Invent. math. 100, 143–165 (1990)

Inventiones mathematicae

Modular Lagrangians and the theta multiplier

Dennis Johnson and John J. Millson*

Mathematics Department, University of California, Los Angeles, Calif. 90024, USA

Dedicated to Armand Borel

Let V be a free module over $\mathbf{Z}/4$ of rank $2n$ and let $\langle\,,\rangle$ be a non-singular skew form on V. Let $\bar{V}$ denote the reduction of V modulo 2 and let Q be a hyperbolic (i.e. Arf invariant zero) quadratic form on $\bar{V}$ with $\langle\,,\rangle$ as associated bilinear form. We let $\Lambda(V)$ denote the set of oriented Lagrangians (i.e. free totally-isotropic submodules of rank n) in V and $\Lambda_0(V)\subset\Lambda(V)$ the subset of those L in $\Lambda(V)$ whose reductions modulo 2 are totally isotropic for Q. We call such an L an oriented *isotropic* Lagrangian. We will study a certain function m on $\Lambda_0(V)\times\Lambda_0(V)$ with values in the group of fourth roots of unity. The function m is skew-symmetric ($m(M,L)=m(L,M)^{-1}$) and is a 1-cocycle. By this we mean that for all triples L, M, N in $\Lambda_0(V)$ we have

$$m(L,M)\,m(M,N)=m(L,N).$$

We will give an explicit formula for m and use the resulting formula to give a remarkably simple formula for λ, the square of the multiplier of the symplectic theta function. Our construction of m and λ is elementary and is independent of the theory of theta functions and the considerations of Sect. 1. We verify directly that our formula for λ defines a character. We then use the considerations of Sect. 1 to relate the character we have obtained to the square of the theta multiplier. It is interesting to note that our function m on $\Lambda_0(V)\times\Lambda_0(V)$ is the modular analogue of the Maslov index m of [4], p. 126. In fact P. Perrin, [7] p. 112, has introduced an analogous function m defined on pairs of Lagrangians over the 2-adic *field*. Our theory is the corresponding 2-adic *integral* theory (in the description above one can replace $\mathbf{Z}/4$ by the 2-adic integers $\mathbf{Z}_2$ and obtain identical results). The integral theory is more delicate since it is necessary to define m on pairs whose intersection is not free. The solution of this problem is one of the main technical achievements of this paper (see §2). We note also that the function of freely intersecting (but not necessarily isotropic) Lagrangian pairs over $\mathbf{Z}_2$ induced by Perrin's function m is not a cocycle. For example assume that L,M and N are mutually transverse. Then Perrin finds that the

* J.J. Millson was supported in part by NSF grant DMS-85-01742

quantity $\tau(L, M, N) = m(L, M)\, m(M, N)\, m(N, L)$ is the square of the Weil index of the quadratic form B on L given by $B(\ell) = \langle \ell, \mu(\ell) \rangle$ where $\mu \in \mathrm{Hom}(L, N)$ satisfies $M = \text{graph } \mu$. However if L, M and N are *isotropic* then B is *even* and nondegenerate and consequently the square of its Weil index is 1 whence $\tau = 1$. We now describe Weil's theory of theta functions and then give our formula for λ.

Let Γ the theta group (the congruence subgroup of $\mathrm{Sp}_{2n}(\mathbf{Z})$ whose reduction modulo 2 preserves a certain Q as above). Then the non-trivial 2-fold central extension $\mathrm{Mp}_{2n}(\mathbf{R})$ of $\mathrm{Sp}_{2n}(\mathbf{R})$ induces a non-trivial 2-fold extension $\tilde{\Gamma}$ of Γ. In his formidable paper [14], Weil showed that the structure of the transformation law of the theta function is best understood in terms of a unitary representation ω of $\mathrm{Mp}_{2n}(\mathbf{R})$ on $L^2(\mathbf{R}^n)$ now called the oscillator or Weil representation. The space of smooth vectors for ω is the Schwartz space $\mathscr{S}(\mathbf{R}^n)$. The Gaussian $\phi_0 \in \mathscr{S}(\mathbf{R}^n)$ defined by:

$$\phi_0(x_1, \ldots, x_n) = e^{-\pi(x_1^2 + \ldots + x_n^2)}$$

is an eigenvector for $\omega | MU(u)$, the maximal compact subgroup of $\mathrm{Mp}_{2n}(\mathbf{R})$ which covers the maximal compact subgroup $U(n)$ of $\mathrm{Sp}_{2n}(\mathbf{R})$. The theta distribution $\Theta = \sum_{\xi \in \mathbf{Z}^n} \delta(x - \xi)$ is an eigenvector for $\omega | \tilde{\Gamma}$ acting by duality on $\mathscr{S}'(\mathbf{R}^n)$, the space of tempered distributions.

We can accordingly define characters α and κ by:

$$\omega(k)\,\phi_0 = \alpha(k)^{-1}\,\phi_0 \qquad \text{for } k \in MU(n)$$

$$\omega(\gamma)\,\Theta = \kappa(\gamma)^{-1}\,\Theta \qquad \text{for } \gamma \in \tilde{\Gamma}.$$

The function $\Theta(g) = \Theta(\omega(g)\,\phi_0)$ on $\mathrm{Mp}_{2n}(\mathbf{R})$ then satisfies the transformation law:

$$\theta(\gamma g k) = \kappa(\gamma)\,\alpha(k)^{-1}\,\theta(g).$$

$\mathrm{Mp}_{2n}(\mathbf{R})$ operates on $\mathfrak{H}_n$, the Siegel space of genus n, via the quotient map to $\mathrm{Sp}_{2n}(\mathbf{R})$ followed by the well-known action of $\mathrm{Sp}_{2n}(\mathbf{R})$ on $\mathfrak{H}_n$. It follows from the Iwasawa decomposition that there is a unique smooth function j on $\mathrm{Mp}_{2n}(\mathbf{R}) \times \mathfrak{H}_n$ which is holomorphic in τ such that:

(i) $j(g_1 g_2, \tau) = j(g_1, g_2 \tau)\, j(g_2, \tau)$

(ii) $j(k, i\,1_n) = \alpha(k)$.

(iii) $j(g, \tau)^2 = \det(c\tau + d)$ if g lies over $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sp}_{2n}(\mathbf{R})$.

Let $g_\tau \in \mathrm{Mp}_{2n}(\mathbf{R})$ be an element transforming $i\,1_n \in \mathfrak{H}_n$ to $\tau \in \mathfrak{H}_n$. We define $\theta(\tau)$ on $\mathfrak{H}_n$ by:

$$\theta(\tau) = j(g_\tau, i\,1_n)\, \theta(g_\tau).$$

Then $\theta(\tau)$ satisfies the transformation law:

$$\theta(\gamma\tau) = \kappa(\gamma)\, j(\gamma, \tau)\, \theta(\tau) \qquad \text{for } \gamma \in \tilde{\Gamma}.$$

One of the main points of Weil [14] is that $\theta(\tau)$ is the symplectic theta function:

$$\theta(\tau) = \sum_{\xi \in \mathbf{Z}^n} e^{i\pi^t \xi \tau \xi}.$$

The character κ of $\tilde{\Gamma}$ is called the theta multiplier.

To describe the transformation law in more classical terms we observe that we may obtain a realization $\tilde{\Gamma}_1$ for $\tilde{\Gamma}$ as a set of pairs $(\eta, f(\eta, \tau))$ where $\eta = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is in Γ and $f(\eta, \tau)$ is a holomorphic determination of $(\det(c\tau + d))^{1/2}$. We define the group law so that the function $j((\eta, f(\eta, \tau)), \tau) = f(\eta, \tau)$ is a cocycle on $\tilde{\Gamma}_1$, see Lion-Vergne [7], p. 80. A holomorphic determination of $(\det(c\tau + d))^{1/2}$ for each $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in Γ gives rise to a set-theoretic section $s: \Gamma \to \tilde{\Gamma}_1$. Pulling back the transformation law above by s we obtain:

$$\theta(\eta\tau) = \alpha(\eta)(\det(c\tau + d))^{1/2}\, \theta(\tau)$$

where $\eta = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is in Γ and $\alpha = \kappa \circ s$ is a function from Γ to the group of eighth roots of unity. From this point of view the algebraic structure of the transformation law is no longer apparent.

The starting point of our work was a letter from Roger Howe to the second author describing a remarkable property of the central extension $\mu_2 \to \tilde{\Gamma} \to \Gamma$. We will use the notation μ_n henceforth to denote the subgroup of the circle S^1 consisting of the n-th roots of unity. Howe showed that the above extension was obtained by taking the square root of a character. More precisely there exists a pull-back diagram:

$$\begin{array}{ccc} \mu_2 & \xrightarrow{\text{id}} & \mu_2 \\ \downarrow & & \downarrow \\ \tilde{\Gamma} & \longrightarrow & S^1 \\ \downarrow & & \downarrow \text{square} \\ \Gamma & \longrightarrow & S^1 \end{array}$$

Central extensions of this type are particularly easy to understand – they are the analogues of abelian coverings in the theory of covering spaces.

Howe pointed out that the existence of the previous diagram followed from the construction of the "lattice model" for ω. The details of his argument are worked out in Millson [8], p. 18. We prove in Sect. 1 of this paper that the abelianization of Γ is $\mathbf{Z}/4$, consequently we obtain a refinement of the previous diagram as follows

$$\begin{array}{ccc} \mu_2 & \xrightarrow{\text{id}} & \mu_2 \\ \downarrow & & \downarrow \\ \tilde{\Gamma} & \xrightarrow{\kappa} & \mu_8 \\ \downarrow & & \downarrow \\ \Gamma & \xrightarrow{\lambda} & \mu_4 \end{array}$$

We remark that λ is the multiplier for the automorphic form θ^2; namely:

$$\theta(\gamma\tau)^2 = \lambda(\gamma)\det(c\tau+d)\,\theta(\tau)^2 \quad \text{for } \gamma\in\Gamma \quad \text{with} \quad \gamma=\begin{pmatrix} a & b \\ c & d\end{pmatrix}.$$

We now describe our formula for λ. In §2 we construct a sign function σ on $\Lambda_0(V)$. By this we mean a function:

$$\sigma\colon \Lambda_0(V)\times\Lambda_0(V)\to\mu_2$$

such that

(i) $$\sigma(-L, M)=\sigma(L, -M)=-\sigma(L, M)$$

(ii) $$\sigma(\gamma L, \gamma M)=\sigma(L, M) \quad \text{for } \gamma\in\Gamma.$$

Here $-L$ is the oriented isotropic Lagrangian which is the same subspace of V as L but has the opposite orientation and we have replaced Γ by its image in $\mathrm{Sp}_{2n}(\mathbf{Z}/4)$.

Now define a function:

$$r\colon \Lambda_0(V)\to\mathbf{Z}$$

by

$$r(L, M)=n-\dim(\bar{L}\cap\bar{M}).$$

Here the superscript bar denotes reduction modulo 2. Finally we define:

$$m\colon \Lambda_0(V)\times\Lambda_0(V)\to\mu_4$$

by

$$m(L, M)=i^{-r(L,M)}\sigma(L, M).$$

We obtain:

Theorem. *m is a 1-cocycle.*

Corollary. *For L fixed the function $\lambda(\gamma)=m(L, \gamma L)$ is a character on Γ.*

We check that λ is the square of the theta multiplier by using results of Sect. 1. We then calculate λ to obtain an explicit formula which we now describe. Recall the definition of the Dirichlet character ε: $\mathbf{Z} \to \mu_2$ given for $m \in \mathbf{Z}$ by:

$$\varepsilon(m) = \begin{cases} 0 & \text{if } m \text{ is even} \\ 1 & \text{if } m \equiv 1 \bmod 4 \\ -1 & \text{if } m \equiv -1 \bmod 4. \end{cases}$$

Now let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Our formula for $\lambda(\gamma)$ consists of a finite set of algebraic formulas parametrized by the rank of $\bar{c}$, the reduction of c modulo 2. We will use r to denote this rank.

Suppose rows $j_1, j_2, \ldots, j_r$ of $\bar{c}$ are linearly independent with $j_1 < j_2 < \ldots < j_r$. Let $i_1, i_2, \ldots, i_{n-r}$ be the complement of $j_1, j_2, \ldots, j_r$ in $1, 2, \ldots, n$. Let A be the square matrix obtained from c by replacing rows $i_1, i_2, \ldots, i_{n-r}$ in c with the corresponding rows from a. The determinant of A is odd by Lemma 4.1. Our main formula is then

$$\lambda\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = i^{-r} \varepsilon(|A|). \qquad (*)$$

Here we use the symbol $|A|$ to denote the determinant of the matrix A.

We illustrate our formula by describing it in the genus 1, genus 2 and genus 3 cases.

Genus 1

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

1. $\bar{c} = 0$

$$\lambda(\gamma) = \varepsilon(a)$$

2. $\bar{c} = 1$

$$\lambda(\gamma) = i^{-1} \varepsilon(c)$$

Genus 2

$$\gamma = \begin{pmatrix} a_{11} & a_{12} & b_{11} & b_{12} \\ a_{21} & a_{22} & b_{21} & b_{22} \\ c_{11} & c_{12} & d_{11} & d_{12} \\ c_{21} & c_{22} & d_{21} & d_{22} \end{pmatrix}$$

1. rank $\bar{c} = 0$

$$\lambda(\gamma) = \varepsilon(a_{11} a_{22} - a_{21} a_{12})$$

2. rank $\bar{c} = 1$

(i) If $\bar{c}_{21} = 0$ and $\bar{c}_{22} = 0$ then:

$$\lambda(\gamma) = i^{-1} \varepsilon(c_{11} a_{22} - c_{12} a_{21})$$

(ii) If $\bar{c}_{11}=0$ and $\bar{c}_{12}=0$ then

$$\lambda(\gamma)=i^{-1}\varepsilon(a_{11}c_{22}-c_{21}a_{12})$$

(iii) If $\bar{c}_{11}=\bar{c}_{21}$ and $\bar{c}_{12}=\bar{c}_{22}$ then

$$\lambda(\gamma)=i^{-1}\varepsilon(c_{11}a_{22}-c_{12}a_{21})$$

3. rank $\bar{c}=2$.

$$\lambda(\gamma)=-\varepsilon(c_{11}c_{22}-c_{21}c_{12}).$$

Genus 3

$$\gamma=\begin{pmatrix} a_{11} & a_{12} & a_{13} & b_{11} & b_{12} & b_{13}\\ a_{21} & a_{22} & a_{23} & b_{21} & b_{22} & b_{23}\\ a_{31} & a_{32} & a_{33} & b_{31} & b_{32} & b_{33}\\ c_{11} & c_{12} & c_{13} & d_{11} & d_{12} & d_{13}\\ c_{21} & c_{22} & c_{23} & d_{21} & d_{22} & d_{23}\\ c_{31} & c_{32} & c_{33} & d_{31} & d_{32} & d_{33} \end{pmatrix}$$

1. rank $\bar{c}=0$

$$\lambda(\gamma)=\varepsilon\left(\begin{vmatrix} a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{vmatrix}\right)$$

2. rank $\bar{c}=1$

(i) If the first row of $\bar{c}$ is non-zero:

$$\lambda(\gamma)=i^{-1}\varepsilon\left(\begin{vmatrix} c_{11} & c_{12} & c_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{vmatrix}\right)$$

(ii) If the second row of $\bar{c}$ is non-zero:

$$\lambda(\gamma)=i^{-1}\varepsilon\left(\begin{vmatrix} a_{11} & a_{12} & a_{13}\\ c_{21} & c_{22} & c_{23}\\ a_{23} & a_{32} & a_{33}\end{vmatrix}\right)$$

(iii) If the third row of $\bar{c}$ is non-zero:

$$\lambda(\gamma)=i^{-1}\varepsilon\left(\begin{vmatrix} a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ c_{31} & c_{32} & c_{33}\end{vmatrix}\right).$$

3. rank $\bar{c}=2$

(i) If the first two rows of $\bar{c}$ are independent:

$$\lambda(\gamma) = -\varepsilon\left(\begin{vmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}\right).$$

(ii) If the first and third rows of $\bar{c}$ are independent:

$$\lambda(\gamma) = -\varepsilon\left(\begin{vmatrix} c_{11} & c_{12} & c_{13} \\ a_{21} & a_{22} & a_{23} \\ c_{31} & c_{32} & c_{33} \end{vmatrix}\right).$$

(iii) If the second and third rows of $\bar{c}$ are independent:

$$\lambda(\gamma) = -\varepsilon\left(\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{vmatrix}\right)$$

4. rank $\bar{c} = 3$

$$\lambda(\gamma) = i\varepsilon\left(\begin{vmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{vmatrix}\right)$$

Our formula may also be of interest in the moduli theory of Riemann surfaces with spin structure. Indeed let $\mathcal{M}_n^0$ be the subgroup of the mapping class group of a topological surface S of genus n which preserves the spin structure on S corresponding to Q. Then $\mathcal{M}_n^0$ maps onto Γ through the action of $\mathcal{M}_n^0$ on $H_1(S)$. It is known that this map induces an isomorphism of abelianizations. Our formula determines explicitly all characters of $\mathcal{M}_n^0$.

This paper is dedicated to Armand Borel. Part of this paper and part of a possible sequel were presented by the second author at a conference at the Institute for Advanced Study in honor of his sixtieth birthday five years ago. The second author is pleased to acknowledge the great influence the ideas of Armand Borel have had on his work.

Results concerning the theta multiplier along more classical lines have been obtained by Stark [11], Styer [13] and Friedberg [3]. See also Igusa [5]. The reader may also find the book of Lion-Vergne [7] to be a valuable reference for background material on the Weil representation. We should point out that our result in Theorem 1-1 (i) concerning the abelianization of Γ is known to researchers in algebraic K-theory although in some references it is incorrectly stated to be $\mathbf{Z}/2$. At the Borel conference Hyman Bass informed us that in [1], he and W. Pardon had proved a more general theorem but that they had not realized the connection with the theta function. We would like to thank Joseph Oesterlé for carefully reading a preliminary version of our paper. His criticisms have been incorporated into this paper. Finally we would like to thank Richard Elman and Robert Steinberg for helpful conversations.

1. The abelianization of the theta group

In this section V is a symplectic space over $\mathbf{R}$ of dimension $2n$. We choose a symplectic basis $\{e_1, \ldots, e_n, f_1, \ldots, f_n\}$ for V and identify V with $\mathbf{R}^{2n}$. Using this basis we obtain a symplectic space over $\mathbf{Z}$ with coordinates $(x_1, \ldots, x_n, y_1, \ldots, y_n)$. Reducing modulo 2 we obtain a symplectic space over $\mathbf{Z}/2$. We again let $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ denote the coordinates. We define $Q: (\mathbf{Z}/2)^{2n} \to \mathbf{Z}/2$ by $Q(x_1, \ldots, x_n, y_1, \ldots, y_n) = \sum_{i=1}^{n} x_i y_i$. Then Γ, the theta group, is the subgroup of $\mathrm{Sp}_{2n}(\mathbf{Z})$ which maps to the isometry group $O(Q)$ of Q upon reduction modulo 2.

In the following theorem we assume $2n \geqq 6$.

Theorem 1.1. (i) *Γ is generated by the conjugacy class of an anisotropic transvection $t \in \Gamma$ (see below) and*

$$\frac{\Gamma}{[\Gamma, \Gamma]} = \mathbf{Z}/4.$$

(ii) *The value of λ, the square of the theta multiplier, on t is given by*

$$\lambda(t) = i.$$

We now begin our proof of the theorem.

Definition. Let $v \in V$, then we define $t_v \in \mathrm{Sp}_{2n}(\mathbf{R})$ called transvection by v, by the formula:

$$t_v(x) = x + \langle v, x\rangle v \qquad \text{for } x \in V.$$

If $v \in L$ then $t_v \in \mathrm{Sp}_{2n}(\mathbf{Z})$. If $Q(\bar{v}) \neq 0$ then $t_v \in \Gamma$. In this case we call t_v an *anisotropic* transvection. If $Q(\bar{v}) = 0$ we call t_v an *isotropic* transvection. We note the formulas:

(i) $t_v^n(x) = x + n\langle v, x\rangle v$ for $n \in \mathbf{Z}$ and $v \in V$

(ii) $g\, t_v\, g^{-1} = t_{gv}$ for $g \in \mathrm{Sp}_{2n}(\mathbf{R})$.

Let $\Gamma(2) \subset \mathrm{Sp}_{2n}(\mathbf{Z})$ be the subgroup defined by:

$$\Gamma(2) = \{\gamma \in \mathrm{Sp}_n(\mathbf{Z}) : \gamma \equiv 1_{2n} \bmod 2\}.$$

We will assume the classical result, to be found in Mumford [9], Proposition A 3, that $\Gamma(2)$ is generated by the squares of primitive transvections. In this section only we let L denote the integral lattice $\mathbf{Z}^{2n} \subset V$ (later it will denote a Lagrangian).

It is convenient to introduce some more definitions. Let P be a symplectic vector space of dimension 2 over $\mathbf{Z}/2$ equipped with a quadratic form Q. Then we will say P is a hyperbolic plane if Q has 3 zeros (so $Q(x, y)$ is equivalent to xy) and an elliptic plane if it has one zero (so $Q(x, y)$ is equivalent to $x^2 + xy + y^2$.) We will see below that if $n \geqq 2$ any anisotropic vector can be embedded in an elliptic plane.

Lemma 1.1. *If $n \geqq 2$ then every element in $\Gamma(2)$ is expressible as a product of anisotropic transvections and their inverses.*

Proof. Let v be a primitive vector with $\bar{v}$ isotropic. We claim there exists a vector w with $\bar{w}$ anisotropic and $\langle v, w\rangle = 0$. We first observe that there exists a primitive vector u with $\langle v, u\rangle = 1$ and $Q(\bar{u}) = 1$. Indeed choose a primitive vector u with $\langle v, u\rangle = 1$ then either $Q(\bar{u}+\bar{v}) = 1$ or $Q(\bar{u}) = 1$ so either $u+v$ or u works. Let P be the **Z**-span of $\{u, v\}$. Then $L = P + P^{\perp}$ and $P^{\perp}$ is a non-zero symplectic space over Z by hypothesis. Then any primitive vector w with $Q(\bar{w}) = 1$ satisfies the claim.

We next observe that $v+w$ and $v-w$ are both anisotropic vectors. The lemma now follows from the relation that if $\langle v, w\rangle = 0$ then

$$t_v^2 = t_w^{-2}\, t_{v+w}\, t_{v-w}$$

which expresses the square of the (arbitrarily chosen) isotropic transvection in terms of anisotropic transvections. Since we know every element of $\Gamma(2)$ is a word in the squares of primitive transvections the lemma is proved. □

We need another classical result to be found in Dieudonné [2], Proposition 14, that $O(Q)$ is generated by the transvections $t_{\bar{v}}$ if $n > 2$. We also need the following lemma.

Lemma 1.2. *$O(Q)$ acts transitively on the unit sphere of Q.*

Proof. In case $\dim V = 2$ the lemma is true by inspection. Otherwise let $\bar{u}$, $\bar{v}$ be given $Q(\bar{u}) = Q(\bar{v}) = 1$. Embed $\bar{u}$ in an elliptic plane P_1. If $\bar{v} \in P_1$ we are done so suppose $\bar{v} \in P_1^{\perp}$. If V has dimension 4 then $P_1^{\perp} = P_2$ is another elliptic plane since the Arf invariant of Q is zero. If $\dim V > 4$ then we can embed $\bar{v}$ into an elliptic plane $P_2 \subset P_1^{\perp}$. But now we extend $\bar{u} \to \bar{v}$ to an isometry interchanging P_1 and P_2 and leaving $(P_1 + P_2)^{\perp}$ fixed. □

Corollary. *$O(Q)$ is generated by a single conjugacy class.*

Lemma 1.3. *Γ is generated by primitive anisotropic transvections.*

Proof. Let $\gamma \in \Gamma$. Then $\bar{\gamma} \in O(Q)$ and we may write $\bar{\gamma} = t_{\bar{v}_1} t_{\bar{v}_2} \dots t_{\bar{v}_m}$. Let $\eta = \gamma \circ (t_{v_1} t_{v_2} \dots t_{v_m})^{-1}$. Then $\eta \in \Gamma(2)$ and by Lemma 1.1 we have $\eta = t_{w_1} t_{w_2} \dots t_{w_k}$ with all w_j anisotropic. □

Lemma 1.4. *All primitive anisotropic transvections are conjugate in Γ.*

Proof. Let $t_v, t_w \in \Gamma$ be given. We may choose $\gamma \in \Gamma$ such that $\bar{\gamma}\bar{v} = \bar{w}$ since $O(Q)$ acts transitively on the unit sphere. Hence replacing v by γv we may assume v and w are congruent modulo 2. But it is an immediate consequence of strong approximation that there exists $\eta \in \Gamma(2)$ such that $\eta v = w$ whenever $v \equiv w \bmod 2$. □

We now know that Γ is generated by a single conjugacy class and hence that $\Gamma/[\Gamma, \Gamma]$ is cyclic. To complete (i) of the theorem we must determine the order of t_v in $\Gamma/[\Gamma, \Gamma]$. Now given v primitive with $Q(\bar{v}) = 1$ embed v in a plane P such that $\bar{P}$ is elliptic. This may be done as follows. Choose u with

$\langle u, v\rangle = 1$. If $Q(\bar{u}) = 1$ we are done. Otherwise split off the $\mathbf{Z}$-module R spanned by $\{u, v\}$ and replace u by $u + w$ with $w \in R^{\perp}$ such that $Q(\bar{w}) = 1$. Then $P = \{u + w, v\}$ is an elliptic plane. Now Γ contains the theta group Λ for P (the set of elements in the integral symplectic group of P which upon reduction modulo 2 preserve Q). But $\Lambda \cong \mathrm{SL}_2(\mathbf{Z})$, since $Q(x, y) = x^2 + xy + y^2$ every symplectic transformation of P preserves Q. But by a classical result:

$$\frac{\Lambda}{[\Lambda, \Lambda]} \cong \mathbf{Z}/12.$$

Since $t_v \in \Lambda \subset \Gamma$ we have t_v^{12} in $\Gamma/[\Gamma, \Gamma]$. We now embed t_v into the theta group of a genus 2 quadratic space to obtain a new relation.

Lemma 1.5. *In any genus* 2 *quadratic space an anisotropic transvection satisfies:*

$$t_v^4 \equiv 1 \text{ modulo commutators.}$$

Proof. A genus 2 quadratic space is either the direct sum of two elliptic planes (Arf invariant zero) or the direct sum of an elliptic plane and a hyperbolic plane (Arf invariant 1). In either case we may assume the plane spanned by $\{e_1, f_1\}$ is an elliptic plane. Thus t_{e_1} is an anisotropic transvection. Now define matrices following Steinberg [12] with $r, s, t \in \mathbf{C}$:

$$x_{\omega_1+\omega_2}(t) = \begin{pmatrix} 1 & 0 & 0 & t \\ 0 & 1 & t & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad x_{\omega_1-\omega_2}(s) = \begin{pmatrix} 1 & s & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -s & 1 \end{pmatrix} \quad x_{2\omega_1}(r) = \begin{pmatrix} 1 & 0 & r & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Here ω_1, ω_2 are the standard coordinate functionals on the Cartan subalgebra of $\mathrm{sp}_2(\mathbf{C})$.

We then have the Steinberg relation which the reader may verify by direct calculation:

$$[x_{\omega_1-\omega_2}(s), x_{\omega_1+\omega_2}(t)] = x_{2\omega_1}(2(st)). \qquad (**)$$

The relevance of (**) to the lemma is that

$$t_{e_1}^n = x_{2\omega_1}(n).$$

Applying (**) with $s = 2$ and $t = 2$ we find that $t_{e_1}^8$ is a commutator of two elements of $\Gamma(2)$. Since we already know $t_v^{12} \equiv 1 \bmod [\Gamma, \Gamma]$ we obtain the lemma.

We have now obtained the following lemma.

Lemma 1.6. *If* $n \geqq 3$ *then* $\Gamma/[\Gamma, \Gamma]$ *is a quotient of* $\mathbf{Z}/4$.

In order to prove Theorem 1.1 it will suffice to exhibit a character of order 4. In fact the remaining sections of this paper are devoted to constructing such a character explicitly. In order to complete the proof at this stage we observe that the existence of such a character follows from Theorem 2.2.37 of Lion-

Vergne [7] (our $\kappa(\gamma)$ is denoted $\varepsilon(\gamma)$ there and our $\lambda(\gamma)$ is denoted $k(\gamma)$ there). Also the value of λ on an anisotropic transvection can be checked from the formulas in [7]. □

2. Construction of the sign function on $\Lambda_0(V)$

We now assume V is a symplectic space over $\mathbf{Z}/4$; that is a free $\mathbf{Z}/4$ module equipped with a non-singular skew-symmetric bilinear form to be denoted $\langle\, , \rangle$ taking values in $\mathbf{Z}/4$. The symplectic group $\mathrm{Sp}(V)$ of V is the group of automorphisms of V preserving $\langle\, , \rangle$. If rank $V=2n$ a Lagrangian (submodule) is a free summand L of V of rank n on which the form $\langle\, , \rangle$ is identically zero. Any two bases of L are related by a unique matrix in $\mathrm{GL}_n(\mathbf{Z}/4)$ with determinant ± 1. Two such bases are said to be in the same orientation if the determinant is $+1$. An *oriented* Lagrangian is a Lagrangian with a preferred orientation class of bases. If L is an oriented Lagrangian we write $-L$ for the opposite oriented Lagrangian. There are just two orientation classes on any Lagrangian. The group $\mathrm{Sp}(V)$ acts transitively on the set of oriented Lagrangians $\Lambda(V)$.

We will assume that $\bar{V}$ is equipped with a quadratic form Q of Arf invariant zero such that $\langle\, , \rangle$ is the bilinear form associated to Q. We call such a triple $V, \langle\, , \rangle, Q$ a quadratic space. Henceforth $\Gamma \subset \mathrm{Sp}(V)$ will denote the subgroup of elements whose reductions modulo 2 are isometries of Q. Then Γ is the image of the theta group under reduction modulo 4.

Definition. Suppose $V=V_1+V_2$ is a direct sum of symplectic spaces and L_1, M_1 and L_2, M_2 are Lagrangian pairs in V_1 and V_2 respectively. Then the direct sum of the two Lagrangian pairs is the pair L_1+L_2, M_1+M_2 in V. We will sometimes use the notation $(L_1, M_1)+(L_2, M_2)$ for this sum. If $V=V_1+V_2$ is a direct sum of quadratic spaces then we have an analogous notion of the direct sum of isotropic Lagrangian pairs.

We also make the following definitions. A pair of oriented Lagrangians L, M is called a congruent pair if $L\equiv M \bmod 2$ and a transverse pair if $L\cap M=\{0\}$. In this later case it is easily seen that $V=L+M$ and L and M are dually paired.

We first construct the sign function $\sigma(L, M)$ for a transverse pair L, M. We define $\sigma(L, M)$ to be 1 if the orientation of L followed by that of M is the natural (symplectic) orientation of V; that is, the orientation of a symplectic basis $\{e_1, \dots, e_n, f_1, \dots, f_n\}$ such that $\langle e_i, e_j\rangle=\langle f_i, f_j\rangle=0$ and $\langle e_i, f_j\rangle=\delta_{ij}$. We define $\sigma(L, M)=-1$ otherwise. We observe that if we are given oriented bases $\{l_i\}$ for L and $\{m_j\}$ for M we have:

$$\sigma(L, M)=\det(\langle l_i, m_j\rangle).$$

The function σ on transverse pairs has the following elementary properties.

Lemma 2.1.

(i) $\sigma(L, M)=(-1)^n\sigma(M, L)$

(ii) *If* L, M *is the direct sum of the transverse pairs* L_1, M_1 *and* L_2, M_2 *then:*

$$\sigma(L, M)=\sigma(L_1, M_1)\,\sigma(L_2, M_2).$$

(iii) *If* $L, M, N \in \Lambda_0(V)$ *are mutually transverse then* n *is necessarily even and if* $n=2k$ *then*

$$\sigma(L, M)\,\sigma(M, N)=(-1)^k\,\sigma(L, N).$$

Proof. We prove the last part of (iii) and leave the rest of the lemma to the reader. We first observe that $\sigma(M, N)$ is the determinant of the natural isomorphism $M \to N^*$ induced by $\langle\ ,\ \rangle$. Let a be the composition $M \to N^* \to L \to M^*$. It is apparent (noting that n is even) that

$$\begin{aligned}\det a &= \sigma(M, N)\,\sigma(L, N)\,\sigma(L, M)\\ &= \sigma(M, N)\,\sigma(N, L)\,\sigma(L, M).\end{aligned}$$

We now recall a standard formula for a. Let P_1 and P_2 be the projections associated to the direct sum decomposition $V=L+N$. Let B be the bilinear form on M defined by $B(m, m')=\langle P_1 m, P_2 m'\rangle$. Then B is symmetric, [7], p. 40, and if $b: M \to M^*$ corresponds to B then $a=b$, [7], p. 67. We claim B is an even form; that is, $B(m, m)\equiv 0 \bmod 2$ all $m\in M$. Indeed we have $Q(\bar{m})-Q(P_1\bar{m})-Q(P_2\bar{m})=\langle P_1\bar{m}, P_2\bar{m}\rangle=B(\bar{m}, \bar{m})$ and since $L, M, N\in\Lambda_0(V)$ the claim follows. Since B is even and non-degenerate we have $\dim M=2k$ and $\det B=\det b=(-1)^k$ by the following elementary result on symmetric bilinear forms over $\mathbf{Z}/4$. □

Lemma 2.2. *Let* B *be a non-degenerate even symmetric bilinear form on a free* $\mathbf{Z}/4$ *module of dimension* n. *Then* $n=2k$ *with* $k\in\mathbf{Z}$ *and* $\det B=(-1)^k$.

Proof. See [6], Theorem 33a. □

We conclude our study of transverse pairs an explicit formula. Given a pair $L, M\in\Lambda_0(V)$ we may choose a symplectic basis $\{e_1, \ldots, e_n, f_1, \ldots, f_n\}$ for V with $Q(\bar{e}_i)=Q(\bar{f}_i)=0$ for all i such that $L=E$, the span of $\{e_1, \ldots, e_n\}$. We may then represent M by a $2n$ by n matrix $S=\begin{pmatrix}a\\c\end{pmatrix}$ (where a and c are n by n blocks) whose columns are the coordinates of a properly oriented basis for M relative the given symplectic basis. We will say that S is a representing matrix for M. We observe that changing the basis of M corresponds to performing a column operation on the matrix S and changing the basis of V corresponds to performing a row operation on S. Finally L, M is a transverse pair if and only if c is invertible in which case $\sigma(L, M)=\det c$.

We now treat the case of a congruent pair L, M. We choose a Lagrangian D which is transverse to L. Then L and D are dually paired hence dually paired modulo 2, hence M and D are dually paired modulo 2 and hence M and D are dually paired. In particular M and D are also transverse. Hence we may attempt to define $\sigma(L, M)$ by:

$$\sigma(L, M)=\sigma(L, D)\,\sigma(M, D).$$

For the moment we will use the notation $\sigma_D(L, M)$ for the right-hand side. In order to prove $\sigma_D(L, M)$ is independent of D we need to investigate $\Lambda_D(M)$, the set of oriented Lagrangians transverse to D. Let P_D denote the subgroup of $\mathrm{Sp}(V)$ consisting of those elements which stabilize D and N_D denote the subgroup of those g such that $g|D=1$.

Lemma 2.3. *P_D acts transitively on $\Lambda_D(V)$.*

Proof. Suppose $L, M\in\Lambda_D(V)$. Let P_1 and P_2 be the projections onto L and D respectively associated to the direct sum decomposition $V=L+D$. Then $P_1|M$ is an isomorphism with inverse $q_D: L\to M$. Put $\mu=P_2\circ q$. Then $\mu: L\to D$ and M is the image of the graph of μ under the natural isomorphism $L\oplus D\to V$. We define $T: V\to V$ by $T|L=I_L+\mu$ and $T|D=I_D$. Then $T(L)=M$ and $T\in N_D$. □

Remarks. In case $L, M, D\in\Lambda_0(V)$ it is easily verified that $T\in\Gamma$. Now let $\mu\in\mathrm{Hom}(L, M)$ with L, M Lagrangian. Then $D=\text{graph}\ \mu$ is Lagrangian if and only if the bilinear form B on L given by $B(\ell_1, \ell_2)=\langle\ell_1, \mu(\ell_2)\rangle$ is symmetric. In this case we will say μ is symmetric. If L and M are also isotropic then $D=\text{graph}\ \mu$ is isotropic if and only if B is even. In this case will say μ is even. Finally we observe that if $m\in M$ and $m\equiv\ell \bmod D$ then $m=q_D(\ell)$.

We can now give another description of $\sigma_D(L, M)$. The map q_D is an isomorphism from the oriented Lagrangian L to the oriented Lagrangian M. We define $\varepsilon(q_D)=+1$ if q_D is orientation preserving and -1 otherwise.

Lemma 2.4. *If L, M are both transverse to D then*

$$\sigma(L, D)\,\sigma(M, D)=\varepsilon(q_D).$$

Proof. We put $\varepsilon_1=\sigma(L, D)$ and $\varepsilon_2=\sigma(M, D)$. Then calculating with volume forms we obtain $\omega_L\wedge\omega_D=\varepsilon_1$ vol and $\omega_M\wedge\omega_D=\varepsilon_2$ vol. We apply T to the first equation to obtain

$$(T\omega_L)\wedge\omega_D=T\omega_L\wedge T\omega_D=T(\omega_L\wedge\omega_D)=\varepsilon_1\ \mathrm{vol}.$$

But $T\omega_L=\varepsilon(q_D)\,\omega_M$ whence

$$(T\omega_L)\wedge\omega_D=\varepsilon(q_D)\,\omega_M\wedge\omega_D=\varepsilon(q_D)\,\varepsilon_2\ \mathrm{vol}.\quad\square$$

Now let D' be another Lagrangian which is transverse to L (hence also transverse to M). We may repeat the construction of Lemma 2.3 using the splitting $L+D'$ to represent M as the graph of an element $\tau\in\mathrm{Hom}(L, D')$ and obtain $q_{D'}: L\to V$ with $q_{D'}=I+\tau$ whence $q_{D'}(L)=M$. In order to prove $\varepsilon(q_D)=\varepsilon(q_{D'})$ we need to investigate the decompositions of V associated to L, D, D' in more detail.

Since $D, D'\in\Lambda_L(V)$ we may write D' as the graph of an element $\nu\in\mathrm{Hom}(D, L)$ and D as the graph of an element $\eta\in\mathrm{Hom}(D', L)$. Let P_1' and P_2' be the projections associated to the direct sum decomposition $V=L+D'$. Then it is easily verified that

$$\nu=-\eta\circ P_2'.$$

Lemma 2.5.

$$q_D=q_{D'}\circ(1-\nu\circ\mu).$$

Proof. By definition we have

$$q_D(\ell)=\ell+\mu(\ell).$$

But $\mu(\ell)\in D$ whence $\mu(\ell)=P_2'(\mu(\ell))+\eta(P_2'(\mu(\ell)))$ and we obtain (since $\eta\circ P_2'=-\nu$)

$$q_D(\ell)=\ell-\nu(\mu(\ell))+P_2'(\mu(\ell)).$$

But the right-hand side of this equation is an element of M congruent to the element $\ell-\nu(\mu(\ell))$ of L modulo D' and is therefore equal to $q_{D'}(\ell-\nu(\mu(\ell)))$. □

We can now prove that $\sigma_D(L, M)$ is well-defined (i.e., is independent of D) for a congruent pair L, M.

Lemma 2.6. *Let $L, M, D, D'\in\Lambda_0(V)$ with $\bar{L}=\bar{M}$ and D and D' both transverse to L. Then*

$$\varepsilon(q_D)=\varepsilon(q_{D'}).$$

Proof. It suffices to prove that $1-\nu\mu$ preserves the orientation of L. We choose bases for L and D dually paired under $\langle\,,\rangle$ and bases for L and D' dually paired under $\langle\,,\rangle$. We then express $\mu\in\operatorname{Hom}(L, D)$ and $\nu\in\operatorname{Hom}(D, L)$ as matrices (μ_{ij}) and (ν_{ij}) respectively. Since $\bar{L}=\bar{M}$ we have $\bar{\mu}_{ij}=0$ all i, j and since L, D and D' are isotropic $\bar{\nu}_{ii}=0$ all i. We obtain

$$\det(1-\nu\mu)=1-\operatorname{tr}\nu\mu=1.$$

The last inequality holds as a consequence of the identity

$$\operatorname{tr}\nu\mu=\sum_{i,j}\nu_{ij}\mu_{ij}=\sum_{i}\nu_{ii}\mu_{ii}+2\sum_{i<j}\nu_{ij}\mu_{ij}. \quad □$$

Corollary. *If L and M are congruent isotropic Lagrangians choose an isotropic Lagrangian D transverse to L and define*

$$\sigma(L, M)=\sigma(L, D)\,\sigma(M, D).$$

Then $\sigma(L, M)$ is independent of the choice of D.

We list some properties of $\sigma(L, M)$ on the subspace of $\Lambda_0(M)^2$ consisting of congruent pairs.

Lemma 2.7. (i) $\sigma(L, M)=\sigma(M, L)$.

(ii) *If L, M is the direct sum of the congruent pairs L_1, M_1 and L_2, M_2 then*

$$\sigma(L, M)=\sigma(L_1, M_1)\,\sigma(L_2, M_2).$$

(iii) *If* M_1, M_2, M_3 *are all congruent modulo* 2 *then*

$$\sigma(M_1, M_2)\,\sigma(M_2, M_3)=\sigma(M_1, M_3).$$

We conclude our study of congruent pairs by the following explicit formula. Suppose $L=E$ and M is represented by the matrix $S=\begin{pmatrix}a\\c\end{pmatrix}$ as above. Then L, M is a congruent pair if and only if $c\equiv 0 \bmod 2$ in which case $\sigma(L, M)=\det a$ (this latter statement is easily proved by choosing $D=F=\operatorname{span}\{f_i, \dots, f_n\}$).

We now prove that the general Lagrangian pair may be decomposed into the direct sum of a congruent pair and a transverse pair.

Definition. A symplectic splitting $V=V_1+V_2$ is adapted to the pair L, M if:
(i) $L=L\cap V_1+L\cap V_2$ and $M=M\cap V_1+M\cap V_2$
(ii) $L\cap V_1$ and $M\cap V_1$ are congruent modulo 2
(iii) $L\cap V_2$ and $M\cap V_2$ are transverse.

Lemma 2.8. *If* L, M *is a Lagrangian pair there exists a splitting of* V *adapted to* L, M.

Proof. Consider $\bar{L}\cap\bar{M}$ which we denote $\bar{C}$ (this is an abuse of notation as there is no C as yet). We choose an oriented isotropic subspace $\bar{D}$ such that $\bar{C}$ and $\bar{D}$ are dually paired. Then $\bar{V}_1=\bar{C}+\bar{D}$ is symplectic. We put $\bar{V}_2=\bar{V}_1^{\perp}$. We define $\bar{E}$ to be the annihilator of $\bar{D}$ in $\bar{L}$ and $\bar{F}$ to be the annihilator of $\bar{D}$ in $\bar{M}$. We have an exact sequence $\bar{E}\to\bar{L}\to\bar{D}^*$ which is split by $\bar{C}$ whence $\bar{L}=\bar{C}+\bar{E}$. Similarly $\bar{M}=\bar{C}+\bar{F}$. By a dimension count $(\bar{C}+\bar{D})^{\perp}=\bar{E}+\bar{F}$.

We now lift the above decomposition to V. Choose free isotropic submodules C_1 and D which lift $\bar{C}$ and $\bar{D}$ respectively such that $C_1\subset L$. Then C_1 and D are dually paired whence $V_1=C_1+D$ is symplectic. Thus if $V_2=V_1^{\perp}$ we have $V=V_1+V_2$. We show that this splitting is adapted to the pair L, M.

Let E be the annihilator of D in L whence $E\subset V_2$. We observe that $L=C_1+E$ because the surjection $\phi\colon L\to D^*$ has kernel E and is split by C_1. Let F be the annihilator of D in M. Then the reductions of E and F modulo 2 are $\bar{E}$ and $\bar{F}$ whence E and F are dually paired and $V_2=E+F$. Let C_2 be the annihilator of E in M. Then $\bar{C}_2=\bar{C}_1$ and (using $M\to E^*$ as above) we have $M=C_2+F$. □

Remark. If L is isotropic (for Q) then any subspace is also isotropic (for Q). Hence if L and M are isotropic Lagrangians then $L\cap V_1$, $M\cap V_1$ and $L\cap V_2$, $M\cap V_2$ are also isotropic Lagrangians.

We can now give a (provisional) definition of the sign invariant $\sigma(L, M)$ for a general pair L, M of isotropic Lagrangians in V. We choose an adapted splitting of V and orientations of $L\cap V_1$ and $M\cap V_1$. We give $L\cap V_2$ and $M\cap V_2$ the respective quotient orientations. We then define

$$\sigma(L, M)=\sigma(L\cap V_1, M\cap V_1)\,\sigma(L\cap V_2, M\cap V_2).$$

We must prove that our definition of $\sigma(L, M)$ is independent of choices. We first observe that as long as we require that $V_2\cap L$ be given the quotient orientation then changing the orientation of $V_1 \cap L$ does not change $\sigma(L, M)$. Of course the analogous statement holds for $V_1 \cap M$. It remains to check that $\sigma(L, M)$ does not depend on the choice of adapted splitting.

Lemma 2.9. *$\sigma(L, M)$ as defined above is independent of the adapted splitting of V.*

Proof. We begin by observing that an adapted splitting of V is equivalent to a pair E, F of (dually-paired) free isotropic subspaces of L and M respectively such that

$$\bar{C}+\bar{E}=\bar{L} \quad \text{and} \quad \bar{C}+\bar{F}=\bar{M}.$$

Given such a pair we define $C_1 = L \cap F^{\perp} = L \cap (E+F)^{\perp}$ and $C_2 = M \cap E^{\perp} = M \cap (E+F)^{\perp}$ and we obtain a decomposition of L, M into the direct sum of the congruent pair C_1, C_2 and the transverse pair E, F by the arguments of Lemma 2.8. Thus it is an equivalent problem to show that σ is independent of the choices of E and F. This we now do.

Changing E does not influence the possible choices left for F since these are determined by the condition $\bar{C}+\bar{F}=\bar{M}$ and $\bar{C}$ is not effected by changing E. Thus to see that $\sigma(E', F')=\sigma(E, F)$ it suffices to check that $\sigma(E, F)=\sigma(E', F)$ i.e. that changing only E or F leaves σ unchanged. The argument for changing F is symmetrical to that for changing E, so we do only the latter.

Let E' be a new choice of E. Since F is not changed C_1 is not changed and E' is a new complement to C_1 in L. Let $\{c_i: 1 \leqq i \leqq r\}$ and $\{e_j: 1 \leqq j \leqq n-r\}$ be oriented bases for C_1 and E respectively. Then we may choose a basis $\{e'_j: 1 \leqq j \leqq n-r\}$ of E' of the form $e'_j = e_j + \sum_{i=1}^{r} A_{ij} c_i$. We now show $\sigma(E', F)$ is independent of the matrix (A_{ij}). We remark that the matrix (A_{ij}) is not arbitrary since we must have $\bar{C}'_2 = \bar{C}$ where $C'_2 = M \cap (E')^{\perp}$. We do not need to compute this condition explicitly.

We now find a corresponding $D' \subset (E'+F)^{\perp}$ which is transverse to C_1. Let $\{d_i: 1 \leqq i \leqq r\}$ be a basis for D which is dual to $\{c_i\}$. Then the transformation $T: V \to V$ which is the identity on C_1 and F and satisfies

$$T(e_j) = e_j + \sum_{i=1}^{r} A_{ij} c_i, \qquad T(d_j) = d_j - \sum_{i=1}^{n-r} A_{ji} f_i$$

is an orthogonal transformation. We put $D' = T(D)$. Since $D \subset (E+F)^{\perp}$ we find on applying T that $D' \subset (E'+F)^{\perp}$. Second if we put $d'_j = T(d_j)$ then we find that the bases $\{c_i\}$ and $\{d'_j\}$ are dual. We consider $C'_2 = M \cap (E')^{\perp}$. We observe that D' is transverse to C_1 and since $\bar{C}'_2 = \bar{C}_1$ it follows that D' is transverse to C'_2. Hence we may use D' to transfer the orientation of C_1 to C'_2. We let $q_{D'}: C_1 \to C'_2$ be the isomorphism introduced in Lemma 2.3. Thus if $c \in C_1$ then $q_{D'}(c)$ is the unique element of C'_2 satisfying:

$$q_{D'}(c) \equiv c \bmod D'.$$

We observe that $q_{D'}(c) \equiv c \bmod 2$. Hence if $c_k'' = q_{D'}(c_k)$ there exists D_{kl} such that:

$$c_k'' = c_k + 2\sum_{l=1}^{r} D_{lk}\, d_l' = c_k + 2\sum_{l=1}^{r} D_{lk}\, d_l - 2\sum_{l=1}^{r}\sum_{i=1}^{n-r} D_{lk}\, A_{li} f_i.$$

Now let $\{c_k'\}$ be the basis of C_2 obtained by transferring $\{c_k\}$ to C_2 via the isomorphism q_D. Then c_k' is determined by the congruence:

$$c_k' \equiv c_k \bmod D.$$

But $c_k'' + 2\sum_{l=1}^{r}\sum_{i=1}^{n-r} D_{lk}\, A_{li} f_i$ satisfies the same congruence and we obtain:

$$c_k' = c_k'' + 2\sum_{l=1}^{r}\sum_{i=1}^{n-r} D_{lk}\, A_{li} f_i.$$

Hence $\{c_k', f_j\colon 1 \leqq k \leqq r,\ 1 \leqq j \leqq n-r\}$ and $\{c_k'', f_j\colon 1 \leqq k \leqq r,\ 1 \leqq j \leqq n-r\}$ are in the same orientation class of bases of M. Thus C_2 and C_2' induce the same orientation on F. If $\{f_1, \ldots, f_{n-r}\}$ is a basis which is in this orientation class then noting $\langle e_i', f_j\rangle = \langle e_i, f_j\rangle$ since $C_1 \subset F^{\perp}$ we have:

$$\sigma(E, F) = \det(\langle e_i', f_j\rangle) = \sigma(E', F)$$

and we have established that $\sigma(L, M)$ is independent of the choice of E and F and consequently of the choice of splitting. □

3. Construction of a 1-cocycle on $\Lambda_0(V)$

Definition. A 1-cocycle on $\Lambda_0(V)$ is a function: $m\colon \Lambda_0(V) \times \Lambda_0(V) \to \mu_4$ such that:

(i) $m(L, M)\, m(M, N) = m(L, N)$

We say m is invariant if $\gamma \in \Gamma$ implies

(ii) $m(\gamma M, \gamma N) = m(M, N)$.

Given any function $m\colon \Lambda_0(V) \times \Lambda_0(V) \to \mu_4$ for $\gamma \in \Gamma$ which satisfies (ii) we choose $L \in \Lambda_0(V)$ and define:

$$\lambda(\gamma) = m(L, \gamma L).$$

Lemma 3.1. *λ is a character if and only if m is a 1-cocycle.*

Proof. We leave the proof to the reader (recall Γ acts transitively on $\Lambda_0(V)$).

We now construct an invariant 1-cocycle m on $\Lambda_0(V)$ using the sign function σ of Sect. 2. Recall that if $L, M \in \Lambda_0(V)$ with $\dim L = \dim M = n$ we have defined $r(L, M)$ by:

$$r(L, M) = n - \dim(\bar{L} \cap \bar{M})$$

and $m(L, M)$ by

$$m(L, M) = i^{-r(L,M)}\, \sigma(L, M).$$

We also define $\tau(L, M, N)$ by

$$\tau(L, M, N) = m(L, M)\, m(M, N)\, m(N, L).$$

Theorem 3.1. *m is a 1-cocycle and the associated character λ is the square of the theta multiplier.*

In order to prove the theorem we need to define the stabilization of a quadratic space V. Let P be a quadratic space of dimension 2 with a symplectic basis $\{e, f\}$ such that $\bar{e}$ and $\bar{f}$ are isotropic for the corresponding quadratic form on $\bar{P}$. We define $\tilde{V}$, the stabilization of V, to be the quadratic space which is the orthogonal sum of V and P. We observe that both $i^{-r(L,M)}$ and $\sigma(L, M)$ are multiplicative for direct sums of pairs. Also $r = \frac{1}{2}\dim V_2$ where $V = V_1 + V_2$ is adapted to L, M. As a consequence of the results in § 2 we have the following lemma.

Lemma 3.2. (i) $m((L_1, M_1) + (L_2, M_2)) = m(L_1, M_1)\, m(L_2, M_2)$
(ii) $m(L, M) = (-1)^{r(L,M)}\, m(M, L) = m(M, L)^{-1}$.

Theorem 3.1 will follow from the next four lemmas.

Lemma 3.3. *The 2-cocycle $\tau(L, M, N)$ is invariant under permutation of its arguments $L, M, N \in \Lambda_0(V)$.*

Proof. τ is obviously invariant under cyclic permutations, hence it suffices to prove $\tau(L, M, N) = \tau(M, L, N)$. Using Lemma 3.2(ii) we find

$$\frac{\tau(L, M, N)}{\tau(M, L, N)} = (-1)^{r(L,M) + r(M,N) + r(N,L)}.$$

Thus it suffices to prove

$$r(L, M) + r(M, N) + r(N, L) \equiv 0 \bmod 2.$$

This latter formula concerns only $\bar{L}$, $\bar{M}$ and $\bar{N}$. It follows easily from the splitting $\bar{V} = \bigoplus_{i=0}^{4} \bar{V}_i$ adapted to the triple $\bar{L}$, $\bar{M}$, $\bar{N}$ of [10], Lemma 2.8. by this we mean $\bar{L} \cap \bar{V}_0 = \bar{M} \cap \bar{V}_0 = \bar{N} \cap V_0$ and $\bar{L} \cap \bar{V}_4$, $\bar{M} \cap \bar{V}_4$, $\bar{N} \cap \bar{V}_4$ are mutually transverse and for $i = 1, 2, 3$ exactly two of $\bar{L} \cap \bar{V}_i$, $\bar{M} \cap \bar{V}_i$ and $\bar{N} \cap \bar{V}_i$ coincide and the remaining one is transverse to the other two. The above congruence then follows from the easy fact that $\dim \bar{V}_4 \equiv 0 \bmod 4$ since it contains three mutually transverse *isotropic* Lagrangians, Lemma 2.1(iii). □

Lemma 3.4. *Let V be a quadratic space and $L, M \in \Lambda_0(V)$ with $r(L, M)$ even. Then there exists $N \in \Lambda_0(V)$ with N transverse to both L and M.*

Proof. We have $(L, M) = (L_1, M_1) + (L_2, M_2)$. Choose $N_1 \in \Lambda_0(V)$ which is transverse to L_1. Since $\bar{L}_1 = \bar{M}_1$ it follows that N_1 is also transverse to M_1. The

Lagrangians L_2 and M_2 are transverse. Let $\mu \in \mathrm{Hom}(L_2, M_2)$ be symmetric, even and invertible. Such a μ exists because $\dim L_2 \equiv 0 \bmod 2$. Let N_2 be the graph of μ, and $N = N_1 + N_2$. □

Lemma 3.5. *Assume $\bar{L} \cap \bar{M} \cap \bar{N} = \{0\}$. Then there exists a splitting of V adapted to L, M which is compatible with N.*

Proof. $\bar{N}$ maps onto $(\bar{L} \cap \bar{M})^*$ so we can choose $\bar{D} \subset \bar{N}$ such that $\bar{D}$ and $\bar{C} = \bar{L} \cap \bar{M}$ are dually paired. We lift $\bar{D}$ to $D \subset N$ and proceed as in Lemma 2.8 to construct a splitting $V = V_1 + V_2$ adapted to L, M. Let G be the annihilator of C_1 in N whence $G \subset (C_1 + D)^{\perp} = V_2$. We have a short exact sequence $G \to N \to C_1^*$ which is split by D whence $N = D + G$ and consequently $N = N \cap V_1 + N \cap V_2$. □

Lemma 3.6. *Suppose $L, M, N \in \Lambda_0(V)$ with at least one of the three possible pairs (L, M), (L, N) and (M, N) transverse. Then*

$$m(L, M)\, m(M, N) = m(L, N).$$

Proof. Since $\tau(L, M, N)$ is invariant under S_3 we may assume that N is transverse to L whence $\bar{L} \cap \bar{N} = \{0\}$ and $\bar{L} \cap \bar{M} \cap \bar{N} = \{0\}$. Let $V = V_1 + V_2$ be a splitting adapted to (L, M) and compatible with N. We obtain a decomposition

$$(L, M, N) = (L_1, M_1, N_1) + (L_2, M_2, N_2).$$

By the multiplicativity property of m for sums it suffices to prove the above formula for each of the summands where it follows after a further splitting of V_2 adapted to (M_2, N_2) and compatible with L_2 from the results in § 2. □

We can now conclude the proof of Theorem 3.1. Let $L, M, N \in \Lambda_0(V)$ be given. Let $V = V_1 + V_2$ be a splitting adapted to L, M. If $\dim V_2 \equiv 0 \bmod 4$ we may choose M' transverse to both L and M. If $\dim V_2 \equiv 2 \bmod 4$ we replace V by its stabilization $\tilde{V}$ and we replace L, M, N by $\tilde{L} = L + (e)$, $\tilde{M} = M + (f)$ and $\tilde{N} = N + (e)$ respectively. Then it is easily seen, that $\tau(\tilde{L}, \tilde{M}, \tilde{N}) = \tau(L, M, N)$ and $\tilde{V} = V_1 + (V_2 + P)$ is adapted to the pair $\tilde{L}$, $\tilde{M}$. Thus to prove $\tau(L, M, N) = 1$ it suffices to consider the case in which $\dim V_2 \equiv 0 \bmod 4$. We may then choose $M' \in \Lambda_0(V)$ which is transverse to both L and M. Then applying Lemma 3.6 three times we have

$$\begin{aligned} m(L, M)\, m(M, N) &= m(L, M)\, m(M, M')\, m(M', N) \\ &= m(L, M')\, m(M', N) \\ &= m(L, N). \end{aligned}$$

This concludes the proof of Theorem 3.1 since it is easily checked the $m(L, tL) = i$ where t is an anisotropic transvection.

4. Calculation of $\lambda(\gamma)$

We choose $L = E$ and calculate $\sigma(E, \gamma E)$ for $\gamma \in \Gamma$. We assume γ is given by:

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Hence $M=E'=\gamma E$ is spanned by the columns of the $2n$ by n matrix $S=\begin{pmatrix} a \\ c \end{pmatrix}$. We assume rank $\bar{c}=r$ and that $c \neq 0$. If $c=0$ then clearly $\lambda(\gamma)=\varepsilon(\det a)$.

We will perform column operations on S corresponding to an orientation preserving change of basis of M and row operations corresponding to a change of symplectic basis for V such that the span of the first n-vectors in the new basis $\{e'_1, \ldots, e'_n, f'_1, \ldots, f'_n\}$ is still E and the reductions modulo 2 of the new basis vectors are isotropic for Q. Since neither of these operations change L, M they do not change $\sigma(L, M)$.

In terms of matrices we left multiply S by $m(\alpha)$ or $n(\beta)$ where

$$m(\alpha)=\begin{pmatrix} {}^t\alpha^{-1} & 0 \\ 0 & \alpha \end{pmatrix} \quad \text{and} \quad n(\beta)=\begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}.$$

Here $\alpha \in \mathrm{GL}_n(\mathbf{Z}/4)$ and β is a symmetric matrix with entries in $\mathbf{Z}/4$ and even diagonal entries.

Recall now that the rank of $\bar{c}$ is r and assume that rows $j_1, j_2, \ldots, j_r$ of $\bar{c}$ are linearly independent. Let $i_1, i_2, \ldots, i_{n-r}$ be the complement of $j_1, j_2, \ldots, j_r$ in $1, 2, \ldots, n$ arranged in increasing order. Let π be the permutation of $1, 2, \ldots, n$ given by:

$$\pi=\begin{pmatrix} j_1 & j_2 \cdots j_r & i_1 \cdots & i_{n-r} \\ 1 & 2 \ldots r & r+1 & n \end{pmatrix}.$$

Let α_1 be the permutation matrix in $\mathrm{GL}_n(\mathbf{Z}/4)$ corresponding to π and let $\gamma'=m(\alpha_1)\gamma$. We have $\lambda(\gamma')=\sigma(\pi)\,\lambda(\gamma)$. Writing $\gamma'=\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$ we find that the first r rows of $\overline{c'}$ are linearly independent and the last $n-r$ rows of $\overline{c'}$ are linear combinations of the first r rows.

We assume the first n columns of γ' are given by $S'=\begin{pmatrix} a' \\ c' \end{pmatrix}$. We decompose S' according to:

$$S'=\begin{pmatrix} a'_1 \\ a'_2 \\ c'_1 \\ c'_2 \end{pmatrix}$$

where a'_1 consists of the first r rows of a', a'_2 the last $n-r$ rows of a', c'_1 the first r rows of c' and c'_2 the last $n-r$ rows of c'. Then $A'=\begin{pmatrix} a'_2 \\ c'_1 \end{pmatrix}$ is an n by n matrix. Note that the rows of A' coincide (up to a permutation) with the rows of the matrix A of the introduction.

In the proof of the next lemma we omit the bars – all the matrices involved will be reduced mod 2.

Lemma 4.1. *$\overline{A'}$ is invertible.*

Proof. We first observe that we may subtract linear combinations of the rows of c'_1 from the rows of c'_2 without changing A'. This is because the inverse transpose of such an operation operating on a' adds the corresponding linear combinations of rows of a'_2 to a'_1. Thus we may assume $c'_2=0$. Also the rank of A' is invariant under column operations of S'. By column operations we can transform c_1 to $(I_r|0)$ since c_1 is of rank r. We obtain:

$$S'=\begin{pmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \\ I & 0 \\ 0 & 0 \end{pmatrix}.$$

Since the columns of S' span a Lagrangian we have ${}^t\begin{pmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$ is symmetric; that is $\begin{pmatrix} {}^t\alpha_1 & 0 \\ {}^t\alpha_2 & 0 \end{pmatrix}$ is symmetric and consequently $\alpha_2=0$. Now

$$n=\operatorname{rank} S'=\operatorname{rank}\begin{pmatrix} \alpha_1 & 0 \\ \beta_1 & \beta_2 \\ \mathrm{I} & 0 \end{pmatrix}=\operatorname{rank}\begin{pmatrix} \beta_1 & \beta_2 \\ I & 0 \end{pmatrix}.$$

But $\begin{pmatrix} \beta_1 & \beta_2 \\ I & 0 \end{pmatrix}$ was obtained from A' via column operations and has the same rank as A'. □

We now return to our standard notation in which a superscript bar denotes reduction modulo 2.

Corollary. det A *is odd.*

We lift the row operations performed in the proof of the preceding lemma to the corresponding row operations over **Z**/4. Since these lifted operations consist of adding a row or its negative to another we obtain a unimodular matrix α_2 such that $\gamma''=m(\alpha_2)\gamma'$ has its last $n-r$ rows even and such that $\lambda(\gamma'')=\lambda(\gamma')=\sigma(\pi)\,\lambda(\gamma)$. We let $S''=\begin{pmatrix} a''_1 \\ a''_2 \\ c''_1 \\ c''_2 \end{pmatrix}$ be the corresponding decomposition of the first n columns of γ''. We regard S'' as the representing matrix of E' relative a new symplectic basis (corresponding to $m(\alpha)$).

We now claim that in order to obtain the formula (*) of the introduction for $\lambda(\gamma)$ it is sufficient to prove the following formula. Let $A''=\begin{pmatrix} a''_2 \\ c''_1 \end{pmatrix}$ where the notation is that of the paragraph above. Then:

$$(**)\qquad \lambda(\gamma'')=i^{-r}(-1)^{r(n-r)}\,\varepsilon(\det A'').$$

In order to prove the claim one has only to check that the determinant of the matrix A in the introduction is related to $\det A''$ by the formula

$$\det A = \sigma(\pi)(-1)^{r(n-r)} \det A''.$$

This is an exercise in determinants which we leave to the reader. We now state the above formula as a lemma and prove it.

Lemma 4.2. $\lambda(\gamma'') = i^{-r}(-1)^{r(n-r)} \varepsilon(\det A'')$.

Proof. Of course the lemma is equivalent to the statement that $\sigma(E, E')$ is given by the product of the last three terms on the right-hand side of (**).

It is elementary that there exist row and column operations as above transforming S'' into the matrix (with η_1, β_2 invertible and $\eta_2 \equiv 0 \bmod 2$)

$$S''' = \begin{pmatrix} a''' \\ c''' \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \beta_2 \\ \eta_1 & 0 \\ 0 & \eta_2 \end{pmatrix}.$$

We may now easily compute $\sigma(E, E')$. We let $\{e_1, \ldots, e_n, f_1, \ldots, f_n\}$ now denote the new symplectic basis – relative to which E' is represented by S'''.

We observe that the pair E, E' is the direct sum of pairs E_1, E'_1 and E_2, E'_2 obtained as follows. Let $V_1 = \operatorname{span}(e_1, \ldots, e_r, f_1, \ldots, f_r)$ and $V_2 = \operatorname{span}(e_{r+1}, \ldots, e_n, f_{r+1}, \ldots, f_n)$. Then $V = V_1 + V_2$, an orthogonal direct sum. Clearly the pair E, E' is sum of its intersections E_1, E'_1 and E_2, E'_2 with V_1 and V_2 respectively. Hence:

$$\sigma(E, E') = \sigma(E_1, E'_1)\, \sigma(E_2, E'_2).$$

But E_1, E'_1 is a transverse pair so $\sigma(E_1, E'_1) = \det \eta_1$. Also E_2, E'_2 is a congruent pair so $\sigma(E_2, E'_2) = \det \beta_2$. The lemma is proved by the elementary observation that:

$$\det \begin{pmatrix} 0 & \beta_2 \\ \eta_1 & 0 \end{pmatrix} = (-1)^{r(n-r)} \det \eta_1 \det \beta_2. \quad \square$$

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Oblatum 29-VIII-1986 & 19-VII-1989