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Oscillatory integrals with polynomial phases*

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1 Introduction

The theory of Fourier integral operators associated to a canonical graph [3] constitutes one of the most elegant and powerful tools of partial differential equations. However many issues in both analysis and geometry lead to Fourier integral operators whose Lagrangians are not local graphs. Already problems of diffraction had led to operators with canonical folding relations [4, 5, 10], while problems of integral geometry [1, 2] and differentiation theory [9, 7] lead to operators with even more severe singularities.

Although it seems unlikely that as succinct a calculus as that of the classical theory can be laid down for singular Fourier integral operators, a more limited theory that can predict say, the nature of TT^* and produce L^2 bounds could already prove to be quite useful. As a first step we need to understand simple models, such as the Airy operator provided for operators with folding canonical relations. In this paper we discuss a basic class of such models, namely oscillatory integral operators with homogeneous polynomial phase functions of arbitrary degree in one dimension. A number of features point to these operators as possibly central to a calculus of singular Fourier integral operators: first, they are natural generalizations of the Airy operator, whose phase is a homogeneous polynomial of degree 3; second, they can essentially be identified with Radon transforms along families of curves with non-vanishing "torsion" in manifolds of dimension equal to the degree of homogeneity; third, in the classical theory of Fourier integral operators, the condition that the Lagrangian $\Lambda \subset T^*(X) \times T^*(Y)$ projects smoothly onto one of the factors is symmetric in X and Y. For singular Lagrangians, the stratification structure of the projections on the two factors may differ, and it is valuable to understand how this affects the L^2 bounds. Such issues have been investigated in [1, 7]. The class of operators we study here are the simplest incorporating this asymmetric behavior; finally, it is also a class which despite its high order of degeneracy nevertheless leads to surprisingly simple answers.

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More precisely, let $S(x, y)$ be a homogeneous polynomial in $(x, y) \in \mathbf{R} \times \mathbf{R}$ of degree n

$$S(x, y) = \sum_{1 \leqq i \leqq n-1} \alpha_{i-1} x^{n-i} y^i$$

and define the operator T by

$$(T\phi)(x) = \int e^{i\lambda S(x,y)} \chi(y) \phi(y) dy, \quad \phi \in C_0^\infty(\mathbf{R})$$

where χ is a fixed C^∞ cut-off function on $\mathbf{R}$ with compact support. The norm of the operator T is unaffected by the addition to $S(x, y)$ of terms proportional to either x^n or y^n, which is why we omit them. Assume that the coefficient α_{n-2} of xy^{n-1} is not 0.

Main Theorem. *The operator T extends to a bounded operator on $L^2(\mathbf{R})$ with operator norm*

$$\| T \| \leqq C |\lambda|^{-1/n}$$

when

(a) *n is even and* $|\alpha_{(n/2)-1}| + \ldots + |\alpha_0| \neq 0$;
(b) *n is odd and* $|\alpha_{(n-1)/2-1}| + \ldots + |\alpha_0| \neq 0$.

Note. There is an equivalent way of stating the estimate of the Main Theorem by considering a variant T' of the operator T which is defined without the cut-off χ and the parameter λ

$$(T'\phi)(x) = \int e^{iS(x,y)} \phi(y) dy, \quad \phi \in C_0^\infty(\mathbf{R}).$$

Then the operator T' is extendable to a bounded operator on $L^2(\mathbf{R})$ to itself if and only if the estimate $\| T \| \leqq C |\lambda|^{-1/n}$ holds. The equivalence is an easy consequence of a rescaling argument.

Switching the roles of x and y would lead to a similar statement for $\alpha_0 \neq 0$, and conditions on $|\alpha_{n-2}| + \ldots + |\alpha_p| \neq 0$, with $p = (n/2) - 1$ for n even and $p = (n-3)/2$ for n odd.

It may be instructive to discuss the cases of lowest n in some detail. For $n = 2$ the only possibility is $S(x, y) = \alpha_0 xy$. The operator T is just the Fourier transform up to a scaling, and we immediately obtain $\| T \| = |\alpha_0|^{-1/2} |\lambda|^{-1/2}$, in agreement with the Main Theorem. The case $n = 3$ is more typical of the problems at hand. The phase functions satisfying the hypotheses of the Main Theorem must be of the form

$$S(x, y) = \alpha_0 xy^2 + \alpha_1 x^2 y = \left(\left(\frac{\alpha_0^2}{3\alpha_1} \right)^{1/3} x + \left(\frac{\alpha_1^2}{3\alpha_0} \right)^{1/3} y \right)^3 - \frac{\alpha_0^2}{3\alpha_1} x^3 - \frac{\alpha_1^2}{3\alpha_0} y^3 .$$

After a rescaling T reduces to convolution with $e^{i\lambda x^3}$, which is essentially the Airy operator. The Main Theorem follows from the easy uniform bound $O(|\lambda|^{-1/3})$ for the corresponding multiplier. The only phases which do not satisfy the hypotheses of the Main Theorem when $n = 3$ are $S(x, y) = \alpha_0 xy^2$ or $S(x, y) = \alpha_1 x^2 y$. In this case the kernel $K_\lambda(x, y)$ of TT^* is immediately seen to be given by $\lambda^{-1/2}(x - y)^{-1/2}$, so that the sharp bound for $\| T \|$ is then the weaker bound $O(|\lambda|^{-1/4})$. The case $n = 4$ is the first unknown case, and requires already some of the techniques developed for the proof of the Main Theorem.

Now the simplest degenerate Fourier integral operators $F \in I^{\mu}(X, Y; \Lambda)$ are associated to a Lagrangian $\Lambda \subset T^*(X) \times T^*(Y)$ which is not a local graph, but a fold in the following sense. Let the projections from Λ on $T^*(X)$ and $T^*(Y)$ be denoted respectively by π_R and π_L:

$$\begin{array}{ccc} & \Lambda & \\ \swarrow & & \searrow \\ T^*(Y) & & T^*(X)\,. \end{array}$$

Let $\Sigma \subset \Lambda$ be the variety along which the kernels of $d\pi_L$ and $d\pi_R$ are not trivial. Then Λ is a left fold if the determinant of $d\pi_L$ vanishes of first order along Σ, and the kernel of $d\pi_L$ is one-dimensional and transversal to Σ. It is a right fold if the same condition holds for π_R instead of π_L. We note that although π_L and π_R are regular if and only if one of them is, the Lagrangian Λ can be a left fold without being a right fold. The phase functions homogeneous of order 3 can be viewed as models for Lagrangians with folds, with the left (respectively, right) fold condition corresponding to the non-vanishing of the coefficient α_0 (respectively α_1). In fact, it is known that Lagrangians which are folds on both sides can be conjugated to $S(x, y) = (x - y)^3$ [4, 5, 10], while Lagrangians with a fold on one side and complete degeneracy on the other essentially correspond to $S(x, y) = xy^2$ [1]. The bounds $|\lambda|^{-1/3}$ and $|\lambda|^{-1/4}$ reflect these two different geometric behaviors.

The phases introduced in the Main Theorem are models of Lagrangians with cusps. This means that the kernel of say π_L, may be tangent to Σ along some subvariety Σ_L^1, and also tangent to Σ_L^1 along another subvariety Σ_L^2, etc. The Lagrangian is said to have left torsion if Σ_L^{n-2} is empty [7]. This corresponds to the condition $\alpha_{n-2} \neq 0$. Arguing as for xy^2 we readily see that the best bound for $\|T\|$ when $S(x, y) = xy^{n-1}$ is $O(|\lambda|^{-1/(2(n-1))})$. The issue is whether this bound can be improved by some information on the right projections π_R, restricted to the various singular varieties Σ_L^l. The Main Theorem asserts that this is the case for the model operators. The bounds $\|T\| \leqq C|\lambda|^{-1/n}$ given there are sharp, as can be readily seen in the case of $S(x, y) = (x - y)^n$ which is a convolution operator. Furthermore, it is especially encouraging that the conditions insuring a better bound are given by the non-vanishing of individual coefficients, rather than say the signature of a quadratic form, so that an eventual generalization in terms of stratifications for the projection π_R is possible.

Our method of proof is based on a close study of the singularities of the kernel $K_\lambda(x, y)$ of TT^*. The kernel can be written as an oscillatory integral in z with phase $S(x, z) - S(y, z)$. The main contributions come from critical points. However here the dependence on the parameters x, y is crucial, and the usual bounds provided by the standard van der Corput lemma can blow up as the critical points come close together. This happens near special values of (x, y) where the polynomial $S_z'(x, z) - S_z'(y, z)$ has multiple roots. Thus the first ingredient of our proof is a sharp form of the method of stationary phase. This refinement which keeps track of the distances between critical points may be of independent interest and is presented in Sect. 2. Next we show that the critical points which are algebraic functions in the projective variables (y/x) and (x/y) cannot come too close together in the average, except under very special circumstances ruled out by the form of the particular polynomial equation at hand. Our approach here is direct, and it may be hoped that an algebraic geometric proof may eventually be found which may pave the way to higher dimensions. The singularities of $K_\lambda(x, y)$ near $x = y$, $x = 0$, or $y = 0$, require a separate treatment. This is done in Sect. 4. Section 3 is devoted to a L^2 boundedness theorem for operators whose kernels satisfy the bounds derived in Sect. 2 and 4. Finally we conclude with a brief discussion of open questions in Sect. 5.

2 A sharp version of the method of stationary phase

The key analytic tool of our arguments is the following sharp version of the method of stationary phase, which keeps careful track of the distances separating the critical points.

We consider the oscillatory integral

$$K_\lambda = \int_a^b e^{i\lambda P(z)} \chi(z) dz \,, \tag{1}$$

where χ is a C^∞ function, and $P(z)$ is a monic real polynomial of degree d. The end points a, b can be infinite, in which case χ is required to have compact support. The critical points of the phase are the real roots of the polynomial $P'(z)$. We begin with the case when $P'(z)$ has no complex roots, which incorporates already the main ideas. Let a_k, $k = 1, \ldots, d-1$ be the roots of $P'(z)$, indexed in increasing order. We define a *cluster* L to be a set of consecutive points a_k's, and the *size* $|L|$ of the cluster to be the number of points a_k's in it. Multiple roots are counted with their multiplicities.

Theorem 1 *The following bound holds for* K_λ

$$|K_\lambda| \leqq C_d \max_{1 \leqq k \leqq d-1} \left[\min_{L \ni k} \left(|\lambda| \prod_{j \notin L} |a_k - a_j| \right)^{-1/(|L|+1)} \right] \tag{2}$$

where the constant C_d *depends only on the degree* d *of* $P'(z)$, *and the sup norms of* χ *and* χ'.

Proof of Theorem 1 We begin by an elementary estimate

$$\left| \int_a^b e^{i\lambda P(z)} \chi(z) dz \right| \leqq C_d \left(|\lambda| \min_{[a,b]} |P'(z)| \right)^{-1} \tag{3}$$

with actually $C_d = (2d+1)(\sup_{[a,b]}(|\chi|, |\chi'|)$. This follows from an integration by parts

$$\begin{aligned} \int_a^b e^{i\lambda P(z)} \chi(z) dz = & - \int_a^b e^{i\lambda P(z)} (i\lambda)^{-1} (P'(z)^{-1})' \chi(z) dz \\ & - \int_a^b e^{i\lambda P(z)} (i\lambda P'(z))^{-1} \chi(z)' dz + (i\lambda P'(z))^{-1} e^{i\lambda P(z)} \chi(z)|_a^b \,. \end{aligned} \tag{4}$$

The second and third terms of (4) are bounded by the right hand side of (3) with 3 as the constant. As for the first term we can estimate it by

$$|\lambda|^{-1} \sup_{[a,b]} |\chi| \int_a^b |(P'(z)^{-1})'| dz \leqq |\lambda|^{-1} \sup_{[a,b]} |\chi| \sum_l \left| \int_{a_l}^{a_{l+1}} (P'(z)^{-1})' dz \right| \tag{5}$$

where $[a,b] = \bigcup_l [a_l, b_l]$ is a decomposition into intervals where $((P'(z))^{-1})'$ does not change sign. Obviously the number of these intervals is less than $d-1$. The integral on each interval $[a_l, a_{l+1}]$ equals $|P'(a_{l+1})^{-1} - P'(a_l)^{-1}|$ and is bounded by $2(\min_{[a,b]}|P'(z)|)^{-1}$. This establishes (3).

Returning to the proof of (2), we assume for convenience of notation that the a_k's are distinct. The argument adapts easily to the case of multiple roots. We divide the real line into intervals I_k, $k = 1, \ldots, d-1$ as follows

$$\begin{aligned} I_1 &= [a,b] \cap (-\infty, (a_1 + a_2)/2] \\ I_k &= [a,b] \cap [(a_{k-1} + a_k)/2, (a_k + a_{k+1})/2], \quad 2 \leqq k \leqq d-2 \\ I_{d-1} &= [a,b] \cap [(a_{d-2} + a_{d-1})/2, +\infty)\,. \end{aligned} \tag{6}$$

It is convenient to introduce also the intervals $I_k^+ = I_k \cap \{z > a_k\}$ and $I_k^- = I_k \cap \{z < a_k\}$. The desired estimate (2) is a consequence of the following localized version at each critical point a_k

$$\left| \int_{I_k} e^{i\lambda P(z)} \chi(z)dz \right| \leqq C_d \min_{L \ni k} \left(|\lambda| \prod_{j \notin L} |a_k - a_j| \right)^{-1/(|L|+1)}. \tag{7}$$

We prove this by induction on the size $|L|$ of the clusters L. For $L = 1$ there is only one cluster $L = \{a_k\}$ which contains a_k, and (7) becomes

$$\left| \int_{I_k} e^{i\lambda P(z)} \chi(z)dz \right| \leqq C_d \left(|\lambda| \prod_{j \neq k} |a_k - a_j| \right)^{-1/2}. \tag{8}$$

To see this let $\delta = (\lambda \prod_{j \neq k} |a_k - a_j|)^{-1/2}$. Evidently the integral over the interval $I_k^\delta = [a_k - \delta, a_k + \delta]$ satisfies the desired inequality. On the other hand, on each interval $I_k^\pm \backslash I_k^\delta$ (which may be empty) we have

$$|P'(z)| \geqq c_d \delta \prod_{j \neq k} |a_k - a_j|\,. \tag{9}$$

In fact we may write $P'(z)$ as

$$P'(z) = \prod_{j=1}^{d-1} (z - a_j)\,.$$

For $z \in I_k^+ \backslash I_k^\delta$ say, we must have $|z - a_k| \geqq \delta$ and $|z - a_j| \geqq |a_k - a_j|$ for those a_j's on the left of a_k. For a_j on the right of a_k, we observe that

$$|z - a_j| = |a_k - a_j| - |z - a_k| \geqq |a_k - a_j| - \tfrac{1}{2}|a_k - a_j| = \tfrac{1}{2}|a_k - a_j|$$

giving (9). The argument is similar for z in $I_k^- \backslash I_k^\delta$, so that (9) holds, and hence (8) in view of (3).

We assume now that (7) holds for clusters of size $n-1$. Let $L = \{a_M, \ldots, a_N\}$ $(N = M + n - 1)$ be any cluster of size n containing a_k, and consider two possibilities:

- L neither begins nor ends at a_k. Then $\{a_{M+1}, \ldots, a_N\}$ is a cluster of size $n-1$ which still contains a_k, and the induction hypothesis gives

$$\left| \int_{I_k} e^{i\lambda P(z)} \chi(z)dz \right| \leqq C_d \left(|\lambda| \prod_{j \neq M+1,\ldots,N} |a_k - a_j| \right)^{-1/n}. \tag{10}$$

If

$$\left(|\lambda| \prod_{j \neq M+1,\dots,N} |a_k - a_j|\right)^{-1/n} \leqq \left(|\lambda| \prod_{j \neq M,\dots,N} |a_k - a_j|\right)^{-1/(n+1)} \tag{11}$$

we are done. Otherwise we obtain a bound for $|a_M - a_k|$

$$|a_M - a_k| < \left(|\lambda| \prod_{j \notin L} |a_k - a_j|\right)^{-1/(n+1)} . \tag{12}$$

Since the interval I_k^- is contained in $[a_M, a_k]$, and the integral on I_k^- is dominated by the length $|I_k^-|$ of I_k^-, the desired estimate on I_k^- follows. Similarly the estimate on I_k^+ is obtained by induction from the $n-1$ cluster $\{a_M, \dots, a_{N-1}\}$.

- Say now that L begins at a_k, so it is of the form $L = \{a_k, \dots, a_N\}$, $(N = k + n - 1)$. By comparing with the estimate for the $n-1$ cluster $\{a_k, \dots, a_{N-1}\}$, we may assume as previously that

$$|a_N - a_k| < \left(|\lambda| \prod_{j \notin L} |a_k - a_j|\right)^{-1/(n+1)} . \tag{13}$$

Since I_k^+ is contained in $[a_k, a_N]$, we need only estimate the integral over I_k^-. Let δ be the right hand side of (13). The integral over $I_k^\delta = [a_k - \delta, a_k + \delta]$ evidently satisfies the desired estimate. For z in $I_k^- \backslash I_k^\delta$, $|z - a_j|$ is greater than δ for $j \in L$, so that

$$\min_{I_k^- \backslash I_k^\delta} |P'(z)| \geqq \delta^n \prod_{j \notin L} |a_k - a_j| . \tag{14}$$

The estimate (6) for the integral over I_k^- is a consequence of (3) and (14). The argument for clusters L which end at a_k is the same, and the induction proof is complete. QED

We note that the second derivative of the phase $P(z)$ at each critical point a_k is given by $P''(a_k) = \prod_{j \neq k} (a_k - a_j)$, so that the standard method of stationary phase produces a contribution of $(|\lambda| \prod_{j \neq k} |a_k - a_j|)^{-1/2}$ from the neighborhood of the critical point a_k. This would coincide with the case of size 1 clusters in Theorem 1. The problem with the standard method, however, is that it provides an asymptotic expansion in λ alone, and the error terms depend on the derivatives of the phase and could blow up. The proof of Theorem 1 actually gives a more general theorem:

Theorem 2 *Let $P(z)$ be any C^2 function which satisfies the following conditions on the intersection of $[a, b]$ and the support of $\chi \in C_0^1(\mathbf{R})$:*

(a) $|P'(z)| \geqq A \prod_{j=1}^{d-1} |z - a_j|$, *where d is some integer, and A is a constant;*
(b) $((P'(z))^{-1})'$ *changes sign a finite number N of times.*

Then the oscillatory integral (1) *can be estimated by*

$$|K_\lambda| \leqq C_{N,d,\chi} \max_{1 \leqq k \leqq d-1} \left[\max_{L \ni k} \left(|\lambda| A \prod_{j \notin L} |a_k - a_j|\right)^{-1/(|L|+1)} \right] \tag{15}$$

with $C_{N,d,\chi}$ depending only on N, d, and the C_0^1 norm of χ.

With Theorem 2 we can easily obtain the most general version of Theorem 1, allowing as well complex roots. Complex roots are important in practice because their imaginary parts may be small, and we need to keep track of their sizes. Let then $\zeta_k = a_k + ib_k$, $k = 1, \ldots, d-1$ be the roots of $P'(z)$. Instead of counting the roots inside clusters L as in Theorem 1, it is more convenient to count the number of roots in $L^c = L \backslash \zeta_k$. There is now no preferred ordering, and we can drop at this point the requirement that clusters consist only of consecutive roots. For each k we set

$$K_\lambda(k,m) \equiv \min_{L^c, |L^c| = m-2} \left(\lambda \prod_{j \notin L} |a_k - \zeta_j| \right)^{-1/m} \tag{16}$$

when ζ_k is real, and

$$K_\lambda(k,m) \equiv \min \left\{ \min_{L^c, |L^c| = m-3} \left(|\lambda| \prod_{j \notin L} |a_k - \zeta_j| \right)^{-1/m}, \min_{L^c, |L^c| = m-2} \left(|\lambda| |b_k| \prod_{j \notin L} |a_k - \zeta_j| \right)^{-1/m}, \min_{L^c, |L^c| = m-1} \left(|\lambda| |b_k|^2 \prod_{j \notin L} |a_k - \zeta_j| \right)^{-1/m} \right\} \tag{17}$$

for $\zeta_k = a_k + ib_k$ complex.

Theorem 3 *For any real monic polynomial $P(z)$ of degree d, we have the following estimate for the oscillating integral* (1)

$$|K_\lambda| \leqq C_d \max_k \left(\min_m K_\lambda(k,m) \right) \tag{18}$$

Proof of Theorem 3 We introduce intervals I_k, $I_k^\pm$ as in (6) with the a_k's the ordered real parts of the complex roots ζ_k. It suffices to show that the integral for K_λ restricted to each interval I_k can be bounded for each n by each of the three terms on the right hand side of (16) (17). We may express $P'(z)$ as

$$P'(z) = \prod_{j=1}^{d-1} (z - \zeta_j) .$$

If $\zeta_k = a_k$ is real, the estimate is a consequence of Theorem 2 and the fact that

$$|P'(z)| \geqq \left(|z - a_k| \prod_{j \in L^c} |z - \operatorname{Re} \zeta_j| \right) \left(\prod_{j \notin L} |a_k - \zeta_j| \right) \tag{19}$$

for $z \in I_k$. If $\zeta_k = a_k + ib_k$ is complex, we note that $|z - \zeta_k|^2 = |z - a_k|^2 + b_k^2$, and hence is greater than $\max(|z-a_k|^2, |z-a_k||b_k|, |b_k|^2)$. The desired estimates are consequences of Theorem 2 and the following lower bounds for $P'(z)$ on I_k:

$$|P'(z)| \geqq \left(|z - a_k|^2 \prod_{j \in L^c} |z - \operatorname{Re} \zeta_j| \right) \left(\prod_{j \notin L} |a_k - \zeta_j| \right)$$

$$|P'(z)| \geqq \left(|z - a_k| \prod_{j \in L^c} |z - \operatorname{Re} \zeta_j| \right) \left(|b_k| \prod_{j \notin L} |a_k - \zeta_j| \right)$$

$$|P'(z)| \geqq \left(\prod_{j \in L^c} |z - \operatorname{Re} \zeta_j| \right) \left(|b_k|^2 \prod_{j \notin L} |a_k - \zeta_j| \right) . \tag{20}$$

This establishes Theorem 3. QED

3 L^p bounds for some singular integral operators

In this section we provide L^2 bounds for a class of operators which will be shown to contain TT^* when the conditions of the Main Theorem are satisfied. A simple version sufficient for our purposes is

Theorem 4 *Let S be an operator from $C_0^\infty(\mathbf{R})$ to $L^1_{\mathrm{loc}}(\mathbf{R})$ with kernel $K(x, y)$. Then S extends to a bounded operator on $L^2(\mathbf{R})$, with norm*

$$\|S\| = O(|\lambda|^{-2/n})$$

under the following conditions

(a) *For some $v \geqq 0$, and some function $L(x, y)$ homogeneous of degree $-1 - vn$ in (x, y)*

$$|K(x, y)| \leqq \min((\lambda|x - y|)^{-1/(n-1)}, |\lambda|^{-v-2/n} L(x, y)) \tag{21}$$

(b) *$L(x, y)$ is continuous on the circle $x^2 + y^2 = 1$, except possibly at a finite number of points θ_i, near which it is of size*

$$O(|\mathrm{Arctan}(y/x) - \theta_i|^{-1+\delta_i}) \tag{22}$$

for some positive δ_i;

(c) *If the singularity θ_i corresponds to a point on either the x or y axis, then*

$$\delta_i > 1/2\,. \tag{23}$$

Proof. We can decompose S into two operators whose kernels are supported respectively in $|x - y| < |\lambda|^{-1/n}$ and $|x - y| > |\lambda|^{-1/n}$. The first operator is bounded on $L^2(\mathbf{R})$ with norm $O(|\lambda|^{-2/n})$ since

$$\int_{|x-y|<|\lambda|^{-1/n}} (|\lambda||x - y|)^{-1/(n-1)} dx + \int_{|x-y|<|\lambda|^{-1/n}} (|\lambda||x - y|)^{-1/(n-1)} dy \leqq O(|\lambda|^{-2/n})$$

As for the second operator, the restriction $|x - y| > |\lambda|^{-1/n}$ implies that its kernel can be estimated by

$$|\lambda|^{-v-2/n} L(x, y) \leqq |\lambda|^{-2/n} M(x, y)$$

with $M(x, y) = (|x| + |y|)^{vn} L(x, y)$. Theorem 4 is now a consequence of the following lemma:

Lemma 1 *Let $M(x, y)$ be a positive homogeneous function of degree -1, continuous on the circle $x^2 + y^2 = 1$, except possibly at a finite number of points where its singularities are bounded by the right hand side of (22) (23). Then the operator U with kernel $M(x, y)$ is bounded on $L^2(\mathbf{R})$.*

Proof. The estimates (22) (23) imply that $M(x, y)$ can be bounded by a finite number of terms of the form

$$\sum_{i=1}^{N} |x - a_i y|^{-1+\delta_i}(|x| + |y|)^{-\delta_i},\ |x|^{-1+\delta'}(|x| + |y|)^{-\delta'},\ |y|^{-1+\delta''}(|x| + |y|)^{-\delta''}$$

for $\delta_i > 0$, δ', $\delta'' > 1/2$, and some constants $a_i \neq 0$. For $\varepsilon > 0$ small enough in the first case and $(1-\delta')/2 < \varepsilon < \delta'/2$ or $(1-\delta'')/2 < \varepsilon\delta''/2$ in the other cases, we obtain

$$\int_{-\infty}^{+\infty} M(x,y)|y|^{-2\varepsilon}dy \leqq C|x|^{-2\varepsilon}$$

$$\int_{-\infty}^{+\infty} M(x,y)|x|^{-2\varepsilon}dx \leqq C|y|^{-2\varepsilon}\,. \tag{24}$$

Combining (24) with Holder's inequality gives

$$\begin{aligned}|Uf(x)|^2 &\leqq \left(\int_{-\infty}^{+\infty} M(x,y)|y|^{2\varepsilon}|f(y)|^2\,dy\right)\left(\int_{-\infty}^{+\infty} M(x,y)|y|^{-2\varepsilon}dy\right)\\ &\leqq C|x|^{-2\varepsilon}\left(\int M(x,y)|y|^{2\varepsilon}|f(y)|^2\,dy\right)\end{aligned} \tag{25}$$

and

$$\begin{aligned}\int |Uf(x)|^2\,dx &\leqq C\int\left(\int |x|^{-2\varepsilon}M(x,y)|dx\right)|y|^{2\varepsilon}|f(y)|^2\,dy\\ &\leqq C\int |f(y)|^2\,dy\,.\ \text{QED}\end{aligned} \tag{26}$$

4 Estimates for the kernel of TT^*

We return to the study of the operator T. It is convenient to set $\alpha_{n-2} = 1$ and to introduce a cut-off function χ which is C^∞ with compact support. The transpose of T is given by

$$(T^*\psi)(z) = \int e^{-i\lambda S(y,z)}\chi(z)\psi(y)dy \tag{27}$$

and the kernel $K_\lambda(x,y)$ of TT^* is given by

$$\begin{aligned}K_\lambda(x,y) &= \int e^{i\lambda(S(x,z)-S(y,z))}\chi(z)^2\,dz\\ &= \int e^{i\lambda(x-y)P(x,y,z)}\chi(z)^2 dz\end{aligned} \tag{28}$$

where $P(x,y,z)$ is the homogeneous polynomial in all three variables of degree $n-1$, monic in z, given by

$$P(x,y,z) = (x-y)^{-1}(S(x,z) - S(y,z))\,. \tag{29}$$

Let $\zeta_k(x,y) = a_k(x,y) + ib_k(x,y)$, $k = 1, \ldots, n-2$ be the (possibly complex) roots of $dP(x,y,z)/dz \equiv P'_z(x,y,z)$ as a polynomial in z. They can be chosen to be homogeneous of degree 1 and continuous on small cones in (x,y) space. More precisely there can be difficulties only when $\operatorname{Arctan}(y/x)$ is close to a finite number of special values θ_0, where the polynomial $P'_z(x,y,z)$ has multiple roots. Near such multiple roots, we can represent the $\zeta_j(x,y)$'s by Puiseux series in the projective variables y/x or x/y [8]. In particular they are continuous up to the multiple roots, and analytic away from them. In view of Theorem 3, $K_\lambda(x, y)$ is bounded by expressions of the form (16), with λ replaced here by $\lambda(x-y)$. Choosing m in (16) to be $m = n - 1$, we obtain immediately the first bound required for the application of Theorem 4:

$$|K_\lambda(x, y)| \leqq C(|\lambda||x-y|)^{-1/(n-1)}. \tag{30}$$

To obtain the second bound in Theorem 4, we note that the right hand side of (16) are functions homogeneous in (x, y) of degree $-(n/m)+1$. For $m \leqq n/2$, the quantity $v \equiv (1/m) - (2/n)$ is non-negative, and the degree of homogeneity $1-(n/m)$ can be reexpressed as $-1-nv$. It remains to check only the integrability on the (x, y) unit sphere in order for Theorem 4 to imply the bound $O(|\lambda|^{-2/n})$ for the norm of TT^* on $L^2(\mathbf{R})$. The regions close to the points $x = y$, $x = 0$, $y = 0$, and away from them have to be investigated separately.

Integrability of $K_\lambda(x, y)$ away from $x = y$, $x = 0$, and $y = 0$

In this region the issue of integrability on the unit sphere is resolved through the following lemmas:

Lemma 2 (a) *Let n be even. If the expression*

$$K_\lambda(k, n/2) \tag{31}$$

of (16) (17) *has a non-integrable singularity at some point (x_0, y_0) on the unit sphere different from $x = y$ for some k, $1 \leqq k \leqq n-2$, then the polynomial $P_z'(x, y, z)$ must have two zeroes which are identical as functions of (x, y).*

(b) *Let n be odd. The same conclusion as in* (a) *holds if for some k the expression*

$$[K_\lambda(k, (n-1)/2) K_\lambda(k, (n+1)/2)]^{1/2} \tag{32}$$

has a non-integrable singularity near some point θ_0 on the sphere $x^2 + y^2 = 1$.

Lemma 3 *Let $S(x, z) = \sum_{i=1}^{n-1} \alpha_{i-1} x^{n-i} z^i$. Then $P_z'(x, y, z)$ given by* (29) *cannot have two identical roots in (x, y) unless $|\alpha_1| + |\alpha_0| = 0$, in which case it admits the obvious double root $z \equiv 0$.*

Proof of Lemma 3. If P' admits two identical roots we may write locally

$$S_z'(x, z) - S_z'(y, z) = (x-y)(z - \zeta(x, y))^2 \Phi(x, y, z)$$

where $\Phi(x, y, z)$ is differentiable in (x, y) except at a finite number of lines. Differentiating with respect to x, and setting $z = \zeta(x, y)$, we see that $\zeta(x, y)$ is actually a zero of $S_{xz}''(x, z)$, and is hence independent of y. Differentiating with respect to y shows that it is also independent of x, and thus 0, since it is homogeneous of degree 1. Evidently 0 is an identical double root for $S_z'(x, z) - S_z'(y, z)$ if and only if $|\alpha_1| \pm |\alpha_0| = 0$. QED

Proof of Lemma 2 We begin with (a), so that n is even. Assume first that the zeroes $\zeta_j(x, y) = a_j(x, y)$ are all real, and that $K_\lambda(k, n/2)$ is not integrable near some θ_0 different from $\pi/4$ for some k. (The assumption $\theta_0 \neq \pi/4$, which on the unit sphere $x^2 + y^2 = 1$, is equivalent to the fact that we are away from $x = y$, is used as follows. In the expression (16) λ is actually replaced by $\lambda(x-y)$; therefore the hypothesis $\theta_0 \neq \pi/4$ guarantees that in a neighborhood of that point $|\lambda(x-y)| \sim |\lambda|$.) Since the $a_j(x, y)$ are Puiseux series near θ_0, their distances have the following behavior near θ_0

$$|a_j(x, y) - a_k(x, y)| \sim |\operatorname{Arctan}(y/x) - \theta_0|^{m_j}, \quad j \neq k \tag{33}$$

for some non-negative rational number m_j. Without loss of generality we may assume that $k = 1$, and that the m_j's are arranged in decreasing order

$$m_2 \geqq m_3 \geqq \ldots \geqq m_{n-2} \geqq 0\,. \tag{34}$$

Since $K_\lambda(1, n/2)$ has a divergent singularity, it follows that for any subset L of $(n/2) - 1$ roots $a_j(x, y)$ containing $a_1(x, y)$, we have

$$\left(\prod_{j \notin L} |a_1(x, y) - a_j(x, y)| \right)^{-2/n} \geqq c|\operatorname{Arctan}(y/x) - \theta_0|^{-1}\,. \tag{35}$$

Choosing L to consist of the $(n/2) - 1$ nearest roots to $a_1(x, y)$, i.e., $a_1(x, y), \ldots, a_{(n/2)-1}$, we can express (35) as

$$m_{n/2} + m_{(n/2)+1} + \ldots + m_{n-2} \geqq n/2\,. \tag{36}$$

To show that two zeroes coincide identically, we shall show that the discriminant Δ of the polynomial $P'_z(x, y, z)$

$$\Delta = \prod_{1 \leqq i < j \leqq n-2} |a_i(x, y) - a_j(x, y)|^2 \tag{37}$$

vanishes identically. Now the discriminant Δ is a polynomial in the coefficients of the corresponding polynomial $P'_z(x, y, z)$ and hence a polynomial in x and y. Furthermore it is homogeneous of degree

$$\text{degree}\, \varDelta = (n-2)(n-3) \tag{38}$$

in the roots and thus homogeneous of the same degree in x and y. Our task reduces to that of showing that $\varDelta$ vanishes of some order $> (n-2)(n-3)$ near θ_0. In view of the fact that

$$\begin{aligned} |a_i(x, y) - a_j(x, y)| &\leqq |a_1(x, y) - a_i(x, y)| + |a_1(x, y) - a_j(x, y)| \\ &\leqq 2|\operatorname{Arctan}(y/x) - \theta_0|^{\min(m_i, m_j)} \end{aligned}$$

we can estimate $\varDelta$ by

$$|\varDelta| \leqq \prod_{j=1}^{n-2} \left(\prod_{i=1}^{j-1} |\operatorname{Arctan}(y/x) - \theta_0|^{2m_j} \right) \tag{39}$$

and the order of vanishing of $\varDelta$ is bounded below by

$$\text{order}\, \varDelta \geqq 2 \sum_{j=1}^{n-2} (j-1) m_j\,. \tag{40}$$

We claim that the minimum of the right hand side of (40), subject to the conditions (34) and (36) is

$$\min \left[2 \sum_{j=1}^{n-2} (j-1) m_j \right] = n(n-3) \tag{41}$$

attained when $m_2 = \ldots = m_{n-2} = n/(n-2)$. In fact the right hand side of (40) is linear in the m_j's with positive coefficients. Its minimum is attained then in the subregion of (34) and (36) with $m_2 = \ldots = m_{(n/2)-1} = m_{n/2}$. Thus we need only minimize the function

$$F \equiv \frac{n}{2}\left(\frac{n}{2} - 1\right) m_{n/2} + 2 \sum_{j=(n/2)+1}^{n-2} (j-1)m_j \tag{42}$$

subject to the conditions

$$\begin{aligned} m_{n/2} + m_{(n/2)+1} + \ldots + m_{n-2} &= \kappa \frac{n}{2} \\ m_{n/2} \geqq m_{(n/2)+1} \geqq \ldots \geqq m_{n-2} &\geqq 0 \end{aligned} \tag{43}$$

for fixed $\kappa \geqq 1$. Now the first condition in (43) defines a hyperplane in $(m_{n/2}, \ldots, m_{n-2})$ space, and the second condition in (43) defines a simplex within this hyperplane. Since F is linear, the minimum is at one of the vertices $\mathbf{v}$ of this simplex. The vertices are obtained by setting all the inequality signs in the second condition in (43) to equality signs, except for one. In this way we obtain

- $\mathbf{v} = \{m_{n/2} = \ldots = m_{n-2} = \kappa n/(n-2)\}$

$$F(\mathbf{v}) = \kappa n(n-3) \tag{44}$$

- $\mathbf{v} = \{m_{n/2} = \ldots = m_l = \kappa(n/2)(l - (n/2) + 1)^{-1}, \quad m_{l+1} = \ldots = m_{n-2} = 0\}$, $(n/2) \leqq l \leqq n-2$

$$F(\mathbf{v}) = \kappa \frac{n}{2} \frac{l(l-1)}{l - \frac{n}{2} + 1} \equiv \kappa \frac{n}{2} G(l)\,. \tag{45}$$

The minimum of the function $G(l)$ over the half line $l - \frac{n}{2} + 1 > 0$ is attained at $l_{\min}$

$$n - 3 < l_{\min} = \frac{n}{2} - 1 + \sqrt{\left(\frac{n}{2} - 1\right)\left(\frac{n}{2} - 2\right)} < n - 2\,. \tag{46}$$

Since l is an integer, the minimum of $G(l)$ over integer values greater than $(n/2) - 1$ is attained at rather $l = n - 3$ and $l = n - 2$ where it takes the same value

$$G(l)_{l=n-2 \text{ or } l=n-3} = 2(n-3)\,. \tag{47}$$

Altogether this leads to the value $\kappa n(n-3)$ for F. Letting κ decrease to 1 establishes our claim (41).

We drop now the assumption that the roots $\zeta_j(x, y)$ are real. In particular the root $\zeta_k(x, y) \equiv a_k + ib_k$ at which $K_\lambda(k, n/2)$ fails to be integrable may have an imaginary part which is non-zero away from θ_0. Since $P_z'(x, y, z)$ has real coefficients, $\bar{\zeta}_k = a - ib_k$ is a root as well. Set

$$\begin{aligned} |a_k(x, y) - \zeta_j(x, y)| &\sim |\mathrm{Arctan}(y/x) - \theta_0|^{m_j} \\ |b_k(x, y)| &\sim |\mathrm{Arctan}(y/x) - \theta_0|^{\mu}\,. \end{aligned} \tag{48}$$

Again it is convenient to choose the indexing of the roots so that $k = 1$, and the remaining roots are respectively $\bar{\zeta}_1$, and ζ_j, $3 \leqq j \leqq n - 2$ with

$$m_3 \geqq \ldots \geqq m_{n/2} \geqq m_{(n/2)+1} \geqq \ldots \geqq m_{n-2} \geqq 0 . \tag{49}$$

Returning to the discriminant Δ we estimate this time by

$$|\Delta| \leqq b_1^2 \prod_{j=3}^{n-2} (|a_1 - \zeta_j|^2 + b_1^2)^2 \left(\prod_{3 \leqq i < j \leqq n-2} |\zeta_i - \zeta_j|^2 \right) . \tag{50}$$

We consider several cases separately.

- $\mu \geqq m_{n/2}$. In this case we can bound b_1, and all $|a_1 - \zeta_k|$ with $j \leqq n/2$ by $|a_1 - \zeta_{n/2}|$. It follows that the order of vanishing of Δ at θ_0 is at least

$$\text{order}\,\Delta \geqq 2 \sum_{j=(n/2)+1}^{n-2} (j-1)m_j + \frac{n}{2}\left(\frac{n}{2} - 1\right) m_{n/2} . \tag{51}$$

  Since $K_\lambda(1, n/2)$ is not integrable none of the three defining terms on the right hand side of (16), (17) is. We take the first term, together with the choice $L^c = \{\zeta_3, \ldots, \zeta_{n/2-1}\}$. Its non-integrability on the (x, y) sphere means that

$$m_{n/2} + m_{n/2+1} + \ldots + m_{n-2} \geqq \frac{n}{2} . \tag{52}$$

  Thus we are led to the same linear optimization problem as in (42) with the condition (43). As a consequence order $\Delta \geqq n(n-3)$.
- $m_{n/2} < \mu \leqq m_{(n/2)+1}$. We bound all $|a_1 - \zeta_k|$ with $j \leqq n/2$ by $|b_1|$. This gives as a lower bound for the order of vanishing of Δ

$$\text{order}\,\Delta \geqq 2 \sum_{j=(n/2)+1}^{n-2} (j-1)m_j + \frac{n}{2}\left(\frac{n}{2} - 1\right) \mu . \tag{53}$$

  The non-integrability of the second term in the defining equation (17) for $K_\lambda(1, n/2)$ with $L^c = \{\zeta_3, \ldots, \zeta_{n/2}\}$ gives

$$\mu + m_{(n/2)+1} + m_{(n/2)+2} + \ldots + m_{n-2} \geqq \frac{n}{2} . \tag{54}$$

  The problem of minimizing the right hand side of (53) subject to the restrictions (54) and $\mu \geqq m_{(n/2)+1} \geqq \ldots \geqq m_{n-2} \geqq 0$ is again the same problem as before, with $m_{n/2}$ replaced by μ. We again conclude that the lower bound in $n(n-3)$.
- $m_{(n/2)+1} > \mu \geqq m_{(n/2)+2}$. We bound all the terms $|a_1 - \zeta_j|$ with $j \leqq (n/2) + 1$ by $|b_1|$. The order of vanishing of Δ is

$$\text{order}\,\Delta \geqq 2 \sum_{j=(n/2)+2}^{n-2} (j-1)m_j + \frac{n}{2}\left(\frac{n}{2} + 1\right) \mu . \tag{55}$$

  To obtain a constraint when minimizing the right hand side of (55) we take the third term in the definition (17) for $K_\lambda(1, n/2)$. The choice $L^c = \{\zeta_3, \ldots, \zeta_{(n/2)+1}\}$ gives

$$2\mu + m_{(n/2)+2} + \ldots + m_{n-2} = \kappa \frac{n}{2} \tag{56}$$

with $\kappa \geqq 1$. Together with the restriction

$$\mu \geqq m_{(n/2)+2} \geqq \dots m_{n-2} \geqq 0 \tag{57}$$

we obtain again a simplex within an $(n/2) - 2$ dimensional hyperplane. The vertices different from ($\mu = m_{(n/2)+2} = \dots = m_{n-2} = \kappa n(n-2)^{-1}$) are given by

$$\mu = \dots = m_l = \kappa \frac{n}{2}\left(l - \frac{n}{2} + 1\right)^{-1}, \quad m_{l+1} = \dots = m_{n-2} = 0, \quad l \geqq \frac{n}{2} + 1$$

and the corresponding value for the right hand side of (55) is the same as in (45)

$$\kappa \left(\frac{n}{2}\right) \frac{l(l-1)}{l - \frac{n}{2} + 1}$$

so the minimum value over l integer is still $\kappa n(n-3)$, and thus $n(n-3)$ as κ tends to 1.

- $m_p > \mu \geqq m_{p+1}$, $p \geqq (n/2) + 2$. The separation of this case from the previous one is more for notational convenience than anything else. We bound all the $|a_1 - \zeta_j|$ with $j \leqq (n/2) + 2$ by $|a_1 - \zeta_{(n/2)+2}|$. This gives as lower bound for the order of vanishing of Δ

$$\text{order}\,\Delta \geqq 2 \sum_{j=(n/2)+3}^{p} (j-3)m_j + 2 \sum_{j=p+1}^{n-2} (j-1)m_j + \frac{n}{2}\left(\frac{n}{2} - 1\right) m_{(n/2)+2} + (4p-6)\mu\,. \tag{58}$$

The right hand side is to be minimized subject to the same constraint (56), together with

$$m_{(n/2)+2} \geqq \dots \geqq m_p \geqq \mu \geqq m_{p+1} \geqq \dots \geqq m_{n-2} \geqq 0\,. \tag{59}$$

The vertices of the corresponding simplex are the common vertex ($\mu = m_{(n/2)+2} = \dots = m_{n-2} = \kappa n(n-2)^{-1}$; the vertices $\mathbf{v}_+ = \{m_{(n/2)+2} = \dots = \mu = m_l = \kappa\,(n/2)(l - (n/2) + 1)^{-1},\ m_{l+1} = \dots = m_{n-2} = 0\}$, $l \geqq p$, where the value of the right hand side of (58) is $\kappa(n/2)\,l(l-1)(l-(n/2)+1)^{-1} \geqq \kappa n(n-3)$; and finally the vertices $\mathbf{v}_- = \{m_{(n/2)+2} = \dots = m_l = \kappa(n/2)(l-(n/2)-1)^{-1}$, $m_{l+1} = \dots = \mu = \dots = m_{n-2} = 0\}$, $(n/2) + 2 \leqq l \leqq p$, where the value of (58) is now $\kappa(n/2)(l-3)(l-2)(l-(n/2)-1)^{-1}$. Shifting variables $l \to l + 2$, we recognize the same function (45) as before, with minimum over integer values equal to $n(n-3)$.

The proof that Δ has a zero of order strictly greater than its degree $(n-2)(n-3)$ and hence must vanish identically is complete for the case of even n. This establishes (a) of Lemma 2.

Next we turn to (b), so n is odd. We have already discussed in detail the case $n = 3$ in the Introduction. Thus we may assume that $n \geqq 5$. The argument here is exactly the same one as for the case of even n. The only difference is a harmless shift of order $(1/2n)$ in the lower bounds obtained before for the order of vanishing of the discriminant Δ, due to the fact that we cannot remove a cluster of $n/2$ critical points in the method of stationary phase, but instead have to take the geometric mean of the bounds obtained by removing clusters of $(n-1)/2$ and $(n+1)/2$ critical points. We shall consequently be brief. Again we begin by the case when all roots $\zeta_j = a_j$ are real. If $[K_\lambda(k,(n-1)/2)K_\lambda(k,(n+1)/2]^{1/2}$ is non-integrable for some k, we may assume $k=1$ and order the a_j in order of increasing $|a_1 - a_j|$, $2 \leqq j \leqq n-2$. In the definition of $K_\lambda(1,(n-1)/2)$ and $K_\lambda(1,(n+1)/2)$, we remove respectively the clusters $L^c = \{a_2, \ldots, a_{(n-1)/2-1}\}$ and $L^c = \{a_2, \ldots, a_{(n-1)/2}\}$. The non-integrability implies

$$\left(\prod_{j \geqq (n-1)/2} |a_1 - a_j|\right)^{-1/(n-1)} \left(\prod_{j \geqq (n+1)/2} |a_1 - a_j|\right)^{-1/(n+1)} \geqq c\,|\mathrm{Arctan}(y/x) - \theta_0|^{-1} \tag{60}$$

or, equivalently, with m_j defined as in (33)

$$\frac{n+1}{2n} m_{(n-1)/2} + m_{(n+1)/2} + \ldots + m_{n-2} = \kappa \frac{n^2-1}{2n} \tag{61}$$

for $\kappa \geqq 1$. The order of vanishing of the discriminant Δ is still bounded from below by (40). Bounding all m_j for $j \leqq m_{(n-1)/2}$ by $m_{(n-1)/2}$, we reduce the problem to that of minimizing

$$F^{\mathrm{odd}} \equiv 2 \sum_{j=(n+1)/2}^{n-2} (j-1)m_j + \left(\frac{n-1}{2}\right)\left(\frac{n-3}{2}\right) m_{(n-1)/2} \tag{62}$$

subjected to the constraints (61) and

$$m_{(n-1)/2} \geqq m_{(n+1)/2} \geqq \ldots \geqq m_{n-2} \geqq 0\,. \tag{63}$$

Again it suffices to evaluate the values of F^{odd} at the vertices of the simplex (61)–(63). At the vertex

$$m_{(n-1)/2} = \ldots = m_{n-2} = \kappa(n+1)(n-1)^{-1}$$

we obtain the value

$$F^{\mathrm{odd}} = \kappa \frac{n+1}{n-1}(n-2)(n-3)\,. \tag{64}$$

The value at the other vertices

$$m_{(n-1)/2} = \ldots = m_l = \kappa \frac{n^2-1}{2n\left(l - \frac{n}{2} + 1 + \frac{1}{2n}\right)},$$

$$m_{l+1} = \ldots = m_{n-2} = 0, \quad \frac{n-1}{2} \leqq l \leqq n-2$$

is

$$F^{\mathrm{odd}} = \kappa \frac{n^2-1}{2n}\left(\frac{l(l-1)}{l - \frac{n}{2} + 1 + \frac{1}{2n}}\right) \equiv \kappa \frac{n^2-1}{2n} G^{\mathrm{odd}}(l)\,. \tag{65}$$

The minimum of G^{odd} as a function of l is at $l_{\min}$

$$n-3-\frac{1}{n}<l_{\min}=\left(\frac{n}{2}-1-\frac{1}{2n}\right)+\sqrt{\left(\frac{n}{2}-1-\frac{1}{2n}\right)\left(\frac{n}{2}-2-\frac{1}{2n}\right)}<n-2-\frac{1}{n}. \tag{66}$$

This time the minimum value over integers can be attained at either $l=n-2$, $l=n-3$, or $l=n-4$. The value at $l=n-2$ is given by (64). At the other vertices it is given by

$$G^{\text{odd}}|_{l=n-3}=\frac{2n(n-3)(n-4)}{n^2-4n+1}, \quad G^{\text{odd}}|_{l=n-4}=\frac{2n(n-5)(n-4)}{n^2-6n+1}. \tag{67}$$

Each of these values is strictly greater than $2n(n-4)(n-1)^{-1}$. Substituting in (65) and letting κ tend to 1 gives the following strict lower bounds for F^{odd} at these vertices

$$F^{\text{odd}}>(n+1)(n-4)\geqq(n-2)(n-3). \tag{68}$$

Altogether we always have the order of vanishing of Δ strictly greater than $(n-2)(n-3)$, which is the desired conclusion.

We turn to the general case where the roots $\zeta_j=a_j+ib_j$ are allowed to be complex, and adopt the same notation and ordering as in (48). The distinct cases to be considered are now:

- $\mu\geqq m_{(n-1)/2}$. As in the case of n even, this case is identical to the case of all roots being real treated previously.
- $m_{(n-1)/2}>\mu\geqq m_{(n+1)/2}$ We bound the order of vanishing of Δ by

$$\operatorname{order}\Delta\geqq 2\sum_{j=(n+1)/2}^{n-2}(j-1)m_j+\tfrac{1}{4}(n^2-4n+3)\mu. \tag{69}$$

The constraint is obtained by choosing the same cluster $L^c=\{\zeta_3,\ldots,\zeta_{(n-1)/2}\}$ for both $K_\lambda(1,(n-1)/2)$ and $K_\lambda(1,(n+1)/2)$, but use the first expression on the right hand side of (17) for $K_\lambda(1,(n+1)/2)$ and the second expression for $K_\lambda(1,(n-1)/2)$. The non-integrability of the resulting expression becomes

$$\frac{n+1}{2n}\mu+m_{(n+1)/2}+m_{(n+3)/2}+\ldots+m_{n-2}=\kappa\frac{n^2-1}{2n} \tag{70}$$

for some $\kappa\geqq 1$. In practice we may set $\kappa=1$. We minimize the right hand side of (69) subject to (70) and the ordering

$$\mu\geqq m_{(n+1)/2}\geqq m_{(n+5)/2}\geqq\ldots\geqq m_{n-2}\geqq 0 \tag{71}$$

At the vertex

$$(\mu=m_{(n+1)/2}=\ldots=m_{n-2}=(n+1)(n-1)^{-1}) \tag{72}$$

the value is given by (64). At the vertex ($\mu=n-1$, $m_{(n+1)/2}=\ldots=m_{n-2}=0$), the value is

$$F^{\text{odd}}=\tfrac{1}{4}(n-1)^2(n-3) \tag{73}$$

and at the vertices ($\mu = m_{(n+1)/2} = \ldots = m_l = (n^2-1)(2n)^{-1}(l-(n/2)+1+1/(2n))^{-1}$) the value of F^{odd} is given by the same expression as in (65). Since the value $(n-1)^2(n-3)/4$ is strictly greater than $(n-2)(n-3)$ for $n \geqq 5$, this establishes the identical vanishing of Δ in this case.

- $m_{(n+1)/2} > \mu \geqq m_{(n+3)/2}$. Here we bound the order of vanishing of Δ by

$$\text{order}\,\Delta \geqq 2 \sum_{j=(n+3)/2}^{n-2} (j-1)m_j + \tfrac{1}{4}(n^2-1)\mu\,. \tag{74}$$

The non-integrability condition we are going to use is the one resulting from using the third expression in (17) for both $K_\lambda(1,(n-1)/2)$ and $K_\lambda(1,(n+1)/2)$, with the respective choices $\{\zeta_3, \ldots, \zeta_{(n+1)/2}\}$ and $\{\zeta_3, \ldots, \zeta_{(n+3)/2}\}$ for L^c. In terms of μ and the m_j's it can be written as

$$2\mu + \frac{n+1}{2n} m_{(n+3)/2} + m_{(n+5)/2} + \ldots + m_{n-2} = \kappa \frac{n^2-1}{2n} \tag{75}$$

for some κ which we may again take to be 1. We need only consider vertices different from (72). At the vertex ($\mu = (n^2-1)/(4n)$, $m_{(n+3)/2} = \ldots = m_{n-2} = 0$) the value of (74) is $(n^2-1)^2/(16n)$ which is strictly greater than $(n-2)(n-3)$ for $n \geqq 5$. At the remaining vertices $\mathbf{v} = (\mu = m_{(n+3)/2} = \ldots = m_l = (n^2-1)\cdot(2n)^{-1}(l-(n/2)+1+1/(2n))^{-1})$, the value of the right hand side of (74) is given by (65) as before, and hence is strictly greater than $(n-2)(n-3)$.

- $m_p > \mu \geqq m_{p+1}$, $p \geqq (n+3)/2$. We bound the order of vanishing of the discriminant from below by

$$\text{order}\,\Delta \geqq 2 \sum_{j=p+1}^{n-2} (j-1)m_j + 2 \sum_{j=(n+5)/2}^{p} (j-3)m_j + \left(\frac{n-3}{2}\right)\left(\frac{n-1}{2}\right) m_{(n+3)/2} + (4p-6)\mu\,. \tag{76}$$

We use the same constraint as in (75) together with

$$m_{(n+3)/2} \geqq \ldots \geqq m_p > \mu \geqq m_{p+1} \geqq \ldots \geqq m_{n-2} \geqq 0\,. \tag{77}$$

Apart from the usual vertex (72), we have the vertices ($m_{(n+3)/2} = \ldots = \mu = m_l = (n^2-1)(2n)^{-1}(l-(n/2)+1+(1/(2n))^{-1}$) where the value of the right hand side of (76) is given by (65), and the vertices ($m_{(n+3)/2} = \ldots = m_l = (n^2-1)(2n)^{-1}(l-(n/2)-1+(1/(2n))^{-1}$, $m_{l+1} = \ldots = \mu = m_{n-2} = 0$), where the value is now $(n^2-1)(2n)^{-1}[(l-2)(l-3)(l-(n/2)-1+(1/(2n))]^{-1}$. Shifting l to $l+2$ in this last expression, we obtain in all cases a lower bound of the form (65), which has been shown to be strictly greater than $(n-2)(n-3)$.

The proof of the vanishing of Δ in the case of odd n, and hence of Lemma 2 is complete. QED

Behaviour of $K_\lambda(x, y)$ near $x = 0$ or $y = 0$

We investigate now the order of singularity of $K_\lambda(x, y)$ for x or y near 0. Evidently $K_\lambda(x, y)$ can only be singular say near $y = 0$, if at $y = 0$ some root in z of the equation $S'_z(x, z) - S'_z(y, z) = 0$ has multiplicity $k \geqq 2$. Say $\zeta(x, 0) = ax$ for some a. Thus we have

$$S'_z(x, z) - S'_z(0, z) = \Psi(z, x)(z - ax)^k$$

with $\Psi(z, x)$ a homogeneous polynomial of degree $n - 1 - k$ with $\Psi(ax, x) \neq 0$. If $a = 0$, it follows that

$$\begin{aligned} S(x, z) &= z^{k+1}\left[\frac{1}{k+1}\Phi(z, x) - z^{-(k+1)}\int_0^z s^{k+1}\Phi'(s, x)ds\right] + Cx^n + S(0, z) \\ &\equiv z^{k+1}\Phi(z, x) + Cx^n + S(0, z)\,. \end{aligned} \tag{78}$$

Since the expression $\Phi(z, x)$ vanishes for $x = 0$, the only terms in (78) of the form x^n or z^n are the last two terms, which must then vanish since we chose $S(x, y)$ to be without such terms in the Main Theorem. Next the hypotheses of the Main Theorem imply that k must be $\leqq n/2$ for n even and $\leqq (n - 1)/2$ for n odd. This means that the multiplicities of the roots $\zeta_j(x, 0)$ are bounded respectively by these bounds, and can all be removed in the same cluster when we estimate $K_\lambda(x, y)$ using Theorem 3. In particular $K_\lambda(x, y)$ has no singularity on the unit sphere near $y = 0$. Thus we may assume that $a \neq 0$, and by a simple scaling, that $a = 1$. It is convenient to write $S'_z(x, z) - S'_z(0, z)$ as

$$S'_z(x, z) - S'_z(0, z) = x^l\Psi(z, x)(z - x)^k$$

for some integer $l \geqq 0$ and homogeneous polynomial $\Psi(z, x)$ of degree $n - 1 - k - l$ satisfying

$$\Psi(z, z) \neq 0, \quad \Psi(z, 0) \neq 0\,. \tag{79}$$

Integrating between z and x gives now

$$S(x, z) = x^l\int_x^z \Psi(s, x)(s - t)^k ds + Cx^n + Dz^n\,.$$

The equation for critical points becomes

$$\begin{aligned} 0 &= S'_z(x, z) - S'_z(y, z) \\ &= x^l\Psi(z, x)(z - x)^k - y^l\Psi(z, y)(z - y)^k\,. \end{aligned} \tag{80}$$

It is convenient to use projective coordinates rather than coordinates on the sphere, so we assume that $x = 1$ and investigate the behaviour of the solutions $\zeta_j(1, y)$ of (80) as y tends to 0. Let $\varepsilon_j, j = 1, \ldots, k$ be the k^{th} roots of unity, $\varepsilon_j^k = 1$. For each fixed ε_j we show that there is a solution to (80) of the form

$$\zeta_j = 1 + \varepsilon_j y^{l/k}\left(\frac{\Psi(1, 0)}{\Psi(1, 1)}\right)^{1/k}[1 + E_j(y)] \tag{81}$$

where $E_j(y)$ is a power series in $y^{1/k}$ without constant term. Dropping the index j for convenience and substituting (81) in (80) gives

$$\Psi\left(1 + \varepsilon y^{l/k}\left(\frac{\Psi(1, 0)}{\Psi(1, 1)}\right)^{1/k}(1 + E), 1\right)\Psi(1, 0)(1 + E)^k$$

$$- \Psi(1,1)\Psi\left(1 + \varepsilon y^{l/k}\left(\frac{\Psi(1,0)}{\Psi(1,1)}\right)^{1/k}(1+E), y\right)$$
$$\cdot\left(1 - \varepsilon y^{l/k}\left(\frac{\Psi(1,0)}{\Psi(1,1)}\right)^{1/k}(1+E) - y\right)^k = 0 .$$

If we set $u = y^{1/k}$ this becomes an analytic equation. At $u = 0$, $E = 0$ is the solution. Also the derivative with respect to E at $u = 0$ and $E = 0$ is non-vanishing. The implicit function theorem establishes our claim. Returning to the critical points $\zeta_j(x, y)$, it follows that

$$|\zeta_i(x, y) - \zeta_j(x, y)| \sim |y|^{l/k} . \tag{82}$$

We assume first that n is even. Evidently we may assume that the multiplicity k is $\geqq n/2$, so that $l \leqq (n/2) - 2$. Now removing a cluster of $(n/2) - 1$ roots when applying Theorem 3 to estimate $K_\lambda(x, y)$ produces the following bound for y near 0

$$|K_\lambda(x, y)| \leqq |\lambda|^{-2/n}|y|^{-\alpha}(|x| + |y|)^{-1+\alpha}$$
$$\alpha = \frac{2}{n}\left[\frac{l}{k}\left(k - \frac{n}{2} + 1\right)\right] . \tag{83}$$

It is now routine to maximize the exponent α over the range $l \leqq (n/2) - 2$. We find $(1/2) - (4/n)$ which is strictly less than 1/2 for $n \geqq 3$. The case n odd can be treated in a similar way, this time by taking the geometric average of extimates obtained by removing clusters of $(n-1)/2$ and $(n+1)/2$ roots, and evaluating the resulting maximum degree of singularity. This degree of singularity is strictly less than 1/2 for $n \geqq 5$. Thus we have shown that under the hypotheses of the Main Theorem $K_\lambda(x, y)$ satisfies the third condition of Theorem 4.

Integrability of $K_\lambda(x, y)$ near $x = y$

We shall require the following lemmas:

Lemma 4 *Let $\zeta_j(x, y)$, $1 \leqq j \leqq n-2$, be the roots of the polynomial in z, $S_z'(x, z) - S_z'(y, z)$, chosen to be continuous near $x = y$. Then $\zeta_j(x, x)$, $1 \leqq j \leqq n-2$ are the roots of the polynomial $S_{xx}''(x, z)$.*

Proof of Lemma 4 We can write

$$S_z'(x, z) - S_z'(y, z) = (x - y)\prod_{j=1}^{n-2}(z - \zeta_j(x, y))$$

Since $\partial\zeta_j/\partial x$ is at worst of order $|x - y|^{-1+\delta}$ for some positive δ, we may differentiate (78) and take the limit as y tends to x. The result is

$$S_{xz}''(x, z) = \prod_{j=1}^{n-2}(z - \zeta_j(x, x))$$

which establishes the lemma. QED

Fix a zero $\zeta_j(x, x)$ of $S_{xz}''(x, z)$ and let k be its multiplicity. By homogeneity $\zeta_j(x, x)$ can be written as ax for some a, and we have

$$S_{xz}''(x, z) = (z - ax)^k\phi(x, z) \tag{84}$$

for some homogeneous polynomial $\phi(x, z)$ of degree $n-2-k$ with $\phi(x, ax) \neq 0$. The cases $a = 0$ and $a \neq 0$ behave differently, so we consider them separately, beginning with $a \neq 0$. Without loss of generality we may then set $a = 1$.

First we note that $S_z'(x, z)$ can be obtained by integrating (84) with respect to x

$$\begin{aligned} S_z'(x, z) &= \int_z^x (z-t)^k \phi(t, z)\,dt + S_z'(z, z) \\ &\equiv (z-x)^{k+1}\psi(x, z) + S_z'(z, z) \,. \end{aligned} \tag{85}$$

Here we have chosen z as the limit of integration. Other choices lead to additional terms in (85) which depend on z only, and thus are irrelevant since they cancel when we consider the difference $S_z'(x, z) - S_z'(y, z)$. A key property of $\psi(x, z)$ is

$$(k+1)\psi(x, x) = -\phi(x, x) \neq 0 \,. \tag{86}$$

Our next task is to determine which exponents can occur in the Puiseux series for the solutions $\zeta_j(x, y)$, $j = 1, \ldots, k$, which tend to the common value $\zeta_j(x, x) = x$ as y tends to x. Set

$$\zeta_j(x, y) \equiv y + c_j(x-y)^{\lambda_j} + \ldots \tag{87}$$

with $c_j \neq 0$. Our first claim is that the exponents λ_j cannot be >1. The explicit form of the equation for $z = \zeta_j(x, y)$ is

$$(z-x)^{k+1}\psi(x, z) - (z-y)^{k+1}\psi(y, z) = 0 \,. \tag{88}$$

For $\lambda_j > 1$ the leading expression in the first term in (88) is $(y-x)^{k+1}\psi(y, y)$, while the second term is of order strictly greater than $k+1$, which is not possible. Next we claim that the exponents λ_j cannot be <1 either. In fact the eq. (88) can be rewritten as

$$[(z-x)^{k+1} - (z-y)^{k+1}]\psi(x, z) + (z-y)^{k+1}(\psi(x, z) - \psi(y, z)) = 0 \,. \tag{89}$$

When $\lambda_j < 1$, we have $z - x \sim c_j(x-y)^{\lambda_j}$, and hence the term between brackets in (89) is of order

$$\begin{aligned} (z-x)^{k+1} - (z-y)^{k+1} &= (y-x)\sum_{i=0}^{k} (z-x)^i (z-y)^{k-i} \\ &\sim (k+1)[c_j(x-y)^{\lambda_j}]^k (y-x) \,. \end{aligned}$$

The remaining term in (89) is evidently of order $\lambda_j(k+1) + 1$, contradicting our assumption that c_j not vanish. Altogether we can conclude that the exponents λ_j, $j = 1, \ldots, k$ are all equal to 1. Set then $\lambda_j = 1$ in (87). The coefficients c_j can now be determined by substituting in (88). We obtain

$$(c_j - 1)^{k+1} = c_j^{k+1}$$

and hence the c_j's are given by

$$c_j = (1 - \varepsilon_j)^{-1}, \quad j = 1, \ldots, k$$

where $(\varepsilon_j)^{k+1} = 1$ are the k roots of unity which are different from 1. As in the previous section we can easily show that each ε_j leads to a solution of the eq. (88). Thus we have proved the first part of the following key lemma:

Lemma 5 *Let the equation*

$$(x-y)^{-1}(S_z'(x, z) - S_z'(y, z)) = 0$$

admit k roots $\zeta_j(x, y)$ *which converge to the same root* $\zeta_1(x, x) = \ldots = \zeta_k(x, x)$ *as y tends to x. If this common value is different from 0, then we have*

$$|\zeta_j(x, y) - \zeta_k(x, y)| \sim |x - y|, \quad \text{for } j \neq k \tag{90}$$

If the common root $\zeta_j(x, x)$ *is 0, then the equation admits exactly k roots* $\zeta_j(x, y)$ *which vanish identically as functions of* (x, y).

The second statement in Lemma 5 is easy to check, since $S''_{xz}(x, z)$ and hence $S'_z(x, z)$, and hence the equation in Lemma 5 contain the factor z^k. QED

We return now to the kernel $K_\lambda(x, y)$, localized to a conic neighborhood of the diagonal $x = y$.

Lemma 6 *Assume the hypotheses of the Main Theorem. Then*

(a) *In a small enough neighborhood of the diagonal, the kernel* $K_\lambda(x, y)$ *satisfies either the hypotheses of Theorem 4, or else the bounds*

$$|K_\lambda(x, y)| \leqq C \min((|\lambda||x - y|)^{-1/(n-1)}, |\lambda|^{-\nu - 2/n}|x - y|^{-1-\nu n}) \tag{91}$$

for some strictly positive number ν;

(b) *Any operator whose kernel satisfies the bounds* (91) *is bounded from* $L^p(\mathbf{R})$ *to* $L^p(\mathbf{R})$, $1 \leqq p \leqq \infty$, *with operator norm* $O(|\lambda|^{-2/n})$.

Proof of Lemma 6. Say n is even, the odd case being similar. Assume that there is a root for $S''_{xz}(x, z)$ of multiplicity k. If $k \leqq (n/2) - 1$, we may remove clusters of $(n/2) - 1$ roots in (16), and the remaining $|a_j - a_k|$ are all bounded away from 0 on the unit (x, y) sphere. Consequently we obtain a singularity of order $|x - y|^{-2/n}$ only, which is integrable. Otherwise assume $k \geqq (n/2)$. We note that due to the hypotheses of the Main Theorem, this multiple root cannot be 0. Thus we may apply Lemma 5 to deduce the bound (90) for the distances separating the roots. Now if we removed in (16) clusters of $(n/2) - 1$ roots, we would obtain the bound $|\lambda|^{-2/n}|x - y|^{-1}$ for $K_\lambda(x, y)$, which is just on the edge of integrability. As long as $(n/2) - 2$ is at least 1, the desired bound follows rather from the removal of say, $(n/2) - 2$ roots, in which case the exponent ν of (91) is given by $\nu = 4(n(n-2))^{-1} > 0$. Thus we have established (91) when $n \geqq 5$. The case $n = 4$ is dealt with by an explicit calculation. The only $S''_{xz}(x, z)$ which admits non vanishing double roots in this case is given up to a multiplicative constant by $(z - \alpha x)^2$ for some $\alpha \neq 0$. Integrating gives the following explicit formula

$$|S'_z(x, z) - S'_z(y, z)| = |x - y| \left[\left(z - \frac{1}{2}\alpha(x + y) \right)^2 + \frac{\alpha^2}{12}(x - y)^2 \right]$$

$$\geqq \frac{\alpha^2}{12}|x - y|^3 . \tag{92}$$

Applying (3) to the integral (28) gives in this case

$$|K_\lambda(x, y)| \leqq C(|\lambda||x - y|^3)^{-1}$$

which is of the form (91) with $\nu = 1/2$. Part (a) is proved. As for part (b) it is an easy consequence of the two bounds in (91)

$$\int_{|x-y| > |\lambda|^{-1/n}} |K_\lambda(x, y)|\, dx + \int_{|x-y| > |\lambda|^{-1/n}} |K_\lambda(x, y)|\, dy \leqq C|\lambda|^{-2/n}$$

Lemma 6 is completely proved. QED

The following lemma which is the main goal of this section is now immediate:

Lemma 7 *Under the hypotheses of the Main Theorem, the kernel $K_\lambda(x, y)$ restricted to a small conic neighborhood of the diagonal $x = y$ gives rise to a bounded operator on $L^2(\mathbf{R})$, with norm $O(|\lambda|^{-2/n})$.*

Proof of the Main Theorem

The two previous sections show that we need study $K_\lambda(x, y)$ only away from the diagonal and from $x = 0$ and $y = 0$. If $|\alpha_0| + |\alpha_1| \neq 0$, then Lemmas 2 and 3 show that the hypotheses of Theorem 4 are satisfied for the kernel $K_\lambda(x, y)$ of TT^*, and the Main Theorem is proved in this case. Assume now that $\alpha_0 = \alpha_1 = 0$, so that among the roots $\zeta_j(x, y)$ there are two roots which are identically 0. We again consider the bounds provided by Theorem 3. First we consider a root say $\zeta_1(x, y)$ which tends to a non-zero multiple root. Ordering the roots by the decreasing order of vanishing of $|\alpha_1 - \zeta_j|$, we have evidently $\zeta_{n-3} \equiv \zeta_{n-2} \equiv 0$. Suppose that the bounds provided by Theorem 3 are non-integrable functions on the (x, y) circle. Then the proof of Lemma 2 shows that with the same notation as (33)(40)

$$(n-2)(n-3) < 2 \sum_{j=1}^{n-2} (j-1)m_j = 2 \sum_{j=1}^{n-4} (j-1)m_j .$$

This last expression is a lower bound for the order of vanishing of

$$|\Delta^*| \equiv \prod_{1 \leqq i < j \leqq n-4} |\zeta_i - \zeta_j|^2$$

which can be viewed as the discriminant of the polynomial

$$z^{-2}(x-y)^{-1}(S_z'(x, z) - S_z'(y, z)) . \tag{93}$$

As such it has degree $(n-4)(n-5)$, and hence must vanish identically. This means that the polynomial itself must have two identical roots, which in turn must be identically 0, in view of Lemma 3. In particular $\alpha_2 = \alpha_3 = 0$. Obviously we can repeat the argument after factoring out z^4 to deduce that the next α coefficients must vanish as well, until the hypotheses of the Main Theorem are violated. It remains to consider roots which tend to 0 as a multiple root. These fall into two types, the identically vanishing roots, and the roots which tend to 0 but are not identically 0. For the remainder of this proof we shall denote the former by $\zeta_j^0(x, y) \equiv 0$, $1 \leqq j \leqq p$ for some p, and the latter by $\zeta_j^*(x, y)$, $1 \leqq j \leqq n-2-p$. It follows that the ζ_j^* are the roots of the following polynomial

$$z^{-p}(x-y)^{-1}(S_z'(x, z) - S_z'(y, z)) \tag{94}$$

with non-identically zero constant coefficient. Furthermore they can only arise in a neighborhood of $x + y = 0$ and when the non-zero coefficient of (94) is of the form $(x^k - y^k)(x - y)^{-1}$ for some even k. Set

$$|\zeta_j^*(x, y)| \sim |x + y|^{m_j^*} .$$

The product of the roots $\zeta_j^*(x, y)$ must be proportional to $|(x^k - y^k) \cdot (x - y)^{-1}| \sim |x + y|$. It follows that

$$\sum_{j=1}^{n-2-p} m_j^* = 1 . \tag{95}$$

Consider now the case of n even, and the bounds provided by Theorem 3 near the identically zero root. The hypotheses of the Main Theorem for n even imply that there are at most $p \leqq (n/2) - 1$ of them. We can apply Lemma 2 to remove the whole $(n/2) - 1$ cluster of zero roots. The contributions of the possibly remaining roots ζ_j^* to $K_\lambda(k, n/2)$ are bounded by

$$|x + y|^{-(2/n)\sum m_j^*} \leqq |x + y|^{-2/n}$$

in view of (95). Thus we obtain an integrable kernel and Theorem 4 applies. Next consider the bounds near a non-identically vanishing root, say $\zeta_1^*(x, y)$. If the bounds provided by Theorem 3 are not integrable, then the proof of Lemma 2 shows that the discriminant Δ must vanish of order strictly greater than $(n-2)(n-3)$. Actually we have shown more, since the proof does not rely on any cancellation between the remaining roots, i.e., $\zeta_i^0 - \zeta_j^0$ is merely bounded by $|\zeta_i^0 - \zeta_1^*| + |\zeta_j^0 - \zeta_1^*| \leqq c|x + y|^{m_1^*}$. We may write

$$\Delta = \prod_{1 \leqq i < j \leqq n-2-p} |\zeta_i^* - \zeta_j^*|^2 \prod_{1 \leqq i < n-2-p} \; \prod_{1 \leqq j \leqq p} |\zeta_i^* - \zeta_j^0|^2 \prod_{1 \leqq i < j \leqq p} |\zeta_i^0 - \zeta_j^0|^2 . \tag{96}$$

The first factor in (96) is the discriminant Δ^* of the polynomial (94), the second reduces to $\prod_{1 \leqq i \leqq n-2-p} |\zeta_i^*|^{2p}$, while the order of vanishing of the third term is counted as $2m_1 p(p-1)$ in the proof of Lemma 2. This together with (95) and the fact that $m_1 \leqq 1$ imply that

$$\begin{aligned} \text{order}\, \Delta^* &> (n-2)(n-3) - (2p) - 2p(p-1) = (n-1)(n-3) - 2p^2 \\ &> (n-2-p)(n-3-p) \end{aligned} \tag{97}$$

for $p \leqq (n/2) - 1$ and $n \geqq 4$. This means that Δ^* vanishes identically. Lemma 3 applied to the polynomial (94) implies that its first two coefficients must vanish, and in particular that the original polynomial admits more than p identically zero roots, contradicting our assumption. Finally the case of n odd is treated in the same way. The difference is that we can allow now only $((n-1)/2) - 1$ identically zero roots, so we can remove them all in a cluster of size $((n-1)/2) - 1$. Note that the bound $K_\lambda(k, (n+1)/2)$ is of no help if there are at least $(n-1)/2$ roots, since the factor $K_\lambda(k, (n-1)/2)$ in (16) is already infinite. The proof of the Main Theorem is complete. QED

5 Some open questions

We would like to mention a few questions which are the most direct outgrowths of the present work.

- The purpose of the Main Theorem was to establish the sharp bound $\|T\| \leqq C|\lambda|^{-1/n}$ in the model case of a Lagrangian having cusps (or "torsion") on both sides. We expect to be able to relax the hypotheses. For example when n is even, and the phase function is a homogeneous polynomial of degree n of the form

$$S(x, y) = x^{n/2} y^{n/2} + O(x^{(n/2)+1}), \text{ or } S(x, y) = x^{n/2} y^{n/2} + O(y^{(n/2)+1}) \tag{98}$$

  the usual van der Corput Lemma gives at once the following bound for the kernel $K_\lambda(x, y)$ of TT^*

$$|K_\lambda(x, y)| \leqq C(|\lambda||x - y|)^{-2/n}(|x| + |y|)^{-1+(2/n)} \tag{99}$$

It follows that the bound $\|T\| \leqq C|\lambda|^{-1/n}$ still holds in this case. A plausible conjecture is that this bound holds if and only if there is at least one non-zero coefficient α_i with $1 \leqq i \leqq n/2$, and at least one non-zero coefficient α_j with $n/2 \leqq j \leqq n-1$. We note that when n is even, this condition allows α_i and α_j to be both equal to $n/2$, which is the case we just discussed. Our present approach relies only on the size of $|K_\lambda(x, y)|$. What is needed is some exploitation of its phase. An indirect such method is provided by dyadic partitions away from the zeroes of the Hessian, which can be used for folds in degree 3 [7].

- A generalization to non-polynomial smooth phase functions would be valuable;
- The present operators are intimately related to Radon transforms along curves with one-sided torsion in dimension n. It would be interesting to obtain L^2 as well as $L^p - L^q$ bounds for these;
- One of the most challenging extensions is to oscillatory integral operators with polynomial phases in higher dimensions. Perhaps a generalization of the refined method of stationary phase of Section 2 is a good starting point.

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