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Inventiones mathematicae

The space of negative scalar curvature metrics

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1 Introduction

The topology of the space of positive scalar curvature metrics $S^+(M)$ on a closed manifold M has been studied by Hitchin, Gromov, Lawson and Carr (cf. [LM, IV, §9]) and it turned out that the topology of $S^+(M)$ is quite complicated; there are manifolds M such that the ith homotopy group $\pi_i(S^+(M))$ is non-trivial for some (probably arbitrarily great) $i \geqq 0$ and even the "moduli space" $S^+(M)/\mathrm{Diff}(M)$ can have infinitely many path components.

In this paper we will have a look at the natural counterpart: the topology of the space of negative scalar curvature metrics $S^-(M^n)$ on a closed manifold M^n of dimension $n \geqq 3$.

We will prove that $S^-(M)$ (which always is non-empty by [A] resp. [KW]) is always *connected* and *aspherical*:

Theorem 1 $\pi_i(S^-(M)) = 0, \quad i = 0, 1, 2, \dots.$

By Theorem 1 using a general result of infinite dimensional topology due to Palais and Whitehead (cf. [P, Theorem 15 and corollary]) we get a complete insight into the topology of $S^-(M)$:

Theorem 2 $S^-(M)$ *is contractible.*

From this we get the same information for the space of metrics with constant negative scalar curvature $= -1$ denoted by $S_{-1}(M)$.

Corollary. $S_{-1}(M)$ *is contractible.*

Note that on the other hand $S^-(M)$ and $S_{-1}(M)$ are never convex (cf. [L1]).

2 Continuous extension

We are only concerned with closed C^∞-manifolds and C^∞-Riemannian metrics defined on them. Once given a manifold M we fix some reference metric g_M on M and consider the space of all C^∞-metrics $\mathcal{M}(M)$ on M equipped with the usual

C^∞-topology which is the Fréchet topology defined by all the C^k-norms $\|\cdot\|_k$ on M. $\|\cdot\|_k$ is defined with respect to g_M, but the topology does not depend on g_M.

Now let $f: S^i \to \overline{S^-(M)}$ be a continuous map, we are looking for an extension of f on $B^{i+1} \equiv B_6(0) \subset \mathbb{R}^{i+1}$. $S^i \equiv \partial B_6(0)$, i.e. a continuous map $F: B^{i+1} \to S^-(M)$ with $F_{|S^i} \equiv f$.

We start our construction of F by some trivial extension F_1 of f defined as follows: Let g_0 be any fixed metric on M and $(x, t) \in S^i \times [0, 6]/S^i \times \{0\} \equiv B^{i+1}$ (polar coordinates) then we define

$$F_1(x, t) := \begin{cases} (1-t)\cdot g_0 + t\cdot f(x) & \text{on } S^i \times [0, 1]/S^i \times \{0\} \\ f(x) & \text{on } S^i \times [1, 6]/S^i \times \{0\}. \end{cases}$$

Obviously F_1 is a continuous map with image lying in $\mathscr{M}(M)$. Our goal will be to find deformations of F_1 inside of $S^i \times [0, 5]/S^i \times \{0\}$ such that the image of the deformed map lies in $S^-(M)$.

3 Main deformation

Let $N_i^n, i = 1, 2$ be closed manifolds of dimension $n \geqq 3$, $p_i \in N_i$ fixed base points, g_i and $\overline{g_i}$ metrics on N_i, g_i with injectivity radius $\mathrm{inj}(N_i, g_i) > 5$. Now we define for $\lambda_i \geqq 1$ new metrics on $N_i \setminus \{p_i\}$ by

$$g(\lambda_i, g_i, \overline{g_i}) = h(d_{\lambda_i^2 \cdot g_i}(p_i, \mathrm{id}_{N_i}))\cdot G_{\lambda_i} + (1 - h(d_{\lambda_i^2 \cdot g_i}(p_i, \mathrm{id}_{N_i}))\cdot \lambda_i^2 \cdot \overline{g_i}$$

$h \in C^\infty(\mathbb{R}, [0, 1])$ with $h \equiv 1$ on $\mathbb{R}^{\leqq 3}$, $h \equiv 0$ on $\mathbb{R}^{\geqq 4}$ and $G_{\lambda_i} := f_{\lambda_i}^*(g_{\mathbb{R}} + g_{S^{n-1}})$ where $f_{\lambda_i}: B_5(p_i)\setminus\{p_i\} \to]0, 5[\times S^{n-1}$ is a diffeomorphism defined as follows: Fix a linear isometry $I_i: (T_{p_i}N_i, g_i) \to (\mathbb{R}^n, g_{\mathrm{eukl.}})$ and consider the usual polar coordinates on $\mathbb{R}^n\setminus\{0\}$: $P: \mathbb{R}^n\setminus\{0\} \to \mathbb{R}^{>0} \times S^{n-1}$, $P(z) = (\|z\|, z/\|z\|)$ and define $f_{\lambda_i}(z) := P(\lambda_i \cdot (I_i \circ (\exp_{p_i}^{\lambda_i})^{-1}(z)))$ where $\exp_{p_i}^{\lambda_i}$ denotes the exponential map in p_i for the metric $\lambda_i^2 \cdot g_i$.

By definition $\partial(N_1\setminus B_1(p_1))$ and $\partial(N_2\setminus B_1(p_2))$ equipped with these metrics are isometric and can be identified by the orientation preserving isometry $i(\lambda_1, \lambda_2) := f_{\lambda_2}^{-1} \circ f_{\lambda_1}$ yielding $N_1 \#_i N_2$ together with a smooth metric denoted by

$$g(\lambda_1, g_1, \overline{g_2}) \#_i g(\lambda_2, g_2, \overline{g_2}).$$

Now we specialize to $N_1 = M$, $g_1 = g_M$ (a fixed reference and base metric, with $\mathrm{inj}(M, g_M) > 5$), $\overline{g_1} = g$ (varying metrics), $\lambda_1 = \lambda$, $p_1 = p$ and $N_2 = S^n$, $g_2 = g^*$ (a fixed metric with $\mathrm{inj}(S^n, g^*) > 5$), $g_2 = g_n$ (a fixed negative scalar curvature metric on S^n, $\lambda_2 = \mu$, $p_2 = q$.

From the construction above it is clear that there is a family of diffeomorphisms $F(\lambda, \mu): M \to M \# S^n$ with $F(\lambda, \mu) \equiv \mathrm{id}$ on $M\setminus B_5(p)$ which can be chosen such that the metrics $G(g, \lambda, \mu) := F(\lambda, \mu)^*(g(\lambda, g_M, g) \#_{i(\lambda,\mu)} g(\mu, g^*, g_n))$ depend *continuously* on λ and μ.

Now we are ready to define for $\lambda_0 \geqq 1$, $\mu_1 \geqq \mu_0 \geqq 1$, $(x, t) \in B^{i+1}$

$$F_2(\lambda_0, \mu_0, \mu_1, x, t) := \begin{cases} f(x) & \text{on } S^i \times [4,6]/S^i \times \{0\} \\ ((4-t)\cdot\lambda_0^2 + (1-(4-t)))\cdot f(x) & \text{on } S^i \times [3,4]/S^i \times \{0\} \\ (3-t)\cdot G(f(x), \lambda_0, \mu_0) + (1-(3-t))\cdot\lambda_0^2\cdot f(x) & \text{on } S^i \times [2,3]/S^i \times \{0\} \\ G(f(x), \lambda_0, (2-t)\cdot\mu_1 + (1-(2-t))\cdot\mu_0) & \text{on } S^i \times [1,2]/S^i \times \{0\} \\ G(F_1(x,t), \lambda_0, \mu_1) & \text{on } S^i \times [0,1]/S^i \times \{0\}. \end{cases}$$

We claim

Proposition 1. *There are λ_0, μ_0, μ_1 such that $(F_2(x,t) :=)F_2(\lambda_0, \mu_0, \mu_1, x, t)$ is a continuous extension of f with*

(i) $F_2(x,t) \equiv f(x)$ *on* $S^i \times [4,6]/S^i \times \{0\}$
(ii) $\mathscr{S}(M, F_2(x,t)) < 0$ *on* B^{i+1} (where $\mathscr{S}(U,g) := \int_U S_g d\mathrm{Vol}_g$).

Proof. The continuity of $F_2(x,t)$ and (i) follow directly from the construction above. It remains to show (ii) for appropiate λ_0, μ_0, μ_1, which is trivial on $S^i \times [3,6]/S^i \times \{0\}$.

The following estimates are easily checked noting $\mathscr{S}(U, \lambda^2 \cdot g) = \lambda^{n-2} \cdot \mathscr{S}(U, g)$, $\lambda > 0$ ($B_r(p)$ with respect to $\lambda^2 \cdot g_M$):

(1) there is a $\mu_0 \geqq 1$, independent of $\lambda \geqq 1$, $x \in S^i$, such that $\mathscr{S}(B_4(p), G(f(x), \lambda, \mu)) < 0$ for $\mu \geqq \mu_0$
(2) given $k > 0$ there is a $\lambda(k) \geqq 1$ such that for $x \in S^i$: $\mathscr{S}(M \setminus B_5(p), \lambda^2(k)\cdot f(x)) < -k$
(3) there is $c > 0$, independent of $\lambda \geqq 1$ and $(x,t) \in S^i \times [2,3]$, such that $\mathscr{S}(B_5(p), (3-t)\cdot G(f(x), \lambda, \mu_0) + (1-(3-t))\cdot\lambda^2 f(x)) < c$
(4) given $K > 0$ there is a $\mu(K) \geqq \mu_0$ independent of $\lambda \geqq 1$ such that $\mathscr{S}(B_5(p), G(F_1(x,t), \lambda, \mu(K)) < -K$ for each $(x,t) \in B^{i+1}$.

Now we verify (ii) on $S^i \times [0,3]/S^i \times \{0\}$ for μ_0 as in (1), $\lambda_0 := \lambda(2c)$, $\mu_1 := \mu(|m|+1)$, where $m := \max_{B^{i+1}} \mathscr{S}(M \setminus B_5(p), \lambda_0^2 \cdot F_1(x,t))$: on

$$S^i \times [2,3]/S^i \times \{0\}: \mathscr{S}(M, (3-t)\cdot G(f(x), \lambda_0, \mu_0) + (1-(3-t))\cdot\lambda_0^2 f(x))$$
$$= \mathscr{S}(B_5(p), \ldots) + \mathscr{S}(M \setminus B_5(p), \ldots) < -c < 0, \quad \text{by (2) and (3)}$$

on

$$S^i \times [1,2]/S^i \times \{0\}: \mathscr{S}(M, G(f(x), \lambda_0, (2-t)\mu_1 + (1-(2-t))\cdot\mu_0)) < 0$$

by (1) and $f(x) \in S^-(M)$, on

$$S^i \times [0,1]/S^i \times \{0\}: \mathscr{S}(M, G(F_1(x,t), \lambda_0, \mu_1))$$
$$\leqq m + \mathscr{S}(B_5(p), G(F_1(x,t), \lambda_0, \mu_1)) < -1 \text{ by (4).}$$ □

4 Eigenvectors of the Conformal Laplacian

The scalar curvature S_g transforms under conformal deformations $g_1 = u^{4/n-2} \cdot g$, $\dim M = n \geqq 3$, according to (cf. [K, (3.2)]:

$$L_g u \equiv -\gamma \cdot \Delta_g u + S_g \cdot u = S_{g_1} \cdot u^\alpha, \quad \gamma = 4\frac{n-1}{n-2}, \quad \alpha = \frac{n+2}{n-2}.$$

We are interested in the linear operator L_g which is sometimes called "conformal Laplacian".

Recall from [K, 3.A], that the first eigenvalue $\lambda_1(g)$ of L_g, which fulfills

$$\lambda_1(g) = \inf_{u \in C^\infty(M),\, u \not\equiv 0} \int_M (\gamma \cdot \| \nabla u \|^2 + S_g \cdot u^2)\, dV_g \Big/ \int_M u^2 \cdot dV_g \equiv \inf J_g(u),$$

has a one dimensional eigenspace generated by a (unique) eigenvector $v(g) \in C^\infty(M)$ with $v(g) > 0$, $\max v(g) = 1$.

For completness we will show the following hardly surprising fact, which is hard to quote explicitly from literature:

Proposition 2 *If $g_n \to g$ with respect to the C^∞-topology, then $\lambda_1(g_n) \to \lambda_1(g)$ and $v(g_n) \to v(g)$ also with respect to the C^∞-topology.*

Proof. From the definition of $J_g(u)$, we get for $\varepsilon > 0$ some n_0, such that: $(1-\varepsilon)|J_{g_n}(u)| \leqq |J_g(u)| \leqq (1+\varepsilon)|J_{g_n}(u)|$ for $n \geqq n_0$ and each $u \in C^\infty(M) \setminus \{0\}$. This implies $\lambda_1(g_n) \to \lambda_1(g)$. Furthermore $0 < v(g_n) \leqq 1$, $g_n \to g$ in the C^∞-topology and $L_{g_n} v(g_n) = \lambda(g_n) \cdot v(g_n)$ imply by standard elliptic theory $\| v(g_n) \|_{C^k_{g_M}} < c_k$, c_k independent of n. From $\lambda_1(g_n) \to \lambda_1(g)$ and the Arzela–Ascoli-Theorem we obtain converging subsequences (by iteration) in $\| \cdot \|_k$ and we take the diagonal sequence of these subsequences. This converges in C^∞ to $\tilde{v} \in C^\infty(M)$, with $L_g \tilde{v} = \lambda_1(g) \cdot \tilde{v}$, $\tilde{v} \geqq 0$, $\max \tilde{v} = 1$ (from [K, 3.A], we conclude again $\tilde{v} > 0$). But this $\tilde{v}$ has to be the unique eigenvector $v(g)$, which implies that a fortiori $v(g_n)$ converges. □

5 Final deformation

Now we are ready to complete the proof of our theorem. Since $\mathscr{S}(M, F_2(x,t)) < 0$, $(x,t) \in B^{i+1}$, we conclude from $\lambda_1(g) = \inf J_g(u)$: $\lambda_1(F_2(x,t)) < 0$ on B^{i+1}. We define

$$F(x,t) = \begin{cases} f(x) & \text{on } S^i \times [5,6]/S^i \times \{0\} \\ ((5-t) \cdot v(f(x)) + (1-(5-t)))^{\frac{4}{n-2}} \cdot f(x) & \text{on } S^i \times [4,5]/S^i \times \{0\} \\ v(F_2(x,t))^{\frac{4}{n-2}} \cdot F_2(x,t) & \text{on } S^i \times [0,4]/S^i \times \{0\} \end{cases}$$

and we claim

Proposition 3. *F is a continuous extension of $f: S^i \to S^-(M)$ with $F(B^{i+1}) \subset S^-(M)$.*

Proof. Propositions 1 and 2 imply the continuity. Now we verify $F(x,t) \in S^-(M)$: On $S^i \times [5,6]/S^i \times \{0\}$ there is nothing to prove, on $S^i \times [4,5]/S^i \times \{0\}$ we calculate:

$$S_{F(x,t)} \cdot ((5-t) \cdot v(f(x)) + (1-(5-t)))^\alpha = L_{f(x)}((5-t) \cdot v(f(x)) + (1-(5-t)))$$
$$= (5-t) \cdot \lambda_1(f(x)) \cdot v(f(x)) + S_{f(x)} \cdot (1-(5-t)) < 0$$

on $S^i \times [0,4]/S^i \times \{0\}$ we obtain:

$$S_{F(x,t)} \cdot v(F_2(x,t))^\alpha = L_{F_2(x,t)} v(F_2(x,t)) = \lambda_1(F_2(x,t)) \cdot v(F_2(x,t)) < 0.$$

Since $(\ldots)^\alpha > 0$, we conclude $S_{F(x,t)} < 0$. □

6 Constant scalar curvature

Finally we will show that $S_{-1}(M)$ is contractible (which implies $\pi_i(S_{-1}(M)) = 0$, $i = 0, 1, \ldots$), this can be deduced from:

Proposition 4. *There is a continuous map* $p: S^-(M) \to S_{-1}(M)$ *with* $p_{|S_{-1}(M)} \equiv \mathrm{id}$.

Proof. Let $g \in S^-(M)$ and u a positive solution of the Yamabe equation $-\gamma \cdot \Delta_g u + S_g \cdot u = -u^\alpha$.

We assert

(i) u is unique

(ii) $p(g) := u^{4/n-2} \cdot g$ fulfills the claims.

(i): Let v be a second positive solution, $u^{4/n-2} \cdot g$ and $v^{4/n-2} \cdot g$ have scalar curvature $\equiv -1$. write $v \equiv w \cdot u$ for some $w > 0$, $w \in C^\infty(M)$. Then w fulfills the Yamabe equation for $g_1 = u^{4/n-2} \cdot g$:

$$-\gamma \cdot \Delta_{g_1} w - w = -w^\alpha,$$

now assume $w \not\equiv 1$: since $\alpha > 1$ we get $\Delta_{g_1} w > 0$ or the maximum of w or $\Delta_{g_1} w < 0$ in the minimum of w, which yields a contradiction.

(ii): From (i) $p_{|S_{-1}(M)} \equiv \mathrm{id}$, so it remains to show $g_n \to g$ in C^∞ implies $u_n \to u$ in C^∞ (u_n, u denote the solutions of the Yamabe equation of g_n, g): $-K_1 < S_{g_n} < -K_2$ for some $K_1 > K_2 > 0$ independent of n yields

$$0 < K_2^{\frac{1}{1-\alpha}} < (\min|S_{g_n}|)^{\frac{1}{1-\alpha}} < u_n < (\max|S_{g_n}|)^{\frac{1}{1-\alpha}} < K_1^{\frac{1}{1-\alpha}}.$$

Now using *both* bounds one can proceed as in Proposition 2 to get C^k-estimates independent of n. Again uniqueness of u as shown in (i) implies convergence of u_n. □

Now let $H: S^-(M) \times [0, 1] \to S^-(M)$ be a contraction to a $g_0 \in S_{-1}(M)$, i.e. $H(\cdot, 0) \equiv \mathrm{id}$, $H(\cdot, 1) \equiv g_0$. Consider $p \circ H_{|S_{-1}(M)} \times [0, 1] \to S_{-1}(M)$. $p \circ H$ is continuous by Proposition 4 and $p \circ H(\cdot, 0)_{|S_{-1}(M)} \equiv \mathrm{id}$, $p \circ H(\cdot, 1) \equiv g_0 \in S_{-1}(M)$, i.e. $S_{-1}(M)$ is contractible.

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Note added in proof. The space of negative Ricci curvature metrics $\mathrm{Ric}^-(M)$ is also non-empty and contractible. Furthermore $\mathrm{Ric}^-(M)$ is dense in $\mathcal{M}(M)$ w.r.t. C^0-topology. This is proved in a more geometric but more intricated and conceptually different way. For details we refer to [L2].