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A new obstruction to embedding Lagrangian tori

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Introduction

Let $\mathbb{R}^{2n}$ be endowed with the canonical symplectic form $\sigma = \sum_{i=1}^{n} dx_i \wedge dy^i$, and consider a Lagrangian embedding of a compact manifold $j: L^n \to (\mathbb{R}^{2n}, \sigma)$.

A well-known result due to Gromov ([G]) states that the cohomology class of σ in $H^2(\mathbb{R}^{2n}, L)$ is non-zero (we identify L with its image in $\mathbb{R}^{2n}$).

An equivalent assertion is that the Liouville form $\lambda = \sum_{i=1}^{n} x_i dy^i$ pulls back to a closed form $j^*\lambda$ on L, whose cohomology class does not vanish.

Gromov's proof uses subtle properties of holomorphic maps into $\mathbb{C}^n$. He shows that there is always a holomorphic disk Δ in $\mathbb{C}^n$ with boundary in L. On such a disk, σ is positive, hence $\int_\Delta \sigma = \int_{\partial\Delta} \lambda > 0$.

In this paper we shall only consider the case $L = T^n$, and give an "elementary" proof of Gromov's result (elementary should be understood in the sense of number theory, that is without using holomorphic functions). In fact our result is somewhat more precise and can be stated as follows.

Theorem A. *If $j: T^n \hookrightarrow (\mathbb{R}^{2n}, \omega_0)$ is a Lagrangian embedding, there exists a loop γ on T^n such that*:

(i) $\langle [j^*\lambda], \gamma \rangle > 0$
(ii) $\langle \mu(j), \gamma \rangle \in [2, n+1]$

where $\mu(j)$ is the Maslov class of j, and $\langle \, , \rangle$ is the pairing between H^1 and H_1.

This result answers a question first raised (as far as we know) by Michèle Audin (cf. [Au 1]). Let us point out that since the torus is orientable, $\langle \mu(j), \gamma \rangle$ is always even, so that in (ii) we can replace $n+1$ by the largest even integer less than $n+1$. In particular, for $n = 2$ we get that $\langle \mu(j), \gamma \rangle = 2$. In section 5 we generalize this result to the case where L has a metric with nonpositive sectional curvature[1].

[1] In March 89, the author received from Leonid V. Polterovitch a manuscript giving a proof of theorem A for $n = 2$, based on a holomorphic curve approach

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Note: The map $\text{LagImm}(L) \to \text{Imm}(L)$ which associates to a Lagrange regular homotopy class of Lagrange immersion in $\mathbb{R}^{2n}$ its regular homotopy class as an immersion, has been thoroughly studied by M. Audin ([Au 2]). In particular, if $n \geqq 2$, any even cohomology class μ in $H^1(T^n) = \mathbb{Z}^n$, is the Maslov class of a Lagrange immersion of the torus regularly homotopic to an embedding. A natural question is: can this regular homotopy be made Lagrangian. Because the Maslov class is invariant through Lagrange regular homotopy, the following corollary of theorem A provides a negative answer.

Corollary 1. *If $\mu \in H^1(T^n) = \mathbb{Z}^n$ is contained in $d \cdot \mathbb{Z}^n$, where $d > n + 1$, there is no Lagrangian embedding of the torus with Maslov class μ.*

Let us also remark that for $n = 2$, regular homotopy classes of Lagrangian immersions are classified by the Maslov class. This is because Lagrange immersions are classified by $[T^2, U(2)]$, the set of homotopy classes of maps from T^2 to $U(2)$. But $U(2) \equiv SU(2) \times S^1 \equiv S^3 \times S^1$, thus $[T^2, U(2)] = [T^2, S^3] \times [T^2, S^1] \equiv [T^2, S^1]$. It is easy to check that $[T^2, S^1]$ is given by half the Maslov class. Thus in this case our theorem tells exactly which Lagrange immersions are regularly homotopic (through Lagrange immersions) to an embedding:

Corollary 2. *A Lagrange immersion of T^2 is Lagrange regularly homotopic to an embedding if and only if the Maslov class is twice a generator of $H^1(T^2)$.*

Another application of our main result is:

Corollary 3. *Let $j: T^n \to T^*T^n$ be a Lagrange embedding such that the degree of $\pi \circ j$ is nonzero (π is the projection of the natural projection $T^*T^n \to T^n$). Then j has vanishing Maslov class.*

Proof. (partially due to M. Herman and L. Polterovitch) We first remark that any compact subset of T^*S^1 can be symplectically embedded into $\mathbb{R}^2 - \{0\}$. Thus any compact subset of $T^*T^n \simeq (T^*S^1)^n$ can be symplectically embedded in $\mathbb{R}^{2n}$. The embedding of course depends on how we write T^n as a product of circles. Now, if i is such an embedding, $i \circ j$ will be a Lagrange embedding of T^n in $(\mathbb{R}^{2n}, \sigma)$. The Maslov class of $i \circ j$ is given by the formula

$$\mu(i \circ j) = j^*(\mu(\tilde{\imath})) + \mu(j)$$

where $\tilde{\imath}$ is the restriction of i to the zero section. Now by our assumption, j^* is an isomorphism, and it is easy to show that $\mu(\tilde{\imath})$ is equal to $2 \sum_{i=1}^n e_i^*$ where e_i^* is the image of $H^1(S^1) \to H^1(T^n)$ induced by the projection $T^n \to S^1$. By composition with a map in $SL(n, \mathbb{Z})$, we can arrange $(e_1, \ldots, e_n)$ to be any basis of $H^1(T^n) \simeq \mathbb{Z}^n$, and thus $\mu(\tilde{\imath})$ to be any class equal to twice a generator. Thus we can choose i so that if we write $\mu(j) = 2k \cdot e$ with e a generator, we have $j^* \mu(\tilde{\imath}) = 2 \cdot e$ hence $\mu(i \circ j) = (2k + 2) \cdot e$. According to theorem A, this implies $2 \leqq 2k + 2 \leqq n + 1$, so k must be bounded. Let us show that in fact k must be zero. Assume not, then by taking a suitable p-fold covering of T^n, inducing a covering of T^*T^n, we find a new embedding j': $T^n \to T^*T^n$ with $\mu(j') = 2kp \cdot e$. For p large enough, this contradicts the above inequality[2].

Our approach to this problem is through the use of periodic solutions of the Hamiltonian system described in section 1. By combining ideas from [V2] with a careful comparison of the Conley-Zehnder index of a characteristic curve of the unit sphere bundle over T^n and the Morse index of the corresponding geodesic, we prove

Proposition B. *If j: $T^n \hookrightarrow \mathbb{R}^{2n}$ is a Lagrangian embedding, there is a loop γ such that:*

(i) $\langle [j^*\lambda], \gamma \rangle \geqq 0$
(ii) $\langle \mu(j), \gamma \rangle \in [2, n+1]$.

Note that the proposition only differs from the theorem by changing in (i) the conclusion $\langle [j^*\lambda], \gamma \rangle$ positive into $\langle [j^*\lambda], \gamma \rangle$ nonnegative. Without (ii), the proposition would be trivial, while the theorem is already a deep fact. Because of condition (ii), we are able to show that the proposition and the theorem are equivalent. We now assume the proposition, and prove the theorem by contradiction. We suppose that any loop for which $\langle \mu(j), \gamma \rangle$ is in $[2, n+1]$ has $\langle j^*\lambda, \gamma \rangle \leqq 0$. We are now going to describe a Lagrangian isotopy j_t such that $j = j_0$, and for some small ε, j_ε satisfies:

(*) for any loop γ on T^n, $\langle \mu(j_\varepsilon), \gamma \rangle \in [2, n+1]$ implies $\langle j_\varepsilon^*\lambda, \gamma \rangle < 0$

which would contradict the proposition. Note that $\langle \mu(j_\varepsilon), \gamma \rangle = \langle \mu(j), \gamma \rangle$ since the Maslov class is invariant by Lagrangian regular homotopy.

We construct j_t as follows. According to Weinstein's theorem, we can symplectically identify a neighborhood U of $j(T^n)$ to a neighborhood of the zero section of T^*T^n. Remember that Lagrangian tori close to $j(T^n)$ can be considered as graphs of closed one forms on T^n. Moreover if j_α is the graph of α, then $\langle j_\alpha^*\lambda, \gamma \rangle = \langle \alpha, \gamma \rangle + \langle j_0^*\lambda, \gamma \rangle$.

It is now clear that if we choose α to be a closed one form in the cohomology class of $-\mu(j_0)$, then setting $j_s = j_{s\alpha}$, we get that for s small enough $j_s(T^n)$ is well defined, and

$$\langle j_s^*\lambda, \gamma \rangle = \langle j_0^*\lambda, \gamma \rangle - s\langle \mu(j_0), \gamma \rangle .$$

Let now γ be such that $\langle \mu(j_0), \gamma \rangle$ is in $[2, n+1]$, then $\langle j_\varepsilon^*\lambda, \gamma \rangle < \langle j_0^*\lambda, \gamma \rangle$ by the above inequality, but by assumption $\langle j_0^*\lambda, \gamma \rangle \leqq 0$, so $\langle j_\varepsilon^*\lambda, \gamma \rangle < 0$, hence j_ε satisfies (*). This concludes our proof.

Remark. Let us point out that the most useful applications of Theorem A follow from the property of the Maslov class being in $[2, n+1]$, rather than from the positivity of the action. In fact one of the most celebrated applications of Gromov's result is the existence of exotic symplectic structures on $\mathbb{R}^{2n}$, for $n \geqq 2$. This can also be proved using the boundedness of the Maslov class: let i_0: $T^n \to (\mathbb{R}^{2n}, \sigma)$ be a Lagrange immersion with zero Maslov class. Consider now T^n as a submanifold of

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[2] We refer to a forthcoming paper of Lalonde and Sikorav for a different proof, and among other results, additional interesting consequences of theorem A

$\mathbb{R}^{2n}$, and denote by j the canonical embedding. Using the Hirsch-Smale theory of immersions we see that for $n \geqq 2$, i_0 extends to an immersion i of $\mathbb{R}^{2n}$ in $\mathbb{R}^{2n}$. Then $\omega = i^*\sigma$ is a symplectic form on $\mathbb{R}^{2n}$, and j is a Lagrange embedding of T^n in $(\mathbb{R}^{2n}, \omega)$ with zero Maslov class, hence ω is an exotic symplectic structure (i.e. there is no symplectic embedding $(\mathbb{R}^{2n}, \omega) \to (\mathbb{R}^{2n}, \sigma)$).

Now the reader only interested in the statement (ii) can avail himself of a simpler proof, along the following lines. Use as before Weinstein's theorem to show that we can embed $\Sigma_\varepsilon = \{(q, p) \in T^* T^n \,||p| = \varepsilon\}$. Now Σ_ε is a contact type hypersurface of $\mathbb{R}^{2n}$ to which we can apply the results of [V2]: Σ_ε has a closed characteristic of Conley-Zehnder index in $[2, n+1]$ (this is not exactly what is proved in [V2], however it follows from Prop. 4.1, Cor. 4.2 and the relationship between the Conley-Zehnder index and the Morse index of the dual action functional). We know that the closed characteristic of Σ_ε are in one to one correspondence with the closed geodesics of the torus. Moreover according to Theorem 3.1, the Conley-Zehnder index of a closed characteristic is related to the Morse index of the corresponding geodesic through the formula

$$i_{cz}(x) = i_M(q) + \langle \mu(j), q \rangle$$

Since on the torus (with the flat metric), $i_M(q)$ is always zero, we get immediately $\langle \mu(j), q \rangle \in [2, n+1]$ as announced.

The paper is organized as follows:

1. Hamiltonian systems localized near a Lagrange submanifold
2. Finite dimensional reduction and applications
3. Comparing the indices
4. Existence of periodic orbits with prescribed index and proof of the proposition
5. A generalization of our theorem

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1. Hamiltonian systems localized near a Lagrangian submanifold

In this section we fix j, a Lagrangian embedding of the n-torus. To simplify our notations, we forget about j, and consider T^n as a submanifold of $\mathbb{R}^{2n}$. Remember that according to Weinstein's theorem, we can actually assume that a neighborhood U of T^n is the symplectic image of a neighborhood of the zero section in $T^* T^n$.

For a point x in U, we shall write $x = (q, p)$ with $p \in T_q^* T^n$.

We now define a family H_ε of Hamiltonians on $(\mathbb{R}^{2n}, \omega_0)$ such that the corresponding system

$$(\mathscr{H}_\varepsilon)\begin{cases}\dot{x} = X_{H_\varepsilon}(x)\\ x(0) = x(2\pi)\end{cases}$$

will have a solution with positive action. We shall show that this solution is actually in U and if we write $x = (q, p)$, q is a closed geodesic of T^n (with the canonical flat metric), and the Conley-Zehnder index of x is in $[2, n + 1]$ (prop. 4.4). We shall see in section 3, that this implies $\langle \mu(j), q\rangle \in [2, n + 1]$.

Let us define H_ε: Choose ρ small enough, so that we can assume $\{(q, p) \in T^*T^n \| p| \leqslant \rho\} = U$, and R large enough so that $U \subset B(0, R)$. Then set $H_\varepsilon(q, p) = h_\varepsilon(|p|)$ where h_ε is a C^∞ function such that

$$\begin{aligned}&h_\varepsilon \text{ is nondecreasing, strictly convex on } [0, \varepsilon], \text{ with } h''(0) = 0\\ &\text{concave on } [\rho/2 - \varepsilon, \rho/2], \text{ linear on } [\varepsilon, \rho/2 - \varepsilon]\,.\end{aligned} \tag{1.1}$$

$$h_\varepsilon(s) = -\varepsilon + cs \text{ for } \varepsilon \leqq s \leqq \rho/2 - \varepsilon\,. \tag{1.2}$$

$$h_\varepsilon(0) = h_\varepsilon'(0) = 0 \quad h_\varepsilon(s) = a \text{ for } s \geqq \rho/2 \text{ where } a \geqq \tfrac{3}{4}R^2 \text{ (and } c \simeq a/\rho)\,. \tag{1.3}$$

As a result $h_\varepsilon'(s)s - h_\varepsilon(s) \leqq \varepsilon$ for all s since this quantity is increasing on $[0, \varepsilon]$ (its derivative is $h_\varepsilon''(s)s$), decreasing on $[\rho/2 - \varepsilon, \rho/2]$ and constantly equal to ε on $[\varepsilon, \rho/2 - \varepsilon]$, the above inequality is clear.

We now extend H_ε to $\mathbb{R}^{2n}$ as follows

$$\text{in } B(0, R) - U,\ H_\varepsilon \equiv a \tag{1.4}$$

$$\text{in } \mathbb{R}^{2n} - B(0, R),\ H_\varepsilon(x) = g(|x|^2) \text{ where } g(r) = \begin{cases} a & \text{for } \leqq R^2\\ \tfrac{3}{4}r & \text{for } \geqq 2R^2\end{cases} \tag{1.5}$$

$$\text{and } g'(r) \leqq \tfrac{3}{4},\ g(r) \geqq \tfrac{3}{4}r \text{ (hence } g'(s)s - g(s) \leqq 0)\,.$$

Note that ρ and R are geometric constants, and thus we can choose a and $c(\sim a/\rho)$ once and for all, and only ε will be allowed to vary. We now define the action functional

$$A_H(x) = \int_0^{2\pi}\left[\frac{1}{2}(Jx, \dot{x}) - H(x)\right]dt$$

for $x \in H^1(S^1, \mathbb{R}^{2n})$. A_H is S^1-invariant for the S^1 action defined by $\theta \cdot x(t) = x(t + \theta)$. The critical points of A_H are the solutions of $(\mathscr{H})$.

As we will prove in section 4, A_H has a critical orbit of strictly positive critical value and of Conley-Zehnder index in $[2, n + 1]$.

It is now an easy fact that, due to the choice of H_ε, a critical orbit of A_{H_ε} with positive critical value must have its trajectory contained in U (see [V2] and [H-Z]) and thus can be written as $x = (q, p)$ with

$$\begin{cases}\dot{q} = h_\varepsilon'(|p|)p/|p|\\ \dot{p} = 0\\ (q(0), p(0)) = (q(2\pi), p(2\pi))\end{cases}$$

or else

$$p_\varepsilon(t) \equiv p_\varepsilon, \qquad \dot{q}_\varepsilon(t) = h'_\varepsilon(|p_\varepsilon|)p_\varepsilon/|p_\varepsilon|$$

and q_ε is a geodesic of T^n with length

$$\int_0^{2\pi} |\dot{q}_\varepsilon(t)|dt = \int_0^{2\pi} h'_\varepsilon(|p_\varepsilon|) = 2\pi h'_\varepsilon(|p_\varepsilon|) \leqq 2\pi c$$

where c is as in (1.2).

Note that since length $(q_\varepsilon) \leqq c$, c is independent of ε, and the set of geodesics of length less than c is compact, we can find a subsequence $\varepsilon_n \to 0$ such that q_{ε_n} converges to some geodesic q_0.

Note also that

$$A_{H_\varepsilon}(x_\varepsilon) = \int_0^{2\pi} p_\varepsilon \dot{q}_\varepsilon + \int_0^{2\pi} q_\varepsilon^* \lambda - \int_0^{2\pi} h_\varepsilon(|p_\varepsilon|)$$

$$= 2\pi(h'_\varepsilon(|p_\varepsilon|)|p_\varepsilon| - h_\varepsilon(|p_\varepsilon|)) + \int_0^{2\pi} q_\varepsilon^* \lambda$$

and using inequality (1.2), we get that

$$A_{H_\varepsilon}(x_\varepsilon) \leqq 2\pi\varepsilon + \int_0^{2\pi} q_\varepsilon^* \lambda \, .$$

This yields for $\varepsilon = \varepsilon_n$

$$A_{H_{\varepsilon_n}}(x_{\varepsilon_n}) \leqq 2\pi\varepsilon_n + \int_0^{2\pi} q_{\varepsilon_n}^* \lambda$$

so that if we show that $A_{H_{\varepsilon_n}}(x_{\varepsilon_n}) \geqq 0$, we get

$$- 2\pi\varepsilon_n \leqq \int_0^{2\pi} q_{\varepsilon_n}^* \lambda \, .$$

and letting n go to infinity, implies (i) of the proposition. In the section 4, we shall prove that $\langle \mu(j), q_0 \rangle \in [2, n + 1]$ hence completing the proof of the proposition.

2. Finite dimensional reduction and applications

Let A_H be the action functional on $H^1(S^1, \mathbb{R}^{2n})$. We want to reduce the equation $dA_H(x) = 0$ to a finite dimensional problem.

For this we recall the approach of Amann, Conley and Zehnder (cf. [A-Z 1], [A-Z 2], [Co-Z]) which consists of a Lyapunov-Schmidt reduction.

For $x \in H^1(S^1, \mathbb{R}^{2n})$, we can write its Fourier decomposition $x(t) = \sum_{k \in \mathbb{Z}} \exp(kJt)x_k$, $x_k \in \mathbb{R}^{2n}$, and then $x = u + v$ where:

$$u(t) = \sum_{-N}^{N} \exp(kJt)x_k$$

$$v(t) = x(t) - u(t) \, .$$

Let now P_N be the projection operator defined by $P_N x = u$ and set $Q_N = 1 - P_N$, $E_N = \operatorname{Im} P_N$.

Assuming that $|H''(x)| \leqq C < \frac{N}{2}$, consider for $u \in E_N$ the equation in v:

$$Q_N DA_H(u + v) = 0 \quad \text{or else} \quad J\dot{v} + Q_N(\nabla H(u + v)) = 0\,.$$

We claim this equation has for every u a unique solution $v(u)$, C^1 dependent of u (we refer to [Co-Z], p. 225 for this result). Moreover $a_N(u) = A_H(u + v(u))$ is a C^2 function on E_N such that u is a critical point of a if and only if $u + v(u)$ is a critical point of A_H, and the corresponding critical levels coïncide.

Note also that the S^1 action on E restricts to an action on E_N for which $v \to v(u)$ is equivariant. Hence a_N is also equivariant. We now wish to understand how the topology of the level sets of a_N changes when N increases. Let us first introduce the following notation. If (X, A) and (Y, B) are pairs of spaces, we define X/A to be the usual quotient space, and $X/A \wedge Y/B$ to be $X \times Y/A \times Y \cup X \times B$. We can now state

Proposition 2.1. *Let $E_N^c = \{u \in E_N | a_N(u) \leqq c\}$, then if $c < d$ and neither is a critical value of a_N, we have that $(E_{N+1}^d/E_{N+1}^c) \simeq (D^{2n}/S^{2n-1}) \wedge (E_N^d/E_N^c)$ where the S^1 action on the product is inherited from the canonical one on E, and the $N + 1$-fold Hopf action on (D^{2n}, S^{2n-1}) (i.e. $\theta \cdot (z_1, \ldots, z_n) = (e^{i(N+1)\theta} z_1, \ldots, e^{i(N+1)\theta} z_n)$.*

The main consequence is obtained by applying Thom's isomorphism,

Corollary 2.2. *If $H_{S^1}^*$ denotes Borel's equivariant cohomology functor, $H_{S^1}^{*-2nN}(E_N^d, E_N^c)$ is canonically isomorphic to $H_{S^1}^{*-2nN'}(E_{N'}^d, E_{N'}^c)$ for N, N' large enough.*

The proof of proposition 2.1 will take up the end of this section. It is mainly based on Conley's theory of isolated invariant sets and Morse indices (cf. [Co]) that we will first try to summarize.

Let ξ be a vector field on a manifold M, φ_t its flow, S an invariant set of the flow. In our applications, ξ will be a pseudogradient vector field of some function, and S some union of critical points and connections between them (heteroclinic orbits).

Let us define an index pair for S to be a pair (N_1, N_2) of subsets of M such that

(i) $S \subset \operatorname{interior}(N_1 \backslash N_2)$
(ii) S is the largest invariant set in $N_1 \backslash N_2$
(iii) $\varphi_t(N_2) \cap N_1 \subset N_2$ for $t \geqq 0$ that is N_2 is an exit set
(iv) if for x in N_1 and positive t we have $\varphi_t(x) \notin N_1$, then for some positive t', $\varphi_{[0,t']}(x) \subset N_1$, and $\varphi_{t'}(x) \in N_2$ (that is, if x eventually exits N_1, it does so through N_2, thus N_2 is the exit set of N_1).

We then set $h(S)$ to be the homotopy type of the quotient space N_1/N_2. Provided the flow of ξ satisfies the Palais-Smale condition, $h(S)$ is indeed independent of the choice of the pair (N_1, N_2) (cf. [Co], p. 50 for the case N_1 compact).

Let us remark that if (N_1, N_2) satisfies (iii) and (iv) with respect to two vector fields ξ and η, then, if S and T are the maximal invariant sets in $N_1 \backslash N_2$, and if they satisfy (i) and (ii), then it is clear that $h(S) = h(T)$.

This last remark can be useful in computing the homotopy index, and shall be one of the ingredients of our proof.

Let us now consider ξ, a pseudogradient vector field for a function f on M, that is

$$df(x)\xi(x) \leqq -\|df(x)\|^2$$

(we should say a minus pseudogradient!).

Then if $a < b$ are two regular values of f, and f satisfies the P.S. condition in $X^b \backslash X^a$, then (X^b, X^a) is an index pair for the isolated invariant set S of critical points and connecting orbits between these critical points contained in $X^b \backslash X^a$.

We shall apply this to a suitably chosen pseudogradient vector field ξ for a_{N+1}, so that $h(S) = [E^d_{N+1}/E^c_{N+1}]$. On the other hand, we prove that for R large enough the pair $(E^d_N, E^c_N) \times (D(R, V^-_{N+1}), S(R, V^-_{N+1}))$ is also an index pair for the set of bounded trajectories of ξ in $E^d_{N+1} - E^c_{N+1}$. Proposition 2.1 follows immediately. Here $D(R, V^-_{N+1})$ and $S(R, V^-_{N+1})$ are the disk and sphere of radius R in

$$V^-_{N+1} = \{v \in E_{N+1} | v(t) = \exp((N+1)Jt)v_{N+1}, v_{N+1} \in \mathbb{R}^{2n}\}$$

and

$$V^+_{N+1} = \{v \in E_{N+1} | v(t) = \exp(-(N+1)Jt)v_{N+1}, v_{N+1} \in \mathbb{R}^{2n}\}$$

(hence $E_{N+1} = E_N \oplus V^+_{N+1} \oplus V^-_{N+1}$).

We should also explain how E_N sits into E_{N+1}: we have the embedding

$$E_N \to E_{N+1}$$

$$u \to u + P_{N+1}(v(u)) \stackrel{\text{def}}{=} u + v_{N+1}(u) .$$

Note that $v_{N+1}(u)$ is the unique solution of $\frac{\partial}{\partial v_{N+1}} a_{N+1}(u + v_{N+1}) = 0$, and that $a_N(u) = a_{N+1}(u + v_{N+1}(u))$, so that a_N is the "reduction" of a_{N+1}.

It will be convenient in the sequel to replace a_{N+1} by $\tilde{a}_{N+1}$ such that

$$\tilde{a}_{N+1}(u + v_{N+1}) = a_{N+1}(u + v_{N+1}(u) + v_{N+1})$$

for $v_{N+1} \in V^+_{N+1} \oplus V^-_{N+1}$.

Clearly, if $\tilde{E}^c_{N+1}$ is associated to $\tilde{a}_{N+1}$, $\tilde{E}^c_{N+1}$ is diffeomorphic to E^c_{N+1}, the diffeomorphism being given by $u + v_{N+1} \to u + v_{N+1}(u) + v_{N+1}$. Also, a_N is a reduction of $\tilde{a}_{N+1}$:

$$\frac{\partial}{\partial v_{N+1}} \tilde{a}_{N+1}(u + v_{N+1}) = 0 \Leftrightarrow v_{N+1} = 0$$

$$a_N(u) = \tilde{a}_{N+1}(u + 0)$$

and the embedding $E_N \to \tilde{E}_{N+1}$ is given by $u \to u + 0$.

We are now going to construct a vector field ξ on $\tilde{E}_{N+1}$ such that

$$(\tilde{E}^d_{N+1}, \tilde{E}^c_{N+1}) \text{ is an index pair for } \xi \qquad (2.3)$$

$(E_N^d, E_N^c) \times (D(R, V_{N+1}^-), S(R, V_{N+1}^-)) \times V_{N+1}^+$ is another index pair for ξ. In fact ξ will be a pseudogradient vector field of $\tilde{a}_{N+1}$ near $\partial \tilde{E}_{N+1}^d \overset{\text{def}}{=} \{x \in \tilde{E}_{N+1} | a_{N+1}(x) = d\}$ and ∂E_{N+1}^c, which will ensure (2.3). (2.4)

We now define ξ

Definition 2.5. Set $\xi_1(u + v_{N+1}) = Dv_{N+1} \overset{\text{def}}{=} J\dot{v}_{N+1}$, and let φ be a cut off function near 0. Then we define

$$\xi(u + v) = (1 - \varphi(|v|^2))\xi_1(u + v) - \varphi(|v|^2)\nabla a_N(u) .$$

Lemma 2.6. (i) *If H is equal to $\frac{k}{2}|x|^2$ outside a compact set with $k \notin \mathbb{Z}$, then ξ satisfies the Palais-Smale condition.*

(ii) *If $c < d$ are regular values of $\tilde{a}_{N+1}$, then $(\tilde{E}_{N+1}^d, \tilde{E}_{N+1}^c)$ is an index pair for ξ.*

Proof. We leave (i) to the reader. For (ii) it will be enough to show that $\nabla a_{N+1}(x)\xi(x) \leqq -|\xi(x)|^2$ and that $\xi(x)$ does not vanish on $\partial \tilde{E}_{N+1}^d$ or $\partial \tilde{E}_{N+1}^c$. (Note that this almost says that ξ is a pseudogradient, except that we replaced on the r.h.s. $\|df(x)\|^2$ by $|\xi(x)|^2$, so that ξ is only a pseudogradient away from its zero set.)

If the above inequality is satisfied, $\partial \tilde{E}_{N+1}^d$ is the entrance set of ξ and $\partial \tilde{E}_{N+1}^c$ its exit set, and one easily checks that conditions (iii) and (iv) are fulfilled.

We now show that $\nabla a_{N+1}(x)\xi(x) \leqq -|\xi(x)|^2$. We first compute

$$\nabla \tilde{a}_{N+1}(u + v_{N+1})\cdot \xi_1(u + v_{N+1}) = \langle -J\dot{v}_{N+1} - J\dot{v}_{N+1}(u) - \nabla H(u + v_{N+1}(u) + v_{N+1} + w(u + v_{N+1}(u) + v_{N+1})), -J\dot{v}_{N+1} \rangle$$

where w: $E_{N+1} \to L^2$ is the map given by the Lyapunov-Schmidt reduction to E_{N+1}. Since $\xi_1(u + v_{N+1}) \in V_{N+1}$, and $\nabla \tilde{a}_{N+1}(u + 0)$ has a vanishing V_{N+1} component, the above quantity is equal to

$$\begin{aligned} &\langle \nabla \tilde{a}_{N+1}(u + v_{N+1}) - \nabla \tilde{a}_{N+1}(u + 0), \xi_1(u + v_{N+1}) \rangle = \\ &\langle -J\dot{v}_{N+1} - \nabla H(u + v_{N+1}(u) + v_{N+1} + w(u + v_{N+1}(u) + v_{N+1})) \\ &+ \nabla H(u + v_{N+1}(u) + w(u + v_{N+1}(u)), J\dot{v}_{N+1} \rangle . \end{aligned} \tag{2.7}$$

In order to estimate this term, it will be enough to estimate the part involving ∇H. We shall need

Lemma 2.8. $\|dw\| \leqq \dfrac{C}{N - C}$.

We first conclude our proof, assuming the lemma. We consider the quantity

$$\begin{aligned} &|\nabla H(u + v_{N+1}(u) + v_{N+1} + w(u + v_{N+1}(u) + v_{N+1})) - \nabla H(u + v_{N+1}(u) \\ &+ w(u + v_{N+1}(u))| \leqq \sup|H''(x)\|v_{N+1} + w(u + v_{N+1}(u) + v_{N+1}) \\ &- w(u + v_{N+1}(u))| \leqq C(1 + \|dw\|)|v_{N+1}| \leqq \frac{CN}{N - C}|v_{N+1}| \end{aligned}$$

so that (2.7) can be bounded by

$$-(N+1)^2|v_{N+1}|^2 + \frac{CN}{N-C}(N+1)|v_{N+1}|^2$$

$$= -(N+1)^2\left[1 - \frac{CN}{(N-C)(N+1)}\right]|v_{N+1}|^2$$

$$\leqq -\frac{1}{2}|Dv_{N+1}|^2 \text{ for } N \text{ large enough} .$$

Thus, we just proved that

$$\nabla\tilde{a}_{N+1}(u+v_{N+1})\cdot\xi_1(u+v_{N+1}) \leqq -\tfrac{1}{2}|\xi_1(u+v_{N+1})|^2 .$$

This is very satisfactory, as long as we are away from E_N, the zero set of ξ_1, and is the reason for adding a second term to get ξ.

Now

$$\langle \nabla\tilde{a}_{N+1}(u+v_{N+1}), \xi(u+v_{N+1})\rangle$$
$$= \varphi(|v_{N+1}|^2)\langle \nabla\tilde{a}_{N+1}(u+v_{N+1}), \xi_1(u+v_{N+1})\rangle$$
$$-(1-\varphi(|v_{N+1}|^2)\langle \nabla\tilde{a}_{N+1}(u+v_{N+1}), \nabla\tilde{a}_{N+1}(u+0)\rangle .$$

The first term, we just proved to be bounded by $-\varphi(|v_{N+1}|^2)\frac{1}{2}|\xi_1(u+v_{N+1})|^2$, we now consider the second term.

Using lemma 2.8, we just proved that

$$|\nabla\tilde{a}_{N+1}(u+v_{N+1}) - \nabla\tilde{a}_{N+1}(u+0)| \leqq C'|v_{N+1}| ,$$

thus our second term will be bounded by

$$-(1-\varphi(|v_{N+1}|^2)\{|\nabla\tilde{a}_{N+1}(u+0)|^2 - C'|v_{N+1}|\}|\nabla\tilde{a}_{N+1}(u+0)|$$
$$\leqq -(1-\varphi(|v_{N+1}|^2)|\nabla\tilde{a}_{N+1}(u+0)|\{|\nabla\tilde{a}_{N+1}(u+0)| - C'|v_{N+1}|\} .$$

Now, because $\tilde{a}_{N+1}$ satisfies the P.S. condition, and ∂E_N^d, ∂E_N^c are regular values of $\tilde{a}_{N+1}$, $|\nabla\tilde{a}_{N+1}(u+0)| = |\nabla a_N(u)|$ is bounded from below on these sets. We can then choose φ so that our second term vanishes if $\inf|\nabla\tilde{a}_{N+1}(u+0)| \leqq 2C'|v_{N+1}|$ (the inf is taken on $\partial E_N^c \cup \partial E_N^d$) so our second term is less than

$$-(1-\varphi(|v_{N+1}|^2)\tfrac{1}{2}|\nabla\tilde{a}_{N+1}(u+0)|^2 .$$

It is now clear that

$$\nabla\tilde{a}_{N+1}(u+v_{N+1})\xi(u+v_{N+1}) \leqq -\tfrac{1}{2}|\xi(u+v_{N+1})|^2$$

and that ξ does not vanish on $\partial E_{N+1}^d \cup \partial E_{N+1}^c$, provided we prove lemma 2.8.

Let $w(x)$ be defined by

$$J\dot{w} + Q_N\nabla H(x+w(x)) = 0 ,$$

by differentiating, we get

$$Jd\dot{w}(x) + Q_N H''(x+w)(dx+dw) = 0$$

that is $[D + Q_N H''(x + w)]dw = -Q_N H''(x + w)dx$. Since $\|D + Q_N H''(x + w)\| \geqq (N - C)$, we get $|dw| \leqq \frac{C}{N - C}$ which concludes our proof of lemma 2.8 and thus of lemma 2.6.

In order to conclude our proof of 2.1, we just need to show that the maximal invariant set of ξ in $\tilde{E}^d_{N+1} \backslash \tilde{E}^c_{N+1}$ is contained in

$$E^d_N \times D(R, V^-_{N+1}) \times V^+_{N+1} \backslash E^c_N \times D(R, V^-_{N+1}) \times V^+_{N+1} \cup E^d_N \times S(R, V^-_{N+1}) \times V^+_{N+1}$$

and that

$$(E^d_N, E^c_N) \times (D(R, V^-_{N+1}), S(R, V^-_{N+1})) \times V^+_{N+1}$$

is an index pair for ξ.

The latter is obvious from the definition of ξ, and we now prove the former.

n We shall actually prove that if φ_t is the flow of ξ, and for some x, $\varphi_t(x)$ stays in $\tilde{E}^d_{N+1} \backslash \tilde{E}^c_{N+1}$ for all t, then $x \in E_N$. This is easily seen by looking at the v_{N+1} coordinate of x, it satisfies

$$\begin{cases} \dfrac{d}{dt} v^+_{N+1} = (N + 1)v^+_{N+1} \\ \dfrac{d}{dt} v^-_{N+1} = -(N + 1)v^-_{N+1} \end{cases}$$

so that if for instance v^+_{N+1} is non-zero, then the v^+_{N+1} coordinate of $\varphi_t(x)$ becomes infinite with t, so we can replace x by a point in $\tilde{E}_{N+1}$ with large v^+_{N+1} coordinate. Then, for such an x, $\varphi_t(x)$ coïncides with the flow of ξ_1, hence $\tilde{a}_{N+1}(\varphi_t(x))$ goes to $-\infty$ as t goes to $+\infty$, thus $\varphi_t(x)$ exits $\tilde{E}^d_{N+1} \backslash \tilde{E}^c_{N+1}$.

As a result, the maximal invariant set of ξ in $\tilde{E}^d_{N+1} \backslash \tilde{E}^c_{N+1}$ is actually contained in $E^d_N \backslash E^c_N$, which proves our claims and concludes the proof of 2.1. □

From the proof corollary 2.2 we easily get

Corollary 2.9. *Let $i_N(x)$ and $\nu_N(x)$ be the index and nullity of $P_N(x)$ as a critical point of a_N. Then one has that $i(x) \overset{\text{def}}{=} i_N(x) - 2n(N + \frac{1}{2})$, $\nu(x) \overset{\text{def}}{=} \nu_N(x)$ are indeed independent of N.*

Proof. To fix future conventions, let us mention that the nullity shall designate the equivariant nullity, that is the maximal dimension of a subspace transverse to the orbit of x and contained in the kernel of $D^2 a(x)$: for a critical orbit which is not a fixed point of the action, this is one less than the usual nullity. As for the proof, the reader is invited to supply his own, using the proof of proposition 2.1, or to look it up in [A-Z 1] (prop. 2.1, prop. 4.5, lemma 7.2). □

Note also that $\nu(x)$ coïncides with the nullity of $D^2 A_H(x)$, that is the dimension of the vector space of solutions of

$$\dot{y} = JH''(x)y, \quad y(0) = y(2\pi)$$

We shall call $i(x)$ the Conley-Zehnder index of x and in the next section shall denote it by $i_{cz}(x)$ to avoid confusion with other indices that shall appear there. Let us mention that $i_{cz}(x)$ is denoted by $j(x)$ in [Co-Z]. We shall give in the next section another definition of the index that can be found in [Co-Z].

It will be important for us to know how a_N depends on H. We now prove:

Proposition 2.10. *If we endow the set*

$$\{H \in C^2(\mathbb{R}^{2n}, \mathbb{R}) \mid H(x) = \tfrac{3}{4}|x|^2 \text{ outside a given compact set, and } |H''(x)| \leqq C\}$$

with the C^0 topology and the set $C^0(E_N, \mathbb{R})$ with the topology of uniform C^0 convergence, then the map $H \to a_N$, which is well defined for $C < N/2$, is continuous.

This is clear once we realize that if we write $E = E_N \oplus F_N^+ \oplus F_N^-$ where F_N^+ (resp. F_N^-) is the direct sum of the V_k^+ (resp. V_k^-) for $k > N$, then A_H is convex in the direction of F_N^+ and concave in the direction of F_N^-. As a result we get an alternate definition of a_N as

$$a_N(u) = \sup_{v^- \in F_N^-} \inf_{v^+ \in F_N^+} A_H(u + v^+ + v^-)$$

Now if $\| H - K \|_{C^0} < \varepsilon$, then $\| A_H - A_K \|_{C^0} < 2\pi\varepsilon$, thus making our statement obvious. □

3. Comparing the indices

Let $H(x) = h(|p|)$ as in the previous section, and let $x = (q, p)$ be a solution of $\mathscr{H}$, the trajectory of which is contained in the neighborhood of T^n, U. Then q is a geodesic of the torus endowed with the flat metric, and thus, as a critical point of the energy functional, $E(q) = \int |\dot{q}|^2 dt$ has a Morse index, $i_M(q)$. This is defined as the maximal dimension of a linear subspace of the set of closed loops in $W^{1,2}(S^1, \mathbb{R}^{2n})$ on which $D^2E(q)$ is negative definite. On the other hand if we view x as a critical point of A_H, we can consider its Conley-Zehnder index $i_{cz}(x)$. The purpose of this section is to compare $i_M(q)$ with $i_{cz}(x)$. The result can be stated as follows:

Theorem 3.1. *Assume x to be contained in the region $U_\varepsilon = \{x \in U \mid |p| < \varepsilon\}$ where h is increasing and strictly convex. Then we have that*

$$i_{cz}(x) = i_M(q) + \langle \mu(j), q \rangle$$

Remark. If we had assumed h to be increasing and concave instead of convex, we should have added 1 to the right hand side.

The idea of the proof, which will take up the rest of this section, is to identify both $i_{cz}(x)$ and $i_M(q)$ with the rotation numbers of a path of Lagrange spaces, as in [D] and [Co-Z]. The difference between the Conley-Zehnder and the Morse index comes from the fact that in the first case the rotation of the Lagrange space is measured against a fixed space in $\mathbb{R}^{2n}$, while in the second case it is measured against the vertical Lagrangian distribution in T^*T^n. It is natural to expect that the difference between these numbers will measure the rotation along q of the vertical distribution of T^*T^n with respect to a fixed Lagrange space in $\mathbb{R}^{2n}$. By definition this is the Maslov number of q.

To begin with, we recall some results of Duistermaat (cf. [D]), on the Morse index of Lagrangian functionals. Consider the functional $E(q) = \int_0^{2\pi} K(q, \dot{q})dt$ defined on the space of 2π periodic loops on a compact manifold L. The critical points of E will have finite Morse index if and only if $\partial^2 K/\partial v^2$ is positive definite. With this same hypothesis, we can define the Lagrange transform of K as $H(q, p) = \langle p, v \rangle - K(q, v)$ where v is an implicit function of p through the equation $p = \partial K/\partial v(q, v)$. Now a critical point of E will satisfy the Euler-Lagrange equation

$$\frac{d}{dt}\frac{\partial K}{\partial v}(q, \dot{q}) - \frac{\partial K}{\partial q}(q, \dot{q}) = 0$$

which is classically equivalent through the change of variable $p(t) = \partial K/\partial v(q, \dot{q})$, to Hamilton's equation in T^*L with periodic boundary condition:

$$\dot{q} = \frac{\partial H}{\partial p}$$

$$\dot{p} = -\frac{\partial H}{\partial q}$$

$$(q(0), p(0)) = (q(2\pi), p(2\pi))$$

Let now Ψ_t be the flow defined by the Hamiltonian vector field of H, and $x = (q, p)$ a fixed point of $\Psi = \Psi_{2\pi}$. We denote by $R(t)$ the linearized map of Ψ_t at $x(0)$ that is $R(t)y = d\Psi_t(x(0))y$. Clearly $R(t)$ is a linear symplectic map from $T_{x(0)}(T^*L)$ to $T_{x(t)}(T^*L)$. We shall now associate to any family $R(t)$ of linear symplectic maps from $T_{x(0)}(T^*L)$ to $T_{x(t)}(T^*L)$ an integer $\mathrm{ind}(R)$.

First of all consider the bundle of symplectic spaces over S^1 with fiber $E_\theta = T_{x(0)}(T^*L) \times T_{x(\theta)}(T^*L)$ with the symplectic form $\pi_1^*(dp \wedge dq) - \pi_2^*(dp \wedge dq)$[3]. Now the graph of $R(t)$ can be considered as an element in Λ_t the set of Lagrange subspaces of E_t. Note that R is not a section of Λ since usually $R(2\pi) \neq R(0)$. Now given a map ϕ: $[0, 2\pi] \to \Lambda$ such that $\phi(t) \in \Lambda_t$ and a section L of Λ, we define an integer $\mathrm{ind}_L(\phi)$ as follows. First assume that $L(0)$ is transverse to $\phi(0)$ and to $\phi(2\pi)$. We can then compute the algebraic intersection number, $[\phi: L]$, of the path ϕ with the hypersurface $\sum_L = \bigcup_{\theta \in S^1} \{\alpha \in \Lambda_\theta | \alpha \cap L(\theta) \neq \theta\}$. Before we define $\mathrm{ind}_L(\phi)$ we need one more definition.

Definition 3.2. ([D]) Let (α, β, γ) be a triple of Lagrange subspaces of a linear symplectic space, such that $\alpha \cap \beta = \gamma \cap \beta = \{0\}$. We can then write γ as the graph of linear map C: $\alpha \to \beta$. We now set $Q(\alpha, \beta; \gamma) = \sigma(Cu, u)$. $Q(\alpha, \beta; \gamma)$ is then a

[3] Of course since $\mathrm{Sp}(n)$ is connected all symplectic vector bundles over S^1 are trivial (this is the point of view of Duistermaat). However the trivialization is not unique since $\pi_1(\mathrm{Sp}(n)) \simeq \mathbb{Z}$, and this is crucial in our case

quadratic form on α. It is in fact the generating function of γ if we identify our symplectic vector space to $T^*\alpha = \alpha \oplus \alpha^*$ by identifying α^* to β.

We can now set:

Definition 3.3. If $L(0)$ is transverse to $\phi(0)$ and $\phi(2\pi)$, we set

$$\operatorname{ind}_L(\phi) = [\phi: L] + \text{index } Q(\phi(2\pi), L(0); \phi(o))$$

Lemma 3.4. $\operatorname{ind}_L(\phi)$ *only depends on the homotopy class of L so that we can extend its definition to any section L.*

Proof. We can always find a trivialization of Λ for which L is constant, i.e. if Λ is identified with $S^1 \times \Lambda(n)$ then $L(t)$ goes to (t, L_0). The proof then follows the argument preceding definition 2.3 of [D] (pages 183–184). Now given two sections L_1 and L_2 of Λ we define the Maslov class of the pair (L_1, L_2), denoted by $\mu(L_1, L_2)$, to be the difference of the Maslov classes of L_1 and L_2 read in the same trivialization of Λ (it does not depend on the choice of the latter). □

We now state

Lemma 3.5. *Let L_1 and L_2 be two sections of Λ. Then for any path ϕ as defined above, we have*

$$\operatorname{ind}_{L_2}\phi - \operatorname{ind}_{L_1}\phi = -\mu(L_2, L_1)$$

The proof is easy and left to the reader. Let us now go back to the bundle $E_\theta = T_{x(0)}(T^*L) \times T_{x(\theta)}(T^*L)$. It has a natural section that is $V(\theta) = \bar{V}(0) \times \bar{V}(\theta)$, where $\bar{V}(\theta)$ is the vertical Lagrange subspace of $T_{x(\theta)}(T^*L)$.

We now can rephrase Proposition 4.6 of [D] as:

Proposition 3.6. *Let ϕ be the graph of the linearized Hamiltonian flow associated to the Legendre dual of K. Then, the Morse index of q as a critical point of the functional $E(q) = \int_0^{2\pi} K(q, \dot{q})dt$ defined on the space of 2π periodic loops on L, is given by*

$$i_M(q) = \operatorname{ind}_V(\phi) - n$$

Let $\bar{C}(\theta)$ be the section of $T_{x(\theta)}(T^*L)$ induced by the constant distribution of $\mathbb{R}^{2n}$, $L_0 = \mathbb{R}^n \times \{0\}$ (i.e. $\bar{C}(\theta) = dj(x)^{-1}(L_0)$), and set $C(\theta) = \bar{C}(0) \times \bar{C}(\theta)$. Then, using lemma 3.5, and the fact that $\mu(V, C) = \langle \mu(j), q \rangle$, we can rewrite 3.6 as:

Proposition 3.7.

$$i_M(q) = \operatorname{ind}_C(\phi) - n + \langle \mu(j), q \rangle$$

Theorem 3.1 will follow from

Proposition 3.8. *Let ϕ be the graph of the linearized Hamiltonian flow along x, then the Conley-Zehnder index of x as a solution of $(\mathcal{H})$ is given by*

$$i_{cz}(x) = \operatorname{ind}_C(\phi) - n$$

Comments. If x is a nondegenerate solution of a time dependent Hamiltonian, then 3.8 is contained in [Co-Z]. However because of the S^1 symmetry, periodic orbits of a time independent Hamiltonian are always degenerate. Moreover in our case we are dealing with the problem of geodesics for a flat metric on the torus, in which case the solutions are also degenerate in the direction transverse to the S^1 action. Of course a perturbation could solve this difficulty but this would only further complicate our argument, while it seems to us that extending 3.8 to the degenerate case is an interesting result in its own right.

The proof of 3.8 will require several steps. First of all, $i_{cz}(x)$ only depends on the linearized flow along x, $R(t)$. This is because if H_0 and H_1 are two time dependent Hamiltonians which coïncide up to order two along a common periodic orbit x, then the Conley-Zehnder index of x as a solution of $\mathscr{H}_0$ is the same as its index as a solution of $\mathscr{H}_1$. This can be seen by considering the family $H_\lambda = (1-\lambda)H_0 + \lambda H_1$. Then H_λ also has x as a periodic orbit, and $D^2 H_\lambda(x(t)) = D^2 H_0(x(t))$ so that the linearized flows coïncide. As a result, the nullity of $D^2 A^N_{H_\lambda}(x)$ does not depend on λ, and so does the index of this quadratic form, thus proving our statement. This implies that $i_{cz}(x)$ is also the normalized index of the quadratic form Q^N which is the Lyapunov-Schmidt reduction of

$$Q(y, y) = \int_0^{2\pi} \left[\frac{1}{2}(Jy, \dot{y}) - (H''(x(t))y, y) \right] dt$$

Since $\dot{R} = JH''(x(t))R$ we also denote this number by $i_{cz}(R)$. In fact any C^1 path in $\mathrm{Sp}(n)$ can be written as the solution of $\dot{R} = JA(t)R$, for some path of symmetric matrices $A(t)$. Thus that $i_{cz}(R)$ can be defined for any path in $\mathrm{Sp}(n)$ with $R(0) = \mathrm{Id}$. Let us see more precisely how $i_{cz}(R)$ depends on the path $R\colon [0, 2\pi] \to \mathrm{Sp}(n)$. Clearly, if we deform R in the space of paths starting from the identity, so that $\dim \ker(R(2\pi) - \mathrm{Id})$ does not change, the dimension of the kernel of the quadratic form – the index of which defines $i_{cz}(R)$ – does not change either, hence $i_{cz}(R)$ remains constant. Note that the same is true for $\mathrm{ind}_C(\phi)$, for $\phi = \mathrm{graph}(R)$, because $\ker(R(2\pi) - \mathrm{Id}) = \{0\}$ is equivalent to $\phi(2\pi) \cap \phi(0) = \phi(2\pi) \cap \Delta$ (here Δ denotes the diagonal in $\mathbb{R}^{2n} \times \mathbb{R}^{2n}$).

Now consider a deformation R_s of paths in $\mathrm{Sp}(n)$, and a value s_0 of the parameter such that $\ker(R_s(2\pi) - \mathrm{Id})$ changes dimension as s goes through s_0. To simplify notations, we shall assume $s_0 = 0$. We now state:

Lemma 3.9. *Let R_s satisfy the equation $\dot{R}_s = JA_s(t)R_s$, and assume that the restriction of $\omega\left(\frac{d}{ds} R_s(2\pi)\Big|_{s=0} y, y\right)$ to $\ker(R_s(2\pi) - \mathrm{Id})$ is a non degenerate quadratic form with signature (l, m). We then have*

$$\begin{aligned} i_{cz}(R_+) - i_{cz}(R_0) &= m \\ i_{cz}(R_-) - i_{cz}(R_0) &= l \\ \mathrm{ind}_C(\phi_+) - \mathrm{ind}_C(\phi_0) &= m \\ \mathrm{ind}_C(\phi_+) - \mathrm{ind}_C(\phi_0) &= l \end{aligned}$$

Proof. Let $Q_s^N(y, y)$ be the Lyapunov-Schmidt reduction of $Q_s(y, y)$, and N_0 be the kernel of $Q_0^N(y, y)$. In order to prove the first part of 3.9, it is enough to show that the restriction of $\frac{d}{ds} Q_s^N(y, y)\,|_{s=0}$ to N_0 has signature (l, m). This is easy to check with Q_s instead of Q_s^N, since if $y = R_0(t)y_0$ is in the kernel of Q_0, we have

$$\frac{d}{ds}Q_s(y,y) = -\frac{d}{ds}\int_0^{2\pi}(A_s(t)R_0(t)y_0, R_0(t)y_0)dt$$

$$= \frac{d}{ds}\int_0^{2\pi}(J\dot{R}_sR_s^{-1}R_0y_0, R_0y_0)dt$$

$$= \int_0^{2\pi}\left(J\frac{\partial^2 R_s}{\partial s\partial t}R_s^{-1}R_0y_0, R_0y_0\right)dt -$$

$$\int_0^{2\pi}\left(J\dot{R}_sR_s^{-1}\frac{\partial R_s}{\partial s}R_s^{-1}R_0y_0, R_0y_0\right)dt$$

For $s = 0$ this becomes:

$$\int_0^{2\pi}\left(J\frac{\partial^2 R_s}{\partial s\partial t}\bigg|_{s=0}y_0, R_0y_0\right)dt - \int_0^{2\pi}\left(J\dot{R}_0R_0^{-1}\frac{\partial R_s}{\partial s}\bigg|_{s=0}y_0, R_0y_0\right)dt$$

Integrating the first term by parts, and using in the second term the fact that $J\dot{R}_0R_0^{-1} = A_0$ is symmetric, we obtain

$$\left[\left(J\frac{\partial R_s}{\partial s}\bigg|_{s=0}y_0, R_0y_0\right)\right]_0^{2\pi} - \int_0^{2\pi}\left(J\frac{\partial R_s}{\partial s}\bigg|_{s=0}y_0, \dot{R}_0y_0\right)dt$$

$$-\int_0^{2\pi}\left(\frac{\partial R_s}{\partial s}\bigg|_{s=0}y_0, J\dot{R}_0y_0\right)dt = \left[\left(J\frac{\partial R_s}{\partial s}\bigg|_{s=0}y_0, R_0y_0\right)\right]_0^{2\pi}$$

$$= \omega\left(\frac{\partial R_s}{\partial s}(2\pi)\bigg|_{s=0}y_0, y_0\right)$$

This proves our claim. It is easy to check that for y in the kernel of Q_0, $\frac{\partial}{\partial s}Q_s(y,y)|_{s=0}$ and $\frac{\partial}{\partial s}Q_s^N(y,y)|_{s=0}$ concide, thus proving the first part of our proposition.

To prove the second part, we may, using symplectic reduction to the subspace $\ker(R_0 - \mathrm{Id})$, assume that $R_0 = \mathrm{Id}$. Let $\bar{\Delta}$ be the graph of $-\mathrm{Id}$. Then $\bar{\Delta}$ is transverse to $\phi_0(0) = \phi_0(2\pi) = \Delta$, and this will be true for s small enough. We can thus compute $\mathrm{ind}(\phi)$ using the formula:

$$\mathrm{ind}(\phi_s) = [\phi_s\colon \bar{\Delta}] + \mathrm{index}\ Q(\phi_s(2\pi), \bar{\Delta}; \phi_s(0))$$

By our assumption, the first term of the right hand side does not change for s small enough, so we have to compute index $\frac{d}{ds}Q(\phi_s(2\pi), \bar{\Delta}; \phi_s(0))$. Now $\phi_s(0) = \Delta$, and Δ is the graph of the map $C_s\colon \phi_s(2\pi) \to \bar{\Delta}$ given by $C_s(x, R_sx) = (\frac{1}{2}(R_sx - x), \frac{1}{2}(x - R_sx))$. Here R_s is to mean $R_s(2\pi)$. We can now compute

$$\frac{\partial}{\partial s}\Omega(C_su, u) = \frac{1}{2}\{\omega(R_sx - x, x) - \omega(x - R_sx, R_sx)\} = \omega\left(\frac{\partial}{\partial s}R_sx, x\right)$$

This concludes the proof of 3.9. The proof of 3.8 is now simple. Because two paths in $\mathrm{Sp}(n)$ can always be connected by a family satisfying the assumptions of lemma 3.9, we know that the difference $i_{cz}(R) - \mathrm{ind}_C(\phi)$ is independent of R. Thus we only have to show it equals n for one path in $\mathrm{Sp}(n)$. Choose the path to be $R(t) = \exp \varepsilon Jt$, it is then easy to see that $i_{cz}(R) = n$ while $\mathrm{ind}(\phi) = 2n$ (the easiest way to check this last point is to use the fact that $\mathrm{ind}(\phi) - n$ is the index of the quadratic form $\int_0^{2\pi} [|\dot{q}|^2 - \varepsilon^2 |q|^2] dt$ defined on the set of periodic paths in $\mathbb{R}^n$). □

Proof of theorem 3.1. Let k be the Fenchel dual of h, that is $k'(s) = t \Leftrightarrow h'(t) = s$, which is well defined for $s < \rho$. Then H is the Legendre dual of $K(q, \dot{q}) = k(|\dot{q}|)$, and we can apply prop. 3.7 and 3.8 to $F(q) = \int_0^{2\pi} K(q, \dot{q}) dt$: if $x = (q, p)$ is a periodic solution of $\mathscr{H}_\varepsilon$ with index $i_{cz}(x)$, then q is a critical point of F of Morse index $i_M(q) = i_{cz}(x) + \langle \mu(j), q \rangle$. But $i_M(q)$ is also the Morse index of q as a critical point of $E(q) = \int |\dot{q}|^2$, because k is strictly convex and increasing, thus concluding our proof and this section.

4. Existence of periodic orbits with prescribed index, and proof of the proposition

The proof of existence of a critical orbit of index in the interval $[2, n+1]$ can essentially be recovered from [V1] and [V2]. For the sake of completeness we give the proof in this section. From [V1] we shall use

Proposition 4.1. *Let f be an S^1 equivariant function on some space X, satisfying the Palais-Smale condition, and $u \in H^d_{S^1}(X^b, X^a)$ be a cohomology class whose image vanishes in $H^d_{S^1}(X^c, X^a)$ for some c. If we set $\kappa = \inf\{c \mid u$ is non-zero in $H^d_{S^1}(X^c, X^a)\}$, then κ is a critical value of f. Moreover if κ is an isolated critical value and the critical set in $f^{-1}(\kappa)$ contains no fixed point, then there is at least one critical point in $f^{-1}(\kappa)$ with index m, nullity v satisfying*

$$d - v \leqq m \leqq d .$$

The other result we need is:

Proposition 4.2. *If H is the Hamiltonian defined in section* 1, *then for some real numbers $c > \alpha > 0$, and any integer $r \in [nN - 1, nN - n]$*

$$H^{2r}_{S^1}(E_N - E^\alpha_N, E_N - E^c_N) \neq 0 .$$

Let us remark that this is nothing else than Corollary 4.2 of [V2] but for the fact that we here use the direct action functional instead of the dual action functional. As in [V2] we rely on

Lemma 4.3. *There are two S^1 invariant subspaces of E_N, V and W such that $V \supsetneqq W^\perp \supset \mathrm{Fix}(S^1)$, and if $S(\varepsilon)$ denote the sphere of small radius, ε in E_N, we have, for α small enough*

(i) *a_N is greater than α on $S(\varepsilon) \cap W$*
(ii) *a_N is bounded on V.*
(iii) $\dim W = 2nN$, $\dim V^\perp = 2nN - 2n$.

Proof. Set $Q^{\lambda}(x) = \int_0^{2\pi} [\frac{1}{2}(Jx, \dot{x}) - \frac{\lambda}{2}|x|^2] d\theta$ for x in E (resp. Q_N^{λ} on E_N). Then for $x \in E_N$ small enough, $H(x) \simeq o(|x|^2)$ (we assume that $0 \in j(T^n)$ which can always be done) and $A_H(x) \simeq Q^0(x)$ and $a_N(x) \simeq Q_N^0(x)$. On the other hand, $\| H(x) - \frac{3}{4}|x|^2 \|$ is bounded. According to proposition 2.10 this implies that $a_N(x) - Q_N^{3/2}(x)$ is bounded as well. We now define $W^{\perp}$ (resp. V) to be the direct sum of the eigenspaces of Q_N^0 (resp. $Q_N^{3/2}$) corresponding to nonpositive (resp. negative) eigenvalues. Assertions (i) and (ii) are clear, while (iii) is easy to check if we write

$$Q_N^{\lambda}(x) = \sum_{-N}^{N} -(k+\lambda)/2 \quad |x_k|^2$$

This proves the lemma. □

Proof of proposition 4.2. Consider the maps

$$S(\varepsilon) \cap W \to E_N - E_N^{\alpha} \to E_N$$

which induce maps in equivariant cohomology:

$$H^*_{S^1}(E_N) \to H^*_{S^1}(E_N - E_N^{\alpha}) \to H^*_{S^1}(S(\varepsilon) \cap W)$$

Now the composition of the above maps is onto for $* \leqq \dim W$ because

$$H^*_{S^1}(S(\varepsilon) \cap W) = \mathbb{Q}[u]/(u^{(\frac{1}{2}\dim W)})$$

where u is the pull-back of the generator of $H^*_{S^1}(E_N) = \mathbb{Q}[u]$ by the map induced in cohomology by the inclusion map. Thus the map

$$H^*_{S^1}(E_N - E_N^{\alpha}) \to H^*_{S^1}(S(\varepsilon) \cap W)$$

is onto. On the other hand $E_N - E_N^c$ is contained in $E - V$ which has the equivariant homotopy type of $S(1) \cap V^{\perp}$, thus

$$H^*_{S^1}(E_N - V) = \mathbb{Q}[u]/(u^{\frac{1}{2}\dim V^{\perp}}) .$$

As a result, the maps

$$E_N - E_N^c \to E_N - V \to E_N$$

induce maps

$$H^*_{S^1}(E_N) \to H^*_{S^1}(E_N - V) \to H^*_{S^1}(E_N - E_N^c)$$

the composition of which vanishes for $* \geqq \dim V^{\perp}$. Finally we see that for r in $[\frac{1}{2}\dim V^{\perp}, \frac{1}{2}(\dim W) - 1]$, we have that u^r is zero in $H^*_{S^1}(E_N - E_N^c)$, but nonzero in $H^*_{S^1}(E_N - E_N^{\alpha})$, so that $H^{2r}_{S^1}(E_N - E_N^{\alpha}, E_N - E_N^c)$ does not vanish for $r \in [nN - n, nN - 1]$ which concludes our proof. □

In order to conclude the proof of Proposition B, we still have to prove that the solution of $\mathcal{H}_{\varepsilon}$ that we obtained is contained in $\{(q,p) \mid |p| < \varepsilon\}$. Now for r in the above interval, we denote by $c_r(H_{\varepsilon})$ the critical level defined as the greatest lower bound of the set of real numbers c such that u^r goes to zero in $H^{2r}_{S^1}(E_N - E_N^c)$. It is easy to check that this is a critical value of A_H and it depends continuously on H for the C^0 topology (cf. prop. 2.10). Now $c_r(H_{\varepsilon})$ is equal to $h'_{\varepsilon}(|p_{\varepsilon}|)|p_{\varepsilon}| - h_{\varepsilon}(|p_{\varepsilon}|) + \int_{S^1} q_{\varepsilon}^* \lambda$ where $(q_{\varepsilon}, p_{\varepsilon})$ is a solution of $\mathcal{H}_{\varepsilon}$. If all the periodic orbits of $\mathcal{H}_{\varepsilon}$ were contained in $|p| > \frac{\varrho}{2} - \varepsilon$, then as ε goes to zero, $c_r(H_{\varepsilon})$ goes to $c_{r,a} = -2\pi a$

$+ \rho/2\, \rho(q_{0,a}) \int_{S^1} q_{0,a}^* \lambda$. Since the $c_r(H_\varepsilon)$ are all positive so is $c_{r,a}$. Also because of the above mentioned continuous dependence, $c_{r,a}$ must depend continuously on a. But for a large enough we reach a contradiction, since $\int_{S^1} q^* \lambda\ \rho/2\, \rho(q_{0,a})$ is in a countable set (hence totally discontinuous) and thus can only be continuous if it is constant. In the above argument, we implicitly assumed that $c_r(H_\varepsilon)$ does not depend on the choice of the finite dimensional reduction. This follows immediately from corollary 2.2.

We are now ready to prove

Proposition 4.4. *For a suitable choice of* ε *and* a *with* ε *arbitrarily small and* a *arbitrarily large, the positive critical level* $c_{2nN-2}(H)$ *contains a critical orbit corresponding to a periodic orbit of* $\mathscr{H}_\varepsilon$ *with Conley-Zehnder index in* $[2, n+1]$ *contained in* $\{(q, p) \,|\, |p| < \varepsilon\}$.

Proof. Combining corollary 2.9 and 4.1 we get a critical point of a_N of Morse coindex d, nullity v such that

$$d \leqq 2nN - 2 \leqq d + v$$

hence since

$$d + m + v = 2n(2N+1) - 1\ ,$$

we get a critical point of index m, nullity v such that

$$2n(2N+1) - 1 - m - v \leqq 2nN - 2 \leqq 2n(2N+1) - 1 - m$$

or

$$m \leqq 2n(N + 1/2) + n + 1 \leqq m + v$$

This corresponds to a solution with Conley-Zehnder index i_{cz} satisfying the inequality

$$i_{cz} \leqq 1 + n \leqq i_{cz} + v\ .$$

For the standard (flat) metric of T^n, $v = n - 1$ thus the above inequality can be read as

$$i_{cz}(x) \in [2, n+1]$$

which is the promised statement. □

5. A generalization of our theorem

It is easy to see that our proof of Theorem A yields the following generalization:

Theorem A′. *Let* L^n *be a compact manifold admitting a Riemannian metric with nonpositive (resp. negative) sectional curvature. Then for any Lagrangian embedding* $j\colon L \to \mathbb{R}^{2n}$, *there is a loop* γ *on* L *such that:*

(i) $\langle j^* \lambda, \gamma \rangle > 0$
(ii) $\langle \mu(j), \gamma \rangle \in [2, n+1]$ (resp. $= 2$)

The proof goes exactly as in the torus' case, by noticing that in a manifold with nonpositive curvature, all geodesics are of Morse index 0 and nullity at most $n - 1$. Moreover in the case of negative curvature the improvement in condition (ii) is due to the fact that all geodesics are non degenerate. By considering c_{2nN-2}, and c_{2nN-4}, we find two solutions, $x_1 = (q_1, p_1)$, $x_2 = (q_2, p_2)$ of $(\mathscr{H})$ of respective Conley-Zehnder indices equal to $n + 1$ and $n + 3$. Then $\gamma = q_2 \cdot q_1^{-1}$ will satisfy (i) and (ii). □

Examples of embedded Lagrange submanifolds of $\mathbb{R}^{2n}$ having a metric with negative sectional curvature are nonorientable surfaces with Euler characteristic a multiple of four other than the Klein bottle. Note that the conclusion of the theorem also applies to products of such surfaces.

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