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On the group of holomorphic automorphisms of $\mathbb{C}^n$

Erik Andersén[1,*] and László Lempert[2,]**

[1] Department of Mathematics, Lund University, Box 118, S-221 00 Lund, Sweden
[2] Department of Mathematics, Purdue University, West Lafayette, IN 47907, USA

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1 Introduction

The group of holomorphic automorphisms of $\mathbb{C}^n$, denoted Aut $\mathbb{C}^n$, consists of those holomorphic mappings $\Phi: \mathbb{C}^n \to \mathbb{C}^n$ that have a holomorphic inverse $\Psi: \mathbb{C}^n \to \mathbb{C}^n$. The group operation is composition. It is well known that with the topology of locally uniform convergence Aut $\mathbb{C}^n$ becomes a topological group (compare Proposition 6.4) Aut $\mathbb{C}^1$ of course consists of linear mappings $az + b$ ($a, b \in \mathbb{C}$, $a \neq 0$), but when $n \geqq 2$ Aut $\mathbb{C}^n$ is infinite dimensional, for it contains mappings of form

$$(1.1) \qquad (z_1, \ldots, z_n) \to (z_1, \ldots, z_{n-1}, f(z_1, \ldots, z_{n-1}) + z_n),$$

or more generally,

$$(1.2) \quad (z_1, \ldots, z_n) \to (z_1, \ldots, z_{n-1}, f(z_1, \ldots, z_{n-1}) + h(z_1, \ldots, z_{n-1}) z_n),$$

with f, h holomorphic functions on $\mathbb{C}^{n-1}$, h nowhere 0. Following [10] automorphisms of form (1.1) will be called shears, and those of form (1.2) will be called overshears. Slightly more generally, the term (over)shears will also be used for mappings of form (1.1) resp. (1.2) with the coordinates $z_1, \ldots, z_n$ permuted.

The first result on the structure of these automorphism groups is due to Jung [8]. He considers algebraic (i.e. polynomial) automorphisms of $\mathbb{C}^2$. By the fundamental theorem of algebra such an automorphisms must have constant Jacobi determinant. By Jung's theorem the group of algebraic automorphisms of $\mathbb{C}^2$ with Jacobi determinant 1 is generated by polynomial shears. Here and in what follows, "generating" will be meant in the algebraic sense, even though the groups involved may carry a topology.

The corresponding algebraic problem in $\mathbb{C}^n$, $n \geqq 3$ is open. The only general result is due to Šafarevič, who defines a "Zariski topology" on the group of

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algebraic automorphisms of $\mathbb{C}^n$, with Jacobi determinant 1, and proves that polynomial shears generate a "Zariski dense" subgroup, see [11].

In [1] the first named author studied the subgroup $\mathrm{Aut}_1\,\mathbb{C}^n \subset \mathrm{Aut}\,\mathbb{C}^n$ consisting of automorphisms with Jacobi determinant 1 ("volume-preserving automorphisms"). He proved that for $n \geqq 2$ shears generate a dense subgroup of $\mathrm{Aut}_1\,\mathbb{C}^n$, but this subgroup, at least for $n = 2$, is not the entire group $\mathrm{Aut}_1\,\mathbb{C}^n$. Indeed, the mapping

$$(z_1, z_2) \to (z_1 e^{z_1 z_2}, z_2 e^{-z_1 z_2}) \tag{1.3}$$

is in $\mathrm{Aut}_1\,\mathbb{C}^2$ but is not a (finite) composition of shears.

The principal aim of this paper is to clarify, for arbitrary $n \geqq 2$, the relation between the group generated by (over) shears and $\mathrm{Aut}_1\,\mathbb{C}^n$ (resp. $\mathrm{Aut}\,\mathbb{C}^n$). Here are the main results, which are valid for any $n \geqq 2$.

Theorem 1.1 *There is an element of* $\mathrm{Aut}_1\,\mathbb{C}^n$ *which is not a finite composition of shears.*

Theorem 1.2 *There is an element of* $\mathrm{Aut}\,\mathbb{C}^n$ *which is not a finite composition of overshears.*

Theorem 1.3 *The subgroup of* $\mathrm{Aut}\,\mathbb{C}^n$ *generated by overshears is dense.*

In connection with this last theorem we shall also prove a Runge type approximation theorem for biholomorphic mappings defined on subdomains of $\mathbb{C}^n$, see Sect. 4. Actually, holomorphicity does not seem to play any role in Theorem 1.3; corresponding theorems hold for the diffeomorphism group of $\mathbb{R}^n$, or even more general manifolds (see Sect. 5).

As explained above, Theorem 1.1 for $n = 2$ is already contained in [1]. The passage from $n = 2$ to $n > 2$ is not obvious. We were able to extend the Nevanlinna theoretic ideas involved in [1] to prove Theorem 1.2 for $n = 2$ (see Sect. 8), but the proof of the general case uses completely different ideas and is not constructive at all. That is, we can not *construct* the automorphisms whose existence is claimed in Theorems 1.1 and 1.2. This raises some obvious questions, which will be discussed in Sect. 9.

2 Approximating biholomorphisms by compositions of (over)shears

Let $G(\mathbb{C}^n)$ (resp. $G_1(\mathbb{C}^n)$) denote the subgroup of $\mathrm{Aut}\,\mathbb{C}^n$ (resp. $\mathrm{Aut}_1\,\mathbb{C}^n$) consisting of finite compositions of overshears (resp. shears). We want to prove a theorem generalizing Theorem 1.3 in which a biholomorphic mapping Φ from some domain $D \subset \mathbb{C}^n$ into $\mathbb{C}^n$ is to be approximated by elements in $G(\mathbb{C}^n)$. It is natural and almost necessary to assume that D is such that any holomorphic function on D can be locally uniformly approximated by entire functions. In this case we shall say that D has the Runge property (rather than saying that D is a Runge domain, which is often reserved for domains of holomorphy). It is easy to prove that if $\Phi(D)$ has the Runge property and if Φ can be approximated by entire holomorphic maps then D must also have the Runge property. Using the mentioned fact that $\mathrm{Aut}\,\mathbb{C}^n$ is a topological group it is easy to prove the converse implication, namely that if D has the Runge property, and Φ can be approximated by elements of $\mathrm{Aut}\,\mathbb{C}^n$ then the image $\Phi(D)$ must have the Runge property too.

It turns out that the Runge property for D and $\Phi(D)$ is not quite sufficient for approximation (see Sect. 5) but a stronger property of D is

Theorem 2.1 *Let $D \subset \mathbb{C}^n$ be a starshaped domain, $\Phi: D \to \mathbb{C}^n$ a biholomorphic mapping whose image $\Phi(D)$ has the Runge property. Then Φ can be approximated by elements of $G(\mathbb{C}^n)$ locally uniformly on D. If the Jacobi determinant of Φ is identically 1, Φ can be even be approximated by elements of $G_1(\mathbb{C}^n)$.*

Remark. It is known (a simple proof is in [9]) that any star-shaped domain has the Runge property.

Theorem 2.1 clearly implies Theorem 1.3.

Let us denote, for a domain $R \subset \mathbb{C}^n$ that has the Runge property, by $B(R)$ the space of biholomorphic mappings of R into $\mathbb{C}^n$, endowed with the topology of locally uniform convergence on R. Put furthermore

$$G(R) = \{\Phi|_R : \Phi \in G(\mathbb{C}^n)\}.$$

We shall endow $G(R) \subset B(R)$ with the subspace topology. The key to the proof of Theorem 2.1 is the definition (and later, identification) of the tangent space $\mathfrak{g}(R)$ of $G(R)$ at id $\in G(R)$, along the same lines as in [1].

Definition 2.2 A holomorphic mapping $F: R \to \mathbb{C}^n$ is said to be a tangent vector to $G(R)$ at id $\in G(R)$ if there is a family $S_\varepsilon \in G(R)$ ($\varepsilon \in \mathbb{C}$) such that

$$S_\varepsilon(z) = z + \varepsilon F(z) + o_z(\varepsilon) \,. \tag{2.1}$$

Here $o_z(\varepsilon)$ means a term with the property $o_z(\varepsilon)/\varepsilon \to 0$ ($\varepsilon \to 0$) locally uniformly in $z \in R$. The set of tangent vectors at id is denoted $\mathfrak{g}(R)$.

Remark 2.3 We emphasize that we are not requiring that the family S_ε should smoothly, or even continuously, depend on ε. Thus we are defining tangent vectors using nondifferentiable curves! In the last analysis it is this liberal definition that will permit us to identify the tangent space $\mathfrak{g}(R)$.

In what follows, we shall always assume that the domain R has the Runge property.

Theorem 2.4 *$\mathfrak{g}(R)$ consists of all holomorphic mappings $F: R \to \mathbb{C}^n$.*

Actually, we shall need a result stronger than Theorem 2.4: a *parametrized* version of this theorem.

Theorem 2.5 *Let Ω be a compact metric space and $\mathscr{R} \subset \mathbb{C}^n \times \Omega$ an open set such that for every $\omega \in \Omega$*

$$R_\omega = \{z \in \mathbb{C}^n : (z, \omega) \in \mathscr{R}\}$$

is a domain that has the Runge property. If $F: \mathscr{R} \to \mathbb{C}^n$ is continuous and for every $\omega \in \Omega$ the mapping $F_\omega(z) = F(z, \omega)$ defines a holomorphic map $F_\omega: R_\omega \to \mathbb{C}^n$ then $F_\omega \in \mathfrak{g}(R_\omega)$ holds "uniformly". That is, there is a family of continuous mappings $S_\varepsilon: \mathscr{R} \to \mathbb{C}^n$ ($\varepsilon \in \mathbb{C}$) such that for $\omega \in \Omega$ the mapping $S_{\varepsilon,\omega}$ defined by $S_{\varepsilon,\omega}(z) = S_\varepsilon(z, \omega)$ is in $G(R_\omega)$, and

$$S_\varepsilon(z, \omega) = z + \varepsilon F(z, \omega) + o_{z,\omega}(\varepsilon) \,. \tag{2.2}$$

Here $o_{z,\omega}(\varepsilon)/\varepsilon \to 0$ as $\varepsilon \to 0$, locally uniformly in $(z, \omega) \in \mathscr{R}$.

Theorems 2.4 and 2.5 will be proved in Sect. 3, and Theorem 2.1 in Sect. 4.

3 The tangent space $\mathfrak{g}(R)$

When $R = \mathbb{C}^n$, $\mathfrak{g}(\mathbb{C}^n)$ becomes the "Lie algebra" of the group $G(\mathbb{C}^n)$. Obviously, the restriction of any $F \in \mathfrak{g}(\mathbb{C}^n)$ to a general R is in $\mathfrak{g}(R)$. We shall first describe $\mathfrak{g}(\mathbb{C}^n)$.

Proposition 3.1 *The general linear group* $\mathrm{GL}(n, \mathbb{C})$ *is contained in* $G(\mathbb{C}^n)$.

Proof. Any lower triangular matrix in $\mathrm{GL}(n, \mathbb{C})$ is a composition of (linear) overshears. The same is true for upper triangular matrices. Since any matrix in $\mathrm{GL}(n, \mathbb{C})$ is a product of lower and upper triangular matrices, the proposition follows.

Proposition 3.2 $\mathfrak{g}(\mathbb{C}^n)$ *is a vector space; if* $F, G \in \mathfrak{g}(\mathbb{C}^n)$, *and* $c \in \mathbb{C}$ *then* $F + G$, $cF \in \mathfrak{g}(\mathbb{C}^n)$.

Proof. If $S_\varepsilon(z) = z + \varepsilon F(z) + o_z(\varepsilon)$ and $T_\varepsilon(z) = z + \varepsilon G(z) + o_z(\varepsilon)$ then

$$S_\varepsilon \circ T_\varepsilon(z) = z + \varepsilon(F(z) + G(z)) + o_z(\varepsilon) ,$$

$$S_{c\varepsilon}(z) = z + \varepsilon c F(z) + o_z(\varepsilon) .$$

Proposition 3.3 *If a mapping* $F : \mathbb{C}^n \to \mathbb{C}^n$ *is in* $\mathfrak{g}(\mathbb{C}^n)$ *and* $L \in \mathrm{GL}(n, \mathbb{C})$ *then* $L^{-1} \circ F \circ L \in \mathfrak{g}(\mathbb{C}^n)$.

Proof. Let $S_\varepsilon \in G(\mathbb{C}^n)$ be as in (2.1). Then, by Proposition 3.1, $L^{-1} \circ S_\varepsilon \circ L \in G(\mathbb{C}^n)$; and

$$L^{-1} \circ S_\varepsilon \circ L(z) = z + \varepsilon L^{-1} \circ F \circ L(z) + o_\varepsilon(z) .$$

Proposition 3.4 *If* $f, g : \mathbb{C}^{n-1} \to \mathbb{C}$ *are holomorphic then the mapping*

(3.1) $\qquad F(z_1, \ldots, z_n) = (0, \ldots, 0, f(z_1, \ldots, z_{n-1}) + g(z_1, \ldots, z_{n-1}) z_n)$

is in $\mathfrak{g}(\mathbb{C}^n)$.

Proof. Observe that the mapping

$$S_\varepsilon(z) = (z_1, \ldots, z_{n-1}, \varepsilon f(z_1, \ldots, z_{n-1}) + e^{\varepsilon g(z_1, \ldots, z_{n-1})} z_n)$$

is in $G(\mathbb{C}^n)$. With this S_ε and F above (2.1) is satisfied.

Proposition 3.5 *If* $P = (P_j) : \mathbb{C}^n \to \mathbb{C}^n$ *is a polynomial map and* $\operatorname{div} P = \Sigma \partial P_j / \partial z_j = 0$ *then* $P \in \mathfrak{g}(\mathbb{C}^n)$.

Proof. See Lemma 5.7 of [1] or Lemma 2 in [11]. Indeed, there it is proved that P is in the "Lie algebra" $\mathfrak{g}_1(\mathbb{C}^n)$ of $G_1(\mathbb{C}^n)$.

To remove the condition $\operatorname{div} P = 0$ from Proposition 3.5 we shall need some simple algebraic facts.

Proposition 3.6 *Suppose* $p(x)$ *is a polynomial of one variable with complex coefficients, of degree* $d \geqq 0$. *Then the polynomials* $p(x), p(x-1), \ldots, p(x-d)$ *span the space of polynomials of degree at most* d.

Proof. By induction on d. Assume the claim is true for polynomials of degree $d - 1$, and let $p(x)$ be as in the proposition. Put $q(x) = p(x) - p(x-1)$; this is of degree $d - 1$, so $q(x), q(x-1), \ldots, q(x-d+1)$ span the space of polynomials of degree

$\leq d-1$. Hence $p(x), q(x), q(x-1), \ldots, q(x-d+1)$ span all polynomials of degree $\leq d$, whence the claim follows.

Proposition 3.7 *Let $p: \mathbb{C}^n \to \mathbb{C}$ be a polynomial of n variables. Then there are a finite number of polynomials $p_1, p_2, \ldots$ of one variable, and linear forms $\ell_1, \ell_2, \ldots$ of n variables such that*

$$p(z) = p_1(\ell_1(z)) + p_2(\ell_2(z)) + \ldots$$

Proof. Given nonnegative integers $\nu \leq \mu$, by Proposition 3.6 there are complex numbers $c_0, \ldots, c_\mu$ such that $x^\nu = \sum_0^\mu c_j(x-j)^\mu$. Writing x/y for x we get

$$x^\nu y^{\mu-\nu} = \sum_0^\mu c_j(x-jy)^\mu ,$$

whence the proposition follows for polynomials of two variables. For general n the proposition follows by an inductive argument.

Proposition 3.8 *If p is a polynomial of one variable and ℓ is a linear from on $\mathbb{C}^n$ then there is a polynomial mapping $P: \mathbb{C}^n \to \mathbb{C}^n$ such that $P \in \mathfrak{g}(\mathbb{C}^n)$ and* $\operatorname{div} P = p \circ \ell$.

Proof. By the $GL(n, \mathbb{C})$ invariance of $\mathfrak{g}(\mathbb{C}^n)$ (see Proposition 3.3) we can assume $\ell(z) = z_1$. By Proposition 3.4 the mapping $P(z) = (0, \ldots, 0, p(z_1)z_n)$ is in $\mathfrak{g}(\mathbb{C}^n)$, and $\operatorname{div} P(z) = p(z_1)$.

Proposition 3.9 *Any polynomial mapping $P: \mathbb{C}^n \to \mathbb{C}^n$ is in $\mathfrak{g}(\mathbb{C}^n)$.*

Proof. Using Proposition 3.7 write $\operatorname{div} P(z) = \Sigma p_j(\ell_j(z))$ with ℓ_j linear forms on $\mathbb{C}^n$ and p_j polynomials of one variable. By Proposition 3.8 there are $P_j \in \mathfrak{g}(\mathbb{C}^n)$ such that $\operatorname{div} P_j = p_j \circ \ell_j$. Then $Q = P - \Sigma P_j$ is a polynomial mapping and $\operatorname{div} Q = 0$. Hence, by Proposition 3.5 $Q \in \mathfrak{g}(\mathbb{C}^n)$. Thus $P = Q + \Sigma P_j \in \mathfrak{g}(\mathbb{C}^n)$.

Proof of Theorem 2.4 Let $F: R \to \mathbb{C}^n$ be holomorphic. There is a sequence of polynomial mappings $P_j: \mathbb{C}^n \to \mathbb{C}^n$ that converges to F locally uniformly on R. By Proposition 3.9 for every j there is a family $S_{\varepsilon, j} \in G(\mathbb{C}^n)$ such that

$$S_{\varepsilon, j}(z) = z + \varepsilon P_j(z) + o_{z, j}(\varepsilon) , \tag{3.2}$$

where for any fixed j we have $o_{z,j}(\varepsilon)/\varepsilon \to 0$ as $\varepsilon \to 0$, locally uniformly in $z \in \mathbb{C}^n$, hence locally uniformly in $z \in R$. A simple diagonal process (cf. Lemma 5.9 in [1]) now shows that there exists a sequence $j(\varepsilon) \to \infty$ (very slowly) as $\varepsilon \to 0$ such that

$$S_{\varepsilon, j(\varepsilon)}(z) = z + \varepsilon F(z) + o_z(\varepsilon) ,$$

with $o_z(\varepsilon)/\varepsilon \to 0$ $(\varepsilon \to 0)$ locally uniformly in $z \in R$. Thus $F \in \mathfrak{g}(R)$.

To prove Theorem 2.5 we shall need a parametrized version of Proposition 3.9. Let Ω be a compact metric space, $P_\omega: \mathbb{C}^n \to \mathbb{C}^n$ $(\omega \in \Omega)$ a family of polynomial mappings of bounded degree such that $(z, \omega) \to P_\omega(z) = P(z, \omega)$ is continuous on $\mathbb{C}^n \times \Omega$.

Proposition 3.10 *There is a family $S_{\varepsilon, \omega} \in G(\mathbb{C}^n)$, $\varepsilon \in \mathbb{C}$, $\omega \in \Omega$, such that for any ε the mapping $(z, \omega) \to S_{\varepsilon, \omega}(z)$ is continuous and*

$$S_{\varepsilon, \omega}(z) = z + \varepsilon P_\omega(z) + o_{z, \omega}(\varepsilon) ,$$

with $o_{z,\omega}(\varepsilon)/\varepsilon \to 0$ as $\varepsilon \to 0$, *locally uniformly in $(z, \omega) \in \mathbb{C}^n \times \Omega$.*

Proof. This follows by examining the proofs that led to Proposition 3.9. For the case when $\Omega = [0, 1] \subset \mathbb{R}$ and P_ω are divergence free, see also Lemma 5.7 of [1].

Proof of Theorem 2.5 Let us exhaust $\mathcal{R}$ by a sequence of compact sets $\mathcal{K}_j$. Thus $\mathcal{K}_j \subset \operatorname{int} \mathcal{K}_{j+1}$, $\bigcup_1^\infty \mathcal{K}_j = \mathcal{R}$.

Fix j. For any $\omega \in \Omega$ there are open sets $D \subset \mathbb{C}^n$, $\Omega' \subset \Omega$, *D bounded, such that*

$$\mathcal{K}_j^\omega = \{(z, \theta) \in \mathcal{K}_j, \theta = \omega\}$$

is covered by $D \times \Omega'$, and $\bar{D} \times \bar{\Omega}' \subset \mathcal{R}$. By the compactness of $\mathcal{K}_j$ then $\mathcal{K}_j^\theta \subset D \times \Omega'$ if θ is sufficiently near to ω. Shrinking Ω', we can in fact assume this holds when $\theta \in \Omega'$. Since Ω is compact we get finitely many open sets $D_\nu \subset \mathbb{C}^n$ and $\Omega_\nu \subset \Omega$, D_ν bounded, such that

$$\bigcup_\nu \Omega_\nu = \Omega, \qquad \bigcup_\nu (\bar{D}_\nu \times \bar{\Omega}_\nu) \subset \mathcal{R}\,,$$

and if $\omega \in \Omega_\nu$ then

$$\mathcal{K}_j^\omega \subset D_\nu \times \Omega_\nu\,. \tag{3.3}$$

By further subdividing the sets Ω_ν we can assume that for $\omega, \theta \in \Omega_\nu$

$$\max_{z \in \bar{D}_\nu} |F_\omega(z) - F_\theta(z)| < 1/j\,. \tag{3.4}$$

Construct a nonnegative partition of unity ϕ_ν subordinate to Ω_ν, and from every Ω_ν pick a point ω_ν. By the Runge property there are polynomial mappings $Q_\nu : \mathbb{C}^n \to \mathbb{C}^n$ such that

$$\max_{z \in \bar{D}_\nu} |Q_\nu(z) - F_{\omega_\nu}(z)| < 1/j\,. \tag{3.5}$$

Put $P_{j,\omega}(z) = \sum_\nu Q_\nu(z)\phi_\nu(\omega)$. This is a polynomial mapping in z and continuous in (z, ω). Also $\sup_\omega \deg P_{j,\omega} < \infty$. If $(z, \omega) \in \mathcal{K}_j$

$$\begin{aligned}
|P_{j,\omega}(z) - F_\omega(z)| &= \left|\sum_\nu Q_\nu(z)\phi_\nu(\omega) - \sum_\nu F_\omega(z)\phi_\nu(\omega)\right| \\
&\leqq \sum_{\nu:\omega \in \Omega_\nu} |Q_\nu(z) - F_\omega(z)|\phi_\nu(\omega) \\
&\leqq \sum_{\nu:\omega \in \Omega_\nu} (|Q_\nu(z) - F_{\omega_\nu}(z)| + |F_{\omega_\nu}(z) - F_\omega(z)|)\phi_\nu(z) \\
&< 2/j\,,
\end{aligned}$$

since $(z, \omega) \in \mathcal{K}_j$ (i.e. $(z, \omega) \in \mathcal{K}_j^\omega$) and $\omega \in \Omega_\nu$ by (3.3) imply $z \in D_\nu$, so that the last inequality follows from (3.4), (3.5).

Thus $P_j(z, \omega) = P_{j,\omega}(z) \to F(z, \omega)$ $(j \to \infty)$ locally uniformly on $\mathcal{R}$. Using Proposition 3.10, choose families $S_{\varepsilon,j,\omega} \in G(\mathbb{C}^n)$ such that for fixed ε, j the map $S_{\varepsilon,j,\omega}(z)$ is continuous in $(z, \omega) \in \mathbb{C}^n \times \Omega$, and

$$S_{\varepsilon,j,\omega}(z) = z + \varepsilon P_{j,\omega}(\varepsilon) + o_{z,j,\omega}(\varepsilon)\,,$$

with $o_{z,j,\omega}(\varepsilon)/\varepsilon \to 0$ $(\varepsilon \to 0)$ for every fixed j, locally uniformly in $(z, \omega) \in \mathcal{C}^n \times \Omega$. At this point the theorem follows as Theorem 2.4 followed from (3.2). We shall omit the details, but refer the reader to [1, Sect. 5], where an analogous statement is proved (with $\Omega = [0, 1] \subset \mathbb{R}$).

4 Proof of Theorem 2.1

We shall need a simple result about the convergence of a general one-step method in the theory of ordinary differential equations. Let $\mathscr{R}$ be an arbitrary open subset of $\mathbb{C}^n \times [0, 1]$. For $t \in [0, 1]$ put $\mathscr{R}_t = \{z \in \mathbb{C}^n : (z, t) \in \mathscr{R}\}$. Suppose $D \subset \mathscr{R}_0$ is open.

Lemma 4.1 *Let $F : \mathscr{R} \to \mathbb{C}^n$ and $Z : D \times [0, 1] \to \mathbb{C}^n$ be continuously differentiable mappings. Assume that $Z(\zeta, t) \in \mathscr{R}_t$ for $\zeta \in D$, $t \in [0, 1]$, and*

$$\frac{\partial Z(\zeta, t)}{\partial t} = F(Z(\zeta, t), t); \qquad Z(\zeta, 0) = \zeta .$$

Let, furthermore $V : \mathscr{R} \times \mathbb{C} \to \mathbb{C}^n$ satisfy $V(z, t, \varepsilon)/\varepsilon \to 0$ as $\varepsilon \to 0$, locally uniformly in $(z, t) \in \mathscr{R}$. With ε a (small) positive number, and $\zeta \in D$, define a sequence $w_k = w_k(\zeta, \varepsilon)$ by

$$w_{j+1} = w_j + \varepsilon F(w_j, j\varepsilon) + V(w_j, j\varepsilon, \varepsilon); \qquad w_0 = \zeta . \tag{4.1}$$

Then for any given $\zeta \in D$, if $\varepsilon > 0$ is sufficiently small, w_k is defined for $k \leqq 1/\varepsilon$, and $w_k(\zeta, t/k) \to Z(\zeta, t)$ as $k \to \infty$. In fact, this convergence is locally uniform in $(\zeta, t) \in D \times [0, 1]$.

Proof. This lemma is just a slight extension of Corollary 4.2 of [1] (where $\mathscr{R} = \mathbb{C}^n \times [0, 1]$, $D = \mathbb{C}^n$ was assumed). The same proof as there will also prove our lemma.

Proof of Theorem 2.1 We can assume that D is starshaped with respect to 0, and $\Phi(0) = 0$, $\Phi'(0) = \text{id}$. Put $\Phi(D) = R$, and define $\Phi_t(\zeta) = \Phi(t\zeta)/t$ $(0 < t \leqq 1)$, $\Phi_0(\zeta) = \zeta$. Then $\Phi_t : \frac{1}{t} D \to \frac{1}{t} R$ are biholomorpic for $0 < t \leqq 1$ while $\Phi_0 : \mathbb{C}^n \to \mathbb{C}^n$. Put $\mathscr{R} = \{(z, t) : 0 \leqq t \leqq 1 \text{ and } tz \in R\}$. Define a smooth mapping $F : \mathscr{R} \to \mathbb{C}^n$ by

$$F(z, t) = \frac{d\Phi_t}{dt}(\Phi_t^{-1}(z)) .$$

It follows that the mapping $Z : D \times [0, 1] \to \mathbb{C}^n$ defined by $Z(\zeta, t) = \Phi_t(\zeta)$ solves the initial value problem

$$\frac{\partial Z(\zeta, t)}{\partial t} = F(Z(\zeta, t), t); \qquad Z(\zeta, 0) = \zeta . \tag{4.2}$$

By virtue of Theorem 2.5 there is a family of continuous mappings $S_\varepsilon : \mathscr{R} \to \mathbb{C}^n$ $(\varepsilon \in \mathbb{C})$ such that for $(\varepsilon, t) \in \mathbb{C} \times [0, 1]$ $S_{\varepsilon, t}(z) = S_\varepsilon(z, t)$ is a composition of overshears, and

$$S_\varepsilon(z, t) = z + \varepsilon F(z, t) + V(z, t, \varepsilon) , \tag{4.3}$$

With $V(z, t, \varepsilon)/\varepsilon \to 0$ as $\varepsilon \to 0$ locally uniformly in $(z, t) \in \mathscr{R}$. Define a sequence $w_k = w_k(\zeta, \varepsilon)$ $(\zeta \in D, \varepsilon > 0)$ as in (4.1). By (4.3) $w_{j+1}(\zeta, \varepsilon) = S_{\varepsilon, j\varepsilon}(w_j(\zeta, \varepsilon))$, so that $\zeta \to w_k(\zeta, 1/k) = \Psi_k(\zeta)$ is a composition of overshears (and so is, in fact, defined for every $\zeta \in \mathbb{C}^n$). By Lemma 4.1 $\Psi_k(\zeta) \to Z(\zeta, 1) = \Phi(\zeta)$ locally uniformly in $\zeta \in D$, as was to be proved.

If the Jacobi determinant of Φ is identically 1, a simple variation of the above proof will yield a composition of shears $\Psi_k \to \Phi$.

Remark 4.2 In formulating Theorem 2.1 we did not aim at covering the most general situation where approximation by compositions of (over)shears is possible. J.P. Rosay pointed out to us that for certain applications a generalization of Theorem 2.1 might be desirable. The above proof almost verbatim also gives the following generalization: Let $D \subset \tilde{D} \subset \mathbb{C}^n$ be open sets such that $tz \in \tilde{D}$ whenever $0 \leqq t \leqq 1$ and $z \in D$. Suppose $\Phi : \tilde{D} \to \mathbb{C}^n$ is a biholomorphic mapping with image $R = \Phi(\tilde{D})$ having the Runge property. Then Φ can be approximated by elements of $G(\mathbb{C}^n)$ *locally uniformly on* D. If $\det \Phi' \equiv 1$, then Φ can even be approximated by elements of $G_1(\mathbb{C}^n)$.

5 Remarks on Theorems 1.3 and 2.1

The proof of these theorems did not very much use holomorphicity. Smooth shears and overshears can be defined on $\mathbb{R}^n$ by formulae (1.1), (1.2), where now $z_1, \ldots, z_n$ are real coordinates and f, h are just smooth (infinitely differentiable) functions. (Of course, permutation of coordinates is again permitted.) If Diff $\mathbb{R}^n$ denotes the group of smooth diffeomorphisms of $\mathbb{R}^n$ endowed with the topology of locally uniform convergence of all derivatives, and $\text{Diff}_1\, \mathbb{R}^n$ denotes the subgroup of orientation and volume preserving diffeomorphisms, a small variation of our arguments in Sects. 3 and 4 yields the following

Theorem 5.1 *Finite compositions of shears are dense in* $\text{Diff}_1\, \mathbb{R}^n$. *Finite compositions of overshears are dense in* Diff $\mathbb{R}^n$.

In fact, this theorem can even be extended to arbitrary (paracompact) smooth manifolds M. Cover M with countably many coordinate charts and call a self-diffeomorphism of M an overshear if it is the identity outside a compact set contained in some coordinate neighborhood, and in this neighborhood it has form (1.2) with $z_1, \ldots, z_n$ local coordinates and f, $h - 1$ smooth, compactly supported. Then we have

Theorem 5.2 *In the arcwise connected component of the identity of* Diff M *finite compositions of overshears are dense.*

An analogous theorem can be formulated for volume preserving self-diffeomorphisms of M, if M is endowed with smooth volume form.

Returning our attention to the holomorphic category we shall now comment on the necessity of the condition in Theorem 2.1. We shall present two examples that show that for approximability by (over) shears it is not enough to assume that both D and $\Phi(D)$ have the Runge property.

Example 5.3 Let

$$D = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1 z_2 - 1| < 1/2\} ,$$

and $\Phi(z_1, z_2) = (z_1, z_2/z_1)$. Then D is a Runge domain (that is, in addition to having the Runge property, it is also a domain of holomorphy), see e.g. [7], and so is its image $\Phi(D) = \{(w_1, w_2) : |w_1^2 w_2 - 1| < 1/2\}$. Φ is biholomorphic on D but cannot be approximated by a composition of overshears.

Indeed, for any $\Psi \in G(\mathbb{C}^n)$ its Jacobi determinant $\det \Psi' : \mathbb{C}^n \to \mathbb{C} \setminus \{0\}$ is homotopic to a constant map. On the other hand $\det \Phi'(z) = 1/z_1$, which, restricted to the curve $K = \{(\zeta, 1/\zeta) : \zeta \in S^1 \subset \mathbb{C}\} \subset D$ is not homotopic to a constant map $K \to \mathbb{C}\setminus\{0\}$. Hence Φ cannot be uniformly approximated on K by overshear compositions $\Psi \in G(\mathbb{C}^n)$.

This example was built on the fact that the fundamental group of $\mathrm{GL}(n, \mathbb{C})$ is not trivial. To exhibit nonapproximable *volume preserving* biholomorphisms we shall have to consider higher homotopy groups since $\mathrm{SL}(n, \mathbb{C})$ is simply connected.

Example 5.4 By Bott periodicity the third homotopy group of the unitary group, $\pi_3(U(m)) \approx \mathbb{Z}$ if m is sufficiently large, and the inclusion $U(m) \subset U(m+1)$ induces an isomorphism $\pi_3(U(m)) \to \pi_3(U(m+1))$, see [3]. Since $U(m)$ and $SU(m) \times S^1$ are homeomorphic, and $\pi_3(S^1) = 0$, the projection $U(m) \to SU(m)$ induces an isomorphism $\pi_3(U(m)) \overset{\approx}{\to} \pi_3(SU(m))$. Furthermore, by Gram–Schmidt orthogonalization, $SU(m)$ is a deformation retract of $\mathrm{SL}(m, \mathbb{C})$, so that finally we obtain $\pi_3(\mathrm{SL}(m, \mathbb{C})) \approx \mathbb{Z}$, and the inclusions $\mathrm{SL}(m, \mathbb{C}) \subset \mathrm{SL}(m+1, \mathbb{C}) \subset \ldots$ induce isomorphisms $\pi_3(\mathrm{SL}(m, \mathbb{C})) \approx \pi_3(\mathrm{SL}(m+1, \mathbb{C})) \approx \ldots$, provided m is sufficiently large. In fact, an inspection of the proof of Bott's periodicity and simple diagram chasing of exact sequences of homotopy groups associated with fibrations (cf. [5]) show that the above holds as soon as $m \geqq 2$.

Choose an m as above and a real analytic mapping $f: S^3 \to \mathrm{SL}(m, \mathbb{C})$ that is not homotopic to a constant map. We shall think of S^3 as a subset of $\mathbb{C}^4$:

$$S^3 = \{z = (z_j) \in \mathbb{C}^4 : \mathrm{Im}\, z_1 = \ldots = \mathrm{Im}\, z_4 = 0, z_1^2 + \ldots + z_4^2 = 1\}\,.$$

First extend f constant along radii to a neighborhood in $\mathbb{R}^4$ of the unit sphere. Then take $F: V \to \mathrm{SL}(m, \mathbb{C})$ to be the unique holomorphic extension of f to a neighborhood V of $S^3 \subset \mathbb{C}^4$. We may assume that V is a Runge domain; indeed it can be of form

$$\left\{z : |e^{iz_j}| < 1 + a, |e^{-iz_j}| < 1 + a, \left|1 - \sum_j z_j^2\right| < a\right\}$$

with a small positive a, in which case it is polynomially convex, hence Runge (see [7]).

Define now a biholomorphic automorphism Φ of $V \times \mathbb{C}^m$ by $\Phi(z, w) = (z, F(z)w)$ $(z \in V, w \in \mathbb{C}^m)$. The Jacobi determinant of Φ is 1. But the Jacobi *matrix* Φ' restricted to $M = S^3 \times \{0\} \subset V \times \mathbb{C}^m$ is

$$\Phi'|_M = \mathrm{id} \oplus f \in \pi_3(\mathrm{SL}(4 + m, \mathbb{C}))\,,$$

and so cannot be deformed to a constant map. This then implies, as in Example 5.3, that Φ cannot be approximated by compositions of (over)shears.

6 Miscellanea

Now we shall turn our attention to Theorems 1.1 and 1.2. They will be proved by a "counting argument". Throughout this section, we shall assume that $n > 1$. Already the results in Sect. 3 indicate that, say, $\mathrm{Aut}\, \mathbb{C}^n$ is almost as big as the space of all holomorphic mappings $\mathbb{C}^n \to \mathbb{C}^n$. We shall make this more precise below. On the other hand, finite overshear compositions are essentially as many as there are finite sequences of functions of $n-1$ variables. The space of such finite sequences is somehow smaller than the space of mappings $\mathbb{C}^n \to \mathbb{C}^n$. The aim of the present section is to prepare an actual proof of the above vague ideas; it will then immediately follow that Aut $\mathbb{C}^n$ is not generated by overshears.

First we shall need a simple result from Baire's category theory.

Proposition 6.1 *Let $f: V_1 \to V_2$ be an open continuous mapping between topological spaces. If $M \subset V_1$ is non-meagre (that is, of second category), then $f(M)$ is also non-meagre.*

Proof. Assume that $f(M) \subset \bigcup N_i$ where N_i are closed. Then $M \subset f^{-1}(f(M)) \subset f^{-1}(\bigcup N_i) = \bigcup f^{-1}(N_i)$, and all sets $f^{-1}(N_i)$ are closed by continuity of f. Since M is non-meagre, some set $f^{-1}(N_i)$, say $f^{-1}(N_1)$, contains a non-empty open set O. Since f is open $f(O)$ is also open and $f(O) \subset f(f^{-1}(N_1)) \subset N_1$. Therefore, $f(M)$ is non-meagre.

We shall also need two results about coefficientwise approximation of arbitrary polynomial mappings by automorphisms.

Proposition 6.2 *Let $P: \mathbb{C}^n \to \mathbb{C}^n$ be a homogeneous polynomial map of degree $m > 1$ with* div $P \equiv 0$. *Then there is an $S \in G_1(\mathbb{C}^n)$ such that*

$$S(z) = z + P(z) + O(|z|^{m+1})(z \to 0),$$

and S can be chosen to depend continuously on the coefficients of P.

Proof. This is Lemma 7.3 in [1] except for the last assertion, which follows immediately from the proof.

Let $A \subset \text{Aut}\, \mathbb{C}^n$ denote the subgroup of automorphisms that fix the origin, and let $A_1 = A \cap \text{Aut}_1 \mathbb{C}^n$. The following tells us that this group A_1 is big.

Proposition 6.3 *Let $P: \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial mapping of degree at most $m \geqq 1$ such that*

(i) $P(0) = 0$ *and*

(ii) $\det P'(z) = 1 + O(|z|^m)$ $(z \to 0)$.

Then there is an automorphism $Q \in A_1$ such that $P(z) - Q(z) = O(|z|^{m+1})$ as $z \to 0$. Also, Q can be chosen to depend continuously on the coefficients of P.

Proof. By induction on m. Assume the claim is true for $m - 1 \geqq 1$ and write P as above in the form $P = p + \pi$ where $\deg p \leqq m - 1$ and π is homogeneous of degree m. By the induction hypothesis there is a $q \in A_1$ satisfying $p(z) - q(z) = O(|z|^m)$. Put

$$P(q^{-1}(z)) = z + P^*(z) + O(|z|^{m+1}),$$

where P^* is homogeneous of degree m. Since

$$1 = \det \partial(P \circ q^{-1})/\partial z + O(|z|^m) = 1 + \text{div}\, P^*(z) + O(|z|^m),$$

div $P^* = 0$ and so by Proposition 6.2 there is $S \in G_1(\mathbb{C}^n)$ such that $S(z) = z + P^*(z) + O(|z|^{m+1})$. Of course, $S \in A_1$. Therefore $Q = S \circ q$ will do. All operations performed are continuous and therefore Q depends continuously on P.

As a last ingredient we shall need a complete metric on the topological space A. For Φ, $\Psi \in A$, and $r = 1, 2, \ldots$ put

$$d_r(\Phi, \Psi) = \max\left\{\max_{|z| \leqq r} |\Phi(z) - \Psi(z)|, \max_{|z| \leqq r} |\Phi^{-1}(z) - \Psi^{-1}(z)|\right\},$$

$$d(\Phi, \Psi) = \sum_{r=1}^{\infty} \min\{1, d_r(\Phi, \Psi)\} 2^{-r}.$$

As is well known, d_r and d are metrics, and convergence in the metric d is equivalent to convergence in each metric d_r.

Proposition 6.4 *The metric d on A induces the topology of locally uniform convergence. Futhermore, d is a complete metric.*

Proof. (i) Convergence in d clearly implies locally uniform convergence. The converse is proved in [2, Theorem 4].

(ii) If Φ_ν is a Cauchy sequence in (A, d) then both locally uniform limits $\lim \Phi_\nu = \Phi$ and $\lim \Phi_\nu^{-1} = \Psi$ exist, with Φ, Ψ holomorphic mappings $\mathbb{C}^n \to \mathbb{C}^n$. Since $\Phi \circ \Psi = \Psi \circ \Phi = \text{id}$, and $\Phi(0) = \Psi(0) = 0$, $\Phi \in A$ and $\Phi_\nu \to \Phi$ in d.

7 Proof of Theorems 1.1 and 1.2

We shall prove the following stronger result:

Theorem 7.1 *There is a volume preserving automorphism in A_1 that is not a composition of overshears in A.*

This will imply both Theorems 1.1 and 1.2. Indeed, if an automorphism $\Phi \in A_1$ fixing 0 is a composition of overshears then it is also a composition of overshears fixing 0. The reason for this is that conjugating an overshear S by a translation T we get another overshear S^*: $T \circ S = S^* \circ T$. Therefore in a given overshear composition $S_1 \circ S_2 \circ \ldots$ we can introduce translations $T_1, T_2, \ldots$ to replace general overshears by ones that fix 0:

$$\begin{aligned} S_1 \circ S_2 \circ \ldots &= S_1 \circ T_1^{-1} \circ T_1 \circ S_2 \circ \ldots = (S_1 \circ T_1^{-1}) \circ (S_2^* \circ T_1) \circ \ldots \\ &= (S_1 \circ T_1^{-1}) \circ (S_2^* \circ T_2^{-1}) \circ \tilde{T} \circ S_3 \circ \ldots = \ldots \end{aligned}$$

Choosing the translations $T_1, T_2, \ldots$ so that $\tilde{S}_1 = S_1 \circ T_1^{-1}$, $\tilde{S}_2 = S_2^* \circ T_2^{-1}, \ldots$ fix the origin, we obtain overshears $S_j \in A$ such that

$$S_1 \circ S_2 \circ \cdots = \tilde{S}_1 \circ \tilde{S}_2 \circ \cdots \circ T$$

with some translation T. Since both $S_1 \circ S_2 \circ \ldots$ and $\tilde{S}_1 \circ \tilde{S}_2 \circ \ldots$ fix 0, so must T, whence $T = \text{id}$.

With a positive integer m, consider the following sets:

$\Pi = \{F: \mathbb{C}^n \to \mathbb{C}^n \text{ holomorphic}, F(0) = 0\}$,
$\Theta = \{f: \mathbb{C}^{n-1} \to \mathbb{C} \text{ holomorphic}\}$,
$\Theta_0 = \{f \in \Theta : f(0) = 0\}$.
$\Pi^m = \{P: \mathbb{C}^n \to \mathbb{C}^n \text{ polynomials}, \deg P \leqq m, P(0) = 0\}$,
$\Omega^m = \{P \in \Pi^m : \det P'(z) = 1 + O(|z|^m) \text{ as } z \to 0\}$,
$\Theta^m = \{p: \mathbb{C}^{n-1} \to \mathbb{C} \text{ polynomial}, \deg p \leqq m\}$,
$\Theta_0^m = \{p \in \Theta^m, p(0) = 0\}$.

In what follows we shall study the dimensions of these spaces and their interrelations.

$\Pi^m \subset \Pi$, $\Theta^m \subset \Theta$, and $\Theta_0^m \subset \Theta_0$ are vector spaces of dimensions

$$\dim \Theta^m = \binom{m+n-1}{n-1}, \quad \dim \Theta_0^m = \binom{m+n-1}{n-1} - 1,$$

(7.1)
$$\dim \Pi^m = n\binom{m+n}{n} - n.$$

Proposition 7.2. *Ω^m is a submanifold of Π^m,*

(7.2)
$$\dim \Omega^m \geqq n\binom{m+n}{n} - n - \binom{m+n-1}{n}.$$

Proof. Ω^m is clearly a subvariety of Π^m. The equations that define Ω^m correspond to the coefficients c_ν in $\det P'(z) = \sum_\nu c_\nu z^\nu$ with $|\nu| \leqq m-1$. There are $\binom{m+n-1}{n}$ of them. Hence the estimate (7.2). To see that Ω^m is in fact smooth, observe that Ω^m is a group with operation composition modulo terms of degree $\geqq m+1$. (The existence of inverse follows, e.g. from Proposition 6.3). Thus Ω^m is homogeneous; since it has smooth points, it must be globally smooth.

Remark. Actually, equality must hold in (7.2) since $\operatorname{div}: \Pi^m \to \mathrm{Pol}_{m-1}(\mathbb{C}^n)$ is onto and its kernel is the tangent space at the identity to Ω^m. However, we will not need this.

Corollary 7.3 *For fixed k and sufficiently large m*

(7.3)
$$\dim \Omega^m > dim(\Theta_0^m \times \Theta^{m-1})^k.$$

Proof. Indeed, by (7.2) and (7.1) the left-hand side of (7.3) is a polynomial in m of degree n while the right-hand side is of degree $n-1$.

Proof of Theorem 7.1 Any overshear S on $\mathbb{C}^n$ is determined by two holomorphic functions on $\mathbb{C}^{n-1}$ and an integer d, $1 \leqq d \leqq n$, corresponding to the direction of the overshear: S does not change the z_j coordinates ($j \neq d$). In this situation we shall say that S is a d-overshear. Call a composition $S_1 \circ S_2 \circ \ldots$ of overshears preferred if S_1 is a 1-overshear, S_2 is a 2-overshear, $\ldots$, S_{n+1} is a 1-overshear, etc. $\ldots$, and every $S_j \in A$. Since id $\in A$ is a d-overshear for any d, it follows that any composition of ℓ overshears $\in A$ is equal to a preferred composition (of length $k \leqq \ell n$). Let C_k denote the set of those automorphisms $\in A$ that can be represented as a preferred overshear composition of length k. We shall define mappings $\Psi_k: (\Theta_0 \times \Theta)^k \to A$ as follows. Given any pair $\phi \in \Theta_0, \psi \in \Theta$, and an integer d, $1 \leqq d \leqq n$, we denote by $S = S_{d,\phi,\psi}: \mathbb{C}^n \to \mathbb{C}^n$ the d-overshear

$$S(z_1, \ldots, z_n) = (z_1, \ldots, z_{d-1}, \phi(z_1, \ldots, z_{d-1}, z_{d+1}, \ldots) + z_d e^{\psi(z_1, \ldots, z_{d-1}, z_{d+1}, \ldots)}, \ldots, z_n).$$

If $(\phi_1, \psi_1, \ldots, \phi_k, \psi_k) \in (\Theta_0 \times \Theta)^k$ then define

$$\Psi_k(\phi_1, \psi_1, \ldots, \phi_k, \psi_k) = S_{1,\phi_1,\psi_1} \circ S_{2,\phi_2,\psi_2} \circ \cdots \circ S_{d_k,\phi_k,\psi_k} \in A.$$

(Here $1 \leqq d_j \leqq n$ is congruent to $j \bmod n$.) Evidently, C_k is the image of Ψ_k.

Lemma 7.4 *$A_1 \cap C_k$ is meagre in A_1 for any $k = 1, 2, \ldots$*

Accepting this lemma the theorem would follow. Indeed, A_1, as a closed subspace of the complete metric space A, itself is complete, so that by Baire's Theorem $A_1 \not\subset \cup C_k$. Thus what we have to do is to prove Lemma 7.4.

Given an integer m, we shall also define a truncation map $\mathrm{Tr}_m: \Pi \to \Pi^m$ that associates with an $F \in \Pi$ its Taylor polynomial (about 0) of degree m, and the analogous map $\mathrm{tr}_m: \Theta \to \Theta^m$.

Proposition 7.5 *The following diagram commutes:*

$$\begin{array}{ccc} (\Theta_0 \times \Theta)^k & \xrightarrow{\Psi_k} & A \supset A_1 \\ (\mathrm{tr}_m \times \mathrm{tr}_{m-1})^k \downarrow & & \mathrm{Tr}_m \downarrow \;\; \mathrm{Tr}_m \downarrow \\ (\Theta_0^m \times \Theta^{m-1})^k & \xrightarrow{g} & \Pi^m \supset \Omega^m , \end{array}$$

where g is (the restriction of) $\mathrm{Tr}_m \circ \Psi_k$.

The proof is obvious.

Proof of Lemma 7.4 Assume $A_1 \cap C_k$ is non-meagre in A_1 for some fixed k and choose m so that (7.3) holds. Put $\Psi_k^{-1}(A_1) = V$ and $V_m = (\mathrm{tr}_m \times \mathrm{tr}_{m-1})^k(V)$. From Proposition 7.5 we obtain the commutative diagram

$$\begin{array}{ccc} V & \xrightarrow{\Psi_k} & A_1 \\ \downarrow & & \downarrow \mathrm{Tr}_m \\ V_m & \xrightarrow{g} & \Omega^m . \end{array} \qquad (7.4)$$

Since $C_k = \Psi_k((\Theta_0 \times \Theta)^k)$, our indirect assumption implies $\Psi_k(V)$ is non-meagre in A_1. By Proposition 6.3, Tr_m has a continuous right inverse and therefore Tr_m is an open mapping. Whence we obtain that $\mathrm{Tr}_m(\Psi_k(V))$ is non-meagre in Ω^m by Proposition 6.1. By the commutativity of (7.4) it follows that $g(V_m)$ is non-meagre in Ω^m. However, $g: (\Theta_0^m \times \Theta^{m-1})^k \to \Omega^m$ is a differentiable mapping. Because of (7.3), the image of g is meagre. This contradiction proves Lemma 7.4, and with it, Theorem 7.1.

8 The case $n = 2$

In this section we shall give an alternative proof of Theorem 7.1 in the case $n = 2$. This proof has the merit of actually exhibiting automorphisms in $\mathrm{Aut}_1 \mathbb{C}^2 \setminus G(\mathbb{C}^2)$.

Theorem 8.1 *For any $\phi: \mathbb{C} \to \mathbb{C}$ holomorhic and nonconstant the volume preserving automorphism of $\mathbb{C}^2$ given by*

$$(z_1, z_2) \to (z_1 e^{\phi(z_1 z_2)}, z_2 e^{-\phi(z_1 z_2)}) \qquad (8.1)$$

is not a finite composition of overshears.

As said in the introduction, in [1] it is proved that when $\phi(z) \equiv z$ the mapping (8.1) is not in $G_1(\mathbb{C}^2)$. The same proof works for general ϕ, but to show that (8.1) is not in $G(\mathbb{C}^2)$ the ideas of [1] have to be extended, as we shall presently explain.

For brevity, put $E = e^{\phi(z_1 z_2)}$. Also, if u and v are holomorphic functions on $\mathbb{C}^2$. let $[u, v]$ denote the $\mathbb{C}$-linear subspace they span in the space of holomorphic functions on $\mathbb{C}^2$.

Proposition 8.2 *If the map* (8.1) *is in* $G(\mathbb{C}^2)$ *then there are an integer* $k > 1$, *a sequence* $u_0, u_1, \ldots, u_k$ *of holomorphic functions on* $\mathbb{C}^2$, *and two sequences* $f_1, \ldots, f_{k-1}$ *and* $g_1, \ldots, g_{k-1}$ *of holomorphic functions on* $\mathbb{C}$ *such that*

(i) $[u_0, u_1] = [z_1, z_2]$ *and* $[u_{k-1}, u_k] = [z_1 E, z_2 E^{-1}]$;

(ii) *for every* $i = 1, \ldots, k$ *the map*

$$z \to (u_{i-1}(z), u_i(z))$$

defines an automorphism of $\mathbb{C}^2$;

(iii) *for every* $i = 1, \ldots, k-1$ *we have* $u_{i+1} = e^{g_i(u_i)} u_{i-1} + f_i(u_i)$;

(iv) *for every* $i = 1, \ldots, k-1$ *the function* g_i *is nonconstant or* f_i *is non-affine.*

Proof. As in Sect. 3.1 of [1] with the modification that in Definition 3.1.1 (rosaries) one skips the condition that the automorphism (u, v) has Jacobi determinant 1, and uses the fact that if $(z_1, z_2) \to (z_1, u(z_1, z_2))$ is bijective then $u(z_1, z_2) = e^{g(z_1)} z_2 + f(z_1)$.

The proof of Theorem 8.1 will depend on the fact that (i) forces u_{k-1} and u_k to have the same order of growth while (iii) requires u_{i+1} to grow substantially faster than u_i. This will give a contradiction showing that (8.1) is not in $G(\mathbb{C}^2)$. To measure the order of growth of a holomorphic function $u: \mathbb{C}^2 \to \mathbb{C}$ we use its Nevanlinna characteristic

$$m(u, r) = \int_{|z|=1} \log^+ |u(rz)| d\sigma(z), \tag{8.2}$$

where r is a positive number, $\log^+ x = \log x$ if $x \geqq 1$ and $\log^+ x = 0$ if $x < 1$, and $d\sigma$ is the area measure on the unit sphere in $\mathbb{C}^2$ (normalized so that the total area is 1). We shall need the following general properties of $m(\cdot, r)$.

Proposition 8.3 *For* $u, v: \mathbb{C}^2 \to \mathbb{C}$ *holomorphic we have*

$$m(u + v, r) \leqq m(u, r) + m(v, r) + \log 2; \tag{8.3}$$

$$m(uv, r) \leqq m(u, r) + m(v, r); \tag{8.4}$$

$$m\left(\frac{1}{u}, r\right) \leqq m(u, r) + O(1); \tag{8.5}$$

$$m(p(u), r) = \deg(p) m(u, r) + O(1) \text{ as } r \to \infty \text{ if } p \text{ is a polynomial}; \tag{8.6}$$

$$m(f(u), r)/m(u, r) \to \infty \text{ if } f \text{ is transcendental and } u \text{ nonconstant}; \tag{8.7}$$

$$m(\partial u/\partial z_j, r) \leqq 2m(u, r) + O(\log r) \text{ except for a set of } r\text{'s of finite measure}; \tag{8.8}$$

$$m(u, r) = O(\log r)\ (r \to \infty) \text{ if and only if } u \text{ is a polynomial.} \tag{8.9}$$

If u *is a polynomial of degree* d, $m(u, r) \sim d \log r$.

Proof. (8.3) and (8.4) are trivial; (8.5) follows from Jensen's formula (or the first main theorem of Nevanlinna theory). Actually the definition (8.2) makes sense when u is a meromorphic function on $\mathbb{C}^2$, and (8.3), (8.4) are true for meromorphic

u, v. (Here, a meromorphic function is the quotient of two entire functions.) For (8.5) we need u holomorphic but $1/u$ can have singularities.

(8.6) can be proved by considering separately the contributions in $m(u, r)$ and $m(p(u), r)$ of the sets where $|u| \geqq C$ and where $|u| < C$ for a big constant C.

(8.7) is a theorem of Clunie when u is a transcendental holomorphic function on $\mathbb{C}$, see [4] or [6]. The same proof works for transcendental functions of several variables, and even for polynomials. (Alternatively, the case of a polynomial u will also follow from (8.9)).

(8.8) follows from the "lemma of the logarithmic derivative" (see [13, (8.4) and (8.9)]). Finally, (8.9) is a theorem of Stoll, see [12].

We shall also need some estimates related to our map $(z_1 E, z_2 E^{-1})$.

Proposition 8.4 *We have*

(8.10) $$m(z_1 E, r) = m(z_2 E^{-1}, r) + O(\log r) \text{ as } r \to \infty.$$

Proof. By (8.4) (valid for meromorphic functions) and (8.5)

$$\begin{aligned} m(z_2 E^{-1}, r) &\leqq m(z_1 z_2, r) + m((z_1 E)^{-1}, r) \\ &\leqq O(\log r) + O(1) + m(z_1 E, r), \end{aligned}$$

which gives one half of the estimate (8.10). The other half is proved similarly.

Proposition 8.5 *If p and q are nonzero polynomials of one variable then*

$$m(p(z_1 E) + q(z_2 E^{-1}), r) = (\deg p + \deg q) m(z_1 E, r) + O(\log r)$$

as $r \to \infty$.

Proof. Put $V_r = \{z \in \mathbb{C}^2 : |z| = r, |E| \geqq 1\}$. On V_r we have

(8.11) $$p(z_1 E) + q(z_2 E^{-1}) = p(z_1 E) + O(r^{\deg q}) .$$

Since for complex numbers a, b

$$\log^+|a| - \log^+|b| - \log 2 \leqq \log^+|a + b| \leqq \log^+|a| + \log^+|b| + \log 2 ,$$

(8.11) implies

$$\begin{aligned} \int_{V_r} \log^+|p(z_1 E) + q(z_2 E^{-1})| d\sigma(z) &= \int_{V_r} \log^+|p(z_1 E)| d\sigma(z) + O(\log r) \\ &= \int_{|z|=r} \log^+|p(z_1 E)| d\sigma(z) + O(\log r) . \end{aligned}$$

This latter is $\deg(p) m(z_1 E, r) + O(\log r)$ by (8.6). Similarly we obtain

$$\int_{\{|z|=1\} \setminus V_r} \log^+|p(z_1 E) + q(z_2 E^{-1})| d\sigma(z) = \deg(q) m(z_2 E^{-1}) + O(\log r) ,$$

whence (8.10) implies the claim.

Now define a partial ordering $\prec$ on holomorphic functions on $\mathbb{C}^2$ by putting $u \prec v$ if there is an $\eta > 1$ and a set $S \subset \mathbb{R}$ of finite measure such that

(8.12) $$m(v, r) > \eta m(u, r) \quad \text{when } r > 0, \quad r \notin S$$

Proposition 8.6 *Let $u_0, \ldots, u_k$ be as in Proposition* 8.2. *If $u_{i-1} \prec u_i$ then $u_i \prec u_{i+1}$ Moreover, unless g_i is constant and f_i is a polynomial, for any η there is an $S \subset \mathbb{R}$ of finite measure such that* (8.12) *holds with $u = u_i$ and $v = u_{i+1}$.*

Proof. The map

$$T:(z_1, z_2) \to (e^{g_i(z_2)} z_1 + f_i(z_2), z_2)$$

is an overshear with Jacobi determinant $e^{g_i(z_2)}$. Furthermore $T \circ (u_{i-1}, u_i) = (u_{i+1}, u_i)$, whence the chain rule gives

$$e^{g_i(u_i)} = \left(\frac{\partial u_i}{\partial z_1}\frac{\partial u_{i+1}}{\partial z_2} - \frac{\partial u_i}{\partial z_2}\frac{\partial u_{i+1}}{\partial z_1}\right) \Bigg/ \left(\frac{\partial u_{i-1}}{\partial z_1}\frac{\partial u_i}{\partial z_2} - \frac{\partial u_{i-1}}{\partial z_2}\frac{\partial u_i}{\partial z_1}\right).$$

Apply Proposition 8.3 to get

$$m(e^{g_i(u_i)}, r) \leqq 8 \sum_{j=i-1}^{i+1} m(u_j, r) + O(\log r)$$

for $r > 0$ in the complement of a set $S_0 \subset \mathbb{R}$ of finite measure. Rearranging gives

$$\frac{m(u_{i+1}, r)}{m(u_i, r)} \geqq \frac{1}{8}\frac{m(e^{g_i(u_i)}, r)}{m(u_i, r)} - 1 - \frac{m(u_{i-1}, r)}{m(u_i, r)} - O\left(\frac{\log r}{m(u_i, r)}\right).$$

Hence if g_i is nonconstant, by virtue of (8.7), the assumption that $u_{i-1} \prec u_i$, and (8.9) we get the result. On the other hand, if g_i is a constant then

$$\frac{m(u_{i+1}, r)}{m(u_i, r)} = \frac{m(f_i(u_i) + cu_{i-1}, r)}{m(u_i, r)}$$

$$\geqq \frac{m(f_i(u_i), r)}{m(u_i, r)} - \frac{m(u_{i-1}, r)}{m(u_i, r)} - \frac{|\log c| + \log 2}{m(u_i, r)},$$

so that the claim now follows from (8.6) or (8.7), and $u_{i-1} \prec u_i$.

Proof of Theorem 8.1 Assume, to get a contradiction, that the map (8.1) is in $G(\mathbb{C}^2)$. Then there are sequences $u_0, \ldots, u_k, f_1, \ldots, f_{k-1}, g_1, \ldots, g_{k-1}$ as in Proposition 8.2. By (i) of this proposition both u_0, u_1 are linear, hence by (iii), (iv) u_2 is not a polynomial of degree 1. Stoll's theorem (8.9) implies then that $u_1 \prec u_2$. By virtue of Proposition 8.6, $u_2 \prec u_3 \prec \ldots \prec u_{k-1} \prec u_k$ follows.

Since both u_{k-1} and u_k are of form $az_1E + bz_2E^{-1}$ with a, b constants, Proposition 8.5 and 8.9 give that $\lim_{r\to\infty} m(u_k, r)/m(u_{k-1}, r) = 1/2, 1,$ or 2. But $u_{k-1} \prec u_k$ so only the value 2 is possible, and this can happen only if $u_{k-1}(z) = cz_1E$ or $u_{k-1}(z) = cz_2E^{-1}$ with some constant c. Assume the first. In view of Proposition 8.6 g_{k-1} must be constant and f_{k-1} must be a polynomial. Since f_{k-1} is non-affine, $\deg f_{k-1} \geqq 2$. Thus we have with some constants a, b, c, d ($d \neq 0$).

$$az_1E + bz_2E^{-1} = u_k = du_{k-2} + f_{k-1}(u_{k-1})$$

$$= du_{k-2} + f_{k-1}(cz_1E),$$

which can also be written as

$$p(z_1E) + \frac{b}{d}z_2E^{-1} = u_{k-2},$$

with p a polynomial of degree at least 2. Using Proposition 8.5 we find

$$\lim_{r \to \infty} m(u_{k-2}, r)/m(u_{k-1}, r) \geqq \deg p \geqq 2 ,$$

which contradicts $u_{k-2} \prec u_{k-1}$.

9 Remarks on Theorems 1.1 and 1.2

The proof of these theorems given in Sect. 7 does not *produce* examples of automorphisms that are not compositions of (over) shears. Indeed, for $n > 2$ no such examples are known. It is worth mentioning that a "stabilization" of (8.1), i.e. the automorphism Φ of $\mathbb{C}^3$ given by

$$\Phi(z_1, z_2, z_3) = (z_1 e^{\phi(z_1 z_2)}, z_2 e^{-\phi(z_1 z_2)}, z_3)$$

is known to be a composition of shears.

Indeed, consider the shears

$$S_1(z_1, z_2, z_3) = (z_1 - e^{z_3} z_2, z_2, z_3) ,$$

$$S_2(z_1, z_2, z_3) = (z_1, z_2 + e^{-z_3} z_1, z_3) ,$$

$$S(z_1, z_2, z_3) = (z_1, z_2, z_3 + \phi(z_1 z_2)) ,$$

and the mapping $L \in \mathrm{SL}(3, \mathbb{C})$ given by

$$L(z_1, z_2, z_3) = (z_2, -z_1, z_3) .$$

According to Theorem A of [1], L is a composition of shears, hence so is $T = S_1 \circ S_2 \circ S_1 \circ L$. We compute

$$T(z_1, z_2, z_3) = (e^{z_3} z_1, e^{-z_3} z_2, z_3) ,$$

whence $T^{-1} S^{-1} T S = \Phi$.

In spite of its nonconstructive nature, our first proof of, say, Theorem 1.1 has certain merits, even in the case $n = 2$. Indeed, the same approach would also prove that shears and mappings of form (8.1) together do not generate the whole group $\mathrm{Aut}_1 \mathbb{C}^2$ either. This could even be pushed further to formulate the following principle: A set of generators for $\mathrm{Aut}_1 \mathbb{C}^n$ (or $\mathrm{Aut}\, \mathbb{C}^n$) cannot be given by countably many expressions involving arbitrary holomorphic functions of $n - 1$ variables. It is possible to convert this principle into a theorem, once a precise meaning is given to "expressions" above, but we shall leave this to the interested reader.

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