & & & \\ & & f^*T_{Y(\mathbb{C})} & \xrightarrow{\sim} & f^*T_{Y(\mathbb{C})} & & & & \\ & & \downarrow & & \downarrow & & & & \\ & & 0 & & 0\,. & & & & \end{array}$$

From (6) and [GS3, Propositions 1.3.2 and 1.3.4], we deduce from this diagram that

$$\widetilde{Td}(f/g)f^*(Td(\bar{T}_{Y(\mathbb{C})}))Td(\bar{N}_{\mathbb{C}}) - i^*(\widetilde{Td}(X/P))Td(\bar{N}_{\mathbb{C}})$$
$$= \widetilde{Td}(f)Td(\bar{N}_{\mathbb{C}}) - i^*(\widetilde{Td}(g)) .$$

From this and (29) we get

$$x = a(f_*(ch(\bar{\mathscr{F}}_{\mathbb{C}})\widetilde{Td}(f/g)) + g_*(\widetilde{ch}(E.)Td(\bar{g}_{\mathbb{C}})))Td(\overline{T_{Y(\mathbb{C})}}) , \tag{30}$$

as was to be shown. The identities (25), (26), (27), and (30) prove 1). In other words, if

$$\tau(\lambda(E_j), h_{\mathrm{Q}})^{(1)} = g_*(\tau(\bar{E}_j) + a(ch(\bar{E}_{j,\mathbb{C}})\widetilde{Td}(g)) - a(ch(E_{j,\mathbb{C}})Td(g_{\mathbb{C}})R(f_{\mathbb{C}})))^{(1)}$$

holds for all $j \geqq 0$, then

$$\tau(\lambda(\mathscr{F}), h_{\mathrm{Q}})^{(1)} = f_*(\tau(\bar{\mathscr{F}}) + a(ch(\bar{\mathscr{F}}_{\mathbb{C}})\widetilde{Td}(f)) - a(ch(\mathscr{F}_{\mathbb{C}})Td(f_{\mathbb{C}})R(f_{\mathbb{C}})))^{(1)} .$$

4.2.3 To prove 1) in case (ii) we can proceed as above, using [BL] and [BGS3]. The argument is somewhat simpler since we deal with Todd classes of the relative tangent spaces and we do not need the discussion about τ in Sect. 3.

More precisely, to prove (12) (the proof of (13) is similar: delete α in the formulas below and replace cap products by cup products), from [GS3, 1.2.5 and (4.8.2)], given any $\alpha \in \widehat{CH}_{\cdot}(Y)_{\mathbb{Q}}$, we have

$$(\hat{c}_1(\lambda(\bar{F})) - \hat{c}_1(\lambda(\bar{E}.))) \cap \alpha = a((-\log \|\sigma\|_{\mathrm{Q}}^2)\omega(\alpha)) \tag{31}$$

and, from [BL, Theorem 0.1],

$$-\log \|\sigma\|_{\mathrm{Q}}^2 = [f_*(ch(F_{\mathbb{C}})R(N_{\mathbb{C}})Td(f_{\mathbb{C}})) + g_*(\widetilde{ch}(E.)Td(\bar{g}_{\mathbb{C}})) \tag{32}$$
$$-f_*(ch(\bar{F}_{\mathbb{C}})\widetilde{Td}(f/g))]^{(0)} .$$

On the other hand, the following equation holds in $\widehat{CH}_{\cdot}(Y)_{\mathbb{Q}}$:

$$f_*(\widehat{ch}(\bar{F})\widehat{Td}(f) \cap f^*(\alpha)) = g_*(\widehat{ch}(\bar{E}.)\widehat{Td}(g) \cap g^*(\alpha)) + a g_*(\widetilde{ch}(E.)Td(\bar{g}_{\mathbb{C}})g^*(\omega(\alpha)))$$
$$+ a f_*(ch(\bar{F}_{\mathbb{C}})\widetilde{Td}(f/g)f^*(\omega(\alpha))) .$$

Indeed, by definition of $\widehat{Td}(f)$ (see 2.6.2) and since $f^* = i^*g^*$, the following holds

$$\widehat{Td}(f) \cap f^*(\alpha) - a(\widetilde{Td}(f/g)f^*(\omega(\alpha))) = \widehat{Td}^{-1}(\bar{N}) \cap i^*(\widehat{Td}(g) \cap g^*(\alpha)) .$$

Let $\gamma = \widehat{Td}(g) \cap g^*(\alpha) \in \widehat{CH}^{\cdot}(P)$. From this, (31) and (32) it follows that (12) is equivalent to the identity

$$g_*(\widehat{ch}(\bar{E}.) \cap \gamma) + a g_*(\widetilde{ch}(E.)\omega(\gamma)) = f_*(\widehat{Td}^{-1}(\bar{N})\widehat{ch}(\bar{F}) \cap i^*(\gamma)) . \tag{33}$$

This equality (33) plays in the l.c.i. case the role of Lemma 21 in Case (i). When f is smooth and Y is regular, (33) is Theorem 4.13 in [BGS3] (with different notations).

The proof of [BGS3] extends as follows to any l.c.i. map f. The immersion $i: X \to P$ being regular, the Grassmannian graph $W = W(E.)$ is isomorphic to the deformation to the normal cone (see 1.1.2), and $|Z| = \mathbb{P}_X(N \oplus 1)$. Let $\tilde{E}.$ be the extension of $E.$ to W as in [BGS3, Lemma 4.3] or 1.1.2 above, and $E.^Z$ its restriction to $|Z|$. Choose a metric on $\tilde{E}.$ whose restriction to $\tilde{P}(\mathbb{C})$ is split acyclic, and which satisfies condition (A) as in [BGS3, 4.12] (or Lemma 21 above). Define

$$\beta = \int_{W(\mathbb{C})/P(\mathbb{C})} ch(\bar{\tilde{E}}.) \log |z|^2$$

in $\tilde{A}^{\cdot}(P_{\mathbb{R}})$. By the same proof as in Lemma 4.12 in [BGS3], we have

$$(34) \qquad g_*(\widehat{ch}(\bar{E}.) \cap \gamma) = f_* \pi_* (\widehat{ch}(\bar{E}.^Z) \cap \pi^* i^*(\gamma)) + a g_*(\beta \omega(\gamma))$$

where $\pi: \mathbb{P}_X(N \oplus 1) \to X$ is the (smooth) projection.

The right-hand side of (34) can be computed using Theorems 3.22 and 4.11 of [BGS3] as in the proof of Theorem 4.13 in [BGS3], i.e. by comparing $E.^Z$ with the Koszul complex $K.(H) \otimes \pi^* F$ considered in Theorem 1 ii) above. Formula (4.36) in [BGS3] has to replaced by the identity

$$\pi_*(\widehat{ch}(\overline{K.(H)} \otimes \pi^* \bar{F}) \cap \pi^* i^*(\gamma)) = \pi_*(\widehat{ch}(\overline{K.(H)}) \cap \pi^* i^*(\gamma)) \cap \widehat{ch}(\bar{F}),$$

which follows from Theorem 4(5). As in the proof of Theorem 4 we may choose a map $h: X \to M$, where M is regular, and a vector bundle N' on M such that $h^*(N') = N$. Since proving (33) for one metric on N implies the result for all metrics, we may assume that there exists a metric on N' such that $\bar{N} = h^*(\bar{N}')$. On $\mathbb{P}_M(N' \oplus 1)$ we consider the canonical hyperplane bundle $H' \subset \pi'^*(N') \oplus 1$, where $\pi': \mathbb{P}_M(N' \oplus 1) \to M$ is the projection, and the Koszul complex $K.(H')$ with the metric induced by $\bar{N}'$. Then, using the projection formula in Lemma 12, we get

$$\pi_*(\widehat{ch}(\overline{K.(H)}) \cap \pi^* i^*(\gamma)) = i^*(\gamma) \cdot_h \pi'_*(\widehat{ch}(\overline{K.(H')})) .$$

Applying (4.37) and (4.38) in [BGS3] on M concludes the proof.

4.2.4 Now we prove 2) in 4.2.1, i.e. Theorem 7 for the projective space over Y. Given any Hermitian vector bundle $\bar{E}$ on $P = Y \times \mathbb{P}^N$ and any $\alpha \in \widehat{CH}_p(Y)_{\mathbb{Q}}$, we let

$$\delta(E) \cap \alpha = \hat{c}_1(\lambda(E), h_Q) \cap \alpha - g_*(\widehat{ch}(\bar{E}) \widehat{Td}(g) \cap g^*(\alpha))_{(p-1)} .$$

As in [GS4], Theorem 1.4 (i)], we see that $\delta(E) \cap \alpha$ depends only on the class of $E_{\mathbb{Q}}$ in the Grothendieck group $K_0(P_{\mathbb{Q}})$ of the generic fiber of P, since it is invariant under change of metric on E and additive on exact sequences (by the results in [BGS1]), and it vanishes on a virtual bundle with support in the special fibers (by the algebraic Riemann–Roch theorem [S1, Theorem 7]). Furthermore, if F is any bundle on Y, we have, by [GS4, Theorem 1.4 (iii)],

$$\delta(E \otimes g^*(F)) \cap \alpha = rk(F) \delta(E) \cap \alpha .$$

As a module over $K_0(Y)$, $K_0(P)$ is generated by the positive powers $\mathcal{O}_P(n)$ of the canonical line bundle. Therefore it is enough to show that $\delta(\mathcal{O}_P(n)) = 0$.

But $\mathcal{O}_P(n)$ is pulled back from $\mathbb{P}^N$ by the second projection, $\lambda(\mathcal{O}_P(n))$ is constant on Y and, if $p: Y \to \mathrm{Spec}(\mathbb{Z})$ is the projection map, we get, using Lemma 12,

$\delta(\mathcal{O}_P(n)) \cap \alpha = \delta(\mathcal{O}_{\mathbb{P}^N}(n) \cdot_p \alpha$. So we may assume that $Y = \mathrm{Spec}(\mathbb{Z})$ and $P = \mathbb{P}^N$, in which case $\delta(\mathcal{O}_{\mathbb{P}^N}(n))$ lies in $\widehat{CH}^1(\mathrm{Spec}(\mathbb{Z})) = \mathbb{R}$..

To prove that $\delta(\mathcal{O}_{\mathbb{P}^N}(n)) = 0$ we proceed by induction on n and N. When $n = 0$ and $N \geqq 1$ the fact that Riemann–Roch holds for the trivial line bundle on $\mathbb{P}^N$ is Theorem 2.1.1 in [GS4] (in fact the power series defining R was computed in order that this fact be true). When $n \geqq 0$ consider the standard inclusion $i: \mathbb{P}^N \to \mathbb{P}^{N+1}$. There are standard exact sequences on $\mathbb{P}^{N+1}$

$$0 \to \mathcal{O}_{\mathbb{P}^{N+1}}(n) \to \mathcal{O}_{\mathbb{P}^{N+1}}(n+1) \to i_* \mathcal{O}_{\mathbb{P}^N}(n) \to 0\,. \tag{35}$$

By induction on N, we may assume that $\delta(\mathcal{O}_{\mathbb{P}^N}(n)) = 0$ for all $n \in \mathbb{N}$ (the case $N = 0$ is trivially true). Using the fact 1) proved in 4.2.2 above (i.e. the compatibility of the statement with immersions), we deduce from (35) that $\delta(\mathcal{O}_{\mathbb{P}^{N+1}}(n)) = 0$ for all $n \geqq 0$. This ends the proof of Theorem 7.

4.2.5 When f is smooth, X and Y are regular, and $\mathscr{F} = F$, Theorem 7 was conjectured in [GS4, Conjecture 1.3]. Special cases were announced in [S3] and [G3]. The statement (i) was announced in [GS6, Theorem 2], but the statement in loc. cit. is not correct in general since we wrote $\hat{c}_1(\lambda(F), h_Q)$ instead of

$$\tau(\lambda(\mathscr{F}), h_Q)^{(1)} = \hat{c}_1(\lambda(F), h_Q) - \hat{c}_1(X)/2$$

for the left-hand side of equation (11), and we forgot the term $a(ch(\bar{\mathscr{F}}_{\mathbb{C}})\widetilde{Td}(f))^{(1)}$ on the right-hand side.

Theorem 7 extends the arithmetic Riemann–Roch theorem in relative dimension one due to Faltings [F1] and Deligne [D]. Deligne considered a smooth map $f: X \to Y$ of relative dimension one. He obtained a canonical isomorphism of Hermitian line bundles between (a power of) $\det Rf_*(E)$ with its Quillen metric and an Hermitian line bundle on Y whose class in $\widehat{\mathrm{Pic}}(Y)_{\mathbb{Q}} = \widehat{CH}^1(Y)_{\mathbb{Q}}$ coincides with (a multiple of) $f_*(\widehat{ch}(\bar{E})\widehat{Td}(f))^{(1)}$ (at least when the set of complex imbeddings of A contains a real imbedding, see [GS3], 4.10). Deligne's isomorphism is true up to some universal constant. Therefore Theorem 7 computes this constant: it comes from the class $R(f)$ and involves $\zeta'(-1)$. For a precise statement, see [S2] and [GS4], 1.5. Notice that Deligne's theorem is stronger than Theorem 7, since our result computes only an isomorphism class of Hermitian line bundles on Y. The algebraic isomorphism in Deligne's result was extended to arbitrary degree and arbitrary relative dimension by Franke [Fr]. It would be interesting to get an arithmetic analog of Franke's version of the Riemann–Roch–Grothendieck theorem.

Faltings gives in [F3] a new proof of Theorem 7, when f is smooth, X and Y are projective and regular and $\mathscr{F} = F$ is locally free. His proof is valid in all degrees in $\widehat{CH}^{\cdot}(Y)$, and not only for the determinant of cohomology. Its analytical part does not use the work of Bismut and Lebeau [BL].

5 Small sections of ample bundles

5.1 The main result

5.1.1 In this paragraph we shall apply (a weak version of) the arithmetic Riemann–Roch–Grothendieck Theorem 7 to produce small sections of symmetric powers of ample vector bundles on arithmetic varieties.

We first state our main result. Let X be a projective flat variety of relative dimension d over $\mathbb{Z}$ with smooth generic fibre $X_{\mathbb{Q}}, \bar{E} = (E, h)$ an Hermitian vector bundle on X, and $\bar{\mathscr{F}} = (\mathscr{F}, h')$ a Hermitian coherent sheaf on X. We make the following assumptions on $\bar{E}$:

A1) E is ample on X, in the sense of [H1, par. 2];
A2) the metric h on $E_{\mathbb{C}}$ is positive in the sense of Griffiths, i.e., any nonzero smooth section e (resp. u) of $E_{\mathbb{C}}$ (resp. $T_{X(\mathbb{C})}$), one has

$$h(R^E(u, \bar{u})(e), e) > 0 ,$$

where R^E is the curvature form of $E_{\mathbb{C}}$ (with values in the endomorphisms of $E_{\mathbb{C}}$).

Denote by $S^n E$ the n-th symmetric power of E, i.e. the degree n part of the quotient of the tensor algebra of E by the ideal generated by the elements $x \otimes y - y \otimes x$ [EGA2, p.14]. We equip $S^n E_{\mathbb{C}}$ with the quotient metric $S^n h$ induced from $h^{\otimes n}$ by the map $E_{\mathbb{C}}^{\otimes n} \to S^n E_{\mathbb{C}}$, and $\mathscr{F}_{\mathbb{C}} \otimes S^n E_{\mathbb{C}}$ with $h' \otimes S^n h$. We look for small sections of $\mathscr{F} \otimes S^n E$ on X when n is big.

Choose a Kähler metric on $X(\mathbb{C})$ (invariant under complex conjugation) and denote by $\chi_{L^2}(S^n E \otimes \mathscr{F})$ the real number

$$\chi_{L^2}(\mathscr{F} \otimes S^n E) = -\log \mathrm{vol}_{L^2}(H^0(X(\mathbb{C}), \mathscr{F}_{\mathbb{C}} \otimes S^n E_{\mathbb{C}})^+ / H^0(X, S^n E \otimes \mathscr{F})) ,$$

the volume being taken for the L^2-metric.

Let r (resp. r') be the rank of $E_{\mathbb{C}}$ (resp. $\mathscr{F}_{\mathbb{C}}$). For any $k \geqq 0$ denote by $\hat{s}'_k(\bar{E}) \in \widehat{CH}^k(X)$ (resp. $s_k(E) \in CH^k(X)$) the arithemetic Segre class of $\bar{E}$ (resp. the algebraic Segre class of E), as in 2.7. We introduce a real number

$$\hat{\sigma}_{d+1}(\bar{E}) = p_*(\hat{s}'_{d+1}(\bar{E}))/(d+r)! \in \mathbb{R} = \widehat{CH}^1(\mathrm{Spec}(\mathbb{Z}))$$

and a rational number

$$\sigma_d(E) = p_*(s_d(E))/(d+r-1)! \in \mathbb{Q} = CH^0(\mathrm{Spec}(\mathbb{Z}))_{\mathbb{Q}}$$

(where $p: X \to \mathrm{Spec}(\mathbb{Z})$ is the map defining X). In particular, when E is a line bundle L, we have

$$\hat{\sigma}_{d+1}(\bar{L}) = p_*(\hat{c}_1(\bar{L})^{d+1})/(d+1)! .$$

We shall use the following notations. Given a map $n \to \phi(n)$ from positive integers to positive real numbers, we write $o(\phi(n))$ (resp. $O(\phi(n))$) for any real valued function of n such that $\lim_{n\to\infty} |o(\phi(n))|/\phi(n) = 0$ (resp. $|O(\phi(n))|/\phi(n)$ is bounded above by some constant independent of n). Our main result is the following:

Theorem 8

As n goes to infinity,

$$\chi_{L^2}(\mathscr{F} \otimes S^n E) = r'\hat{\sigma}_{d+1}(\bar{E})n^{d+r} + (r'/4)(d-r+1)\sigma_d(E)n^{d+r-1}\log(n) + An^{d+r-1} + o(n^{d+r-1})$$

where A is given by formula (47) *below.*

5.1.2 Let $P = \mathbb{P}(E)$ be the projective space of E, $f: P \to X$ the projection, and $\mathcal{O}(1)$ the standard line bundle on P. According to [EGA2, 3.3] and [H2, Proposition 7.11], the cup-product

$$f_*(\mathcal{O}(1))^{\otimes n} \to f_*\mathcal{O}(n)$$

induces a canonical isomorphism

$$\alpha : S^n E \xrightarrow{\sim} f_* \mathcal{O}(n) . \tag{36}$$

Furthermore $R^q f_*(\mathcal{O}(n)) = 0$ if $q > 0$ and $n \geqq 0$. To prove Theorem 8 we shall first use the restriction of α to the generic fibre $X_{\mathbb{Q}}$ to get the following lemma. Let

$$ch : K_0(X_{\mathbb{Q}})_{\mathbb{Q}} \to CH^{\cdot}(X_{\mathbb{Q}})_{\mathbb{Q}}$$

be the Chern character isomorphism for the Grothendieck group of $X_{\mathbb{Q}}$ and $[\mathcal{F}_{\mathbb{Q}} \otimes S^n E_{\mathbb{Q}}] \in K_0(X_{\mathbb{Q}})_{\mathbb{Q}}$ the class of the restriction of $\mathcal{F} \otimes S^n E$ to $X_{\mathbb{Q}}$

Lemma 29 *There exist elements $a_i \in K_0(X_{\mathbb{Q}})_{\mathbb{Q}}$ independent of n such that*

$$[\mathcal{F}_{\mathbb{Q}} \otimes S^n E_{\mathbb{Q}}] = \sum_{i=0}^{d+r-1} a_i n^i$$

and

$$ch(a_{d+r-1}) = r' s_d(E)/(d+r-1)! .$$

Proof. From (36) we deduce that

$$\mathcal{F}_{\mathbb{Q}} \otimes S^n E_{\mathbb{Q}} = (f_* \mathcal{F} \otimes (\mathcal{O}(n)))_{\mathbb{Q}} = f_*(f^*(\mathcal{F})(n)_{\mathbb{Q}}) .$$

Therefore, the Riemann–Roch–Grothendieck theorem for f [BGI] gives

$$\begin{aligned} ch(\mathcal{F}_{\mathbb{Q}} \otimes S^n E_{\mathbb{Q}}) &= ch(f_*(f^* \mathcal{F}(n)_{\mathbb{Q}})) = f_*(ch(f^* \mathcal{F}(n)_{\mathbb{Q}}) Td(f)) \\ &= f_*(\exp(nc_1(\mathcal{O}(1))) Td(f)) ch(\mathcal{F}_{\mathbb{Q}}) = \sum_{i=0}^{d+r-1} b_i n^i \end{aligned}$$

since $CH^k(P) = 0$ when $k > d + r - 1$. Furthermore

$$b_{d+r-1} = r' f_*(c_1(\mathcal{O}(1))^{d+r-1})/(d+r-1)! = r' s_d(E)/(d+r-1)!$$

by definition of Segre classes. Since ch is an isomorphism [BGI] the lemma follows.

5.1.3 Since E is ample on X by A1), we can take n big enough so that

$$H^q(X, \mathcal{F} \otimes S^n E) = 0 \quad \text{when } q > 0 .$$

From (36) we get an isomorphism

$$H^0(X, \mathcal{F} \otimes S^n E) \xrightarrow{\sim} H^0(P, f^*(\mathcal{F})(n)) .$$

From the metric on $E_{\mathbb{C}}$ we deduce a metric on $\mathcal{O}(1)$ (a quotient of $f^*(E_{\mathbb{C}})$), hence on $\mathcal{O}(n)$. We endow $f^*(\mathcal{F})(n)$ with the tensor product of this metric with $f^*(h)$. Let us choose an arbitrary Kähler metric k_P on P.

Now we apply the arithmetic Riemann–Roch Theorem 7 (formulated as in 4.1.5) to $f^*(\mathcal{F})(n)$ on P. Since $H^0(P, f^*(\mathcal{F})(n))$ is the only nonvanishing cohomology group we have

$$\begin{aligned} \chi_{Q}(f^*(\bar{\mathcal{F}})(n)) = &- \log \mathrm{vol}_{L^2}(H^0(P(\mathbb{C}), f^*(\mathcal{F}_{\mathbb{C}})(n))^+ / H^0(P, f^*(\mathcal{F})(n))) \\ &- (1/2)\, T(P(\mathbb{C}), f^*(\mathcal{F}_{\mathbb{C}})(n)) , \end{aligned} \tag{37}$$

and

$$\chi_{\mathrm{Q}}(f^*(\bar{\mathscr{F}})(n)) = g_*(\tau(f^*(\bar{\mathscr{F}})(n)))^{(1)} - a(g_*(ch(f^*(\mathscr{F}_{\mathbb{C}})(n))R(g)Td(g)))^{(1)},$$

with $g = p \circ f: P \to \mathrm{Spec}(\mathbb{Z})$. Since

$$\tau(f^*(\bar{\mathscr{F}})(n)) = \tau(f^*(\bar{\mathscr{F}}))\widehat{ch}(\overline{\mathscr{O}(n)}) = \tau(f^*(\bar{\mathscr{F}}))\exp(n\hat{c}_1(\overline{\mathscr{O}(1)})),$$

we may write $\chi_{\mathrm{Q}}(f^*(\bar{\mathscr{F}})(n))$ as a polynomial in n:

$$\begin{aligned}\chi_{\mathrm{Q}}(f^*(\bar{\mathscr{F}})(n)) = {} & g_*(\tau(f^*(\bar{\mathscr{F}}))\hat{c}_1(\overline{\mathscr{O}(1)})^{d+r})^{(1)} n^{d+r}/(d+r)! \\ & + g_*(\tau(f^*(\bar{\mathscr{F}}))\hat{c}_1(\overline{\mathscr{O}(1)})^{d+r-1})^{(1)} n^{d+r-1}/(d+r-1)! \\ & - a(g_*(ch(f^*(\mathscr{F}_{\mathbb{C}}))c_1(\mathscr{O}(1))^{d+r-1}R(g)Td(g)))^{(1)} \\ & \times n^{d+r-1}/(d+r-1)! + \sum_{k \leqq d+r-2} \alpha_k n^k .\end{aligned} \tag{38}$$

We compute

$$\begin{aligned} g_*(\tau(f^*(\bar{\mathscr{F}}))\hat{c}_1(\overline{\mathscr{O}(1)})^{d+r})^{(1)}/(d+r)! &= r' g_*(\hat{c}_1(\overline{\mathscr{O}(1)})^{d+r})/(d+r)! \\ &= r'\hat{\sigma}_{d+1}(\bar{E}), \end{aligned} \tag{39}$$

$$g_*(\tau(f^*(\bar{\mathscr{F}}))\hat{c}_1(\overline{\mathscr{O}(1)})^{d+r-1})^{(1)} = g_*(\tau(f^*(\bar{\mathscr{F}}), k_P)^{(1)}\hat{c}_1(\overline{\mathscr{O}(1)})^{d+r-1}). \tag{40}$$

(where we write $\tau(f^*(\bar{\mathscr{F}}), k_P)$ instead of $\tau(f^*(\bar{\mathscr{F}}))$ to indicate the dependence on the metric on P), and

$$a(g_*(ch(f^*(\mathscr{F}))c_1(\mathscr{O}(1))^{d+r-1}R(g)Td(g))^{(1)} = 0 \tag{41}$$

since $R(g)$ has positive degree.

Bismut and Vasserot [BV] computed the asymptotics of analytic torsion under twisting by a positive line bundle. The hypothesis A2) on the curvature form of $\bar{E}_{\mathbb{C}}$ is equivalent to the fact that the curvature form R of $\overline{\mathscr{O}(1)}_{\mathbb{C}}$ on $P(\mathbb{C})$ is positive. Let $\mathring{R}$ be the endomorphism of the tangent space of $P(\mathbb{C})$ attached to R by the formula

$$R(u, v) = \langle u, \mathring{R}(v) \rangle$$

where u and v are two tangent vectors and $\langle\, , \rangle$ our chosen metric on $P(\mathbb{C})$. Define a functional in the metric h on $E_{\mathbb{C}}$ and the metric k_P on $P(\mathbb{C})$ by the formula

$$BV(h, k_P) = \int_{P(\mathbb{C})} \log\det\left(\frac{\mathring{R}}{2\pi}\right)\exp\left(\frac{i}{2\pi}R\right).$$

Then, by [BV, Theorem 8], as n goes to infinity,

$$\begin{aligned} T(P(\mathbb{C}), f^*(\mathscr{F}_{\mathbb{C}})(n)) &= \frac{r'}{2}\int_{P(\mathbb{C})} \log\det\left(\frac{n\mathring{R}}{2\pi}\right)\exp\left(\frac{in}{2\pi}R\right) + o(n^{d+r-1}) \\ &= (r'/2)\sigma_d(E)(d+r-1)n^{d+r-1}\log(n) \\ &\quad + \frac{r'}{2}BV(h, k_P)n^{d+r-1} + o(n^{d+r-1}). \end{aligned} \tag{42}$$

It follows from (37), (38), (39), (40), (41) and (42), that the number $\chi_{L^2}(f^*(\mathscr{F}_{\mathbb{C}})(n), k_P) = -\log \mathrm{vol}_{L^2}(H^0(P(\mathbb{C}), f^*(\mathscr{F}_{\mathbb{C}})(n))^+/H^0(P, f^*(\mathscr{F})(n)))$ satisfies

$$\begin{aligned}\chi_{L^2}(f^*(\mathscr{F}_{\mathbb{C}})(n), k_P) = {} & r'\hat{\sigma}_{d+1}(\bar{E})n^{d+r} + (r'/4)\sigma_d(E)(d+r-1)n^{d+r-1}\log(n) \\ & + (r'/4)BV(h, k_P)n^{d+r-1} \\ & + g_*(\tau(f^*(\bar{\mathscr{F}}), k_P)^{(1)}\hat{c}_1(\overline{\mathcal{O}(1)})^{d+r-1})n^{d+r-1}/(d+r-1)! \\ & + o(n^{d+r-1}) . \end{aligned} \tag{43}$$

To relate this number to $\chi_{L^2}(S^n E \otimes \mathscr{F})$ we introduce the following metric on $TP(\mathbb{C})$. The Hermitian holomorphic connection on $E_{\mathbb{C}}$ gives a splitting

$$TP(\mathbb{C}) = Tf_{\mathbb{C}} \oplus f^* TX(\mathbb{C}) .$$

Denote by h_P the metic whose associated (1, 1)-form is the direct sum of the restriction of $c_1(\overline{\mathcal{O}(1)})$ to $Tf_{\mathbb{C}}$ with the inverse image by f^* of the Kähler form on $X(\mathbb{C})$ (the metric h_P needs not be Kähler). The volume form on $TP(\mathbb{C})$ attached to h_P is the product of the Fubini-Study volume form of $Tf_{\mathbb{C}}$ with the volume form on $f^* TX(\mathbb{C})$. Therefore, if we endow $f_*(\mathcal{O}(n))_{\mathbb{C}}$ with the L^2 metric along the fibers of f, we get

$$\chi_{L^2}(f^*(\mathscr{F}_{\mathbb{C}})(n), h_P) = \chi_{L^2}(\mathscr{F} \otimes f_*(\mathcal{O}(n))_{\mathbb{C}}) . \tag{44}$$

To compute the norm of the isomorphism

$$\alpha : S^n E \xrightarrow{\sim} f_*(\mathcal{O}(n))$$

we may assume that $X(\mathbb{C})$ is a point. One gets that

$$\|\alpha(x)\|^2 = \frac{n!(r-1)!}{(n+r-1)!}\|x\|^2 ;$$

Indeed, when $X(\mathbb{C})$ is a point, if $x = e_1^{\otimes \alpha_1} \otimes e_2^{\otimes \alpha_2} \ldots \otimes e_r^{\alpha_r}$ is a generator of $S^n E_{\mathbb{C}}$, with e_i an orthonormal basis of $E_{\mathbb{C}}$, by definition of the quotient metric, we get

$$\|x\|^2 = \frac{\alpha_1! \ldots \alpha_r!}{n!} .$$

On the other hand, if μ is the invariant volume form of total volume one on $\mathbb{P}^{r-1}(\mathbb{C})$, we get

$$\begin{aligned}\|\alpha(x)\|^2 & = \int_{\mathbb{P}^{r-1}(\mathbb{C})} \frac{|z_1|^{2\alpha_1} \ldots |z_r|^{2\alpha_r}}{(|z_1|^2 + \ldots + |z_r|^2)^n}\mu \\ & = \frac{(\alpha_1! \ldots \alpha_r!)(r-1)!}{(n+r-1)!} .\end{aligned}$$

It follows that α multiplies the norms by $n!(r-1)!/(n+r-1)!$.

Notice that $H^0(X(\mathbb{C}), \mathscr{F}_{\mathbb{C}} \otimes S^n E_{\mathbb{C}})$ has rank

$$\chi(X(\mathbb{C}), \mathscr{F}_{\mathbb{C}} \otimes S^n E_{\mathbb{C}}) = r'\sigma_d(E)n^{d+r-1} + O(n^{d+r-2}) .$$

Therefore, by (44) and the Stirling formula, we get

$$\chi_{L^2}(\mathscr{F}\otimes S^nE)-\chi_{L^2}(f^*(\mathscr{F}_{\mathbb{C}})(n),h_P)$$

$$=(1/2)\chi(X(\mathbb{C}),\mathscr{F}_{\mathbb{C}}\otimes S^n_{\mathbb{C}})\log(n!(r-1)!/(n+r-1)!)$$

$$=-(r'/2)(r-1)\sigma_d(E)n^{d+r-1}\log n+(r'/2)\log((r-1)!)\sigma_d(E)n^{d+r-1}$$

$$+o(n^{d+r-1})\,. \tag{45}$$

According to [BV, Theorem 10],

$$\chi_{L^2}(f^*(\mathscr{F}_{\mathbb{C}})(n),h_P)-\chi_{L^2}(f^*(\mathscr{F}_{\mathbb{C}})(n),k_P)=(r'/2)(BV(h,h_P)-BV(h,k_P))n^{d+r-1}+o(n^{d+r-1})\,. \tag{46}$$

Furthermore we compute

$$g_*(\tau(f^*(\bar{\mathscr{F}}),h_P)^{(1)}\hat{c}_1(\overline{\mathscr{O}(1)})^{d+r-1})-g_*(\tau(f^*(\bar{\mathscr{F}}),k_P)^{(1)}\hat{c}_1(\overline{\mathscr{O}(1)})^{d+r-1})$$

$$=g_*(ch(f^*(\bar{\mathscr{F}}_{\mathbb{C}}))\widetilde{Td}(h_P,k_P)^{(0)}c_1(\overline{\mathscr{O}(1)})^{d+r-1})$$

$$=(r'/4)(BV(h,h_P)-BV(h,k_P))(d+r-1)!\,.$$

Combining this fact with (43), (45) and (46), Theorem 8 follows with

$$A=g_*(\tau(f^*(\bar{\mathscr{F}}),h_P)^{(1)}\hat{c}_1(\overline{\mathscr{O}(1)})^{d+r-1})/(d+r-1)!+\frac{r'}{4}BV(h,h_P)+(r'/2)\log((r-1)!)\sigma_d(E)\,. \tag{47}$$

q.e.d.

Remark. Notice that, in the proof of Theorem 8, the exact form of the arithmetic Riemann–Roch–Grothendieck Theorem 7 is not used. One needs only to know the curvature of the determinant line bundle [BGS1] in addition to the algebraic Riemann–Roch–Grothendieck theorem, since this implies that the defect $\delta(S^nE\otimes\mathscr{F})$, defined as in 4.2.4, depends only on the class $[\mathscr{F}_{\mathbb{Q}}\otimes S^nE_{\mathbb{Q}}]\in K_0(X_{\mathbb{Q}})_{\mathbb{Q}}$, to which Lemma 29 applies (see [GS6]).

5.2 *Small sections*

5.2.1 We keep the notations of Sect. 5.1. Fix a positive real number $\varepsilon>0$. Denote by N_ε the number of sections $s\in H^0(X,\mathscr{F}\otimes S^nE)=\Lambda$ such that, for every point $x\in X(\mathbb{C})$,

$$\|s(x)\|\leqq\exp(n(\hat{\sigma}_{d+1}(\bar{E})-\varepsilon)/\sigma_d(E))\,,$$

and by N_ε^* the number of elements $\lambda\in\operatorname{Hom}(\Lambda,\mathbb{Z})$ such that

$$\sup|\lambda(s)|\leqq\exp(n(\varepsilon-\hat{\sigma}_{d+1}(\bar{E}))/\sigma_d(E))\,,$$

where the sup is taken over all sections s such that $\|s(x)\|\leqq 1$ for all x in $X(\mathbb{C})$.

Theorem 9 *As n goes to infinity*

$$\log(N_\varepsilon) - \log(N_\varepsilon^*) = r'\varepsilon n^{d+r} + O(n^{d+r-1}\log(n)) .$$

5.2.2 To prove Theorem 9 we apply a variant of Minkowski's theorem proved in [GS5] to the lattice Λ equipped with the norm

$$|s| = \sup_{x \in X(\mathbb{C})} (\|s(x)\|)\exp(-n(\hat{\sigma}_{d+1}(\bar{E}) - \varepsilon)/\sigma_d(E))) .$$

From [GS5, Theorem 1] we have

$$(48) \quad |\log(N_\varepsilon) - \log(N_\varepsilon^*) + \log \mathrm{vol}_{|\cdot|}(\Lambda \otimes \mathbb{R}/\Lambda)| \leqq 6\log(\chi) - \log \mathrm{vol}(B) ,$$

where χ is the rank of Λ, B the unit ball in $\mathbb{R}^\chi$ and the covolume $\mathrm{vol}_{\|\cdot\|}(\Lambda \otimes \mathbb{R}/\Lambda)$ is taken for the norm $|\cdot|$. From Lemma 29 we get (when n is big enough)

$$(49) \quad \chi = \chi(X(\mathbb{C}), \mathscr{F}_{\mathbb{C}} \otimes S^n E_{\mathbb{C}}) = r'\sigma_d(E)n^{d+r-1} + O(n^{d+r-2}) .$$

Therefore, by Stirling's formula,

$$(50) \quad -\log \mathrm{vol}(B) = O(\chi \log(\chi)) = O(n^{d+r-1}\log(n))$$

and, if $\mathrm{vol}_{\sup}$ is the volume for the sup norm on Λ, we have

$$(51) \quad \begin{aligned} -\log \mathrm{vol}_{|\cdot|}(\Lambda \otimes \mathbb{R}/\Lambda) + \log \mathrm{vol}_{\sup}(\Lambda \otimes \mathbb{R}/\Lambda) &= \chi.(n(\hat{\sigma}_{d+1}(\bar{E}) - \varepsilon)/\sigma_d(E)) \\ &= r'(\varepsilon - \hat{\sigma}_{d+1}(\bar{E}))n^{d+r} \\ &\quad + O(n^{d+r-1}) . \end{aligned}$$

From Lemma 30 below we may compare the sup norm and the L^2 norm on Λ. First, the L^2 norm on Λ is less than or equal to the sup norm. On the other hand the sup norm on Λ, using the isomorphism α of (36), is bounded above by $\frac{n!(r-1)!}{(n+r-1)!}$ times the sup norm of $P(\mathbb{C})$ of $f^*(\mathscr{F}_{\mathbb{C}})(n)$. By Lemma 30 below this is bounded above by a constant multiple of n^{d+r-1} times the L^2 norm on $P(\mathbb{C})$ of $f^*(\mathscr{F}_{\mathbb{C}})(n)$ (using h_P). But, by definition of the metric on $f_*(\mathscr{O}(n))_{\mathbb{C}}$ and Fubini's theorem, this L^2 norm is also the L^2 norm on Λ. It follows that

$$(52) \quad \begin{aligned} &|\log \mathrm{vol}_{\sup}(\Lambda \otimes \mathbb{R}/\Lambda) - \log \mathrm{vol}_{L^2}(\Lambda \otimes \mathbb{R}/\Lambda)| \\ &\qquad = \chi.O(\log n) = O(n^{d+r-1}\log(n)) . \end{aligned}$$

Finally, from Theorem 8, we have

$$(53) \quad -\log \mathrm{vol}_{L^2}(\Lambda \otimes \mathbb{R}/\Lambda) = r'\hat{\sigma}_{d+1}(\bar{E})n^{d+r} + O(n^{d+r-1}\log(n)) .$$

Combining (51), (52) and (53) we get

$$(54) \quad -\log \mathrm{vol}_{|\cdot|}(\Lambda \otimes \mathbb{R}/\Lambda) = r'\varepsilon n^{d+r} + O(n^{d+r-1}\log(n)) .$$

From (48), (49), (50) and (54) we deduce Theorem 9. q.e.d.

5.2.3 To compare the sup norm and the L^2 norm on Λ we use the following result, that was explained to us by Gromov. Let M be a compact Riemannian complex manifold, V an Hermitian complex vector bundle and L an Hermitian line bundle on M. Let $d = \dim_{\mathbb{C}} M$. Denote by $\|\cdot\|_{\sup}$ and $\|\cdot\|_{L^2}$ the sup norm and the L^2 norm of sections of bundles on M.

Lemma 30 *There is a positive constant $C > 0$ (independent on n) such that, for any section s of $V \otimes L^{\otimes n}$ on M,*

$$\|s\|_{\sup} \leqq C n^d \|s\|_{L^2} .$$

Proof. We first prove a local statement. Let B be the open ball $B = \{z \in \mathbb{C}^d / |z| < 3\}$. Assume we are given two smooth strictly positive real functions g and p on B, and a positive definite k by k Hermitian matrix valued function $h = (h_{ij})$ on B. Then there exists a constant $C(p, h, g) > 0$ such that, for any k-tuple $(f_1, \ldots, f_k)$ of holomorphic functions on B, any integer $n \geqq 0$, and any $w \in B$ with $|w| < 1$,

$$\int_{|z-w| \leqq 1} \left(\sum_{i,j} h_{ij}(z) f_i(z) \bar{f}_j(z) \right) p(z)^n g(z) dx_1 dy_1 \ldots dy_d$$
$$\geqq C(p, h, g) \left(\sum_{i,j} h_{ij}(w) f_i(w) \overline{f_j(w)} \right) p(w)^n n^{-2d} .$$

To prove this, let $g_0 = \mathrm{Inf}_{|z| \leqq 2}\, g(z)$, and let $M(h)$ (resp. $m(h)$) be the supremum (resp. the infimum) of the largest (resp. smallest) eigenvalue of $(h_{ij}(z))$, $|z| \leqq 2$. Furthermore, let dp be the differential of p,

$$c' = \left(\operatorname*{Sup}_{|z| \leqq 2} \|dp(z)\| \right) \Big/ \left(\operatorname*{Inf}_{|z| \leqq 2} p(z) \right),$$

and $c = \mathrm{Sup}(c', 1)$. By the mean value theorem, we have

$$p(z) \geqq p(w)(1 - c|z - w|)$$

when $|z| \leqq 2$ and $|w| \leqq 2$.

With these definitions we obtain, where $dx = dx_1 dy_1 \ldots dy_d$,

$$I = \int_{|z-w| \leqq 1} \left(\sum_{i,j} h_{ij}(z) f_i(z) \bar{f}_j(z) \right) p(z)^n g(z)\, dx$$
$$\geqq m(h) g_0 \int_{|z-w| \leqq 1} \sum_{i=1}^{k} |f_i(z)|^2 p(z)^n\, dx$$
$$\geqq m(h) g_0 \int_{|z-w| \leqq 1} \left(\sum_{i=1}^{k} |f_i(z)|^2 \right) (1 - c|z - w|)^n p(w)^n\, dx .$$

Since the f_i's are holomorphic, for any positive real number r, the average value of $|f_i(z)|^2$ on the sphere $|z - w| = r$ is bounded below by $|f_i(w)|^2$. Therefore, if S^{2d-1} is the unit sphere, we get

$$I \geqq m(h) g_0 \left(\sum_{i=1}^{k} |f_i(w)|^2 \right) p(w)^n \operatorname{vol}(S^{2d-1}) \int_{r=0}^{1/c} (1 - cr)^n r^{2d-1}\, dr$$
$$\geqq \frac{m(h)}{M(h)} g_0 \left(\sum_{i,j} h_{ij}(w) f_i(w) \bar{f}_j(w) \right) p(w)^n \operatorname{vol}(S^{2d-1}) c^{-2d} n^{-2d} ,$$

which proves our assertion.

To deduce the lemma from this fact, choose a finite open cover $\Omega_\alpha \subset M$, $\alpha \in A$, biholomorphic isomorphisms $\varphi_\alpha : B \to \Omega_\alpha$, and trivializations of $\varphi_\alpha^* L$ and $\varphi_\alpha^* E$ on B. We assume that

$$M = \bigcup_{\alpha \in A} \varphi_\alpha(\{w \in B \text{ such that } |w| \leqq 1\}) .$$

Write (h_{ij}^α) and p^α for the functions on B induced by the metrics on E and L, and our choice of trivializations. The measure $d\mu$ on M defines a positive function g^α on B by the formula $\varphi_\alpha^*(d\mu) = g^\alpha(z)\,dx$. Let $C = \inf_{\alpha \in A} \sqrt{C(p^\alpha, h^\alpha, g^\alpha)}$. Given any holomorphic section s of $V \otimes L^{\otimes n}$ on M, let x_0 be the point at which the norm of s is maximum. Choose $\alpha \in A$ and $w \in B$, $|w| \leqq 1$, such that $\varphi_\alpha(w) = x_0$. Then, from our choice of trivializations, we get a k-tuple of holomorphic functions $(f_1, \ldots, f_k)$ on B such that $\varphi_\alpha^*(s) = (f_1, \ldots, f_k)$. Therefore, by the result above

$$\begin{aligned} \|s\|_{L^2}^2 &= \int_M \|s(x)\|^2 \, d\mu(x) \\ &\geqq \int_{|z-w| \leqq 1} \|\varphi_\alpha^* s\|^2 g^\alpha \, dx \\ &\geqq C^2 n^{-2d} \left(\sum_{i,j} h_{ij}(w) f_i(w) \bar{f}_j(w) \right) p(w)^n \\ &= C^2 n^{-2d} \|s(x_0)\|^2 = C^2 n^{-2d} \|s\|_{\sup}^2 . \end{aligned}$$

q.e.d.

5.3 Variants

5.3.1 There are other variants of Theorem 9. For instance, if one is only interested in bounding N_ε from below, one may replace the hypothesis A1) of ampleness of E by the vanishing of the even cohomology groups $H^{2k}(X, \mathscr{F} \otimes S^n E)$, $k > 0$, $n \gg 0$. According to (14) this will be enough to get an estimate from below for $\chi_{\mathbb{Q}}(f^*(\bar{\mathscr{F}})(n))$ since, by A2), the cohomology groups $H^k(X(\mathbb{C}), \mathscr{F}_{\mathbb{C}} \otimes S^n E_{\mathbb{C}})$ will vanish for $k > 0$ and $n \gg 0$.

This is an argument which has been used by Vojta in [Vo]. Furthermore, in loc. cit., the hypothesis A2) is also replaced by a weaker assumption.

One could also replace $S^n E$ by $S^{n_1} E_1 \otimes \ldots \otimes S^{n_k} E_k$, or replace Spec($\mathbb{Z}$) by a more general base scheme Y.

5.3.2 One may wonder if $\mathscr{F}$ could be any algebraic coherent sheaf on X, not necessarily locally free on $X_{\mathbb{Q}}$. This raises the question of defining Hermitian metrics on arbitrary coherent sheaves, and we do not know whether this can be done in general.

However, given a Hermitian coherent sheaf $\bar{\mathscr{F}}$ on X (in the sense of Definition 5 in Par. 2.5.) and a subsheaf $\mathscr{I} \subset \mathscr{F}$, one may ask whether there are nontrivial sections s in $H^0(X, \mathscr{I} \otimes S^n E)$ which are bounded in $H^0(X, \mathscr{F} \otimes S^n E)$. Using the exact sequence

$$0 \to H^0(X, \mathscr{I} \otimes S^n E) \to H^0(X, \mathscr{F} \otimes S^n E) \xrightarrow{\pi} H^0(X, (\mathscr{F}/\mathscr{I}) \otimes S^n E)$$

one may apply our lower estimate on N_ε and the Dirichlet box principle to produce such a section s. Indeed, if N_ε is bigger than the cardinality of $\pi(B_\varepsilon)$, where B_ε is the set of sections of $\mathscr{F} \otimes S^n E$ satisfying the first inequality in 5.2.1., we may find two sections s_1, s_2 in B_ε, $s_1 \neq s_2$, having the same image by π. The difference $s = s_1 - s_2$ is then a bounded section of $\mathscr{I} \otimes S^n E$. For further discussion see [L].

For arithmetic surfaces Zhang, using Theorem 9, gets in [Z] an arithmetic analog of the Nakai–Moishezon criterion for ampleness.

5.3.3 When X is a projective space and $E = \mathcal{O}(1)$, Theorem 9 amounts to producing small homogeneous polynomials with integral coefficients and, when the support of $\mathscr{F}/\mathscr{I}$ is flat and finite over $\mathbb{Z}$, 5.3.2. amounts to asking that some partial derivatives of these polynomials vanish at some points. This has been solved classically in the theory of diophantine approximation (Siegel's lemma). For more general X's, Faltings [F2] and Bombieri [Bo] have shown how to replace the use of the arithmetic Riemann–Roch theorem in the work of Vojta [Vo] by a more direct approach, inspired by Siegel's lemma.

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