THE MOTION OF A LUNAR SATELLITE

CLAUS OESTERWINTER

U.S. Naval Weapons Laboratory, Dahlgren, Va., U.S.A.

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Abstract. Presented in this theory is a semianalytical solution for the problem of the motion of a satellite in orbit around the moon. The principal perturbations on such a body are due to the non-spherical gravity field of the moon, the attraction of the earth, and, to a lesser degree, the attraction of the sun. The major part of the problem is solved by means of the celebrated von Zeipel Method, first successfully applied to the motion of an artificial earth satellite by Brouwer in 1959. After eliminating from the Hamiltonian all terms with the period of the satellite and those with the period of the moon, it is suggested to solve the remaining problem with the aid of numerical integration of the modified equations of motion.

This theory was written in 1964 and presented as a dissertation to Yale University in 1965. Since then a great deal has been learned about the gravity field of the moon. It seems that quite a number of recently determined gravity coefficients would qualify as small quantities of order two. Hence, according to the truncation criteria employed, they should be considered in the present theory. However, the author has not endeavored to update the work accordingly. The final results, therefore, are incomplete in the lunar gravitational perturbations. Nevertheless, the theory does give the largest such variations and it does present the methods by which perturbations may be derived for any gravity terms not actually developed.

1. Introduction

The largest perturbation on a close lunar satellite is caused by the zonal harmonic of degree two in the oblate gravity field of the moon. Its coefficient, labeled c_{20}, is of order 10^{-4}. It is known from Brouwer's (1959) work that this coefficient will appear as a divisor in developing the terms of long period, namely those with the arguments g and h. Hence, in order to achieve adequate accuracy in the final results, it will be necessary to develop the disturbing function to a fairly high degree of precision. After some experiments, it has been decided to include all terms $\geqslant 10^{-8}$ in the disturbing function such that the final results, allowing for divisors of order 10^{-4}, are accurate to about 10^{-4}.

The coefficients of the leading zonal harmonic, c_{20}, and the leading longitudinal harmonic, c_{22}, are known with sufficient accuracy. Their numerical values are 2.0×10^{-4} and 0.25×10^{-4}, respectively. However, nothing is known about the coefficients of higher order terms. It is conceivable that quite a number of these may exceed 10^{-8}. Nevertheless, since there is no way of assigning numerical values to coefficients of degree larger than two, the present theory is restricted to terms of degree $\leqslant 2$ in the triaxial potential of the moon. Since both zonal and longitudinal harmonics are treated, a later extension should not pose any particular difficulties. In dealing with the perturbations due to the moon's figure, it was found that the leading terms of the moon's physical libration should be considered. They contribute terms to the disturbing function somewhat in excess of 10^{-8}.

The largest perturbation due to the earth is seen to be about 4×10^{-5}, while that due

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to the sun is approximately 2×10^{-7}. In addition to the small parameters already quoted, there will appear the following quantities:

$\gamma_☾ = 8.9 \times 10^{-2}$, the inclination of the lunar orbit to the ecliptic;
$e_☾ = 5.5 \times 10^{-2}$, the eccentricity of the lunar orbit;
$I_☾ = 2.7 \times 10^{-2}$, the inclination of the lunar equator to the ecliptic;
$\varepsilon_1 = 9.3 \times 10^{-4}$, a coefficient associated with the physical libration;
$\tau_1 = 4.9 \times 10^{-4}$, a coefficient associated with the physical libration.

Hence there is a total of nine small parameters. It would be undesirable to assign orders of magnitude to each quantity separately since, upon forming squares, cubes, and products of these numbers, there would result some overlap between terms belonging to neighboring orders of magnitude. In order to avoid this difficulty, the disturbing function is first developed completely, and only then are orders of magnitude assigned to the various coefficients. These coefficients may be products of two or three of the small parameters enumerated above.

In the development of terms of intermediate period, small divisors of order 10^{-2} appear. This feature suggests, prompted by the table of small parameters above, to designate terms of approximately 10^{-2} to be of order one. Roughly, then, order two is 10^{-4}, order three is 10^{-6}, and order four about 10^{-8}. A more precise definition will be found in a later part of this paper.

The derivations will reveal that the periodic perturbations of the lunar orbital elements have been ignored, while the secular variations are included. Since some of the former lead to perturbative forces somewhat greater than 10^{-8}, there is an inconsistency insofar as terms due to physical libration, of comparable size or even a little smaller, are included. However, some restrictions had to be imposed in order to be able to cope with the amount of algebra. It was felt that physical libration was the more interesting of these two features to treat.

The elimination of short-period terms presents no difficulties. It is easily understood that there are no terms of first order. In keeping with the desired accuracy of 10^{-4} for the final results, the short-period terms of second order are calculated, but those of third order are not needed. Here as throughout the development, no expansion in the satellite's eccentricity is required.

As briefly mentioned before, in deriving the perturbations of intermediate period, namely those containing the mean longitude of the moon, a small divisor of the first order is encountered. This divisor, of course, is $n_☾/n$, where $n_☾$ and n are the mean motions of the moon and the satellite, respectively. Hence there will appear terms of first order, and, in order to obtain the variations of second order, it will be necessary to consider the third-order part of the Hamiltonian at this point.

After this step, there remain in the Hamiltonian the mean longitude of the sun and the nodes and pericenters of the satellite and the moon. It will be seen that the Hamiltonian is now of such a form that the solution cannot be obtained by successive approximations. It is suggested that this part of the problem be solved by numerical integration. This approach should prove to yield a theory valid for a considerable length of time. The integration step size would not be less than several days. For this purpose, the Hamiltonian, freed of terms of short and intermediate period, is given through order three, and the method for obtaining the fourth-order part is outlined.

The author apologizes for his scanty knowledge of the Russian astronomical literature. It was not until recently that he became aware of Brumberg's (1962) important work on the motion of lunar satellites. Timely knowledge of Brumberg's theory would undoubtedly have been beneficial in the planning and execution of this work.

2. Development of the Disturbing Function

In this section the force function will be stated and developed by taking into account the perturbative accelerations due to earth, sun, and the moon's gravitational field greater than 10^{-8}. When dealing with the latter, particular attention is given to the physical libration of the moon.

A. THE INITIAL FORCE FUNCTION

The equations of motion with their force function will be given valid in an inertial frame. The coordinate system will then be changed to have its origin in the center of the moon. The force function will be expressed in the relative coordinates and put in such a form that the corresponding equations of motion are canonical.

The equations of motion of four point masses in inertial space are

$$\ddot{\bar{\varrho}}_i = k^2 \sum_{\substack{j=0 \\ j \neq i}}^{3} m_j \frac{\bar{\varrho}_j - \bar{\varrho}_i}{r_{ij}^3}, \quad j = 0, 1, 2, 3$$

where

$$\bar{\varrho} = \begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix}$$

and

$$r_{ij}^2 = (\xi_i - \xi_j)^2 + (\eta_i - \eta_j)^2 + (\zeta_i - \zeta_j)^2 .$$

Upon introducing the force function

$$F'' = k^2 \left[\frac{m_0 m_1}{r_{01}} + \frac{m_0 m_2}{r_{02}} + \frac{m_0 m_3}{r_{03}} + \frac{m_1 m_2}{r_{12}} + \frac{m_1 m_3}{r_{13}} + \frac{m_2 m_3}{r_{23}} \right], \tag{1}$$

the equations may be written more conveniently in the form

$$m_j \ddot{\bar{\varrho}}_j = \tilde{\nabla} F'', \quad j = 0, 1, 2, 3 \tag{2}$$

where $\tilde{\nabla}$ is the operator

$$\tilde{\nabla} = \begin{pmatrix} \dfrac{\partial}{\partial \xi} \\ \dfrac{\partial}{\partial \eta} \\ \dfrac{\partial}{\partial \zeta} \end{pmatrix}$$

The notation F'' has been chosen so that the symbol F may be available for a different function.

Let now the subscript 0 refer to the moon, 1 refer to the satellite, 2 refer to the earth, and 3 refer to the sun, and let the coordinates of satellite and earth be referred to the center of the moon, but those of the sun to the barycenter of the earth-moon system. These new coordinates will be represented by the symbol $\bar{r}$. Then

$$\begin{aligned} \bar{r}_1 &= \bar{\varrho}_1 - \bar{\varrho}_0 , \\ \bar{r}_2 &= \bar{\varrho}_2 - \bar{\varrho}_0 , \\ \bar{r}'_3 &= \bar{\varrho}_3 - \frac{m_0\bar{\varrho}_0 + m_2\bar{\varrho}_2}{m_0 + m_2} . \end{aligned} \tag{3}$$

The prime attached to the last vector calls attention to the fact that the solar coordinates are not referred to the center of the moon. It is clear that the axes of the new coordinate frame are parallel to those of the inertial one.

In order to obtain the acceleration of m_1 with respect to the new frame, differentiate the first of (3) twice with respect to time, and then employ (2):

$$\begin{aligned} \ddot{\bar{r}}_1 &= \ddot{\bar{\varrho}}_1 - \ddot{\bar{\varrho}}_0 \\ &= \frac{1}{m_1} \tilde{\nabla}_1 F'' - \frac{1}{m_0} \tilde{\nabla}_0 F'' . \end{aligned} \tag{4}$$

In order to express (4) in terms of the new coordinates, consider

$$F'' = F''(\bar{r}_1, \bar{r}_2, \bar{r}'_3) .$$

Hence

$$\frac{\partial F''}{\partial \xi_j} = \frac{\partial F''}{\partial x_1}\frac{\partial x_1}{\partial \xi_j} + \frac{\partial F''}{\partial x_2}\frac{\partial x_2}{\partial \xi_j} + \frac{\partial F''}{\partial x'_3}\frac{\partial x'_3}{\partial \xi_j}, \quad j = 0, 1 \tag{5}$$

and similar equations for the other components of $\tilde{\nabla}_j F''$. With the aid of (3), one obtains

$$\begin{aligned} &\frac{\partial x_1}{\partial \xi_1} = 1 \quad \frac{\partial x_2}{\partial \xi_1} = 0 \quad \frac{\partial x'_3}{\partial \xi_1} = 0 \\ &\frac{\partial x_1}{\partial \xi_0} = -1 \quad \frac{\partial x_2}{\partial \xi_0} = -1 \quad \frac{\partial x'_3}{\partial \xi_0} = -\frac{m_0}{m_0 + m_2} \end{aligned} \tag{6}$$

and again similar results for y and z. Upon putting (6) into (5),

$$\tilde{\nabla}_1 F'' = \nabla_1 F'' \tag{7}$$

and

$$\tilde{\nabla}_0 F'' = -\nabla_1 F'' - \nabla_2 F'' - \frac{m_0}{m_0 + m_2} \nabla'_3 F'' ,$$

where now

$$\nabla = \begin{pmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial z} \end{pmatrix}.$$

Substitution of (7) into (4) leads to

$$\ddot{\bar{r}}_1 = \frac{m_0 + m_1}{m_0 m_1} \nabla_1 F'' + \frac{1}{m_0} \nabla_2 F'' + \frac{1}{m_0 + m_2} \nabla_3' F''. \tag{8}$$

F'' is still expressed in terms of the old coordinates ξ, η, and ζ. In order to make F'' a function of the new variables, first note that one is free to choose the center of mass of the entire system as origin of the inertial frame. For reasons of legibility this is not depicted in Figure 1. With this choice

$$\sum_{j=0}^{3} m_j \bar{\varrho}_j = \bar{0}. \tag{9}$$

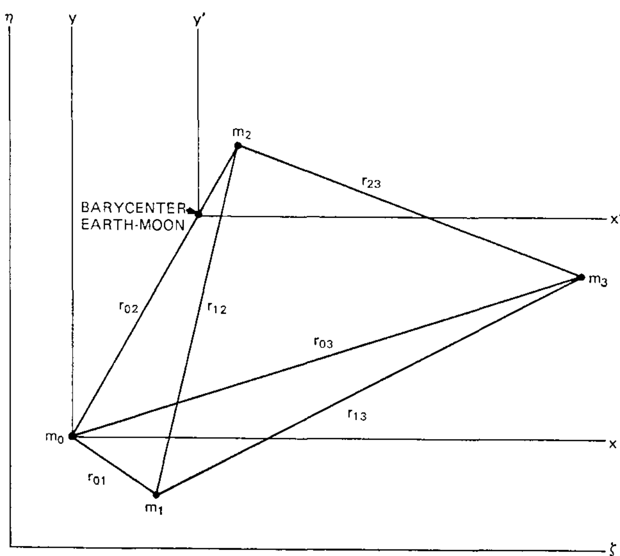

Fig. 1. The diagram shows the inertial and relative coordinate frames. Note that the four bodies in general do not lie in the same plane.

Inasmuch as the satellite's mass is negligible, it is permissible to put $m_1 = 0$ for this purpose; some labor can be saved by doing so. Combining formulae (3) and (9) and solving for the $\bar{\varrho}_i$ gives the relations

$$\bar{\varrho}_0 = -\frac{m_2}{m_0 + m_2} \bar{r}_2 - \frac{m_3}{m_0 + m_2 + m_3} \bar{r}_3'$$

$$\bar{\varrho}_1 = \bar{r}_1 - \frac{m_2}{m_0 + m_2}\,\bar{r}_2 - \frac{m_3}{m_0 + m_2 + m_3}\,\bar{r}'_3$$

$$\bar{\varrho}_2 = \frac{m_0}{m_0 + m_2}\,\bar{r}_2 - \frac{m_3}{m_0 + m_2 + m_3}\,\bar{r}'_3$$

$$\bar{\varrho}_3 = \frac{m_0 + m_2}{m_0 + m_2 + m_3}\,\bar{r}'_3 .$$

Appealing now to

$$r_{ij}^2 = (\xi_i - \xi_j)^2 + (\eta_i - \eta_j)^2 + (\zeta_i - \zeta_j)^2 ,$$

there follows

$$\begin{aligned}
r_{01}^2 &= r_1^2 \\
r_{02}^2 &= r_2^2 \\
r_{03}^2 &= r_3'^2 + \frac{2m_2}{m_0 + m_2}\,\bar{r}_2\cdot\bar{r}'_3 + \left(\frac{m_2}{m_0 + m_2}\right)^2 r_2^2 \\
r_{12}^2 &= r_2^2 - 2\bar{r}_1\cdot\bar{r}_2 + r_1^2 \\
r_{13}^2 &= r_3'^2 + \frac{2m_2}{m_0 + m_2}\,\bar{r}_2\cdot\bar{r}'_3 + \left(\frac{m_2}{m_0 + m_2}\right)^2 r_2^2 \\
&\qquad - 2\bar{r}_1\cdot\bar{r}'_3 - \frac{2m_2}{m_0 + m_2}\,\bar{r}_1\cdot\bar{r}_2 + r_1^2 \\
r_{23}^2 &= r_3'^2 - \frac{2m_0}{m_0 + m_2}\,\bar{r}_2\cdot\bar{r}'_3 + \left(\frac{m_0}{m_0 + m_2}\right)^2 r_2^2 .
\end{aligned} \tag{10}$$

With these relations one can express F'', given by (1), entirely in terms of the new coordinates.

It will next be necessary to calculate $\ddot{\bar{r}}_1$, given by (8), explicitly in terms of the coordinates. Appealing to (1) and (8), there follows

$$\begin{aligned}
\ddot{\bar{r}}_1 &= \frac{m_0 + m_1}{m_0 m_1}\,k^2\left[m_0 m_1 \nabla_1 \frac{1}{r_1} + m_1 m_2 \nabla_1 \frac{1}{r_{12}} + m_1 m_3 \nabla_1 \frac{1}{r_{13}}\right] \\
&\quad + \frac{1}{m_0}\,k^2\left[m_0 m_2 \nabla_2 \frac{1}{r_2} + m_0 m_3 \nabla_2 \frac{1}{r_{03}} + m_2 m_3 \nabla_2 \frac{1}{r_{23}}\right] \\
&\quad + \frac{1}{m_0 + m_2}\,k^2\left[m_0 m_3 \nabla'_3 \frac{1}{r_{03}} + m_2 m_3 \nabla'_3 \frac{1}{r_{23}}\right],
\end{aligned} \tag{11}$$

where some terms have vanished because $m_1 = 0$. In computing the partial derivatives, it is advantageous to make use of

$$\nabla \frac{1}{r} = \frac{-1}{2r^3}\,\nabla r^2$$

since the r_{ij}^2 are conveniently available in (10). The necessary partials are

$$\nabla_1 \frac{1}{r_1} = \frac{-\bar{r}_1}{r_1^3}, \qquad \nabla_2 \frac{1}{r_2} = \frac{-\bar{r}_2}{r_2^3},$$

$$\nabla_1 \frac{1}{r_{12}} = \frac{\bar{r}_2 - \bar{r}_1}{r_{12}^3}, \qquad \nabla_2 \frac{1}{r_{03}} = \frac{-m_2}{(m_0+m_2)\, r_{03}^3}\left(\frac{m_2}{m_0+m_2}\bar{r}_2 + \bar{r}_3'\right),$$

$$\nabla_1 \frac{1}{r_{13}} = \frac{-1}{r_{13}^3}\left(\bar{r}_1 - \frac{m_2}{m_0+m_2}\bar{r}_2 - \bar{r}_3'\right), \qquad \nabla_2 \frac{1}{r_{23}} = \frac{-m_0}{(m_0+m_2)\, r_{23}^3}\left(\frac{m_0}{m_0+m_2}\bar{r}_2 - \bar{r}_3'\right),$$

$$\nabla_3' \frac{1}{r_{03}} = \frac{-1}{r_{03}^3}\left(\bar{r}_3' + \frac{m_2}{m_0+m_2}\bar{r}_2\right), \qquad \nabla_3' \frac{1}{r_{23}} = \frac{-1}{r_{23}^3}\left(\bar{r}_3' - \frac{m_0}{m_0+m_2}\bar{r}_2\right).$$

After substitution of these into (11) and some simplification, there follows

$$\ddot{\bar{r}}_1 = -k^2\left[m_0\frac{\bar{r}_1}{r_1^3} + m_2\frac{\bar{r}_2}{r_2^3} + m_2\frac{\bar{r}_1-\bar{r}_2}{r_{12}^3} + \frac{m_3}{r_{13}^3}\left(\bar{r}_1 - \frac{m_2}{m_0+m_2}\bar{r}_2 - \bar{r}_3'\right) + \frac{m_3}{r_{03}^3}\left(\frac{m_2}{m_0+m_2}\bar{r}_2 + \bar{r}_3'\right)\right]. \tag{12}$$

With the partial derivatives given above, and noting that

$$\nabla_i(\bar{r}_i\cdot\bar{r}_j) = \bar{r}_j,$$

it can be seen by inspection that

$$\ddot{\bar{r}}_1 = \nabla_1 F' \tag{13}$$

if

$$F' = k^2\left[\frac{m_0}{r_1} - \frac{m_2}{r_2^3}\bar{r}_1\cdot\bar{r}_2 + \frac{m_2}{r_{12}} + \frac{m_3}{r_{13}} - \frac{m_3}{r_{03}^3}\left(\frac{m_2}{m_0+m_2}\bar{r}_1\cdot\bar{r}_2 + \bar{r}_1\cdot\bar{r}_3'\right)\right]. \tag{14}$$

The equations of motion, (13), are now in the desired canonical form. This will permit an immediate transition from these equations to the equations of motion in terms of the Delaunay variables which are to be used later. Note that the two functions F'' and F' do not have the same physical dimensions.

B. THE TERMS DUE TO EARTH AND SUN

The next step in the development of F' requires the expansions of

$$\frac{1}{r_{12}}, \frac{1}{r_{13}}, \text{ and } \frac{1}{r_{03}^3}$$

in form of power series in the ratios r_1/r_2, r_1/r_3, and r_2/r_3. Defining three angles by

$$\bar{r}_1\cdot\bar{r}_2 = r_1 r_2 \cos S_{12}$$
$$\bar{r}_1\cdot\bar{r}_3' = r_1 r_3' \cos S_{13}'$$
$$\bar{r}_2\cdot\bar{r}_3' = r_2 r_3' \cos S_{23}',$$

one has, with the aid of Equations (10),

$$r_{12}^2 = r_2^2\left[1 - 2\frac{r_1}{r_2}\cos S_{12} + \left(\frac{r_1}{r_2}\right)^2\right]$$

$$r_{13}^2 = r_3'^2\left[1 + \frac{2m_2}{m_0 + m_2}\frac{r_2}{r_3'}\cos S_{23}' + \left(\frac{m_2}{m_0 + m_2}\right)^2\left(\frac{r_2}{r_3'}\right)^2 - 2\frac{r_1}{r_3'}\cos S_{13}' - \frac{2m_2}{m_0 + m_2}\frac{r_1 r_2}{r_3'^2}\cos S_{12} + \left(\frac{r_1}{r_3'}\right)^2\right]$$

$$r_{03}^2 = r_3'^2\left[1 + \frac{2m_2}{m_0 + m_2}\frac{r_2}{r_3'}\cos S_{23}' + \left(\frac{m_2}{m_0 + m_2}\right)^2\left(\frac{r_2}{r_3'}\right)^2\right].$$

Clearly the first and third of these may be expanded with the aid of Legendre polynomials. For the second, a carefully executed binomial expansion will serve. In order to determine the point of truncation for each series, the numerical values of their coefficients are computed based on an assumed radius of the satellite orbit of 3000 km. This does not restrict the validity of the theory to radii of less than 3000 km, but the exact numerical compatibility of the various external perturbations at this altitude will deteriorate in either direction from it. In units of r_1 and m_0, one finds, approximately,

$$\begin{aligned} r_1 &= 1 & m_0 &= 1 \\ r_2 &= 1.3 \times 10^2 & m_2 &= 0.81 \times 10^2 \\ r_3' &= 5.0 \times 10^4 & m_3 &= 2.7 \times 10^7. \end{aligned} \tag{15}$$

Using these numbers, it is easily verified that

$$\frac{m_2}{r_{12}} \approx \frac{m_2}{r_2} = 0.62$$

$$\frac{m_3}{r_{13}} \approx \frac{m_3}{r_3'} = 0.54 \times 10^3$$

$$\frac{m_3}{r_{03}^3}\frac{m_2}{m_0 + m_2}\bar{r}_1\cdot\bar{r}_2 \approx \frac{m_3 m_2}{m_0 + m_2}\frac{r_1 r_2}{r_3'^3} = 2.8 \times 10^{-5}$$

$$\frac{m_3}{r_{03}^3}\bar{r}_1\cdot\bar{r}_3' \approx m_3\frac{r_1}{r_3'^2} = 1.1 \times 10^{-2}.$$

With these figures it is clear how far the expansions have to be carried. Although the development will later be restricted to terms $\geqslant 10^{-8}$, at this stage all terms $\geqslant 10^{-10}$ shall be retained. Consequently, it is necessary to expand

$$\frac{1}{r_{12}} \quad \text{through} \quad \frac{10^{-10}}{0.62} = 1.6 \times 10^{-10}$$

$$\frac{1}{r_{13}} \quad \text{through} \quad \frac{10^{-10}}{0.54 \times 10^3} = 1.9 \times 10^{-13}$$

$$\frac{1}{r_{03}^3} \quad \text{through} \quad \frac{10^{-10}}{1.1 \times 10^{-2}} = 0.91 \times 10^{-8}.$$

The final results of the three expansions are:

$$\frac{1}{r_{12}} = \frac{1}{r_2} + \frac{r_1}{r_2^2}\cos S_{12} + \frac{3}{2}\frac{r_1^2}{r_2^3}\cos^2 S_{12} - \frac{1}{2}\frac{r_1^2}{r_2^3} + \frac{5}{2}\frac{r_1^3}{r_2^4}\cos^3 S_{12} - \frac{3}{2}\frac{r_1^3}{r_2^4}\cos S_{12} + \frac{35}{8}\frac{r_1^4}{r_2^5}\cos^4 S_{12} - \frac{15}{4}\frac{r_1^4}{r_2^5}\cos^2 S_{12} + \frac{3}{8}\frac{r_1^4}{r_2^5}$$

$$\begin{aligned}\frac{1}{r_{13}} = {} & \frac{1}{r_3'} - M\frac{r_2}{r_3'^2}\cos S_{23}' + \frac{r_1}{r_3'^2}\cos S_{13}' + \frac{3}{2}M^2\frac{r_2^2}{r_3'^3}\cos^2 S_{23}' - \frac{1}{2}M^2\frac{r_2^2}{r_3'^3} \\
& - 3M\frac{r_1 r_2}{r_3'^3}\cos S_{13}'\cos S_{23}' + M\frac{r_1 r_2}{r_3'^3}\cos S_{12} - \frac{5}{2}M^3\frac{r_2^3}{r_3'^4}\cos^3 S_{23}' \\
& + \frac{3}{2}M^3\frac{r_2^3}{r_3'^4}\cos S_{23}' + \frac{15}{2}M^2\frac{r_1 r_2^2}{r_3'^4}\cos S_{13}'\cos^2 S_{23}' + \frac{3}{2}\frac{r_1^2}{r_3'^3}\cos^2 S_{13}' \\
& - 3M^2\frac{r_1 r_2^2}{r_3'^4}\cos S_{12}\cos S_{23}' - \frac{3}{2}M^2\frac{r_1 r_2^2}{r_3'^4}\cos S_{13}' - \frac{1}{2}\frac{r_1^2}{r_3'^3} \\
& + \frac{35}{8}M^4\frac{r_2^4}{r_3'^5}\cos^4 S_{23}' - \frac{15}{4}M^4\frac{r_2^4}{r_3'^5}\cos^2 S_{23}' + \frac{3}{8}M^4\frac{r_2^4}{r_3'^5} \\
& - \frac{15}{2}M\frac{r_1^2 r_2}{r_3'^4}\cos^2 S_{13}'\cos S_{23}' - \frac{35}{2}M^3\frac{r_1 r_2^3}{r_3'^5}\cos S_{13}'\cos^3 S_{23}' \\
& + 3M\frac{r_1^2 r_2}{r_3'^4}\cos S_{12}\cos S_{13}' + \frac{15}{2}M^3\frac{r_1 r_2^3}{r_3'^5}\cos S_{12}\cos^2 S_{23}' \\
& + \frac{15}{2}M^3\frac{r_1 r_2^3}{r_3'^5}\cos S_{13}'\cos S_{23}' + \frac{3}{2}M\frac{r_1^2 r_2}{r_3'^4}\cos S_{23}' \\
& + \frac{35}{4}M^5\frac{r_2^5}{r_3'^6}\cos^3 S_{23}' - \frac{63}{8}M^5\frac{r_2^5}{r_3'^6}\cos^5 S_{23}' - \frac{3}{2}M^3\frac{r_1 r_2^3}{r_3'^5}\cos S_{12} \\
& - \frac{15}{8}M^5\frac{r_2^5}{r_3'^6}\cos S_{23}'\end{aligned}$$

$$\begin{aligned}\frac{1}{r_{03}^3} = {} & \frac{1}{r_3'^3} - 3M\frac{r_2}{r_3'^4}\cos S_{23}' + \frac{15}{2}M^2\frac{r_2^2}{r_3'^5}\cos^2 S_{23}' - \frac{3}{2}M^2\frac{r_2^2}{r_3'^5} \\
& - \frac{35}{2}M^3\frac{r_2^3}{r_3'^6}\cos^3 S_{23}' + \frac{15}{2}M^3\frac{r_2^3}{r_3'^6}\cos S_{23}',\end{aligned}$$

where

$$M = \frac{m_2}{m_0 + m_2}.$$

In the above expressions, terms with identical coefficients have purposely been separated to facilitate cancellation at a subsequent step. It may be seen that 11 terms in the last two series do not depend on the coordinates of the satellite. They may be eliminated. Upon substitution of all three series into (14), a considerable amount of cancellation occurs. There remains only

$$F' = k^2\left[\frac{m_0}{r_1} + \frac{1}{2} m_2 \frac{r_1^2}{r_2^3}(3\cos^2 S_{12} - 1) + \frac{1}{2} m_2 \frac{r_1^3}{r_2^4}(5\cos^3 S_{12} - 3\cos S_{12}) + \frac{1}{8} m_2 \frac{r_1^4}{r_2^5}(35\cos^4 S_{12} - 30\cos^2 S_{12} + 3) + \frac{1}{2} m_3 \frac{r_1^2}{r_3'^3}(3\cos^2 S'_{13} - 1) + \frac{3}{2} m_3 M \frac{r_1^2 r_2}{r_3'^4}(-5\cos^2 S'_{13}\cos S'_{23} + 2\cos S_{12}\cos S'_{13} + \cos S'_{23})\right].$$

The numerical values of the coefficients appearing above are easily established with the aid of set (15). They are:

$$\frac{m_0}{r_1} = 1 \qquad m_3 \frac{r_1^2}{r_3'^3} = 2.2 \times 10^{-7}$$

$$m_2 \frac{r_1^2}{r^3} = 3.7 \times 10^{-5} \qquad m_2 \frac{r_1^4}{r^5} = 2.2 \times 10^{-9}$$

$$m_2 \frac{r_1^3}{r_2^4} = 2.8 \times 10^{-7} \qquad m_3 \frac{r_1^2 r_2}{r_3'^4} = 0.57 \times 10^{-9}.$$

It will be seen at a much later stage in the development of the theory that small divisors of order 10^{-2} and 10^{-4} will occur. Hence, in order for the theory to be good to about 10^{-4}, it will be necessary to retain all terms $\geqslant 10^{-8}$ in the force function at this point. Consequently, the force function to be retained is

$$F' = k^2\left[\frac{m_0}{r_1} + \frac{1}{2} m_2 \frac{r_1^2}{r_2^3}(3\cos^2 S_{12} - 1) + \frac{1}{2} m_2 \frac{r_1^3}{r_2^4}(5\cos^3 S_{12} - 3\cos S_{12}) + \frac{1}{2} m_3 \frac{r_1^2}{r_3'^3}(3\cos^2 S'_{13} - 1)\right]. \qquad (16)$$

It can easily be seen that the earth may be treated as a point mass. The leading oblateness term in the earth's potential acting upon the lunar satellite would contribute to (16) an additional member with coefficient

$$m_2 \frac{r_1^2}{r_2^3} J_2 \left(\frac{R_E}{r_2}\right)^2,$$

where R_E is the equatorial radius of the earth. Since $J_2 \approx 1.1 \times 10^{-3}$, the numerical value of this term is about 1.1×10^{-11}.

C. THE OBLATE FORCE FIELD OF THE MOON

In (16), the moon still appears as a point mass. It is the purpose of this section to allow for the principal effects of the moon's non-spherical gravity field.

The potential of a triaxial body, such as the moon, can be written in the form

$$U = \frac{\mu}{r}\left[1 + \sum_{n=1}^{\infty}\sum_{m=0}^{n}\left(\frac{R}{r}\right)^n P_n^m(\sin\beta) \times (c_{nm}\cos m\lambda + s_{nm}\sin m\lambda)\right], \tag{17}$$

where, in the case on hand, $\mu = k^2 m_0$, $r = r_1$, $R = R_{☾}$ is the lunar mean equatorial radius, $P_n^m(\sin\beta)$ is the associated Legendre polynomial, c_{nm} and s_{nm} are numerical coefficients, λ is the selenocentric longitude, and β is the selenocentric latitude.

Suppose the reference frame is chosen such that its origin coincides with the center of mass. Then $c_{1m} = s_{1m} = 0$. Moreover, let the reference frame be oriented so that its axes coincide with the principal axes of the body. Then the part of U with $n = 2$ can be written in terms of the principal moments of inertia A, B, and C, as

$$U_2 = \frac{k^2}{2r^3}\left[\left(C - \frac{A+B}{2}\right)(1 - 3\sin^2\beta) - \tfrac{3}{2}(A - B)\cos^2\beta\cos 2\lambda\right] \tag{18}$$

if

$$\begin{aligned} x_1' &= r_1\cos\beta\cos\lambda \\ y_1' &= r_1\cos\beta\sin\lambda \\ z_1' &= r_1\sin\beta \end{aligned} \tag{19}$$

and

$$r_1^2 = x_1'^2 + y_1'^2 + z_1'^2 .$$

Consider now the part of (17) with $n = 2$ and compare to (18). There follows

$$\begin{aligned} s_{20} &= c_{21} = s_{21} = s_{22} = 0 \\ c_{20} &= \frac{A + B - 2C}{2m_0 R_{☾}^2} \\ c_{22} &= \frac{B - A}{4m_0 R_{☾}^2} . \end{aligned}$$

Taking, with Brouwer (1963),

$$\begin{aligned} \frac{C - B}{m_0 R_{☾}^2} &= 1.497 \times 10^{-4} \\ \frac{C - A}{m_0 R_{☾}^2} &= 2.495 \times 10^{-4} \\ \frac{B - A}{m_0 R_{☾}^2} &= 0.998 \times 10^{-4}, \end{aligned}$$

one obtains

$$\begin{aligned} c_{20} &= -\,2.00 \times 10^{-4} \\ c_{22} &= 0.25 \times 10^{-4} . \end{aligned}$$

There exists, at present, no way of obtaining an estimate of the numerical values of higher order coefficients. There is some reason to believe that a number of these may exceed 10^{-8}. Due to lack of information the present theory will be restricted to the terms with $n=2$. In any event, the leading terms are being included, and the development illustrates the treatment of both zonal and longitudinal harmonics.

With the aid of (17), the lunar part of the force function may then be written as

$$\frac{k^2 m_0}{r_1}\left\{1+\left(\frac{R_{☾}}{r_1}\right)^2\left[\frac{c_{20}}{2}(3\sin^2\beta-1)+3c_{22}(1-\sin^2\beta)\cos 2\lambda\right]\right\}. \tag{20}$$

This can easily be put in terms of the new rectangular coordinates, fixed in the moon. From (19),

$$\begin{aligned} 3\sin^2\beta-1 &= \frac{1}{r_1^2}(3z_1'^2-r_1^2) \\ (1-\sin^2\beta)\cos 2\lambda &= \frac{1}{r_1^2}(x_1'^2-y_1'^2). \end{aligned} \tag{21}$$

Define now

$$\begin{aligned} c_{18} &= 3c_{22}-\frac{c_{20}}{2} \\ c_{19} &= -\,3c_{22}-\frac{c_{20}}{2} \end{aligned} \tag{22}$$

and substitute (21) and (22) into (20). The subscripts 18 and 19 have no significance other than to provide a symmetric form, as seen below. Upon inserting (20) into (16), the augmented force function becomes

$$\begin{aligned} F' = {} & \frac{k^2 m_0}{r_1}\left[1+\frac{R_{☾}^2}{r_1^4}(c_{18}x_1'^2+c_{19}y_1'^2+c_{20}z_1'^2)\right] \\ & +\frac{k^2 m_2}{2}\frac{r_1^2}{r_2^3}(3\cos^2 S_{12}-1)+\frac{k^2 m_2}{2}\frac{r_1^3}{r_2^4}(5\cos^3 S_{12}-3\cos S_{12}) \\ & +\frac{k^2 m_3}{2}\frac{r_1^2}{r_3'^3}(3\cos^2 S_{13}'-1). \end{aligned} \tag{23}$$

D. SELECTION OF THE REFERENCE FRAME

At this point in the development it becomes necessary to choose a suitable reference frame. It seems desirable to select a frame that can be considered inertial for all practical purposes. The plane of the lunar orbit appears undesirable because of the relatively rapid motion of the lunar node. The same is true for the plane defined by the lunar equator; its precessional rate is the same as that of the lunar orbit. The two remaining natural planes are the ecliptic and the earth's equator. The former will be selected or, to be more precise, a plane parallel to the ecliptic containing the moon at all times. The advantages of this plane over that of the earth's equator are the small inclinations of the lunar equator and the apparent earth's orbit with respect to it. Full advantage of these small angles will be taken. The positive x-axis is taken to point toward the vernal equinox at a time when the moon passes through the ecliptic. The z-axis points toward the north pole of the ecliptic, and the y-axis completes a right-handed frame.

The next steps are to express the moon-fixed coordinates x_1', y_1', z_1' in (23) in terms of the new x_1, y_1, z_1, and subsequently the latter in terms of the orbital elements of the satellite. The first of these steps requires the introduction of the moon's physical libration.

E. THE MOON'S PHYSICAL LIBRATION

Appealing to Koziel (1962), Cassini's laws can be expressed geometrically as shown in Figure 2. Taking into account the small deviations from these laws, namely the physical librations, Figure 2 must be replaced by Figure 3. In these diagrams, $\lambda_☾$ is the mean longitude of moon, $\Omega_☾$ is the longitude of ascending node of lunar orbit, $I_☾$ is the mean inclination of lunar equator to ecliptic, τ is the physical libration in longitude, σ is the physical libration in node, and ϱ is the physical libration in inclination.

Since it is customary to view the geometry at the point of the ascending node, this change is made and illustrated in Figure 4. With the aid of this figure it is easily verified that the relations between the equatorial and ecliptic coordinates are given by

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = A \begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

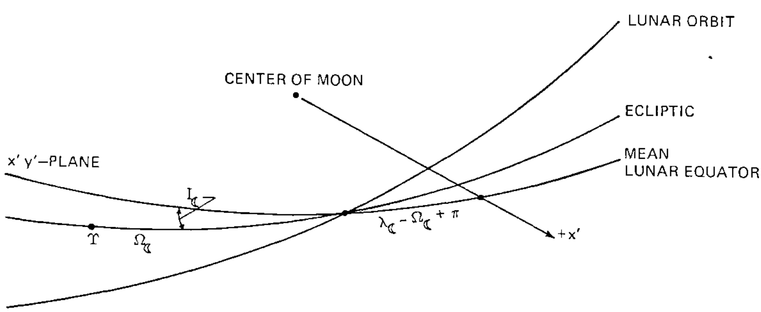

Fig. 2. Geometrical relations between ecliptic, the moon's orbit, equator, and first radius. This figure is consistent with Cassini's laws and, hence, only an approximation.

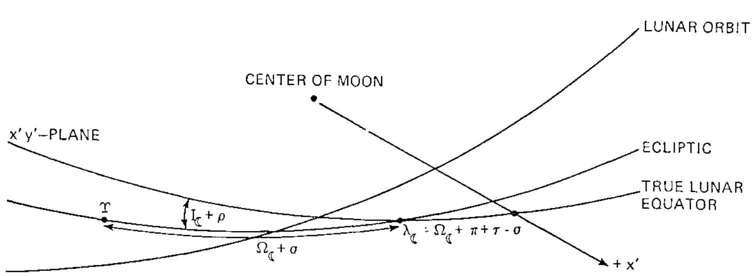

Fig. 3. Figure 2 modified for the effects of physical libration. The relations are now rigorous.

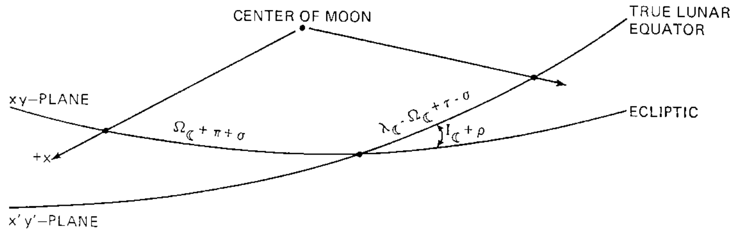

Fig. 4. The relations between the $x'y'$ and xy-planes as seen from the ascending node of the lunar equator on the ecliptic.

where $A =$

$$\begin{pmatrix} \cos\alpha\cos\gamma - \sin\alpha\cos\beta\ \sin\gamma & \sin\alpha\cos\gamma + \cos\alpha\cos\beta\ \sin\gamma & \sin\beta\ \sin\gamma \\ -\cos\alpha\ \sin\gamma - \sin\alpha\cos\beta\cos\gamma & -\sin\alpha\ \sin\gamma + \cos\alpha\cos\beta\cos\gamma & \sin\beta\cos\gamma \\ \sin\alpha\ \sin\beta & -\cos\alpha\ \sin\beta & \cos\beta \end{pmatrix} \tag{24}$$

if

$$\begin{aligned} \Omega_{☾} + \pi + \sigma &= \alpha \\ I_{☾} + \varrho &= \beta \\ \lambda_{☾} + \Omega_{☾} + \tau - \sigma &= \gamma . \end{aligned} \tag{25}$$

Again appealing to Koziel (1962), it is seen that the free libration is negligible. Those terms of the forced libration which will exceed 10^{-8} in the disturbing function are found to be only

$$\begin{aligned} \tau &= \tau_1 \sin l_{\odot} \\ \varrho &= \varrho_1 \cos l_{☾} \\ I_{☾}\sigma &= \sigma_1 \sin l_{☾} . \end{aligned} \tag{26}$$

By choosing $(C-B)/(C-A)=0.60$, as does Brouwer (1963), the above coefficients are, according to Koziel,

$$\begin{aligned} \tau_1 &= +\ 101''.4 = +\ 4.9 \times 10^{-4}\ \text{rad} \\ \varrho_1 &= -\ \ 95''.6 = -\ 4.6 \times 10^{-4}\ \text{rad} \\ \sigma_1 &= -\ \ 97''.5 = -\ 4.7 \times 10^{-4}\ \text{rad} . \end{aligned} \tag{27}$$

Inasmuch as these coefficients are multiplied by parameters such as $c_{20}=0(10^{-4})$, it is seen that their contribution may exceed 10^{-8}.

The relations (25) are now substituted into (24). The trigonometric functions of sums of angles are written as products of trigonometric functions, where appropriate, so that the sin and cos of the small angles $I_{☾}$, τ, ϱ, and $I_{☾}\,\sigma$ may be expanded. Upon expansion, all terms $\leqslant 10^{-4}$ are discarded since they would contribute $< 10^{-8}$ to the disturbing function. This is easily accomplished by using (27) and by noting that

$$I_{☾} = 2.7 \times 10^{-2} .$$

Designating the elements of the rotation matrix (24) temporarily by

$$A = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix},$$

the result of above operations is found to be:

$$\begin{aligned}
A_{11} &= -\cos\lambda_☾ + \tau\sin\lambda_☾ + \tfrac{1}{4}I_☾^2[\cos\lambda_☾ - \cos(\lambda_☾ - 2\Omega_☾)] \\
A_{12} &= -\sin\lambda_☾ - \tau\cos\lambda_☾ + \tfrac{1}{4}I_☾^2[\sin\lambda_☾ + \sin(\lambda_☾ - 2\Omega_☾)] \\
A_{13} &= (I_☾ + \varrho)\sin(\lambda_☾ - \Omega_☾) - I_☾\sigma\cos(\lambda_☾ - \Omega_☾) \\
A_{21} &= \sin\lambda_☾ + \tau\cos\lambda_☾ - \tfrac{1}{4}I_☾^2[\sin\lambda_☾ - \sin(\lambda_☾ - 2\Omega_☾)] \\
A_{22} &= -\cos\lambda_☾ + \tau\sin\lambda_☾ + \tfrac{1}{4}I_☾^2[\cos\lambda_☾ + \cos(\lambda_☾ - 2\Omega_☾)] \\
A_{23} &= (I_☾ + \varrho)\cos(\lambda_☾ - \Omega_☾) + I_☾\sigma\sin(\lambda_☾ - \Omega_☾) \\
A_{31} &= -(I_☾ + \varrho)\sin\Omega_☾ - I_☾\sigma\cos\Omega_☾ \\
A_{32} &= (I_☾ + \varrho)\cos\Omega_☾ - I_☾\sigma\sin\Omega_☾ \\
A_{33} &= 1 - \tfrac{1}{2}I_☾^2 .
\end{aligned}$$

Having A, the coordinates x_1', y_1', z_1' can now be computed and inserted into the second member of (23). In doing so, introduce

$$\varrho_1 + \sigma_1 = \varepsilon_1$$

and note that $\varrho_1 - \sigma_1$ is negligible. Employ also (26). There results, in terms of the ecliptic coordinates,

$$\begin{aligned}
\frac{c_{18}x_1'^2 + c_{19}y_1'^2 + c_{20}z_1'^2}{r_1^2} = \frac{x_1^2}{r_1^2}\,&[c_{20}(-\tfrac{1}{2} + \tfrac{3}{4}I_☾^2) + c_{22}(3 - \tfrac{3}{2}I_☾^2)\cos 2\lambda_☾ \\
&- 3c_{22}\tau_1\cos(2\lambda_☾ - l_\odot) + 3c_{22}\tau_1\cos(2\lambda_☾ + l_\odot) \\
&+ \tfrac{3}{2}c_{22}I_☾^2\cos(2\lambda_☾ - 2\Omega_☾) - \tfrac{3}{4}c_{20}I_☾^2\cos 2\Omega_☾] \\
+ \frac{y_1^2}{r_1^2}\,&[c_{20}(-\tfrac{1}{2} + \tfrac{3}{4}I_☾^2) - c_{22}(3 - \tfrac{3}{2}I_☾^2)\cos 2\lambda_☾ \\
&+ 3c_{22}\tau_1\cos(2\lambda_☾ - l_\odot) - 3c_{22}\tau_1\cos(2\lambda_☾ + l_\odot) \\
&+ \tfrac{3}{2}c_{22}I_☾^2\cos(2\lambda_☾ - 2\Omega_☾) + \tfrac{3}{4}c_{20}I_☾^2\cos 2\Omega_☾] \\
+ \frac{z_1^2}{r_1^2}\,&[c_{20}(1 - \tfrac{3}{2}I_☾^2) - 3c_{22}I_☾^2\cos(2\lambda_☾ - 2\Omega_☾)] \\
+ \frac{x_1y_1}{r_1^2}\,&[c_{22}(6 - 3I_☾^2)\sin 2\lambda_☾ + 6c_{22}\tau_1\sin(2\lambda_☾ + l_\odot) \\
&- 6c_{22}\tau_1\sin(2\lambda_☾ - l_\odot) - \tfrac{3}{2}c_{20}I_☾^2\sin 2\Omega_☾] \\
+ \frac{x_1z_1}{r_1^2}\,&[-6c_{22}I_☾\sin(2\lambda_☾ - \Omega_☾) - \tfrac{3}{2}c_{20}\varepsilon_1\sin(l_☾ + \Omega_☾) \\
&- 3c_{22}\varepsilon_1\sin(\lambda_☾ + \omega_☾) - 3c_{20}I_☾\sin\Omega_☾]
\end{aligned}$$

$$+ \frac{y_1 z_1}{r_1^2} [6c_{22} I_☾ \cos(2\lambda_☾ - \Omega_☾) + \tfrac{3}{2} c_{20} \varepsilon_1 \cos(l_☾ + \Omega_☾) + 3c_{22}\varepsilon_1 \cos(\lambda_☾ + \omega_☾) + 3c_{20} I_☾ \cos \Omega_☾]. \quad (28)$$

F. THE FORCE FUNCTION IN TERMS OF ELLIPTIC ELEMENTS

The force function (23) will now be expressed in terms of the orbital elements of the satellite and the known elements of moon and sun.

The second member of (23), given in an intermediate form by (28), is readily expressed in the satellite's elements with the aid of

$$\bar{r}_1 = r_1 B \begin{pmatrix} \cos f \\ \sin f \\ 0 \end{pmatrix},$$

where f is the true anomaly and $B =$

$$\begin{pmatrix} \cos h \cos g - \cos I \sin h \sin g & -\cos h \sin g - \cos I \sin h \cos g & \sin I \sin h \\ \sin h \cos g + \cos I \cos h \sin g & -\sin h \sin g + \cos I \cos h \cos g & -\sin I \cos h \\ \sin I \sin g & \sin I \cos g & \cos I \end{pmatrix}.$$

The matrix B is given completely although its last column is evidently not needed. Note that the elements without subscripts are those of the satellite. After some algebra, it follows that

$$\begin{aligned}
\frac{c_{18}x_1'^2 + c_{19}y_1'^2 + c_{20}z_1'^2}{r_1^2} &= (-\tfrac{1}{2} + \tfrac{3}{4} I_☾^2)\, c_{20} \\
&+ (\tfrac{3}{4} - \tfrac{9}{8} I_☾^2)\, c_{20} \sin^2 I (1 - \cos 2u) + \tfrac{3}{2} I_☾^2 c_{22} \cos(2\lambda_☾ - 2\Omega_☾) \\
&- \tfrac{9}{4} I_☾^2 c_{22} \sin^2 I (1 - \cos 2u) \cos(2\lambda_☾ - 2\Omega_☾) \\
&+ (\tfrac{3}{2} - \tfrac{3}{4} I_☾^2)\, c_{22} \sin^2 I \cos(2\lambda_☾ - 2h) \\
&+ (\tfrac{3}{2} - \tfrac{3}{4} I_☾^2)\, c_{22} (1 + \cos^2 I) \cos 2u \cos(2\lambda_☾ - 2h) \\
&+ (3 - \tfrac{3}{2} I_☾^2)\, c_{22} \cos I \sin 2u \sin(2\lambda_☾ - 2h) \\
&+ \tfrac{3}{2} \tau_1 c_{22} \sin^2 I [\cos(2\lambda_☾ + l_\odot - 2h) - (2\lambda_☾ - l_\odot - 2h)] \\
&+ \tfrac{3}{2} \tau_1 c_{22} (1 + \cos^2 I) \cos 2u [\cos(2\lambda_☾ + l_\odot - 2h) \\
&- \cos(2\lambda_☾ - l_\odot - 2h)] + 3\tau_1 c_{22} \cos I \sin 2u [\sin(2\lambda_☾ + l_\odot - 2h) \\
&- \sin(2\lambda_☾ - l_\odot - 2h)] - \tfrac{3}{8} I_☾^2 c_{20} \sin^2 I \cos(2\Omega_☾ - 2h) \\
&- \tfrac{3}{4} I_☾^2 c_{20} \cos I \sin 2u \sin(2\Omega_☾ - 2h) \\
&- \tfrac{3}{8} I_☾^2 c_{20} (1 + \cos^2 I) \cos 2u \cos(2\Omega_☾ - 2h) + I_☾ \sin 2I (1 - \cos 2u) \\
&\times [\tfrac{3}{2} c_{22} \cos(2\lambda_☾ - \Omega_☾ - h) + \tfrac{3}{4} c_{20} \cos(\Omega_☾ - h)] - I_☾ \sin I \sin 2u \\
&\times [3 c_{22} \sin(2\lambda_☾ - \Omega_☾ - h) + \tfrac{3}{2} c_{20} \sin(\Omega_☾ - h)] \\
&+ \varepsilon_1 \sin 2I (1 - \cos 2u) [\tfrac{3}{4} c_{22} \cos(\lambda_☾ + \omega_☾ - h) \\
&+ \tfrac{3}{8} c_{20} \cos(\lambda_☾ - \omega_☾ - h)] - \varepsilon_1 \sin I \sin 2u [\tfrac{3}{2} c_{22} \sin(\lambda_☾ + \omega_☾ - h) \\
&+ \tfrac{3}{4} c_{20} \sin(\lambda_☾ - \omega_☾ - h)]. \qquad (29)
\end{aligned}$$

Here as in many places in the subsequent development it will be found convenient to retain products of trigonometric functions rather than converting them into sums and differences. Note also that the argument of latitude

$$u = g + f$$

has been introduced.

Attention is focused now on the functions $\cos S_{12}$ and $\cos S'_{13}$ which appear in the third, fourth, and fifth, members of (23). In order to express these in terms of orbital elements, consider first Figure 5. This diagram as well as the following formulae per-

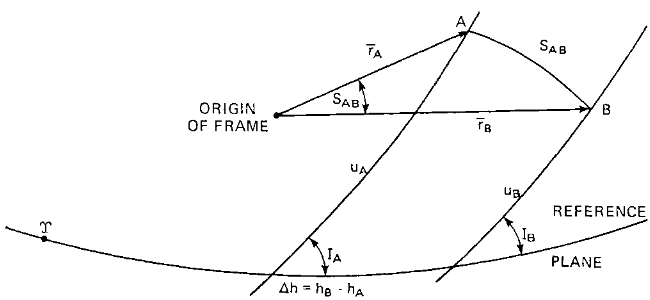

Fig. 5. Geometric relations between the angle S_{AB} and the orbital elements of any two orbits A and B.

mit the computation of the angle S_{AB} for any two orbits A and B as long as they are referred to the same frame. It is readily seen that

$$\hat{r}_A = \begin{pmatrix} \cos u_A \\ \cos I_A \ \sin u_A \\ \sin I_A \ \sin u_A \end{pmatrix}$$

$$\hat{r}_B = \begin{pmatrix} \cos \Delta h \cos u_B - \cos I_B \sin \Delta h \sin u_B \\ \sin \Delta h \cos u_B + \cos I_B \cos \Delta h \sin u_B \\ \sin I_B \sin u_B \end{pmatrix},$$

where $\hat{r}$ is a unit vector. Hence

$$\begin{aligned} \cos S_{AB} = {} & \cos \Delta h \cos u_A \cos u_B - \cos I_B \sin \Delta h \cos u_A \sin u_B \\ & + \cos I_A \sin \Delta h \sin u_A \cos u_B + \cos I_A \cos I_B \cos \Delta h \sin u_A \sin u_B \\ & + \sin I_A \sin I_B \sin u_A \sin u_B . \end{aligned} \tag{30}$$

For the work on hand the angles S_{12} and S'_{13} are needed. It will be shown below that the latter can be replaced by S_{13}. All three angles are depicted in Figure 6.

In the development of $\cos S_{12}$, $\cos^2 S_{12}$, $\cos^3 S_{12}$, and $\cos S'_{13}$, it is again necessary to retain all terms whose contribution to the force function (23) exceeds 10^{-8}. Recall that

$$m_2 \frac{r_1^2}{r_2^3} = 3.7 \times 10^{-5}, \quad m_2 \frac{r_1^3}{r_2^4} = 2.8 \times 10^{-7}, \quad m_3 \frac{r_1^2}{r_3'^3} = 2.2 \times 10^{-7}.$$

Hence it is clear that $\cos^2 S_{12}$ is to be developed through

$$\frac{10^{-8}}{3.7 \times 10^{-5}} = 0.0003,$$

$\cos S_{12}$ and $\cos^3 S_{12}$ are to be developed through

$$\frac{10^{-8}}{2.8 \times 10^{-7}} = 0.04,$$

$\cos^2 S'_{13}$ is to be developed through

$$\frac{10^{-8}}{2.2 \times 10^{-7}} = 0.05.$$

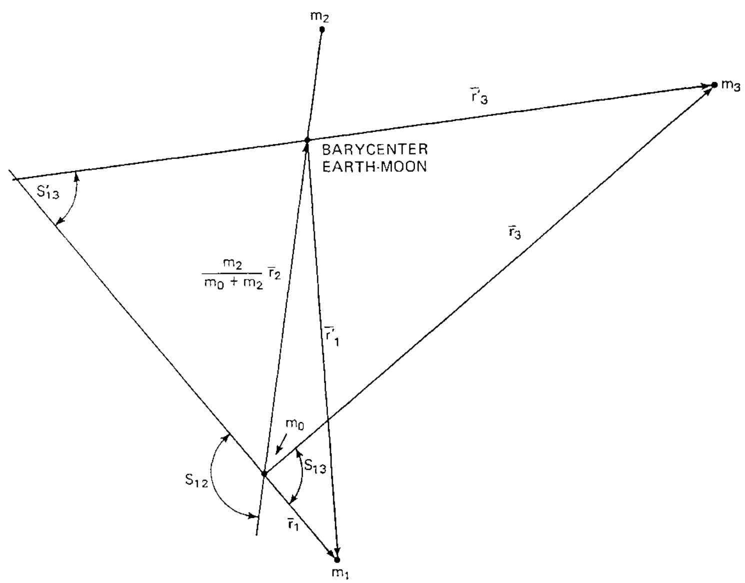

Fig. 6. The angles S_{12}, S_{13}, and S'_{13}.

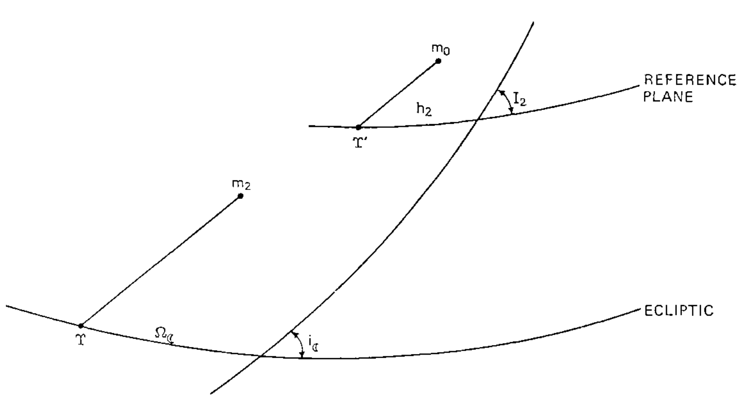

Fig. 7. The relations between h_2 and $\Omega_☾$ as well as I_2 and $i_☾$. Note that Y' was chosen such that $m_0 - Y'$ is parallel to $m_2 - Y$.

Upon applying (30) to $\cos S_{12}$, there will appear h_2, I_2, and u_2. Aided by Figure 7, it follows that

$$h_2 = \Omega_☾, \quad I_2 = i_☾, \quad u_2 = u_☾ + \pi,$$

where $i_☾$ is the mean inclination of the lunar orbit on the ecliptic. It will be found convenient to introduce

$$\gamma_☾ = \sin i_☾$$

and

$$\psi = f + g + h.$$

In truncating the various expressions, note that

$$\gamma_☾ = 0.089.$$

Turning now to $\cos S'_{13}$, Figure 6 will verify that

$$\begin{aligned}\cos S'_{13} &= \hat{r}_1 \cdot \hat{r}'_3 \\ &= \hat{r}_1 \cdot \left(\frac{\bar{r}_3}{r'_3} - \frac{m_2}{m_0 + m_2} \frac{\bar{r}_2}{r'_3} \right).\end{aligned}$$

The second term is negligible, as is easily demonstrated. Hence

$$\cos S'_{13} = \frac{r_3}{r'_3} \cos S_{13}.$$

But, within the required tolerances, $r_3/r'_3 = 1$. Therefore

$$\cos S'_{13} = \cos S_{13}.$$

Simple calculations will also verify that the sun deviates less than 1′ from the reference plane, and that ψ_3 differs from $\psi_\odot$ by less than 0.0025. Hence, within the required accuracy,

$$I_3 = 0$$

and

$$\psi_3 = \psi_\odot.$$

With these preparations, the results can now be stated:

$$\begin{aligned}\cos^2 S_{12} = {} & \tfrac{1}{4}(1 + \cos^2 I) + \tfrac{1}{4} \sin^2 I \cos 2u + \tfrac{1}{4} [\sin^2 I \\ & + (1 + \cos^2 I) \cos 2u] \cos(2\psi_☾ - 2h) + \tfrac{1}{2} \cos I \sin 2u \sin(2\psi_☾ - 2h) \\ & + \tfrac{1}{2}\gamma_☾ \sin I \sin 2u [\sin(2\psi_☾ - \Omega_☾ - h) - \sin(\Omega_☾ - h)] \\ & - \tfrac{1}{4}\gamma_☾ \sin 2I (1 - \cos 2u) [\cos(2\psi_☾ - \Omega_☾ - h) - \cos(\Omega_☾ - h)] \\ & + \tfrac{1}{8}\gamma_☾^2 [(1 - 3\cos^2 I) - 3 \sin^2 I \cos 2u] [1 - \cos(2\psi_☾ - 2\Omega_☾)] \\ & - \tfrac{1}{8}\gamma_☾^2 [\sin^2 I + (1 + \cos^2 I) \cos 2u] [\cos(2\psi_☾ - 2h) \\ & - \cos(2\Omega_☾ - 2h)] - \tfrac{1}{4}\gamma_☾^2 \cos I \sin 2u [\sin(2\psi_☾ - 2h) \\ & - \sin(2\Omega_☾ - 2h)] + \tfrac{1}{4}\gamma_☾^3 \sin I \sin 2u [\sin(\Omega_☾ - h)\end{aligned}$$

$$- \tfrac{1}{2}\sin(2\psi_{☾} - \Omega_{☾} - h) + \tfrac{1}{2}\sin(2\psi_{☾} - 3\Omega_{☾} + h)]$$
$$- \tfrac{1}{8}\gamma_{☾}^3 \sin 2I(1 - \cos 2u)\,[\cos(\Omega_{☾} - h)$$
$$- \tfrac{1}{2}\cos(2\psi_{☾} - \Omega_{☾} - h) - \tfrac{1}{2}\cos(2\psi_{☾} - 3\Omega_{☾} + h)]\,. \tag{31}$$

$$\cos S_{12} = -\cos u\cos(\psi_{☾} - h) - \cos I\sin u\sin(\psi_{☾} - h) - \gamma_{☾}\sin I\sin u\sin(\psi_{☾} - \Omega_{☾}) \tag{32}$$

$$\begin{aligned}\cos^3 S_{12} = &- \tfrac{3}{16}[(4 - \sin^2 I)\cos u + \sin^2 I\cos 3u]\cos(\psi_{☾} - h)\\ &- \tfrac{1}{16}[3\sin^2 I\cos u + (4 - 3\sin^2 I)\cos 3u]\cos(3\psi_{☾} - 3h)\\ &- \tfrac{3}{16}\cos I[(4 - 3\sin^2 I)\sin u + \sin^2 I\sin 3u]\sin(\psi_{☾} - h)\\ &- \tfrac{1}{16}\cos I[3\sin^2 I\sin u + (4 - \sin^2 I)\sin 3u]\sin(3\psi_{☾} - 3h)\\ &- \tfrac{3}{8}\gamma_{☾}\sin I[(1 + 3\cos^2 I)\sin u + \sin^2 I\sin 3u]\sin(\psi_{☾} - \Omega_{☾})\\ &+ \tfrac{3}{16}\gamma_{☾}\sin 2I(\cos u - \cos 3u)\,[\cos(3\psi_{☾} - \Omega_{☾} - 2h)\\ &- \cos(\psi_{☾} + \Omega_{☾} - 2h)] - \tfrac{3}{16}\gamma_{☾}\sin I[(1 - 3\cos^2 I)\sin u\\ &+ (1 + \cos^2 I)\sin 3u]\,[\sin(3\psi_{☾} - \Omega_{☾} - 2h) - \sin(\psi_{☾} + \Omega_{☾} - 2h)]\end{aligned} \tag{33}$$

$$\cos^2 S'_{13} = \tfrac{1}{4}(1 + \cos^2 I) + \tfrac{1}{4}\sin^2 I\cos 2u + \tfrac{1}{4}[\sin^2 I + (1 + \cos^2 I) \times \cos 2u]\cos(2\psi_{\odot} - 2h) + \tfrac{1}{2}\cos I\sin 2u\sin(2\psi_{\odot} - 2h)\,. \tag{34}$$

In the final step of the development of the force function, the radii vectors of the moon and sun, as well as their true longitudes, must be expressed in terms of their semi-major axes, eccentricities, and mean longitudes so that the latter can immediately be written as functions of the time. This is accomplished with the aid of the equation of the ellipse and the equation of the center. In doing so, note that

$$r_2 = r_{☾}, \quad r'_3 = r_{\odot}\,.$$

Since

$$e_{☾} = 0.055, \quad e_{\odot} = 0.017\,,$$

it will be seen that $e_{☾}^2$ must be included in certain parts of the third member of (23). There will also appear terms in $e_{☾}\gamma_{☾}$ and $e_{☾}\gamma_{☾}^2$. For the fourth member of (23), the desired accuracy requires inclusion of terms in $e_{☾}$. In the fifth member, finally, the sun's eccentricity is entirely negligible.

These modifications are made in (23) and (31)–(34). Then the formulae (29) and (31)–(34) are substituted into (23). Since the lunar and solar elements are all clearly identified, the subscript 1, designating the satellite, is no longer needed. After some algebra, the force function takes on the following form:

$$\begin{aligned}F' = &\frac{k^2 m_0}{r} + \tfrac{1}{4}c_{20}k^2 m_0\frac{R_{☾}^2}{r^3}\{[(1 - 3\cos^2 I) - 3\sin^2 I\cos 2u]\}\\ &+ 3c_{22}k^2 m_0\frac{R_{☾}^2}{r^3}\{\tfrac{1}{2}[\sin^2 I + (1 + \cos^2 I)\cos 2u]\cos(2\lambda_{☾} - 2h)\\ &+ \cos I\sin 2u\sin(2\lambda_{☾} - 2h)\}\end{aligned}$$

$$
\begin{aligned}
&+ \tfrac{3}{2} I_{☾} c_{20} k^2 m_0 \frac{R_{☾}^2}{r^3} \{\tfrac{1}{2} \sin 2I (1 - \cos 2u) \cos(\Omega_{☾} - h) \\
&\qquad - \sin I \sin 2u \sin(\Omega_{☾} - h)\} \\
&+ 3 I_{☾} c_{22} k^2 m_0 \frac{R_{☾}^2}{r^3} \{\tfrac{1}{2} \sin 2I (1 - \cos 2u) \cos(2\lambda_{☾} - \Omega_{☾} - h) \\
&\qquad - \sin I \sin 2u \sin(2\lambda_{☾} - \Omega_{☾} - h)\} \\
&+ \tfrac{3}{4} \varepsilon_1 c_{20} k^2 m_0 \frac{R_{☾}^2}{r^3} \{\tfrac{1}{2} \sin 2I (1 - \cos 2u) \cos(\lambda_{☾} - \tilde{\omega}_{☾} + \Omega_{☾} - h) \\
&\qquad - \sin I \sin 2u \sin(\lambda_{☾} - \tilde{\omega}_{☾} + \Omega_{☾} - h)\} \\
&+ \tfrac{3}{4} I_{☾}^2 c_{20} k^2 m_0 \frac{R_{☾}^2}{r^3} \{- \tfrac{1}{2} [(1 - 3\cos^2 I) - 3 \sin^2 I \cos 2u] \\
&\qquad - \tfrac{1}{2} [\sin^2 I + (1 + \cos^2 I) \cos 2u] \cos(2\Omega_{☾} - 2h) \\
&\qquad - \cos I \sin 2u \sin(2\Omega_{☾} - 2h)\} \\
&+ \tfrac{3}{2} \varepsilon_1 c_{22} k^2 m_0 \frac{R_{☾}^2}{r^3} \{\tfrac{1}{2} \sin 2I (1 - \cos 2u) \cos(\lambda_{☾} + \tilde{\omega}_{☾} - \Omega_{☾} - h) \\
&\qquad - \sin I \sin 2u \sin(\lambda_{☾} + \tilde{\omega}_{☾} - \Omega_{☾} - h)\} \\
&+ \tfrac{3}{2} I_{☾}^2 c_{22} k^2 m_0 \frac{R_{☾}^2}{r^3} \{- \tfrac{1}{2} [(1 - 3\cos^2 I) \\
&\qquad - 3 \sin^2 I \cos 2u] \cos(2\lambda_{☾} - 2\Omega_{☾}) \\
&\qquad - \tfrac{1}{2} [\sin^2 I + (1 + \cos^2 I) \cos 2u] \cos(2\lambda_{☾} - 2h) \\
&\qquad - \cos I \sin 2u \sin(2\lambda_{☾} - 2h)\} \\
&+ 3 \tau_1 c_{22} k^2 m_0 \frac{R_{☾}^2}{r^3} \{\tfrac{1}{2} [\sin^2 I + (1 + \cos^2 I) \cos 2u] \\
&\qquad \times [\cos(2\lambda_{☾} + l_{\odot} - 2h) - \cos(2\lambda_{☾} - l_{\odot} - 2h)] \\
&\qquad + \cos I \sin 2u [\sin(2\lambda_{☾} + l_{\odot} - 2h) - \sin(2\lambda_{☾} - l_{\odot} - 2h)]\} \\
&+ \tfrac{3}{4} k^2 m_2 \frac{r^2}{a_{☾}^3} \{- \tfrac{1}{6} [(1 - 3\cos^2 I) - 3 \sin^2 I \cos 2u] \\
&\qquad + \tfrac{1}{2} [\sin^2 I + (1 + \cos^2 I) \cos 2u] \cos(2\lambda_{☾} - 2h) \\
&\qquad + \cos I \sin 2u \sin(2\lambda_{☾} - 2h)\} \\
&+ \tfrac{3}{8} e_{☾} k^2 m_2 \frac{r^2}{a_{☾}^3} \{- [(1 - 3\cos^2 I) - 3 \sin^2 I \cos 2u] \cos(\lambda_{☾} - \tilde{\omega}_{☾}) \\
&\qquad + \tfrac{1}{2} [\sin^2 I + (1 + \cos^2 I) \cos 2u] \\
&\qquad \times [7 \cos(3\lambda_{☾} - \tilde{\omega}_{☾} - 2h) - \cos(\lambda_{☾} + \tilde{\omega}_{☾} - 2h)] \\
&\qquad + \cos I \sin 2u [7 \sin(3\lambda_{☾} - \tilde{\omega}_{☾} - 2h) - \sin(\lambda_{☾} + \tilde{\omega}_{☾} - 2h)]\} \\
&+ \tfrac{3}{8} e_{☾}^2 k^2 m_2 \frac{r^2}{a_{☾}^3} \{- \tfrac{1}{2} [(1 - 3\cos^2 I) - 3 \sin^2 I \cos 2u] \\
&\qquad \times [1 + 3 \cos(2\lambda_{☾} - 2\tilde{\omega}_{☾})] + \tfrac{1}{2} [\sin^2 I + (1 + \cos^2 I) \cos 2u] \\
&\qquad \times [17 \cos(4\lambda_{☾} - 2\tilde{\omega}_{☾} - 2h) - 5 \cos(2\lambda_{☾} - 2h)] \\
&\qquad + \cos I \sin 2u [17 \sin(4\lambda_{☾} - 2\tilde{\omega}_{☾} - 2h) - 5(2\lambda_{☾} - 2h)]\}
\end{aligned}
$$

$$
\begin{aligned}
&+ \tfrac{3}{4}\gamma_{\text{☾}} k^2 m_2 \frac{r^2}{a_{\text{☾}}^3} \{-\tfrac{1}{2}\sin 2I(1-\cos 2u) \\
&\qquad \times [\cos(2\lambda_{\text{☾}} - \Omega_{\text{☾}} - h) - \cos(\Omega_{\text{☾}} - h)] \\
&\qquad + \sin I \sin 2u [\sin(2\lambda_{\text{☾}} - \Omega_{\text{☾}} - h) - \sin(\Omega_{\text{☾}} - h)]\} \\
&+ \tfrac{3}{8} e_{\text{☾}}\gamma_{\text{☾}} k^2 m_2 \frac{r^2}{a_{\text{☾}}^3} \{-\tfrac{1}{2}\sin 2I(1-\cos 2u)[7\cos(3\lambda_{\text{☾}} - \tilde{\omega}_{\text{☾}} - \Omega_{\text{☾}} - h) \\
&\qquad - \cos(\lambda_{\text{☾}} + \tilde{\omega}_{\text{☾}} - \Omega_{\text{☾}} - h) - 3\cos(\lambda_{\text{☾}} - \tilde{\omega}_{\text{☾}} + \Omega_{\text{☾}} - h) \\
&\qquad - 3\cos(\lambda_{\text{☾}} - \tilde{\omega}_{\text{☾}} - \Omega_{\text{☾}} + h)] \\
&\qquad + \sin I \sin 2u [7\sin(3\lambda_{\text{☾}} - \tilde{\omega}_{\text{☾}} - \Omega_{\text{☾}} - h) \\
&\qquad - \sin(\lambda_{\text{☾}} + \tilde{\omega}_{\text{☾}} - \Omega_{\text{☾}} - h) - 3\sin(\lambda_{\text{☾}} - \tilde{\omega}_{\text{☾}} + \Omega_{\text{☾}} - h) \\
&\qquad + 3\sin(\lambda_{\text{☾}} - \tilde{\omega}_{\text{☾}} - \Omega_{\text{☾}} + h)]\} \\
&+ \tfrac{3}{8}\gamma_{\text{☾}}^2 k^2 m_2 \frac{r^2}{a_{\text{☾}}^3} \{\tfrac{1}{2}[(1 - 3\cos^2 I) - 3\sin^2 I \cos 2u] \\
&\qquad \times [1 - \cos(2\lambda_{\text{☾}} - 2\Omega_{\text{☾}})] \\
&\qquad - \tfrac{1}{2}[\sin^2 I + (1 + \cos^2 I)\cos 2u] \\
&\qquad \times [\cos(2\lambda_{\text{☾}} - 2h) - \cos(2\Omega_{\text{☾}} - 2h)] \\
&\qquad - \cos I \sin 2u [\sin(2\lambda_{\text{☾}} - 2h) - \sin(2\Omega_{\text{☾}} - 2h)]\} \\
&+ \tfrac{3}{16}\gamma_{\text{☾}}^3 k^2 m_2 \frac{r^2}{a_{\text{☾}}^3} \{-\tfrac{1}{2}\sin 2I(1-\cos 2u)[2\cos(\Omega_{\text{☾}} - h) \\
&\qquad - \cos(2\lambda_{\text{☾}} - \Omega_{\text{☾}} - h) - \cos(2\lambda_{\text{☾}} - 3\Omega_{\text{☾}} + h)] \\
&\qquad + \sin I \sin 2u [2\sin(\Omega_{\text{☾}} - h) \\
&\qquad - \sin(2\lambda_{\text{☾}} - \Omega_{\text{☾}} - h) + \sin(2\lambda_{\text{☾}} - 3\Omega_{\text{☾}} + h)]\} \\
&+ \tfrac{3}{16} e_{\text{☾}}\gamma_{\text{☾}}^2 k^2 m_2 \frac{r^2}{a_{\text{☾}}^3} \{\tfrac{1}{2}[(1 - 3\cos^2 I) - 3\sin^2 I \cos 2u] \\
&\qquad \times [6\cos(\lambda_{\text{☾}} - \tilde{\omega}_{\text{☾}}) + \cos(\lambda_{\text{☾}} + \tilde{\omega}_{\text{☾}} - 2\Omega_{\text{☾}}) \\
&\qquad - 7\cos(3\lambda_{\text{☾}} - \tilde{\omega}_{\text{☾}} - 2\Omega_{\text{☾}})] - \tfrac{1}{2}[\sin^2 I + (1 + \cos^2 I)\cos 2u] \\
&\qquad \times [7\cos(3\lambda_{\text{☾}} - \tilde{\omega}_{\text{☾}} - 2h) - \cos(\lambda_{\text{☾}} + \tilde{\omega}_{\text{☾}} - 2h) \\
&\qquad - 3\cos(\lambda_{\text{☾}} - \tilde{\omega}_{\text{☾}} + 2\Omega_{\text{☾}} - 2h) - 3\cos(\lambda_{\text{☾}} - \tilde{\omega}_{\text{☾}} - 2\Omega_{\text{☾}} + 2h)] \\
&\qquad - \cos I \sin 2u [7\sin(3\lambda_{\text{☾}} - \tilde{\omega}_{\text{☾}} - 2h) - \sin(\lambda_{\text{☾}} + \tilde{\omega}_{\text{☾}} - 2h) \\
&\qquad - 3\sin(\lambda_{\text{☾}} - \tilde{\omega}_{\text{☾}} + 2\Omega_{\text{☾}} - 2h) + 3\sin(\lambda_{\text{☾}} - \tilde{\omega}_{\text{☾}} - 2\Omega_{\text{☾}} + 2h)]\} \\
&- \tfrac{1}{32} k^2 m_2 \frac{r^3}{a_{\text{☾}}^4} \{3[(4 - 5\sin^2 I)\cos u + 5\sin^2 I \cos 3u]\cos(\lambda_{\text{☾}} - h) \\
&\qquad + 5[3\sin^2 I \cos u + (4 - 3\sin^2 I)\cos 3u]\cos(3\lambda_{\text{☾}} - 3h) \\
&\qquad + 3\cos I[(4 - 15\sin^2 I)\sin u + 5\sin^2 I \sin 3u]\sin(\lambda_{\text{☾}} - h) \\
&\qquad + 5\cos I[3\sin^2 I \sin u + (4 - \sin^2 I)\sin 3u]\sin(3\lambda_{\text{☾}} - 3h)\} \\
&\tfrac{3}{16}\gamma_{\text{☾}} k^2 m_2 \frac{r^3}{a_{\text{☾}}^4} \{\sin I[(3 - 15\cos^2 I)\sin u \\
&\qquad - 5\sin^2 I \sin 3u]\sin(\lambda_{\text{☾}} - \Omega_{\text{☾}}) + \tfrac{5}{2}\sin 2I(\cos u - \cos 3u)
\end{aligned}
$$

$$
\begin{aligned}
&\times [\cos(3\lambda_{\mathbb{C}} - \Omega_{\mathbb{C}} - 2h) - \cos(\lambda_{\mathbb{C}} + \Omega_{\mathbb{C}} - 2h)] \\
&\quad + \tfrac{5}{2} \sin I [(1 - 3\cos^2 I)\sin u + (1 + \cos^2 I)\sin 3u] \\
&\quad \times [\sin(\lambda_{\mathbb{C}} + \Omega_{\mathbb{C}} - 2h) - \sin(3\lambda_{\mathbb{C}} - \Omega_{\mathbb{C}} - 2h)]\} \\
&- \tfrac{1}{32} e_{\mathbb{C}} k^2 m_2 \frac{r^3}{a^4} \{3[(4 - 5\sin^2 I)\cos u + 5\sin^2 I \cos 3u] \\
&\qquad \times [3\cos(2\lambda_{\mathbb{C}} - \tilde{\omega}_{\mathbb{C}} - h) + \cos(\tilde{\omega}_{\mathbb{C}} - h)] \\
&\qquad + 5[3\sin^2 I \cos u + (4 - 3\sin^2 I)\cos 3u] \\
&\qquad \times [5\cos(4\lambda_{\mathbb{C}} - \tilde{\omega}_{\mathbb{C}} - 3h) - \cos(2\lambda_{\mathbb{C}} + \tilde{\omega}_{\mathbb{C}} - 3h)] \\
&\qquad + 3\cos I[(4 - 15\sin^2 I)\sin u + 5\sin^2 I \sin 3u] \\
&\qquad \times [3\sin(2\lambda_{\mathbb{C}} - \omega_{\mathbb{C}} - h) + \sin(\tilde{\omega}_{\mathbb{C}} - h)] \\
&\qquad + 5\cos I[3\sin^2 I \sin u + (4 - \sin^2 I)\sin 3u] \\
&\qquad \times [5\sin(4\lambda_{\mathbb{C}} - \tilde{\omega}_{\mathbb{C}} - 3h) - \sin(2\lambda_{\mathbb{C}} + \tilde{\omega}_{\mathbb{C}} - 3h)]\} \\
&+ \tfrac{1}{4} k^2 m_3 \frac{r^2}{a_{\odot}^3} \{-\tfrac{1}{2}[(1 - 3\cos^2 I) - 3\sin^2 I \cos 2u] \\
&\quad + \tfrac{3}{2}[\sin^2 I + (1 + \cos^2 I)\cos 2u]\cos(2\lambda_{\odot} - 2h) \\
&\quad + 3\cos I \sin 2u \sin(2\lambda_{\odot} - 2h)\}\,. \qquad (35)
\end{aligned}
$$

Orders of magnitude are still to be assigned to all the members of F'. In order to examine the approximate numerical values of the coefficients, it may be well to compile the following table:

$$
\begin{aligned}
&\left(\frac{R_{\mathbb{C}}}{r}\right)_{\text{MAX}} = 1 && \frac{m_2}{m_0} = 0.81 \times 10^2 \\
&\left(\frac{r^3}{a_{\mathbb{C}}^3}\right)_{\text{MAX}} = 0.49 \times 10^{-6} && \frac{m_3}{m_0} = 2.7 \times 10^7 \\
&\left(\frac{r^4}{a_{\mathbb{C}}^4}\right)_{\text{MAX}} = 0.39 \times 10^{-8} && c_{20} = 2.0 \times 10^{-4} \\
&\left(\frac{r^3}{a_{\odot}^3}\right)_{\text{MAX}} = 0.8 \times 10^{-14} && c_{22} = 0.25 \times 10^{-4} \\
&\varepsilon_1 = 9.3 \times 10^{-4} && \tau_1 = 4.9 \times 10^{-4} \\
&I_{\mathbb{C}} = 2.7 \times 10^{-2} \quad \gamma_{\mathbb{C}} = 8.9 \times 10^{-2} && e_{\mathbb{C}} = 5.5 \times 10^{-2}.
\end{aligned}
$$

With these figures in hand, the numerical values of the coefficients in (35) can be obtained by dividing them by the leading term, namely $k^2 m_0/r$. This is done without regard to the various numerical factors, such as 3, 3/2, 3/4, etc., which appear in (35). There result 21 different quotients. In the same order as in (35), they are:

$$
\begin{aligned}
&Q_1 = 2.0 \times 10^{-4} && Q_8 = 1.8 \times 10^{-8} && Q_{15} = 3.2 \times 10^{-7} \\
&Q_2 = 2.5 \times 10^{-5} && Q_9 = 1.2 \times 10^{-8} && Q_{16} = 2.8 \times 10^{-8} \\
&Q_3 = 0.54 \times 10^{-5} && Q_{10} = 4.0 \times 10^{-5} && Q_{17} = 1.7 \times 10^{-8} \\
&Q_4 = 0.68 \times 10^{-6} && Q_{11} = 2.2 \times 10^{-6} && Q_{18} = 3.2 \times 10^{-7} \qquad (36) \\
&Q_5 = 1.9 \times 10^{-7} && Q_{12} = 1.2 \times 10^{-7} && Q_{19} = 2.8 \times 10^{-8} \\
&Q_6 = 1.5 \times 10^{-7} && Q_{13} = 3.6 \times 10^{-6} && Q_{20} = 1.8 \times 10^{-8} \\
&Q_7 = 2.3 \times 10^{-8} && Q_{14} = 2.0 \times 10^{-7} && Q_{21} = 2.2 \times 10^{-7}
\end{aligned}
$$

A careful check will reveal that the magnitudes of all oblateness terms were computed for a satellite orbit with $r = R_☾$, while those due to earth and sun are based on an orbit with $r = 3000$ km. The latter number was employed in earlier parts of this work.

By hindsight, the bounds on the various orders of magnitude are established by requiring that the range of small quantities of the first order extends from 1.8×10^{-2} to 5.0×10^{-3}. By taking the second, third, and fourth powers of these numbers, the limits of the higher orders of magnitude are easily found. They are listed in Table I. In the adjacent column are given the smallest and largest term which will appear in each order, according to (36).

TABLE I

Order	Range of order	Terms present in F'
1	from 1.8×10^{-2}	–
	to 5.0×10^{-3}	–
2	from 3.2×10^{-4}	from 2.0×10^{-4}
	to 2.5×10^{-5}	to 2.5×10^{-5}
3	from 5.8×10^{-6}	from 5.4×10^{-6}
	to 1.2×10^{-7}	to 1.2×10^{-7}
4	from 1.0×10^{-7}	from 2.8×10^{-8}
	to 6.2×10^{-10}	to 1.2×10^{-8}

There is only a small degree of arbitrariness in establishing these bounds. First, it is required that the small divisor $n_☾/n$ be of order one. For an orbit of $r = 3000$ km, $n_☾/n = 0.62 \times 10^{-2}$. Since c_{22} is of the order of $(n_☾/n)^2$, it should be a quantity of order two. For c_{20}, one finds $c_{20} \approx (1/30)(n_☾/n)$, but $c \approx 8c_{22}$. Hence it is also taken to be a second-order quantity. These designations pretty well establish the basic structure of the table. Using the above criteria, all terms in (35) are now classified to be of second, third, or fourth order. Within each order, they will be rearranged in descending order according to the sizes of their coefficients, and they will be listed, separated into F_2, F_3, and F_4, in the next section.

Before giving these functions F_i, there remains one more modification to be made. The quantities

$$\lambda_☾, \tilde{\omega}_☾, \Omega_☾, \lambda_\odot, l_\odot$$

must be expressed as functions of the time. They are all available in the form

$$\alpha = \alpha_0 + \alpha_1 T + \alpha_2 T^2 + \cdots .$$

It is clear that all terms linear in the time must be taken into account. A look at the coefficients of T^2 for these five elements will show that α_2 for $\tilde{\omega}_☾$ is the largest one, namely 1.7×10^{-4}, if T is expressed in centuries. This does take into consideration that $\lambda_☾$ appears with factors up to four. With these numbers available, a simple calculation will verify that the terms quadratic in T will produce contributions of order 10^{-8} only after 75 years or so. They are consequently ignored.

It will be found more convenient to replace the various trigonometric arguments, composed of the above five elements of moon and sun, by the following set:

$$
\begin{array}{rr}
\lambda_{\mathbb{C}} = \phi_1 & 2\lambda_{\mathbb{C}} = \phi_{20} \\
\lambda_{\mathbb{C}} + \tilde{\omega}_{\mathbb{C}} = \phi_2 & 2\lambda_{\mathbb{C}} + \tilde{\omega}_{\mathbb{C}} = \phi_{21} \\
\lambda_{\mathbb{C}} - \tilde{\omega}_{\mathbb{C}} = \phi_3 & 2\lambda_{\mathbb{C}} - \tilde{\omega}_{\mathbb{C}} = \phi_{22} \\
\lambda_{\mathbb{C}} + \Omega_{\mathbb{C}} = \phi_4 & 2\lambda_{\mathbb{C}} - \Omega_{\mathbb{C}} = \phi_{23} \\
\lambda_{\mathbb{C}} - \Omega_{\mathbb{C}} = \phi_5 & 2\lambda_{\mathbb{C}} - 2\tilde{\omega}_{\mathbb{C}} = \phi_{24} \\
\lambda_{\mathbb{C}} + \tilde{\omega}_{\mathbb{C}} - \Omega_{\mathbb{C}} = \phi_6 & 2\lambda_{\mathbb{C}} - 2\Omega_{\mathbb{C}} = \phi_{25} \\
\lambda_{\mathbb{C}} - \tilde{\omega}_{\mathbb{C}} + \Omega_{\mathbb{C}} = \phi_7 & 2\lambda_{\mathbb{C}} - 3\Omega_{\mathbb{C}} = \phi_{26} \\
\lambda_{\mathbb{C}} - \tilde{\omega}_{\mathbb{C}} - \Omega_{\mathbb{C}} = \phi_8 & 2\lambda_{\mathbb{C}} + l_{\odot} = \phi_{27} \\
\lambda_{\mathbb{C}} + \tilde{\omega}_{\mathbb{C}} - 2\Omega_{\mathbb{C}} = \phi_9 & 2\lambda_{\mathbb{C}} - l_{\odot} = \phi_{28} \\
\lambda_{\mathbb{C}} - \tilde{\omega}_{\mathbb{C}} + 2\Omega_{\mathbb{C}} = \phi_{10} & 4\lambda_{\mathbb{C}} - \tilde{\omega}_{\mathbb{C}} = \phi_{40} \\
\lambda_{\mathbb{C}} - \tilde{\omega}_{\mathbb{C}} - 2\Omega_{\mathbb{C}} = \phi_{11} & 4\lambda_{\mathbb{C}} - 2\tilde{\omega}_{\mathbb{C}} = \phi_{41} \\
3\lambda_{\mathbb{C}} = \phi_{30} & \\
3\lambda_{\mathbb{C}} - \tilde{\omega}_{\mathbb{C}} = \phi_{31} & 2\lambda_{\odot} = \phi_{50} \\
3\lambda_{\mathbb{C}} - \Omega_{\mathbb{C}} = \phi_{32} & \tilde{\omega}_{\mathbb{C}} = \phi_{51} \\
3\lambda_{\mathbb{C}} - \tilde{\omega}_{\mathbb{C}} - \Omega_{\mathbb{C}} = \phi_{33} & \Omega_{\mathbb{C}} = \phi_{52} \\
3\lambda_{\mathbb{C}} - \tilde{\omega}_{\mathbb{C}} - 2\Omega_{\mathbb{C}} = \phi_{34} & 2\Omega_{\mathbb{C}} = \phi_{53}
\end{array}
\tag{37}
$$

Note that the subscript is indicative of the frequency of any particular term. Each argument ϕ_i has the form

$$\phi_i = \phi_{i0} + n_i t ,$$

where ϕ_{i0} and its mean motion n_i are known. Most of these mean motions will later appear as divisors.

3. Solution of the Problem by von Zeipel's Method

The major part of the problem is solved analytically by von Zeipel's Method. In a first transformation the mean anomaly of the satellite is eliminated. The second transformation does not eliminate any particular canonical variable but rather all terms which contain the longitude of the moon. There remains a Hamiltonian, containing periodic terms with a minimum period of one half year, whose character is such that the von Zeipel treatment or any other method of successive approximations is not applicable. It is suggested that this part of the problem be solved by numerical integration.

A. EQUATIONS OF MOTION AND THE HAMILTONIAN

The force function (35) is of the form

$$F' = \frac{k^2 m_0}{r} + R ,$$

where R is the disturbing function. R depends explicitly on the time. One way in which this problem may be attacked is to introduce the new canonical variable

$$d = \text{time}.$$

The canonical conjugate to d is D. The validity of this approach is given elegantly and concisely by Brouwer and Clemence (1961). As a consequence, the Hamiltonian of the system becomes

$$F = \frac{\mu_0^2}{2L^2} - D + R = \text{constant}, \tag{38}$$

and the differential equations, in Delaunay variables, are

$$\begin{aligned} \frac{\mathrm{d}L}{\mathrm{d}t} &= \frac{\partial F}{\partial l} & \frac{\mathrm{d}l}{\mathrm{d}t} &= -\frac{\partial F}{\partial L} \\ \frac{\mathrm{d}G}{\mathrm{d}t} &= \frac{\partial F}{\partial g} & \frac{\mathrm{d}g}{\mathrm{d}t} &= -\frac{\partial F}{\partial G} \\ \frac{\mathrm{d}H}{\mathrm{d}t} &= \frac{\partial F}{\partial h} & \frac{\mathrm{d}h}{\mathrm{d}t} &= -\frac{\partial F}{\partial H} \\ \frac{\mathrm{d}D}{\mathrm{d}t} &= \frac{\partial F}{\partial d} & \frac{\mathrm{d}d}{\mathrm{d}t} &= -\frac{\partial F}{\partial D}. \end{aligned} \tag{39}$$

It is probably more customary to put d equal to the mean longitude of the disturbing body, if only one such exists in the problem, so that the additional term in the Hamiltonian is $n'D$ rather than D. Here n' is meant to be the mean motion of the disturbing body. However, since there are two disturbing bodies as well as the rates of three other elements in the problem on hand, the above choice of the additional variable was made for reasons of symmetry.

The Delaunay variables are defined by

$$\begin{aligned} L &= \sqrt{\mu_0 a} & l &= \text{mean anomaly} \\ G &= L\sqrt{1 - e^2} & g &= \text{argument of pericenter} \\ H &= G \cos I & h &= \text{longitude of ascending node}. \end{aligned} \tag{40}$$

Here use is made of the definition

$$\mu_0 = k^2 m_0.$$

Again for reasons of symmetry, let also

$$\mu_2 = k^2 m_2 \quad \text{and} \quad \mu_3 = k^2 m_3.$$

The quantity $\mu_0^2/2L^2$ is clearly of order zero. Throughout the work it is desired that numerical subscripts attached to the various functions F and S are indicative of their order of magnitude. It will be necessary to take D as a quantity of order one so that this may hold. The justification for this assignment will become quite clear in the next section.

With the aid of (35), (37), and (38), the explicit Hamiltonian, separated into orders of magnitude, will be:

$$F_0 = \frac{\mu_0^2}{2L^2} \tag{41}$$

$$F_1 = -D \tag{42}$$

$$\begin{aligned}
F_2 = {} & \tfrac{1}{4}c_{20}\mu_0 \frac{R_☾^2}{r^3}\{(1-3\cos^2 I) - 3\sin^2 I \cos 2u\} \\
& + 3c_{22}\mu_0 \frac{R_☾^2}{r^3}\{\tfrac{1}{2}[\sin^2 I + (1+\cos^2 I)\cos 2u]\cos(\phi_{20}-2h) \\
& \quad + \cos I \sin 2u \sin(\phi_{20}-2h)\} \\
& + \tfrac{3}{4}\mu_2 \frac{r^2}{a_☾^3}\{-\tfrac{1}{6}[(1-3\cos^2 I) - 3\sin^2 I\cos 2u] \\
& \quad + \tfrac{1}{2}[\sin^2 I + (1+\cos^2 I)\cos 2u]\cos(\phi_{20}-2h) \\
& \quad + \cos I \sin 2u \sin(\phi_{20}-2h)\}
\end{aligned} \tag{43}$$

$$\begin{aligned}
F_3 = {} & \tfrac{3}{2}I_☾ c_{20}\mu_0 \frac{R_☾^2}{r^3}\{\tfrac{1}{2}\sin 2I(1-\cos 2u)\cos(\phi_{52}-h) \\
& \quad - \sin I \sin 2u \sin(\phi_{52}-h)\} \\
& + 3I_☾ c_{22}\mu_0 \frac{R_☾^2}{r^3}\{\tfrac{1}{2}\sin 2I(1-\cos 2u)\cos(\phi_{23}-h) \\
& \quad - \sin I \sin 2u \sin(\phi_{23}-h)\} \\
& - \tfrac{3}{4}\gamma_☾\mu_2 \frac{r^2}{a_☾^3}\{\tfrac{1}{2}\sin 2I(1-\cos 2u)[\cos(\phi_{23}-h) - \cos(\phi_{52}-h)] \\
& \quad - \sin I \sin 2u[\sin(\phi_{23}-h) - \sin(\phi_{52}-h)]\} \\
& - \tfrac{3}{8}e_☾\mu_2 \frac{r^2}{a_☾^3}\{[(1-3\cos^2 I) - 3\sin^2 I\cos 2u]\cos\phi_3 \\
& \quad - \tfrac{1}{2}[\sin^2 I + (1+\cos^2 I)\cos 2u] \\
& \quad \times [7\cos(\phi_{31}-2h) - \cos(\phi_2-2h)] \\
& \quad - \cos I \sin 2u[7\sin(\phi_{31}-2h) - \sin(\phi_2-2h)]\} \\
& - \tfrac{1}{32}\mu_2 \frac{r^3}{a_☾^4}\{3[(4-5\sin^2 I)\cos u + 5\sin^2 I\cos 3u]\cos(\phi_1-h) \\
& \quad + 5[3\sin^2 I\cos u + (4-3\sin^2 I)\cos 3u]\cos(\phi_{30}-3h) \\
& \quad + 3\cos I[(4-15\sin^2 I)\sin u + 5\sin^2 I \sin 3u]\sin(\phi_1-h) \\
& \quad + 5\cos I[3\sin^2 I\sin u + (4-\sin^2 I)\sin 3u]\sin(\phi_{30}-3h)\} \\
& + \tfrac{3}{8}\gamma_☾^2\mu_2 \frac{r^2}{a_☾^3}\{\tfrac{1}{2}[(1-3\cos^2 I) - 3\sin^2 I\cos 2u](1-\cos\phi_{25}) \\
& \quad - \tfrac{1}{2}[\sin^2 I + (1+\cos^2 I)\cos 2u] \\
& \quad \times [\cos(\phi_{20}-2h) - \cos(\phi_{53}-2h)] \\
& \quad - \cos I \sin 2u[\sin(\phi_{20}-2h) - \sin(\phi_{53}-2h)]\}
\end{aligned}$$

$$
\begin{aligned}
&- \tfrac{3}{8} e_{\leftmoon} \gamma_{\leftmoon} \mu_2 \frac{r^2}{a_{\leftmoon}^3} \{ \tfrac{1}{2} \sin 2I (1 - \cos 2u) \\
&\qquad \times [7 \cos(\phi_{33} - h) - \cos(\phi_6 - h) - 3 \cos(\phi_7 - h) \\
&\qquad - 3 \cos(\phi_8 + h)] - \sin I \sin 2u \, [7 \sin(\phi_{33} - h) \\
&\qquad - \sin(\phi_6 - h) - 3 \sin(\phi_7 - h) + 3 \sin(\phi_8 + h)]\} \\
&- \tfrac{1}{4} \mu_3 \frac{r^2}{a_{\odot}^3} \{ \tfrac{1}{2} [(1 - 3 \cos^2 I) - 3 \sin^2 I \cos 2u] \\
&\qquad - \tfrac{3}{2} [\sin^2 I + (1 + \cos^2 I) \cos 2u] \cos(\phi_{50} - 2h) \\
&\qquad - 3 \cos I \sin 2u \sin(\phi_{50} - 2h)\} \\
&+ \tfrac{3}{4} \varepsilon_1 c_{20} \mu_0 \frac{R_{\leftmoon}^2}{r^3} \{ \tfrac{1}{2} \sin 2I (1 - \cos 2u) \cos(\phi_7 - h) \\
&\qquad - \sin I \sin 2u \sin(\phi_7 - h)\} \\
&- \tfrac{3}{4} I_{\leftmoon}^2 c_{20} \mu_0 \frac{R_{\leftmoon}^2}{r^3} \{ \tfrac{1}{2} [(1 - 3 \cos^2 I) - 3 \sin^2 I \cos 2u] \\
&\qquad + \tfrac{1}{2} [\sin^2 I + (1 + \cos^2 I) \cos 2u] \cos(\phi_{53} - 2h) \\
&\qquad + \cos I \sin 2u \sin(\phi_{53} - 2h)\} \\
&- \tfrac{3}{8} e_{\leftmoon}^2 \mu_2 \frac{r^2}{a_{\leftmoon}^3} \{ \tfrac{1}{2} [(1 - 3 \cos^2 I) - 3 \sin^2 I \cos 2u] \, (1 + 3 \cos \phi_{24}) \\
&\qquad - \tfrac{1}{2} [\sin^2 I + (1 + \cos^2 I) \cos 2u] \\
&\qquad \times [17 \cos(\phi_{41} - 2h) - 5 \cos(\phi_{20} - 2h)] \\
&\qquad - \cos I \sin 2u \, [17 \sin(\phi_{41} - 2h) - 5 \sin(\phi_{20} - 2h)]\} \qquad (44)
\end{aligned}
$$

$$
\begin{aligned}
F_4 = {} & \tfrac{3}{2} \varepsilon_1 c_{22} \mu_0 \frac{R_{\leftmoon}^2}{r^3} \{ \tfrac{1}{2} \sin 2I (1 - \cos 2u) \cos(\phi_6 - h) \\
&\qquad - \sin I \sin 2u \sin(\phi_6 - h)\} \\
&- \tfrac{3}{2} I_{\leftmoon}^2 c_{22} \mu_0 \frac{R_{\leftmoon}^2}{r^3} \{ \tfrac{1}{2} [(1 - 3 \cos^2 I) - 3 \sin^2 I \cos 2u] \cos \phi_{25} \\
&\qquad + \tfrac{1}{2} [\sin^2 I + (1 + \cos^2 I) \cos 2u] \cos(\phi_{20} - 2h) \\
&\qquad + \cos I \sin 2u \sin(\phi_{20} - 2h)\} \\
&+ 3 \tau_1 c_{22} \mu_0 \frac{R_{\leftmoon}^2}{r^3} \{ \tfrac{1}{2} [\sin^2 I + (1 + \cos^2 I) \cos 2u] \\
&\qquad \times [\cos(\phi_{27} - 2h) - \cos(\phi_{28} - 2h)] \\
&\qquad + \cos I \sin 2u \, [\sin(\phi_{27} - 2h) - \sin(\phi_{28} - 2h)]\} \\
&- \tfrac{3}{16} \gamma_{\leftmoon}^3 \mu_2 \frac{r^2}{a_{\leftmoon}^3} \{ \tfrac{1}{2} \sin 2I (1 - \cos 2u) \\
&\qquad \times [2 \cos(\phi_{52} - h) - \cos(\phi_{23} - h) - \cos(\phi_{26} + h)] \\
&\qquad - \sin I \sin 2u \, [2 \sin(\phi_{52} - h) \\
&\qquad - \sin(\phi_{23} - h) + \sin(\phi_{26} + h)]\}
\end{aligned}
$$

$$+ \tfrac{3}{16} e_{☾} \gamma_{☾}^2 \mu_2 \frac{r^2}{a_{☾}^3} \{\tfrac{1}{2}[(1 - 3\cos^2 I) - 3\sin^2 I \cos 2u]$$
$$\times (6\cos\phi_3 + \cos\phi_9 - 7\cos\phi_{34})$$
$$- \tfrac{1}{2}[\sin^2 I + (1 + \cos^2 I)\cos 2u]$$
$$\times [7\cos(\phi_{31} - 2h) - \cos(\phi_2 - 2h)$$
$$- 3\cos(\phi_{10} - 2h) - 3\cos(\phi_{11} + 2h)]$$
$$- \cos I \sin 2u [7\sin(\phi_{31} - 2h) - \sin(\phi_2 - 2h)$$
$$- 3\sin(\phi_{10} - 2h) + 3\sin(\phi_{11} + 2h)]\}$$
$$+ \tfrac{3}{16} \gamma_{☾} \mu_2 \frac{r^3}{a_{☾}^4} \{\sin I [(3 - 15\cos^2 I)\sin u - 5\sin^2 I \sin 3u] \sin\phi_5$$
$$+ \tfrac{5}{2}\sin 2I (\cos u - \cos 3u)[\cos(\phi_{32} - 2h) - \cos(\phi_4 - 2h)]$$
$$- \tfrac{5}{2}\sin I [(1 - 3\cos^2 I)\sin u + (1 + \cos^2 I)\sin 3u]$$
$$\times [\sin(\phi_{32} - 2h) - \sin(\phi_4 - 2h)]\}$$
$$- \tfrac{1}{32} e_{☾} \mu_2 \frac{r^3}{a_{☾}^4} \{3[(4 - 5\sin^2 I)\cos u + 5\sin^2 I \cos 3u]$$
$$\times [3\cos(\phi_{22} - h) + \cos(\phi_{51} - h)]$$
$$+ 5[3\sin^2 I \cos u + (4 - 3\sin^2 I)\cos 3u]$$
$$\times [5\cos(\phi_{40} - 3h) - \cos(\phi_{21} - 3h)]$$
$$+ 3\cos I [(4 - 15\sin^2 I)\sin u + 5\sin^2 I \sin 3u]$$
$$\times [3\sin(\phi_{22} - h) + \sin(\phi_{51} - h)]$$
$$+ 5\cos I [3\sin^2 I \sin u + (4 - \sin^2 I)\sin 3u]$$
$$\times [5\sin(\phi_{40} - 3h) - \sin(\phi_{21} - 3h)]\}. \tag{45}$$

B. PERTURBATIONS OF SHORT PERIOD

In this section the determining function S_2 will be developed from which the short-periodic perturbation can be obtained.

It is desired to make a transformation to new variables during which the Hamiltonian F is changed into a new Hamiltonian F^* that no longer depends on the fast variable l. To state this more carefully,

$$F(L, G, H, D, l, g, h, d) = F^*(L', G', H', D', -, g', h', d'). \tag{46}$$

The old unprimed variables and the new primed ones are to be related through the transformation equations

$$\begin{aligned} L &= \frac{\partial S}{\partial l} & l' &= \frac{\partial S}{\partial L'} \\ G &= \frac{\partial S}{\partial g} & g' &= \frac{\partial S}{\partial G'} \\ H &= \frac{\partial S}{\partial h} & h' &= \frac{\partial S}{\partial H'} \\ D &= \frac{\partial S}{\partial d} & d' &= \frac{\partial S}{\partial D'}. \end{aligned} \tag{47}$$

The determining function S must clearly depend partly on the old and partly on the new variables, that is,

$$S = S(L', G', H', D', l, g, h, d).$$

Since the solution is to be found by successive approximations, it is required to separate S into its various orders of magnitude, namely

$$S = S_0 + S_1 + S_2 + \cdots. \tag{48}$$

If S_0 is taken as

$$S_0 = L'l + G'g + H'h + D'd, \tag{49}$$

then, with (47) and (48),

$$L = L' + \frac{\partial S_1}{\partial l} + \frac{\partial S_2}{\partial l} + \cdots,$$

and similar equations for the other variables.

Suppose now the problem is solved by restricting F to

$$F = F_0 + F_1.$$

Then Equations (39) are integrable immediately and S reduces to S_0, given by (49), which yields the identity transformation. Hence $S_1 = 0$. For this argument see also Hori (1963).

Moreover, the theory is restricted to periodic terms of order two. Hence S_3 and higher order parts of S are not needed. Consequently S reduces to

$$S = S_0 + S_2. \tag{50}$$

Therefore the transformation equations become

$$\begin{aligned} L &= L' + \frac{\partial S_2}{\partial l} & l' &= l + \frac{\partial S_2}{\partial L'} \\ G &= G' + \frac{\partial S_2}{\partial g} & g' &= g + \frac{\partial S_2}{\partial G'} \\ H &= H' + \frac{\partial S_2}{\partial h} & h' &= h + \frac{\partial S_2}{\partial H'} \\ D &= D' + \frac{\partial S_2}{\partial d} & d' &= d. \end{aligned} \tag{51}$$

The last of these relations will be seen to hold since S_2 will not contain D'.

Now separate (46) into its various orders of magnitude. It then reads

$$\begin{aligned} &F_0(L) + F_1(D) + F_2(L, G, H, -, l, g, h, d) + F_3(L, G, H, -, l, g, h, d) \\ &\quad + F_4(L, G, H, -, l, g, h, d) = F_0^*(L') + F_1^*(D') \\ &\quad + F_2^*(L', G', H', -, -, g', h', d') + F_3^*(L', G', H', -, -, g', h', d') \\ &\quad + F_4^*(L', G', H', -, -, g', h', d'). \end{aligned} \tag{52}$$

Use is made of the fact that D appears only in F_1. In preparation of a Taylor expansion about the point

$$L', G', H', D', l, g, h, d,$$

the momenta L, G, H, and D on the left of (52) and the angular variables g', h', and d' on the right of (52) are replaced with the aid of (51). Note that substitution of the identity transformation into F_3 and F_4 suffices since no terms of order higher than four are needed. With these substitutions, (52) becomes

$$\begin{aligned} &F_0\left(L' + \frac{\partial S_2}{\partial l}\right) + F_1\left(D' + \frac{\partial S_2}{\partial d}\right) \\ &\quad + F_2\left(L' + \frac{\partial S_2}{\partial l}, G' + \frac{\partial S_2}{\partial g}, H' + \frac{\partial S_2}{\partial h}, -, l, g, h, d\right) \\ &\quad + F_3(L', G', H', -, l, g, h, d) + F_4(L', G', H', -, l, g, h, d) \\ &= F_0^*(L') + F_1^*(D') + F_2^*\left(L', G', H', -, -, g + \frac{\partial S_2}{\partial G'}, h + \frac{\partial S_2}{\partial H'}, d\right) \\ &\quad + F_3^*(L', G', H', - -, g, h, d) + F_4^*(L', G', H', -, -, g, h, d). \end{aligned} \tag{53}$$

Expand now everywhere by Taylor's theorem retaining all terms through order four. Note that the notation

$$\frac{\partial F}{\partial L'} \quad \text{means} \quad \left(\frac{\partial F}{\partial L}\right)_{L=L'}$$

and

$$\frac{\partial F^*}{\partial g} \quad \text{means} \quad \left(\frac{\partial F^*}{\partial g'}\right)_{g'=g}$$

and similarly for all other terms. Observe also that

$$\frac{\partial^2 F_1}{\partial D'^2} = 0.$$

After completing the expansion, members on either side of the equation of identical order are equated. There results:

$$\text{Order 0:} \quad F_0(L') = F_0^*(L') \tag{54}$$

$$\text{Order 1:} \quad F_1(D') = F_1^*(D') \tag{55}$$

$$\begin{aligned} \text{Order 2:} \quad &\frac{\partial F_0}{\partial L'}\frac{\partial S_2}{\partial l} + F_2(L', G', H', -, l, g, h, d) \\ &= F_2^*(L', G', H', -, -, g, h, d) \end{aligned} \tag{56}$$

$$\begin{aligned} \text{Order 3:} \quad &\frac{\partial F_1}{\partial D'}\frac{\partial S_2}{\partial d} + F_3(L', G', H', -, l, g, h, d) \\ &= F_3^*(L', G', H', -, -, g, h, d) \end{aligned} \tag{57}$$

$$\text{Order 4:}\quad \frac{1}{2}\frac{\partial^2 F_0}{\partial L'^2}\left(\frac{\partial S_2}{\partial l}\right)^2 + \frac{\partial F_2}{\partial L'}\frac{\partial S_2}{\partial l} + \frac{\partial F_2}{\partial G'}\frac{\partial S_2}{\partial g} + \frac{\partial F_2}{\partial H'}\frac{\partial S_2}{\partial h}$$
$$+ F_4(L', G', H', -, l, g, h, d) = \frac{\partial F_2^*}{\partial g}\frac{\partial S_2}{\partial G'} + \frac{\partial F_2^*}{\partial h}\frac{\partial S_2}{\partial H'}$$
$$+ F_4^*(L', G', H', -, -, g, h, d). \tag{58}$$

The determining function S_2 is to be obtained from (56). Let the subscript S indicate the secular part of any function F, which is the part free of l, and the subscript P the periodic part, namely that containing l. Demanding that

$$F_{2S}(L', G', H', -, -, g, h, d) = F_2^*(L', G', H', -, -, g, h, d), \tag{59}$$

formula (56) leads to

$$-\frac{\partial F_0}{\partial L'}\frac{\partial S_2}{\partial l} = F_{2P}(L', G', H', -, l, g, h, d). \tag{60}$$

Before separating F_2 into its secular and periodic content, it will be advantageous to rewrite F_2, given by (43), in the following form:

$$F_2 = \mu_0 \frac{R_{\text{☾}}^2}{a^3}\left[\tfrac{1}{4}c_{20}(1 - 3\cos^2 I) + \tfrac{3}{2}c_{22}\sin^2 I\cos(\phi_{20} - 2h)\right]\frac{a^3}{r^3}$$
$$- \mu_0 \frac{R_{\text{☾}}^2}{a^3}\left[\tfrac{3}{4}c_{20}\sin^2 I - \tfrac{3}{2}c_{22}(1 + \cos^2 I)\cos(\phi_{20} - 2h)\right]\frac{a^3}{r^3}\cos 2u$$
$$+ 3c_{22}\mu_0 \frac{R_{\text{☾}}^2}{a^3}\left[\cos I\sin(\phi_{20} - 2h)\right]\frac{a^3}{r^3}\sin 2u$$
$$- \mu_2 \frac{a^2}{a_{\text{☾}}^3}\left[\tfrac{1}{8}(1 - 3\cos^2 I) - \tfrac{3}{8}\sin^2 I\cos(\phi_{20} - 2h)\right]\frac{r^2}{a^2}$$
$$+ \mu_2 \frac{a^2}{a_{\text{☾}}^3}\left[\tfrac{3}{8}\sin^2 I + \tfrac{3}{8}(1 + \cos^2 I)\cos(\phi_{20} - 2h)\right]\frac{r^2}{a^2}\cos 2u$$
$$+ \tfrac{3}{4}\mu_2 \frac{a^2}{a_{\text{☾}}^3}\left[\cos I\sin(\phi_{20} - 2h)\right]\frac{r^2}{a^2}\sin 2u. \tag{61}$$

Although the variables I and a should properly be primed, according to (56), this refinement will be deferred until the final results are written down. The six functions to be split into their secular and periodic parts are then

$$\frac{a^3}{r^3} \qquad \frac{a^3}{r^3}\cos 2u \qquad \frac{a^3}{r^3}\sin 2u$$
$$\frac{r^2}{a^2} \qquad \frac{r^2}{a^2}\cos 2u \qquad \frac{r^2}{a^2}\sin 2u.$$

The secular part of any of these functions θ is given by

$$\theta_s = \frac{1}{2\pi}\int_0^{2\pi} \theta \, dl . \tag{62}$$

It will be found advantageous to integrate the first three functions with respect to f, the remaining three with respect to E, the eccentric anomaly. The substitutions needed in (62) are

$$dl = \frac{L}{G}\frac{r^2}{a^2}\, df$$

and

$$dl = \frac{r}{a}\, dE .$$

In order to find the first three average values, one may also make use of the work of Brouwer (1959) and Kozai (1959). Letting the purely periodic parts of the six functions θ_i be designated by σ_i, the results of operation (62) are

$$\begin{aligned} \frac{a^3}{r^3} &= \frac{L^3}{G^3} + \sigma_1 & \frac{r^2}{a^2} &= (1 + \tfrac{3}{2}e^2) + \sigma_4 \\ \frac{a^3}{r^3}\cos 2u &= \sigma_2 & \frac{r^2}{a^2}\cos 2u &= \tfrac{5}{2}e^2 \cos 2g + \sigma_5 \\ \frac{a^3}{r^3}\sin 2u &= \sigma_3 & \frac{r^2}{a^2}\sin 2u &= \tfrac{5}{2}e^2 \sin 2g + \sigma_6 . \end{aligned} \tag{64}$$

Since the second and third of these contain no secular parts, F_{2S} and F_{2P} may be written in the form

$$F_{2S} = A_1 \frac{L^3}{G^3} + A_4(1 + \tfrac{3}{2}e^2) + \tfrac{5}{2}A_5 e^2 \cos 2g + \tfrac{5}{2}A_6 e^2 \sin 2g \tag{65}$$

$$F_{2P} = \sum_{i=1}^{6} A_i \sigma_i . \tag{66}$$

The temporary abbreviations A_i are clearly defined by comparison with (61).

Turning now to (60), first note that, with (41),

$$-\frac{\partial F_0}{\partial L'} = \frac{\mu^2}{L'^3} .$$

With this, (60), and (66), S_2 becomes

$$S_2 = \frac{L'^3}{\mu^2} \sum_{i=1}^{6} A_i \int \sigma_i \, dl . \tag{67}$$

The first two of these integrals are given by Brouwer (1959). The other four are readily obtained by integrating with respect to f or E, as explained above. Recall that the σ_i are defined by (64). The constants of integration may be disregarded since only partial derivatives of S_2 with respect to the Delaunay variables are required. The results are, again ignoring primes:

$$\int \sigma_1 \, dl = \frac{L^3}{G^3}(f - l + e \sin f)$$

$$\int \sigma_2 \, dl = \frac{1}{2}\frac{L^3}{G^3}\left[\sin(2g + 2f) + e \sin(2g + f) + \frac{e}{3}\sin(2g + 3f)\right]$$

$$\int \sigma_3 \, dl = -\frac{1}{2}\frac{L^3}{G^3}\left[\cos(2g + 2f) + e \cos(2g + f) + \frac{e}{3}\cos(2g + 3f)\right]$$

$$\int \sigma_4 \, dl = -\left(2e - \frac{3}{4}e^3\right)\sin E + \frac{3}{4}e^2 \sin 2E - \frac{e^3}{12}\sin 3E$$

$$\begin{aligned}\int \sigma_5 \, dl = &\left(\frac{1}{2} + \frac{e^2}{2}\right)\frac{G}{L}\sin 2g \cos 2E \\ &+ \left(\frac{1}{2} + \frac{e^2}{4}\right)\cos 2g \sin 2E - \frac{e}{6}\frac{G}{L}\sin 2g(15 \cos E + \cos 3E) \\ &- \left(\frac{e}{6} - \frac{e^3}{12}\right)\cos 2g(15 \sin E + \sin 3E)\end{aligned}$$

$$\begin{aligned}\int \sigma_6 \, dl = &-\left(\frac{1}{2} + \frac{e^2}{2}\right)\frac{G}{L}\cos 2g \cos 2E + \left(\frac{1}{2} + \frac{e^2}{4}\right)\sin 2g \sin 2E \\ &+ \frac{e}{6}\frac{G}{L}\cos 2g(15 \cos E + \cos 3E) \\ &- \left(\frac{e}{6} - \frac{e^3}{12}\right)\sin 2g(15 \sin E + \sin 3E). \qquad (68)\end{aligned}$$

A few comments on the form of these functions are called for. First, expressing the latter three in terms of E is a logical choice because the integrands were of the form $(r/a)^n$. There is nothing to be gained by changing from E to f. Next, retaining products of trigonometric functions in the last two permits a more compact form. Finally, judicious choice of Keplerian or Delaunay variables helps to abbreviate many expressions. It is often desirable to replace $\cos I$ by H/G, but it seldom pays to substitute Delaunay elements for e. The last two notes hold for much of the subsequent work.

Formulae (67) and (68), aided by (61), permit S_2 to be written down at once:

$$\begin{aligned}S_2 = \frac{\mu_0^2 R_{\text{☾}}^2}{G'^3}&\left[\frac{1}{4}c_{20}(1 - 3\cos^2 I') + \frac{3}{2}c_{22}\sin^2 I' \cos(\phi_{20} - 2h)\right] \\ &\times (f - l + e' \sin f)\end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{2}\frac{\mu_0^2 R_{☾}^2}{G'^3}\left[\frac{3}{4}c_{20}\sin^2 I' - \frac{3}{2}c_{22}(1+\cos^2 I')\cos(\phi_{20}-2h)\right] \\
&\qquad \times\left[\sin(2g+2f) + e'\sin(2g+f) + \frac{e'}{3}\sin(2g+3f)\right] \\
&-\frac{3}{2}c_{22}\frac{\mu_0^2 R_{☾}^2}{G'^3}\cos I' \sin(\phi_{20}-2h) \\
&\qquad \times\left[\cos(2g+2f) + e'\cos(2g+f) + \frac{e'}{3}\cos(2g+3f)\right] \\
&+\frac{\mu_2 L'^7}{\mu_0^4 a_{☾}^3}\left[\frac{1}{8}(1-3\cos^2 I') - \frac{3}{8}\sin^2 I'\cos(\phi_{20}-2h)\right] \\
&\qquad \times\left[\left(2e' - \frac{3}{4}e'^3\right)\sin E - \frac{3}{4}e'^2\sin 2E + \frac{e'^3}{12}\sin 3E\right] \\
&+\frac{\mu_2 L'^7}{\mu_0^4 a_{☾}^3}\left[\frac{3}{8}\sin^2 I' + \frac{3}{8}(1+\cos^2 I')\cos(\phi_{20}-2h)\right] \\
&\qquad \times\left[\left(\frac{1}{2}+\frac{e'^2}{2}\right)\frac{G'}{L'}\sin 2g\cos 2E\right. \\
&\qquad + \left(\frac{1}{2}+\frac{e'^2}{4}\right)\cos 2g\sin 2E - \frac{e'}{6}\frac{G'}{L'}\sin 2g(15\cos E + \cos 3E) \\
&\qquad \left. - \left(\frac{e'}{6}-\frac{e'^3}{12}\right)\cos 2g(15\sin E + \sin 3E)\right] \\
&-\frac{3}{4}\frac{\mu_2 L'^7}{\mu_0^4 a_{☾}^3}\cos I'\sin(\phi_{20}-2h)\left[\left(\frac{1}{2}+\frac{e'^2}{2}\right)\frac{G'}{L'}\cos 2g\cos 2E\right. \\
&\qquad - \left(\frac{1}{2}+\frac{e'^2}{4}\right)\sin 2g\sin 2E - \frac{e'}{6}\frac{G'}{L'}\cos 2g(15\cos E + \cos 3E) \\
&\qquad \left. + \left(\frac{e'}{6}-\frac{e'^3}{12}\right)\sin 2g(15\sin E + \sin 3E)\right]. \qquad (69)
\end{aligned}
$$

From S_2 the short-periodic perturbations may be calculated as indicated by (51). They will be listed in a later section.

C. PERTURBATIONS OF INTERMEDIATE PERIOD

The determining functions S_1^* and S_2^* will be developed in this section. A small divisor of order one will appear.

At this point the problem has been reduced to the solution of a dynamical system with three degrees of freedom since the variable l no longer appears in F^*. The new equations of motions are

$$\begin{aligned} \frac{dL'}{dt} &= 0 & \frac{dl'}{dt} &= -\frac{\partial F^*}{\partial L'} \\ \frac{dG'}{dt} &= \frac{\partial F^*}{\partial g'} & \frac{dg'}{dt} &= -\frac{\partial F^*}{\partial G'} \\ \frac{dH'}{dt} &= \frac{\partial F^*}{\partial h'} & \frac{dh'}{dt} &= -\frac{\partial F^*}{\partial H'} \\ \frac{dD'}{dt} &= \frac{\partial F^*}{\partial d'} & \frac{dd'}{dt} &= -\frac{\partial F^*}{\partial D'}\,. \end{aligned} \tag{70}$$

Note that now

$$L' = \text{constant}\,.$$

The various parts of F^* are found as follows. F_0^* and F_1^* are given by (54) and (55) with the aid of (41) and (42). They will be listed together with F_2^* and F_3^* farther below. F_2^* is obtained from (59) with (65) and (61). Here, as well as in other parts of the work, the variables g, h, and d are replaced by mechanical substitution with g', h', and d'. F_3^*, however, requires some preparation. From (57) it is seen that F_3^* is to be taken as

$$F_3^* = F_{3S} + \left[\frac{\partial F_1}{\partial D'}\frac{\partial S_2}{\partial d}\right]_S.$$

But since the bracket is zero,

$$F_3^* = F_{3S}\,. \tag{71}$$

F_3 is given by (44) of which the secular part is to be taken. There is a total of ten short-periodic functions in F_3. Six of these have already been averaged, with results in (64), and the other four are

$$\begin{aligned} &\frac{r^3}{a^3}\cos u & &\frac{r^3}{a^3}\cos 3u \\ &\frac{r^3}{a^3}\sin u & &\frac{r^3}{a^3}\sin 3u\,. \end{aligned}$$

Their averages are easily found as outlined before. The results are

$$\begin{aligned} \left[\frac{r^3}{a^3}\cos u\right]_S &= -\left(\tfrac{5}{2}e + \tfrac{15}{8}e^3\right)\cos g \\ \left[\frac{r^3}{a^3}\sin u\right]_S &= -\left(\tfrac{5}{2}e + \tfrac{15}{8}e^3\right)\sin g \\ \left[\frac{r^3}{a^3}\cos 3u\right]_S &= -\tfrac{35}{8}e^3\cos 3g \\ \left[\frac{r^3}{a^3}\sin 3u\right]_S &= -\tfrac{35}{8}e^3\sin 3g\,. \end{aligned} \tag{72}$$

F_3^* can now be written down. F_4^* is not needed as this stage. The four required parts of F^* are:

$$F_0^* = \frac{\mu_0^2}{2L'^2} \tag{73}$$

$$F_1^* = -D' \tag{74}$$

$$\begin{aligned}F_2^* &= \frac{1}{4} c_{20} \frac{\mu_0^4 R_☾^2}{L'^3 G'^3}(1 - 3\cos^2 I') - \frac{1}{8}\frac{\mu_2 L'^4}{\mu_0^2 a_☾^3}\left(1 + \frac{3}{2}e'^2\right)(1 - 3\cos^2 I') \\ &+ \frac{15}{16}\frac{\mu_2 L'^4}{\mu_0^2 a_☾^3} e'^2 \sin^2 I' \cos 2g' + \left[\frac{3}{2} c_{22} \frac{\mu_0^4 R_☾^2}{L'^3 G'^3}\sin^2 I'\right. \\ &\left. + \frac{3}{8}\frac{\mu_2 L'^4}{\mu_0^2 a_☾^3}\left(1 + \frac{3}{2}e'^2\right)\sin^2 I' + \frac{15}{16}\frac{\mu_2 L'^4}{\mu_0^2 a_☾^3} e'^2 (1 + \cos^2 I')\cos 2g'\right] \\ &\times \cos(\phi'_{20} - 2h') + \frac{15}{8}\frac{\mu_2 L'^4}{\mu_0^2 a_☾^3} e'^2 \cos I' \sin 2g' \sin(\phi'_{20} - 2h')\end{aligned} \tag{75}$$

$$\begin{aligned}F_3^* &= \frac{3}{4} I_☾ c_{20} \frac{\mu_0^4 R_☾^2}{L'^3 G'^3} \sin 2I' \cos(\phi'_{52} - h') \\ &+ \frac{3}{2} I_☾ c_{22} \frac{\mu_0^4 R_☾^2}{L'^3 G'^3} \sin 2I' \cos(\phi'_{23} - h') \\ &+ \frac{3}{8} \varepsilon_1 c_{20} \frac{\mu_0^4 R_☾^2}{L'^3 G'^3} \sin 2I' \cos(\phi'_7 - h') \\ &- \frac{3}{8} I_☾^2 c_{20} \frac{\mu_0^4 R_☾^2 \cdot}{L'^3 G'^3} [(1 - 3\cos^2 I')\sin^2 I' \cos(\phi'_{53} - 2h')] \\ &- \frac{3}{8}\gamma_☾ \frac{\mu_2 L'^4}{\mu_0^2 a_☾^3}\left\{\sin 2I'\left(1 + \frac{3}{2}e'^2 - \frac{5}{2}e'^2\cos 2g'\right)\right. \\ &\qquad \times [\cos(\phi'_{23} - h') - \cos(\phi'_{52} - h')] \\ &\qquad \left. - 5e'^2 \sin I' \sin 2g' [\sin(\phi'_{23} - h') - \sin(\phi'_{52} - h')]\right\} \\ &- \frac{3}{8} e_☾ \frac{\mu_2 L'^4}{\mu_0^2 a_☾^3}\left\{\left[\left(1 + \frac{3}{2}e'^2\right)(1 - 3\cos^2 I')\right.\right. \\ &\qquad \left. - \frac{15}{2}e'^2 \sin^2 I' \cos 2g'\right]\cos\phi'_3 - \frac{1}{2}\left[\left(1 + \frac{3}{2}e'^2\right)\sin^2 I'\right. \\ &\qquad \left. + \frac{5}{2}e'^2(1 + \cos^2 I')\cos 2g'\right] \\ &\qquad \times [7\cos(\phi'_{31} - 2h') - \cos(\phi'_2 - 2h')] \\ &\qquad \left. - \frac{5}{2}e'^2 \cos I' \sin 2g' [7\sin(\phi'_{31} - 2h') - \sin(\phi'_2 - 2h')]\right\} \\ &+ \frac{3}{16}\gamma_☾^2 \frac{\mu_2 L'^4}{\mu_0^2 a_☾^3}\left\{\left[\left(1 + \frac{3}{2}e'^2\right)(1 - 3\cos^2 I')\right.\right.\end{aligned}$$

$$- \frac{15}{2} e'^2 \sin^2 I' \cos 2g' \Big] (1 - \cos \phi'_{25}) - \bigg[\Big(1 + \frac{3}{2} e'^2\Big) \sin^2 I'$$

$$+ \frac{5}{2} e'^2 (1 + \cos^2 I') \cos 2g' \bigg]$$

$$\times [\cos(\phi'_{20} - 2h') - \cos(\phi'_{53} - 2h')]$$

$$- 5e'^2 \cos I' \sin 2g' [\sin(\phi'_{20} - 2h') - \sin(\phi'_{53} - 2h')] \Big\}$$

$$- \frac{3}{16} e_{☾} \gamma_{☾} \frac{\mu_2 L'^4}{\mu_0^2 a_{☾}^3} \Big\{ \sin 2I' \Big(1 + \frac{3}{2} e'^2 - \frac{5}{2} e'^2 \cos 2g'\Big)$$

$$\times [7 \cos(\phi'_{33} - h') - \cos(\phi'_6 - h') - 3 \cos(\phi'_7 - h')$$

$$- 3 \cos(\phi'_8 + h')] - 5e'^2 \sin I' \sin 2g' [7 \sin(\phi'_{33} - h')$$

$$- \sin(\phi'_6 - h') - 3 \sin(\phi'_7 - h') + 3 \sin(\phi'_8 + h')] \Big\}$$

$$- \frac{1}{8} \frac{\mu_3 L'^4}{\mu_0^2 a_{\odot}^3} \bigg\{ \bigg[\Big(1 + \frac{3}{2} e'^2\Big) (1 - 3 \cos^2 I') - \frac{15}{2} e'^2 \sin^2 I' \cos 2g' \bigg]$$

$$- 3 \bigg[\Big(1 + \frac{3}{2} e'^2\Big) \sin^2 I' + \frac{5}{2} e'^2 (1 + \cos^2 I') \cos 2g' \bigg]$$

$$\times \cos(\phi'_{50} - 2h') - 15 e'^2 \cos I' \sin 2g' \sin(\phi'_{50} - 2h') \bigg\}$$

$$- \frac{3}{16} e_{☾}^2 \frac{\mu_2 L'^4}{\mu_0^2 a_{☾}^3} \bigg\{ \bigg[\Big(1 + \frac{3}{2} e'^2\Big) (1 - 3 \cos^2 I') - \frac{15}{2} e'^2 \sin^2 I' \cos 2g' \bigg]$$

$$\times (1 + 3 \cos \phi'_{24}) - \bigg[\Big(1 + \frac{3}{2} e'^2\Big) \sin^2 I'$$

$$+ \frac{5}{2} e'^2 (1 + \cos^2 I') \cos 2g' \bigg] [17 \cos(\phi'_{41} - 2h')$$

$$- 5 \cos(\phi'_{20} - 2h')] - 5e'^2 \cos I' \sin 2g'$$

$$\times [17 \sin(\phi'_{41} - 2h') - 5 \sin(\phi'_{20} - 2h')] \bigg\}$$

$$+ \frac{5}{64} \frac{\mu_2 L'^6}{\mu_0^3 a_{☾}^4} \bigg\{ 3 \bigg[\Big(e' + \frac{3}{4} e'^3\Big) (4 - 5 \sin^2 I') \cos g'$$

$$+ \frac{35}{4} e'^3 \sin^2 I' \cos 3g' \bigg] \cos(\phi'_1 - h')$$

$$+ 5 \bigg[3 \Big(e' + \frac{3}{4} e'^3\Big) \sin^2 I' \cos g'$$

$$+ \frac{7}{4} e'^3 (4 - 3 \sin^2 I') \cos 3g' \bigg] \cos(\phi'_{30} - 3h')$$

$$+ 3\cos I'\left[\left(e' + \frac{3}{4}e'^3\right)(4 - 15\sin^2 I')\sin g' + \frac{35}{4}e'^3\sin^2 I'\sin 3g'\right]\sin(\phi_1' - h') + 5\cos I'\left[3\left(e' + \frac{3}{4}e'^3\right)\sin^2 I'\sin g' + \frac{7}{4}e'^3(4 - \sin^2 I')\sin 3g'\right]\sin(\phi_{30}' - 3h')\Big\}. \quad (76)$$

Note that the symbol ϕ' is used instead of ϕ for reasons of symmetry only. Nevertheless, it is well to recall that $\phi' \equiv \phi$ since $d' \equiv d \equiv t$.

Similar to the previous section, a transformation is being made where the old and new Hamiltonians are related through

$$F^*(L', G', H', D', -, g', h', d') = F^{**}(L'', G'', H'', D'', -, g'', h'', d''). \quad (77)$$

In contrast to Section B, as mentioned earlier, no particular canonical variable is eliminated but rather all periodic terms which contain the longitude of the moon. The transformation equations are

$$L' = \frac{\partial S^*}{\partial l'} \qquad l'' = \frac{\partial S^*}{\partial L''}$$
$$G' = \frac{\partial S^*}{\partial g'} \qquad g'' = \frac{\partial S^*}{\partial G''}$$
$$H' = \frac{\partial S^*}{\partial h'} \qquad h'' = \frac{\partial S^*}{\partial H''} \quad (78)$$
$$D' = \frac{\partial S^*}{\partial d'} \qquad d'' = \frac{\partial S^*}{\partial D''}$$

where $S^* = S^*(L'', G'', H'', D'', -, g', h', d')$.
Again let

$$S^* = S_0^* + S_1^* + S_2^* + \cdots$$

and take

$$S_0^* = L''l' + G''g' + H''h' + D''d'.$$

Again in contrast to the previous section, S_1^* will appear because of the small divisor. Also, since third order terms are not needed, S_3^* and higher order parts of S^* are ignored. Hence

$$S^* = S_0^* + S_1^* + S_2^*. \quad (79)$$

The transformation equations become

$$
\begin{aligned}
L' &= L'' & l'' &= l' + \frac{\partial S_1^*}{\partial L''} + \frac{\partial S_2^*}{\partial L''} \\
G' &= G'' + \frac{\partial S_1^*}{\partial g'} + \frac{\partial S_2^*}{\partial g'} & g'' &= g' + \frac{\partial S_1^*}{\partial G''} + \frac{\partial S_2^*}{\partial G''} \\
H' &= H'' + \frac{\partial S_1^*}{\partial h'} + \frac{\partial S_2^*}{\partial h'} & h'' &= h' + \frac{\partial S_1^*}{\partial H''} + \frac{\partial S_2^*}{\partial H''} \\
D' &= D'' + \frac{\partial S_1^*}{\partial d'} + \frac{\partial S_2^*}{\partial d'} & d'' &= d' .
\end{aligned}
\tag{80}
$$

S_1^* and S_2^*, of course, cannot contain l' and, as before, will not depend on D''. Quite similar to (52), (77) is split into its various orders of magnitude. Upon substituting the transformation equations, there follows:

$$
\begin{aligned}
& F_0^*(L') + F_1^*\left(D'' + \frac{\partial S_1^*}{\partial d'} + \frac{\partial S_2^*}{\partial d'}\right) \\
& \quad + F_2^*\left(L', G'' + \frac{\partial S_1^*}{\partial g'} + \frac{\partial S_2^*}{\partial g'}, H'' + \frac{\partial S_1^*}{\partial h'} + \frac{\partial S_2^*}{\partial h'}, -, -, g', h', d'\right) \\
& \quad + F_3^*\left(L', G'' + \frac{\partial S_1^*}{\partial g'}, H'' + \frac{\partial S_1^*}{\partial h'}, -, -, g', h', d'\right) \\
& \quad + F_4^*(L', G'', H'', -, -, g', h', d') = F_0^{**}(L') + F_1^{**}(D'') \\
& \quad + F_2^{**}\left(L', G'', H'', -, -, g' + \frac{\partial S_1^*}{\partial G''} + \frac{\partial S_2^*}{\partial G''}, h' + \frac{\partial S_1^*}{\partial H''} + \frac{\partial S_2^*}{\partial H''}, d'\right) \\
& \quad + F_3^{**}\left(L', G'', H'', -, -, g' + \frac{\partial S_1^*}{\partial G}, h' + \frac{\partial S_1^*}{\partial H''}, d'\right) \\
& \quad + F_4^{**}(L', G'', H'', -, -, g', h', d').
\end{aligned}
\tag{81}
$$

This time the Taylor expansion is about the point

$$L', G'', H'', D'', -, g', h', d'.$$

After the expansion, again terms of identical order are equated. The result is:

Order 0: $F_0^*(L') = F_0^{**}(L')$ (82)

Order 1: $F_1^*(D'') = F_1^{**}(D'')$ (83)

Order 2:
$$\frac{\partial F_1^*}{\partial D''}\frac{\partial S_1^*}{\partial d'} + F_2^*(L', G'', H'', -, -, g', h', d') = F_2^{**}(L', G'', H'', -, -, g', -, -) \tag{84}$$

Order 3:
$$
\begin{aligned}
& \frac{\partial F_1^*}{\partial D''}\frac{\partial S_2^*}{\partial d'} + \frac{\partial F_2^*}{\partial G''}\frac{\partial S_1^*}{\partial g'} + \frac{\partial F_2^*}{\partial H''}\frac{\partial S_1^*}{\partial h'} \\
& \quad + F_3^*(L', G'', H'', -, -, g', h', d') = \frac{\partial F_2^{**}}{\partial g'}\frac{\partial S_1^*}{\partial G''} \\
& \quad + F_3^{**}(L', G'', H'', -, -, g', h', d')
\end{aligned}
\tag{85}
$$

$$\text{Order 4:}\quad \frac{\partial F_2^*}{\partial G''}\frac{\partial S_2^*}{\partial g'} + \frac{\partial F_2^*}{\partial H''}\frac{\partial S_2^*}{\partial h'} + \frac{1}{2}\frac{\partial^2 F_2^*}{\partial G''^2}\left(\frac{\partial S_1^*}{\partial g'}\right)^2 + \frac{1}{2}\frac{\partial^2 F_2^*}{\partial H''^2}\left(\frac{\partial S_1^*}{\partial h'}\right)^2$$
$$+ \frac{\partial^2 F_2^*}{\partial G''\partial H''}\frac{\partial S_1^*}{\partial g'}\frac{\partial S_1^*}{\partial h'} + \frac{\partial F_3^*}{\partial G''}\frac{\partial S_1^*}{\partial g'} + \frac{\partial F_3^*}{\partial H''}\frac{\partial S_1^*}{\partial h'}$$
$$+ F_4^*(L'', G'', H'', -, -, g', h', d') = \frac{\partial F_2^{**}}{\partial g'}\frac{\partial S_2^*}{\partial G''}$$
$$+ \frac{1}{2}\frac{\partial^2 F_2^{**}}{\partial g'^2}\left(\frac{\partial S_1^*}{\partial G''}\right)^2 + \frac{\partial F_3^{**}}{\partial g'}\frac{\partial S_1^*}{\partial G''} + \frac{\partial F_3^{**}}{\partial h'}\frac{\partial S_1^*}{\partial H''}$$
$$+ F_4^{**}(L'', G'', H'', -, -, g', h', d'). \tag{86}$$

Let here the subscripts $\bar{S}$ and $\bar{P}$ designate the secular and periodic parts of any function, where a periodic term is now one that contains the longitude of the moon. They are easily recognized since they bear subscripts <50 attached to ϕ.

Putting

$$F_{2\bar{S}}(L'', G'', H'', -, -, g', -, -) = F_2^{**}(L'', G'', H'', -, -, g', -, -), \tag{87}$$

formula (84) yields, with the aid of (74),

$$\frac{\partial S_1^*}{\partial d'} = F_{2\bar{P}}^*(L'', G'', H'', -, -, g', h', d'). \tag{88}$$

Employing (75), there follows at once the first-order part of the determining function:

$$S_1^* = \left\{\frac{3}{2}\frac{c_{22}}{n_{20}}\frac{\mu_0^4 R_☾^2}{L''^3 G''^3}\sin^2 I'' + \frac{3}{8}\frac{\mu_2 L''^4}{n_{20}\mu_0^2 a_☾^3}\left[\left(1 + \frac{3}{2}e''^2\right)\sin^2 I''\right.\right.$$
$$\left.\left. + \frac{5}{2}e''^2(1 + \cos^2 I'')\cos 2g'\right]\right\}\sin(\phi'_{20} - 2h')$$
$$- \frac{15}{8}\frac{\mu_2 L''^4}{n_{20}\mu_0^2 a_☾^3}e''^2\cos I''\sin 2g'\cos(\phi'_{20} - 2h'). \tag{89}$$

Equation (85) shows how S_2^* is to be obtained. Again, all secular terms are equated to F_3^{**} so that (85) splits into two formulae:

$$F_3^{**} = \left[\frac{\partial F_2^*}{\partial G''}\frac{\partial S_1^*}{\partial g'}\right]_{\bar{S}} + \left[\frac{\partial F_2^*}{\partial H''}\frac{\partial S_1^*}{\partial h'}\right]_{\bar{S}} + F_{3\bar{S}}^* \tag{90}$$

$$\frac{\partial S_2^*}{\partial d'} = \left[\frac{\partial F_2^*}{\partial G''}\frac{\partial S_1^*}{\partial g'}\right]_{\bar{P}} + \left[\frac{\partial F_2^*}{\partial H''}\frac{\partial S_1^*}{\partial h'}\right]_{\bar{P}} - \left[\frac{\partial F_2^{**}}{\partial g'}\frac{\partial S_1^*}{\partial G''}\right]_{\bar{P}} + F_{3\bar{P}}^*, \tag{91}$$

where use is made of the fact that

$$\left[\frac{\partial F_2^{**}}{\partial g'}\frac{\partial S_1^*}{\partial G''}\right]_{\bar{S}} = 0,$$

which can be verified later. Derivation of the three quantities in brackets involves a good deal of algebra. However, these intermediate results need not be recorded here. Upon separating above functions into their secular and periodic content, the calculation of S_2^* from (91) presents no difficulties. $F_{3\bar{P}}^*$ is, of course, readily extracted from (76). The final result for S_2^* becomes:

$$
\begin{aligned}
S_2^* = {} & \frac{3}{2} I_{☾} \frac{c_{22}}{n_{23}} \frac{\mu_0^4 R_{☾}^2}{L''^3 G''^3} \sin^2 I'' \sin(\phi'_{23} - h') \\
& + \frac{3}{8} \varepsilon_1 \frac{c_{22}}{n_7} \frac{\mu_0^4 R_{☾}^2}{L''^3 G''^3} \sin 2I'' \sin(\phi'_7 - h') \\
& - \frac{3}{8} \gamma_{☾} \frac{\mu_2 L''^4}{n_{23} \mu_0^2 a_{☾}^3} \left[\sin 2I'' \left(1 + \frac{3}{2} e''^2 - \frac{5}{2} e''^2 \cos 2g'\right) \right. \\
& \qquad \left. \times \sin(\phi'_{23} - h') + 5e''^2 \sin I'' \sin 2g' \cos(\phi'_{23} - h') \right] \\
& - \frac{3}{8} e_{☾} \frac{\mu_2 L''^4}{\mu_0^2 a_{☾}^3} \left\{ \frac{1}{n_3} \left[\left(1 + \frac{3}{2} e''^2\right)(1 - 3\cos^2 I'') \right. \right. \\
& \qquad \left. - \frac{15}{2} e''^2 \sin^2 I'' \cos 2g' \right] \sin \phi'_3 \\
& \qquad - \frac{1}{2} \left[\left(1 + \frac{3}{2} e''^2\right) \sin^2 I'' + \frac{5}{2} e''^2 (1 + \cos^2 I'') \cos 2g' \right] \\
& \qquad \times \left[\frac{7}{n_{31}} \sin(\phi'_{31} - 2h') - \frac{1}{n_2} \sin(\phi'_2 - 2h') \right] \\
& \qquad + \frac{5}{2} e''^2 \cos I'' \sin 2g' \left[\frac{7}{n_{31}} \cos(\phi'_{31} - 2h') \right. \\
& \qquad \left. \left. - \frac{1}{n_2} \cos(\phi'_2 - 2h') \right] \right\} \\
& - \frac{3}{16} \gamma_{☾}^2 \frac{\mu_2 L''^4}{\mu_0^2 a_{☾}^3} \left\{ \frac{1}{n_{25}} \left[\left(1 + \frac{3}{2} e''^2\right)(1 - 3\cos^2 I'') \right. \right. \\
& \qquad \left. - \frac{15}{2} e''^2 \sin^2 I'' \cos 2g' \right] \sin \phi'_{25} + \frac{1}{n_{20}} \left[\left(1 + \frac{3}{2} e''^2\right) \sin^2 I'' \right. \\
& \qquad \left. + \frac{5}{2} e''^2 (1 + \cos^2 I'') \cos 2g' \right] \sin(\phi'_{20} - 2h') \\
& \qquad \left. - \frac{5}{n_{20}} e''^2 \cos I'' \sin 2g' \cos(\phi'_{20} - 2h') \right\} \\
& - \frac{3}{16} e_{☾} \gamma_{☾} \frac{\mu_2 L''^4}{\mu_0^2 a_{☾}^3} \left\{ \sin 2I'' \left[\left(1 + \frac{3}{2} e''^2 - \frac{5}{2} e''^2 \cos 2g'\right) \right. \right. \\
& \qquad \times \left[\frac{7}{n_{33}} \sin(\phi'_{33} - h') - \frac{1}{n_6} \sin(\phi'_6 - h') - \frac{3}{n_7} \sin(\phi'_7 - h') \right. \\
& \qquad \left. - \frac{3}{n_8} \sin(\phi'_8 + h') \right] + 5e''^2 \sin I'' \sin 2g' \left[\frac{7}{n_{33}} \cos(\phi'_{33} - h') \right.
\end{aligned}
$$

$$- \frac{1}{n_6} \cos(\phi_6' - h') - \frac{3}{n_7} \cos(\phi_7' - h') + \frac{3}{n_8} \cos(\phi_8' + h')\Bigg]\Bigg\}$$

$$- \frac{3}{16} e_{\mathbb{C}}^2 \frac{\mu_2 L''^4}{\mu_0^2 a_{\mathbb{C}}^3} \Bigg\{\frac{3}{n_{24}} \Bigg[\Bigg(1 + \frac{3}{2} e''^2\Bigg)(1 - 3\cos^2 I'')$$

$$- \frac{15}{2} e''^2 \sin^2 I'' \cos 2g'\Bigg] \sin \phi_{24}' - \Bigg[\Bigg(1 + \frac{3}{2} e''^2\Bigg) \sin^2 I''$$

$$+ \frac{5}{2} e''^2 (1 + \cos^2 I'') \cos 2g'\Bigg]\Bigg[\frac{17}{n_{41}} \sin(\phi_{41}' - 2h')$$

$$- \frac{5}{n_{20}} \sin(\phi_{20}' - 2h')\Bigg] + 5e''^2 \cos I'' \sin 2g'$$

$$\times \Bigg[\frac{17}{n_{41}} \cos(\phi_{41}' - 2h') - \frac{5}{n_{20}} \cos(\phi_{20}' - 2h')\Bigg]\Bigg\}$$

$$+ \frac{5}{64} \frac{\mu_2 L''^6}{\mu_0^3 a_{\mathbb{C}}^4} \Bigg\{\frac{3}{n_1} \Bigg[\Bigg(e'' + \frac{3}{4} e''^3\Bigg)(4 - 5\sin^2 I'') \cos g'$$

$$+ \frac{35}{4} e''^3 \sin^2 I'' \cos 3g'\Bigg] \sin(\phi_1' - h')$$

$$+ \frac{5}{n_{30}} \Bigg[3\Bigg(e'' + \frac{3}{4} e''^3\Bigg) \sin^2 I'' \cos g'$$

$$+ \frac{7}{4} e''^3 (4 - 3\sin^2 I'') \cos 3g'\Bigg] \sin(\phi_{30}' - 3h')$$

$$- \frac{3}{n_1} \cos I'' \Bigg[\Bigg(e'' + \frac{3}{4} e''^3\Bigg)(4 - 15\sin^2 I'') \sin g'$$

$$+ \frac{35}{4} e''^3 \sin^2 I'' \sin 3g'\Bigg] \cos(\phi_1' - h')$$

$$- \frac{5}{n_{30}} \cos I'' \Bigg[3\Bigg(e'' + \frac{3}{4} e''^3\Bigg) \sin^2 I'' \sin g'$$

$$+ \frac{7}{4} e''^3 (4 - \sin^2 I'') \sin 3g'\Bigg] \cos(\phi_{30}' - 3h')\Bigg\}$$

$$+ \frac{9}{2} \frac{c_{22}}{n_{20}^2} \mu_0^8 R_{\mathbb{C}}^4 \frac{H''}{L''^6 G''^8}$$

$$\times \sin^2 I'' \Bigg[c_{20} \sin(\phi_{20}' - 2h') + \frac{1}{2} c_{22} \sin(2\phi_{20}' - 4h')\Bigg]$$

$$+ \frac{45}{32} \frac{c_{20}}{n_{20}^2} \frac{\mu_0^2 \mu_2 R_{\mathbb{C}}^2}{a_{\mathbb{C}}^3} \frac{L''}{G''^4}$$

$$\times e''^2 (4 - 10\sin^2 I'' + 5\sin^4 I'') \sin 2g' \cos(\phi_{20}' - 2h')$$

$$+ \frac{9}{16} \frac{c_{20}}{n_{20}^2} \frac{\mu_0^2 \mu_2 R_{\mathbb{C}}^2}{a_{\mathbb{C}}^3} \frac{L'' H''}{G''^5} e''^2 \Bigg[\Bigg(\frac{2}{e''^2} + 3\Bigg) \sin^2 I''$$

$$- 10(1 - 2\sin^2 I'')\cos 2g' \Big] \sin(\phi'_{20} - 2h')$$

$$- \frac{45}{32}\frac{c_{22}}{n_{20}^2}\frac{\mu_0^2\mu_2 R_☾^2}{a_☾^3}\frac{L''}{G''^4} e''^2 \sin^2 I'' \sin 2g' \Big[(4 - 10\sin^2 I'')$$

$$\times \cos(\phi'_{20} - 2h') + \frac{1}{2}(6 - 5\sin^2 I'')\cos(2\phi'_{20} - 4h') \Big]$$

$$- \frac{9}{16}\frac{c_{22}}{n_{20}^2}\frac{\mu_0^2\mu_2 R_☾^2}{a_☾^3}\frac{L''H''}{G''^5}$$

$$\times e''^2 \sin^2 I'' \Big[\left(\frac{4}{e''^2} + 6 - 10\cos 2g' \right) \sin(\phi'_{20} - 2h')$$

$$- \frac{1}{2}\left(\frac{4}{e''^2} + 6 + 15\cos 2g' \right) \sin(2\phi'_{20} - 4h') \Big]$$

$$- \frac{45}{16}\frac{\mu_2^2}{n_{20}^2\mu_0^4 a_☾^6} L''^6 G'' e''^2 (1 - 2\sin^2 I'')\sin 2g' \cos(\phi'_{20} - 2h')$$

$$- \frac{9}{32}\frac{\mu_2^2}{n_{20}^2\mu_0^4 a_☾^6} L''^6 H'' e''^2 \Big[\left(\frac{2}{e''^2} - 17 \right) \sin^2 I''$$

$$- 5(2 - 3\sin^2 I'')\cos 2g' \Big] \sin(\phi'_{20} - 2h')$$

$$- \frac{45}{256}\frac{\mu_2^2}{n_{20}^2\mu_0^4 a_☾^6}\frac{L''^8}{G''} e''^2 \Big[2\left(3\frac{G''^2}{L''^2} - 5\sin^2 I'' \right) \sin^2 I'' \sin 2g'$$

$$+ \left(20\frac{G''^2}{L''^2} - 20\sin^2 I'' + \frac{5}{2}\sin^4 I'' + 15 e''^2 \sin^2 I'' \right) \sin 4g' \Big]$$

$$\times \cos(2\phi'_{20} - 4h')$$

$$+ \frac{9}{256}\frac{\mu_2^2}{n_{20}^2\mu_0^4 a_☾^6}\frac{L''^8 H''}{G''^2} e''^2 \Big[\left(\frac{4}{e''^2} + 12 + 9e''^2 \right) \sin^2 I''$$

$$+ 30\frac{G''^2}{L''^2}\sin^2 I'' \cos 2g' + 25\left(4\frac{G''^2}{L''^2} - 2\sin^2 I'' \right.$$

$$\left. + e''^2 \sin^2 I'' \right) \cos 4g' \Big] \sin(2\phi'_{20} - 4h'). \qquad (92)$$

Expressions for the perturbations of intermediate period are derived from S_1^* and S_2^*. They will be listed in a later section.

D. LONG-PERIODIC AND SECULAR-PERTURBATIONS

The new Hamiltonian F^{**} is given, and it is proposed that the remainder of the problem is solved by numerical integration.

The new equations of motion are

$$\begin{aligned}
\frac{\mathrm{d}L''}{\mathrm{d}t} &= 0 & \frac{\mathrm{d}l''}{\mathrm{d}t} &= -\frac{\partial F^{**}}{\partial L''} \\
\frac{\mathrm{d}G''}{\mathrm{d}t} &= \frac{\partial F^{**}}{\partial g''} & \frac{\mathrm{d}g''}{\mathrm{d}t} &= -\frac{\partial F^{**}}{\partial G''} \\
\frac{\mathrm{d}H''}{\mathrm{d}t} &= \frac{\partial F^{**}}{\partial h''} & \frac{\mathrm{d}h''}{\mathrm{d}t} &= -\frac{\partial F^{**}}{\partial H''} \\
\frac{\mathrm{d}D''}{\mathrm{d}t} &= \frac{\partial F^{**}}{\partial d''} & \frac{\mathrm{d}d''}{\mathrm{d}t} &= -\frac{\partial F^{**}}{\partial D''} .
\end{aligned} \tag{93}$$

The different parts of F^{**} are found as follows. F_0^{**} and F_1^{**} are given by (82) and (83) with (73) and (74). F_2^{**} follows from (87) and (75). F_3^{**} is defined by (90). F_4^{**} will be discussed later. The individual functions are:

$$F_0^{**} = \frac{\mu_0^2}{2L''^2} \tag{94}$$

$$F_1^{**} = -D'' \tag{95}$$

$$\begin{aligned}
F_2^{**} = {} & \frac{1}{4} c_{20} \frac{\mu_0^4 R_☾^2}{L''^3 G''^3} (1 - 3\cos^2 I'') - \frac{1}{8} \frac{\mu_2 L''^4}{\mu_0^2 a_☾^3} \left(1 + \frac{3}{2} e''^2\right)(1 - 3\cos^2 I'') \\
& + \frac{15}{16} \frac{\mu_2 L''^4}{\mu_0^2 a_☾^3} e''^2 \sin^2 I'' \cos 2g'' .
\end{aligned} \tag{96}$$

$$\begin{aligned}
F_3^{**} = {} & \frac{3}{4} I_☾ c_{20} \frac{\mu_0^4 R_☾^2}{L''^3 G''^3} \sin 2I'' \cos(\phi''_{52} - h'') \\
& - \frac{3}{8} I_☾^2 c_{20} \frac{\mu_0^4 R_☾^2}{L''^3 G''^3} [(1 - 3\cos^2 I'') + \sin^2 I'' \cos(\phi''_{53} - 2h'')] \\
& + \frac{3}{8} \gamma_☾ \frac{\mu_2 L''^4}{\mu_0^2 a_☾^3} \left[\sin 2I'' \left(1 + \frac{3}{2} e''^2 - \frac{5}{2} e''^2 \cos 2g''\right) \cos(\phi''_{52} - h'') \right. \\
& \qquad \left. - 5e''^2 \sin I'' \sin 2g'' \sin(\phi''_{52} - h'') \right] \\
& + \frac{3}{16} \gamma_☾^2 \frac{\mu_2 L''^4}{\mu_0^2 a_☾^3} \left\{ \left[\left(1 + \frac{3}{2} e''^2\right)(1 - 3\cos^2 I'') - \frac{15}{2} e''^2 \sin^2 I'' \cos 2g'' \right] \right. \\
& \qquad + \left[\left(1 + \frac{3}{2} e''^2\right) \sin^2 I'' + \frac{5}{2} e''^2 (1 + \cos^2 I'') \cos 2g'' \right] \\
& \qquad \left. \times \cos(\phi''_{53} - 2h'') + 5e''^2 \cos I'' \sin 2g'' \sin(\phi''_{53} - 2h'') \right\} \\
& - \frac{1}{8} \frac{\mu_3 L''^4}{\mu_0^2 a_\odot^3} \left\{ \left[\left(1 + \frac{3}{2} e''^2\right)(1 - 3\cos^2 I'') - \frac{15}{2} e''^2 \sin^2 I'' \cos 2g'' \right] \right.
\end{aligned}$$

$$- 3\left[\left(1 + \frac{3}{2} e''^2\right)\sin^2 I'' + \frac{5}{2} e''^2 (1 + \cos^2 I'')\cos 2g''\right]$$

$$\times \cos(\phi''_{50} - 2h'') - 15e''^2 \cos I'' \sin 2g'' \sin(\phi''_{50} - 2h'')\Big\}$$

$$- \frac{3}{16} e_{\text{☾}}^2 \frac{\mu_2 L''^4}{\mu_0^2 a_{\text{☾}}^3}\left[\left(1 + \frac{3}{2} e''^2\right)(1 - 3\cos^2 I'') - \frac{15}{2} e''^2 \sin^2 I'' \cos 2g''\right]$$

$$+ \frac{9}{2}\frac{c_{22}^2}{n_{20}} \mu_0^8 R_{\text{☾}}^4 \frac{H''}{L''^6 G''^8} \sin^2 I'' + \frac{9}{8}\frac{c_{22}}{n_{20}} \frac{\mu_0^2 \mu_2 R_{\text{☾}}^2}{a_{\text{☾}}^3} \frac{L'' H''}{G''^5}$$

$$\times \sin^2 I'' \left(2 + 3e''^2 + \frac{15}{2} e''^2 \cos 2g''\right) + \frac{9}{64} \frac{\mu_2^2}{n_{20} \mu_0^4 a_{\text{☾}}^6}$$

$$\times L''^6 H'' e''^2 \left[50 + \frac{\sin^2 I''}{e''^2}(2 - 17e''^2) + 15 \sin^2 I'' \cos 2g''\right]. \tag{97}$$

The problem is now of such a nature that it can no longer be solved by any method of successive approximations. This may be seen in a number of ways. It suffices to say that the von Zeipel technique is not applicable since the Hamiltonian has no term containing G'' of lower order than the $\cos 2g''$ term in (96). This may be verified by trying to produce the counterpart of (84) in order to get an S^{**}. The problem is not a new one in celestial mechanics, but it has probably first been pointed out for the lunar orbiter by Kozai (1963).

An analytical solution by different means is not attempted in this paper. Since the shortest period is one half year, it is felt that numerical integration of Equations (93) is not only feasible but may be desirable as far as practical applications are concerned. The integration interval can be taken quite large and should hardly be less than several days. It is believed that a simple Runge-Kutta algorithm would suffice for most any application, and only very high precision and very long intervals of time may require something like the Adams process.

Numerical integration of the differential Equations (93) in no way alters the character of the solution. In particular, the small divisors of order two, which would arise in an analytical solution, will manifest themselves just the same in the numerical results. For example, numerical integration of (93) with $F^{**} = F_3^{**}$ will lead to variations of order one. If one desires to verify the appearance of such divisors analytically, this is readily accomplished. It is only necessary to perform the initial steps of a solution of this problem with $F_1^{**} = 0$ and replacing the right-hand member of (96) by $-D''$. After the counterpart of formula (85) has been obtained, the small divisors are self-evident.

The above list of the various parts of F^{**} is seen to lack F_4^{**}. In principle F_4^{**} is easily obtained. It is defined by (86) with all members known. However, any attempt at its derivation will soon show that the amount of algebra required is indeed formidable. Before embarking on such a venture, it would be well to ascertain whether the resulting increase in accuracy is really needed. Nevertheless, the route for its derivation will be outlined very briefly.

Consider (86). It follows immediately that

$$F_4^{**} = \left[\frac{\partial F_2^*}{\partial G''}\frac{\partial S_2^*}{\partial g'}\right]_S + \left[\frac{\partial F_2^*}{\partial H''}\frac{\partial S_2^*}{\partial h'}\right]_S + \frac{1}{2}\left[\frac{\partial^2 F_2^*}{\partial G''^2}\left(\frac{\partial S_1^*}{\partial g'}\right)^2\right]_S$$
$$+ \frac{1}{2}\left[\frac{\partial^2 F_2^*}{\partial H''^2}\left(\frac{\partial S_1^*}{\partial h'}\right)^2\right]_S + \left[\frac{\partial^2 F_2^*}{\partial G''\partial H''}\frac{\partial S_1^*}{\partial g'}\frac{\partial S_1^*}{\partial h'}\right]_S + \left[\frac{\partial F_3^*}{\partial G''}\frac{\partial S_1^*}{\partial g'}\right]_S$$
$$+ \left[\frac{\partial F_3^*}{\partial H''}\frac{\partial S_1^*}{\partial h'}\right]_S - \left[\frac{\partial F_2^{**}}{\partial g'}\frac{\partial S_2^*}{\partial G''}\right]_S - \frac{1}{2}\left[\frac{\partial^2 F_2^{**}}{\partial g'^2}\left(\frac{\partial S_1^*}{\partial G''}\right)^2\right]_S$$
$$- \left[\frac{\partial F_3^{**}}{\partial g'}\frac{\partial S_1^*}{\partial G''}\right]_S - \left[\frac{\partial F_3^{**}}{\partial h'}\frac{\partial S_1^*}{\partial H''}\right]_S + F_{4S}^*.$$

F_{4S}^* is found with the aid of (58) to be

$$F_{4S}^* = \frac{1}{2}\left[\frac{\partial^2 F_0}{\partial L'^2}\left(\frac{\partial S_2}{\partial l}\right)^2\right]_{SS} + \left[\frac{\partial F_2}{\partial L'}\frac{\partial S_2}{\partial l}\right]_{SS} + \left[\frac{\partial F_2}{\partial G'}\frac{\partial S_2}{\partial g}\right]_{SS}$$
$$+ \left[\frac{\partial F_2}{\partial H'}\frac{\partial S_2}{\partial h}\right]_{SS} - \left[\frac{\partial F_2^*}{\partial g}\frac{\partial S_2}{\partial G'}\right]_{SS} - \left[\frac{\partial F_2^*}{\partial h}\frac{\partial S_2}{\partial H'}\right]_{SS} + F_{4SS}.$$

In calculating the latter of these two equations, many new averages of the type

$$\left[\frac{r^n}{a^n}\begin{matrix}\cos\\ \sin\end{matrix}\, mu\right]_S,$$

n both positive and negative, have to be found. It may be advantageous to employ some of the relations given by Kozai (1962).

4. Final Results and Sequence of Computations

The explicit expressions for the various perturbations are given, and the sequence of computations for practical application of the theory is outlined.

A. FORMULAE FOR THE PERTURBATIONS OF SHORT AND INTERMEDIATE PERIOD

The perturbations of short period are clearly defined by the transformation Equations (51), those of intermediate period by (80). All there is left to be done is to take the partial derivatives. In doing so, the following relations will be found useful:

$$\frac{\partial e}{\partial L} = \frac{1}{e}\frac{G^2}{L^3} \qquad \frac{\partial e}{\partial G} = -\frac{1}{e}\frac{G}{L^2}$$
$$\frac{\partial f}{\partial L} = \frac{\sin f}{eL}\left(\frac{G^2}{L^2}\frac{a}{r}+1\right) \qquad \frac{\partial f}{\partial G} = \frac{-\sin f}{eG}\left(\frac{G^2}{L^2}\frac{a}{r}+1\right)$$
$$\frac{\partial E}{\partial L} = \frac{1}{e}\frac{G^2}{L^3}\frac{a}{r}\sin E \qquad \frac{\partial E}{\partial G} = -\frac{1}{e}\frac{G}{L^2}\frac{a}{r}\sin E.$$

The short-periodic terms follow immediately from S_2 as given in (69). Note that $\partial S_2/\partial l$ need not be computed as it is available through (60). For the terms of intermediate period (89) and (92) are used. The complete list of partial derivatives follows.

$$
\begin{aligned}
\frac{\partial S_2}{\partial l} = {} & \frac{\mu_0^2 R_☾^2}{L'^3}\left[\frac{1}{4} c_{20}(1 - 3\cos^2 I') + \frac{3}{2} c_{22}\sin^2 I' \cos(\phi_{20} - 2h)\right] \\
& \times \left(\frac{a'^3}{r^3} - \frac{L'^3}{G'^3}\right) \\
& - \frac{\mu_0^2 R_☾^2}{L'^3}\left[\frac{3}{4} c_{20}\sin^2 I' - \frac{3}{2} c_{22}(1 + \cos^2 I')\cos(\phi_{20} - 2h)\right] \\
& \times \frac{a'^3}{r^3}\cos(2g + 2f) \\
& + 3c_{22}\frac{\mu_0^2 R_☾^2}{L'^3}\cos I' \sin(\phi_{20} - 2h)\frac{a'^3}{r^3}\sin(2g + 2f) \\
& - \frac{\mu_2 L'^7}{\mu_0^4 a_☾^3}\left[\frac{1}{8}(1 - 3\cos^2 I') - \frac{3}{8}\sin^2 I' \cos(\phi_{20} - 2h)\right] \\
& \times \left(\frac{r^2}{a'^2} - 1 - \frac{3}{2}e'^2\right) \\
& + \frac{3}{8}\frac{\mu_2 L'^7}{\mu_0^4 a_☾^3}[\sin^2 I' + (1 + \cos^2 I')\cos(\phi_{20} - 2h)] \\
& \times \left[\frac{r^2}{a'^2}\cos(2g + 2f) - \frac{5}{2}e'^2\cos 2g\right] \\
& + \frac{3}{4}\frac{\mu_2 L'^7}{\mu_0^4 a_☾^3}\cos I' \sin(\phi_{20} - 2h)\left[\frac{r^2}{a'^2}\sin(2g + 2f) - \frac{5}{2}e'^2\sin 2g\right]
\end{aligned}
\tag{98}
$$

$$
\begin{aligned}
\frac{\partial S_2}{\partial g} = {} & -2\frac{\mu_0^2 R_☾^2}{G'^3}\left[\frac{3}{8} c_{20}\sin^2 I' - \frac{3}{4} c_{22}(1 + \cos^2 I')\cos(\phi_{20} - 2h)\right] \\
& \times \left[\cos(2g + 2f) + e'\cos(2g + f) + \frac{e'}{3}\cos(2g + 3f)\right] \\
& + 3c_{22}\mu_0^2 R_☾^2 \frac{H'}{G'^4}\sin(\phi_{20} - 2h) \\
& \times \left[\sin(2g + 2f) + e'\sin(2g + f) + \frac{e'}{3}\sin(2g + 3f)\right] \\
& + \frac{3}{8}\frac{\mu_2 L'^7}{\mu_0^4 a_☾^3}[\sin^2 I' + (1 + \cos^2 I')\cos(\phi_{20} - 2h)] \\
& \times \left[(1 + e'^2)\frac{G'}{L'}\cos 2g \cos 2E - \left(1 + \frac{e'^2}{2}\right)\sin 2g \sin 2E\right.
\end{aligned}
$$

$$- \frac{e'}{3}\frac{G'}{L'}\cos 2g\,(15\cos E + \cos 3E)$$

$$+ \left(\frac{e'}{3} - \frac{e'^3}{6}\right)\sin 2g\,(15\sin E + \sin 3E)\Bigg]$$

$$+ \frac{3}{4}\frac{\mu_2 L'^7}{\mu_0^4 a_☾^3}\cos I'\sin(\phi_{20} - 2h)\left[(1 + e'^2)\frac{G'}{L'}\sin 2g\cos 2E\right.$$

$$+ \left(1 + \frac{e'^2}{2}\right)\cos 2g\sin 2E - \frac{e'}{3}\frac{G'}{L'}\sin 2g\,(15\cos E + \cos 3E)$$

$$\left. - \left(\frac{e'}{3} - \frac{e'^3}{6}\right)\cos 2g\,(15\sin E + \sin 3E)\right]. \tag{99}$$

$$\frac{\partial S_2}{\partial h} = 3c_{22}\frac{\mu_0^2 R_☾^2}{G'^3}\sin^2 I'\sin(\phi_{20} - 2h)\,[f - l + e'\sin f]$$

$$+ \frac{3}{2}c_{22}\frac{\mu_0^2 R_☾^2}{G'^3}(1 + \cos^2 I')\sin(\phi_{20} - 2h)$$

$$\times\left[\sin(2g + 2f) + e'\sin(2g + f) + \frac{e'}{3}\sin(2g + 3f)\right]$$

$$+ 3c_{22}\mu_0^2 R_☾^2\frac{H'}{G'^4}\cos(\phi_{20} + 2h)$$

$$\times\left[\cos(2g + 2f) + e'\cos(2g + f) + \frac{e'}{3}\cos(2g + 3f)\right]$$

$$- \frac{3}{4}\frac{\mu_2 L'^7}{\mu_0^4 a_☾^3}\sin^2 I'\sin(\phi_{20} - 2h)$$

$$\times\left[\left(2e' - \frac{3}{4}e'^3\right)\sin E - \frac{3}{4}e'^2\sin 2E + \frac{e'^3}{12}\sin 3E\right]$$

$$+ \frac{3}{8}\frac{\mu_2 L'^7}{\mu_0^4 a_☾^3}(1 + \cos^2 I')\sin(\phi_{20} - 2h)$$

$$\times\left[(1 + e'^2)\frac{G'}{L'}\sin 2g\cos 2E + \left(1 + \frac{e'^2}{2}\right)\cos 2g\sin 2E\right.$$

$$- \frac{e'}{3}\frac{G'}{L'}\sin 2g\,(15\cos E + \cos 3E)$$

$$\left. - \left(\frac{e'}{3} - \frac{e'^3}{6}\right)\cos 2g\,(15\sin E + \sin 3E)\right]$$

$$+ \frac{3}{4}\frac{\mu_2 L'^7}{\mu_0^4 a_☾^3}\cos I'\cos(\phi_{20} - 2h)$$

$$\times\left[(1 + e'^2)\frac{G'}{L'}\cos 2g\cos 2E - \left(1 - \frac{e'^2}{2}\right)\sin 2g\sin 2E\right.$$

$$- \frac{e'}{3}\frac{G'}{L'}\cos 2g(15\cos E + \cos 3E)$$

$$+ \left(\frac{e'}{3} - \frac{e'^3}{6}\right)\sin 2g(15\sin E + \sin 3E)\Bigg]. \tag{100}$$

$\dfrac{\partial S_2}{\partial d}$ is not needed.

$$\frac{\partial S_2}{\partial L'} = \frac{1}{4}\frac{\mu_0^2 R_☾^2}{e'L'^3G'}[c_{20}(1 - 3\cos^2 I') + 6c_{22}\sin^2 I'\cos(\phi_{20} - 2h)]$$

$$\times\left(\frac{G'^2}{L'^2}\frac{a'^2}{r^2} + \frac{a'}{r} + 1\right)\sin f$$

$$- \frac{3}{8}\frac{\mu_0^2 R_☾^2}{e'L'^3G'}[c_{20}\sin^2 I' - 2c_{22}(1 + \cos^2 I')\cos(\phi_{20} - 2h)]$$

$$\times\Bigg\{[2\cos(2g + 2f) + e'\cos(2g + f) + e'\cos(2g + 3f)]$$

$$\times\left(\frac{a'}{r} + \frac{L'^2}{G'^2}\right)\sin f + \sin(2g + f) + \frac{1}{3}\sin(2g + 3f)\Bigg\}$$

$$+ \frac{3}{2}c_{22}\mu_0^2 R_☾^2\frac{H'}{e'L'^3G'^2}\sin(\phi_{20} - 2h)$$

$$\times\Bigg\{[2\sin(2g + 2f) + e'\sin(2g + f) + e'\sin(2g + 3f)]$$

$$\times\left(\frac{a'}{r} + \frac{L'^2}{G'^2}\right)\sin f - \cos(2g + f) - \frac{1}{3}\cos(2g + 3f)\Bigg\}$$

$$+ \frac{1}{8}\frac{\mu_2}{\mu_0^4 a_☾^3}\frac{L'^6}{e'}[(1 - 3\cos^2 I') - 3\sin^2 I'\cos(\phi_{20} - 2h)]$$

$$\times\Bigg\{\frac{1}{4}(8 + 39e'^2 - 12e'^4)\sin E - \frac{3}{4}(2e' + 5e'^3)\sin 2E$$

$$+ \left(\frac{1}{4}e'^2 + \frac{1}{3}e'^4\right)\sin 3E + \Bigg[\left(2e' - \frac{3}{4}e'^3\right)\cos E$$

$$- \frac{3}{2}e'^2\cos 2E + \frac{1}{4}e'^3\cos 3E\Bigg]\frac{G'^2}{L'^2}\frac{a'}{r}\sin E\Bigg\}$$

$$+ \frac{3}{16}\frac{\mu_2}{\mu_0^4 a_☾^3}\frac{L'^6}{e'}[\sin^2 I' + (1 + \cos^2 I')\cos(\phi_{20} - 2h)]$$

$$\times\Bigg\{4(2e' + e'^3)\frac{G'}{L'}\sin 2g\cos 2E$$

$$+ \frac{1}{2}(16e' + 5e'^3)\cos 2g\sin 2E$$

$$
\begin{aligned}
&- \frac{1}{3}(1 + 5e'^2)\frac{G'}{L'}\sin 2g(15\cos E + \cos 3E) \\
&- \frac{1}{6}(2 + 9e'^2 - 4e'^4)\cos 2g(15\sin E + \sin 3E) \\
&- \Bigg[2(1 + e'^2)\frac{G'}{L'}\sin 2g\sin 2E - (2 + e'^2)\cos 2g\cos 2E \\
&- e'\frac{G'}{L'}\sin 2g(5\sin E + \sin 3E) \\
&+ \left(e' + \frac{e'^3}{2}\right)\cos 2g(5\cos E + \cos 3E)\Bigg]\frac{G'^2}{L'^2}\frac{a'}{r}\sin E\Bigg\} \\
- \frac{3}{8}\frac{\mu_2}{\mu_0^4 a_{☾}^3}\frac{L'^6}{e'}\cos I'\sin(\phi_{20} - 2h)&\Bigg\{4(2e' + e'^3)\frac{G'}{L'}\cos 2g\cos 2E \\
&- \frac{1}{2}(16e' + 5e'^3)\sin 2g\sin 2E - \frac{1}{3}(1 + 5e'^2)\frac{G'}{L'} \\
&\times \cos 2g(15\cos E + \cos 3E) + \frac{1}{6}(2 + 9e'^2 - 4e'^4) \\
&\times \sin 2g(15\sin E + \sin 3E) - \Bigg[2(1 + e'^2)\frac{G'}{L'}\cos 2g\sin 2E \\
&+ (2 + e'^2)\sin 2g\cos 2E - e'\frac{G'}{L'}\cos 2g(5\sin E + \sin 3E) \\
&- \left(e' - \frac{e'^3}{2}\right)\sin 2g(5\cos E + \cos 3E)\Bigg]\frac{G'^2}{L'^2}\frac{a'}{r}\sin E\Bigg\} \qquad (101)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial S_2}{\partial G'} = &- \frac{3}{4}\frac{\mu_0^2 R_{☾}^2}{G'^4}[c_{20}(1 - 5\cos^2 I') + 2c_{22}(3 - 5\cos^2 I')\cos(\phi_{20} - 2h)] \\
&\times [f - l + e'\sin f] \\
&- \frac{1}{4}\frac{\mu_0^2 R_{☾}^2}{e'L'^2G'^2}[c_{20}(1 - 3\cos^2 I') + 6c_{22}\sin^2 I'\cos(\phi_{20} - 2h)] \\
&\times \left(\frac{G'^2}{L'^2}\frac{a'^2}{r^2} + \frac{a'}{r} + 1\right)\sin f \\
&+ \frac{3}{8}\frac{\mu_0^2 R_{☾}^2}{G'^4}[c_{20}(3 - 5\cos^2 I') - 2c_{22}(3 + 5\cos^2 I')\cos(\phi_{20} - 2h)] \\
&\times \left[\sin(2g + 2f) + e'\sin(2g + f) + \frac{e'}{3}\sin(2g + 3f)\right] \\
&+ \frac{3}{8}\frac{\mu_0^2 R_{☾}^2}{e'L'^2G'^2}[c_{20}\sin^2 I' - 2c_{22}(1 + \cos^2 I')\cos(\phi_{20} - 2h)] \\
&\times \Big\{[2\cos(2g + 2f) + e'\cos(2g + f) + e'\cos(2g + 3f)]
\end{aligned}
$$

$$\times\left(\frac{a'}{r}+\frac{L'^2}{G'^2}\right)\sin f+\sin(2g+f)+\frac{1}{3}\sin(2g+3f)\Bigg\}$$

$$+\,6c_{22}\mu_0^2R_{☾}^2\,\frac{H'}{G'^5}\sin(\phi_{20}-2h)$$

$$\times\left[\cos(2g+2f)+e'\cos(2g+f)+\frac{e'}{3}\cos(2g+3f)\right]$$

$$-\,\frac{3}{2}\,c_{22}\mu_0^2R_{☾}^2\,\frac{H'}{e'L'^2G'^3}\sin(\phi_{20}-2h)$$

$$\times\Bigg\{\left[2\sin(2g+2f)+e'\sin(2g+f)+e'\sin(2g+3f)\right]$$

$$\times\left(\frac{a'}{r}+\frac{L'^2}{G'^2}\right)\sin f-\cos(2g+f)-\frac{1}{3}\cos(2g+3f)\Bigg\}$$

$$+\,\frac{3}{4}\,\frac{\mu_2}{\mu_0^4a_{☾}^3}\,\frac{L'^7H'^2}{G'^3}\left[1-\cos(\phi_{20}-2h)\right]$$

$$\times\left[\left(2e'-\frac{3}{4}e'^3\right)\sin E-\frac{3}{4}e'^2\sin 2E+\frac{e'^3}{12}\sin 3E\right]$$

$$-\,\frac{1}{8}\,\frac{\mu_2}{\mu_0^4a_{☾}^3}\,\frac{L'^5G'}{e'}\left[(1-3\cos^2 I')-3\sin^2 I'\cos(\phi_{20}-2h)\right]$$

$$\times\Bigg\{\left(2-\frac{9}{4}e'^2\right)\sin E-\frac{3}{2}e'\sin 2E+\frac{1}{4}e'^2\sin 3E$$

$$+\Bigg[\left(2e'-\frac{3}{4}e'^3\right)\cos E-\frac{3}{2}e'^2\cos 2E$$

$$+\,\frac{1}{4}e'^3\cos 3E\Bigg]\frac{a'}{r}\sin E\Bigg\}$$

$$+\,\frac{3}{8}\,\frac{\mu_2}{\mu_0^4a_{☾}^3}\,\frac{L'^7H'^2}{G'^3}\left[1-\cos(\phi_{20}-2h)\right]\Bigg[(1+e'^2)\,\frac{G'}{L'}\sin 2g\cos 2E$$

$$+\left(1+\frac{e'^2}{2}\right)\cos 2g\sin 2E-\frac{e'}{3}\,\frac{G'}{L'}\sin 2g(15\cos E+\cos 3E)$$

$$-\,\frac{1}{6}(2e'-e'^3)\cos 2g(15\sin E+\sin 3E)\Bigg]$$

$$-\,\frac{3}{16}\,\frac{\mu_2}{\mu_0^4a_{☾}^3}\,\frac{L'^6}{e'}\left[\sin^2 I'+(1+\cos^2 I')\cos(\phi_{20}-2h)\right]$$

$$\times\Bigg\{(e'-3e'^3)\sin 2g\cos 2E+e'\,\frac{G'}{L'}\cos 2g\sin 2E$$

$$-\,\frac{1}{3}(1-2e'^2)\sin 2g(15\cos E+\cos 3E)$$

$$- \left(\frac{1}{3} - \frac{e'^2}{2}\right) \frac{G'}{L'} \cos 2g\,(15 \sin E + \sin 3E)$$

$$- \left[2(1 + e'^2) \frac{G'}{L'} \sin 2g \sin 2E - (2 + e'^2) \cos 2g \cos 2E \right.$$

$$- e' \frac{G'}{L'} \sin 2g\,(5 \sin E + \sin 3E)$$

$$\left. + \left(e' - \frac{e'^3}{2}\right) \cos 2g\,(5 \cos E + \cos 3E) \right] \frac{G'}{L'} \frac{a'}{r} \sin E \Bigg\}$$

$$+ \frac{3}{8} \frac{\mu_2}{\mu_0^4 a_☾^3} \frac{L'^7 H'}{G'^2} \sin(\phi_{20} - 2h) \left[(1 + e'^2) \frac{G'}{L'} \cos 2g \cos 2E \right.$$

$$- \left(1 + \frac{e'^2}{2}\right) \sin 2g \sin 2E - \frac{e'}{3} \frac{G'}{L'} \cos 2g\,(15 \cos E + \cos 3E)$$

$$\left. + \frac{1}{6}(2e' - e'^3) \sin 2g\,(15 \sin E + \sin 3E) \right]$$

$$+ \frac{3}{8} \frac{\mu_2}{\mu_0^4 a_☾^3} \frac{L'^6}{e'} \cos I' \sin(\phi_{20} - 2h)$$

$$\times \Bigg\{ (e' - 3e'^3) \cos 2g \cos 2E - e' \frac{G'}{L'} \sin 2g \sin 2E$$

$$- \frac{1}{3}(1 - 2e'^2) \cos 2g\,(15 \cos E + \cos 3E)$$

$$+ \left(\frac{1}{3} - \frac{e'^2}{2}\right) \frac{G'}{L'} \sin 2g\,(15 \sin E + \sin 3E)$$

$$- \left[2(1 + e'^2) \frac{G'}{L'} \cos 2g \sin 2E + (2 + e'^2) \sin 2g \cos 2E \right.$$

$$- e' \frac{G'}{L'} \cos 2g\,(5 \sin E + \sin 3E) - \left(e' - \frac{e'^3}{2}\right)$$

$$\left. \times \sin 2g\,(5 \cos E + \cos 3E) \right] \frac{G'}{L'} \frac{a'}{r} \sin E \Bigg\} \qquad (102)$$

$$\frac{\partial S_2}{\partial H'} = - \frac{3}{2} \mu_0^2 R_☾^2 \frac{H'}{G'^5} [c_{20} + 2c_{22} \cos(\phi_{20} - 2h)]\,[f - l + e' \sin f]$$

$$+ \frac{3}{4} \mu_0^2 R_☾^2 \frac{H'}{G'^5} [c_{20} + 2c_{22} \cos(\phi_{20} - 2h)]$$

$$\times \left[\sin(2g + 2f) + e' \sin(2g + f) + \frac{e'}{3} \sin(2g + 3f) \right]$$

$$- \frac{3}{2} c_{22} \frac{\mu_0^2 R_☾^2}{G'^4} \sin(\phi_{20} - 2h)$$

$$\times\left[\cos(2g+2f)+e'\cos(2g+f)+\frac{e'}{3}\cos(2g+3f)\right]$$

$$-\frac{3}{4}\frac{\mu_2}{\mu_0^4 a_☾^3}\frac{L'^7 H'}{G'^2}[1-\cos(\phi_{20}-2h)]$$

$$\times\left[\left(2e'-\frac{3}{4}e'^3\right)\sin E-\frac{3}{4}e'^2\sin 2E+\frac{e'^3}{12}\sin 3E\right]$$

$$-\frac{3}{8}\frac{\mu_2}{\mu_0^4 a_☾^3}\frac{L'^7 H'}{G'^2}[1-\cos(\phi_{20}-2h)]$$

$$\times\left[(1+e'^2)\frac{G'}{L'}\sin 2g\cos 2E+\left(1+\frac{e'^2}{2}\right)\cos 2g\sin 2E\right.$$

$$-\frac{e'}{3}\frac{G'}{L'}\sin 2g(15\cos E+\cos 3E)-\left(\frac{e'}{3}-\frac{e'^3}{6}\right)$$

$$\left.\times\cos 2g(15\sin E+\sin 3E)\right]$$

$$-\frac{3}{8}\frac{\mu_2}{\mu_0^4 a_☾^3}\frac{L'^7}{G'}\sin(\phi_{20}-2h)\left[(1+e'^2)\frac{G'}{L'}\cos 2g\cos 2E\right.$$

$$-\left(1+\frac{e'^2}{2}\right)\sin 2g\sin 2E-\frac{e'}{3}\frac{G'}{L'}\cos 2g(15\cos E+\cos 3E)$$

$$\left.+\left(\frac{e'}{3}-\frac{e'^3}{6}\right)\sin 2g(15\sin E+\sin 3E)\right] \quad (103)$$

$$\frac{\partial S_2}{\partial D'}=0 \quad (104)$$

$$\frac{\partial S_1^*}{\partial l'}=0 \quad (105)$$

$$\frac{\partial S_1^*}{\partial g'}=-\frac{15}{8}\frac{\mu_2 L''^4 e''^2}{n_{20}\mu_0^2 a_☾^3}[(1+\cos^2 I'')\sin 2g'\sin(\phi'_{20}-2h')$$

$$+2\cos I''\cos 2g'\cos(\phi'_{20}-2h')] \quad (106)$$

$$\frac{\partial S_1^*}{\partial h'}=-\frac{3}{4}\left\{4\frac{c_{22}}{n_{20}}\frac{\mu_0^4 R_☾^2}{L''^3 G''^3}\sin^2 I''+\frac{\mu_2 L''^4}{n_{20}\mu_0^2 a_☾^3}\left[\left(1+\frac{3}{2}e''^2\right)\sin^2 I''\right.\right.$$

$$\left.\left.+\frac{5}{2}e''^2(1+\cos^2 I'')\cos 2g'\right]\right\}\cos(\phi'_{20}-2h') \quad (107)$$

$\dfrac{\partial S_1^*}{\partial d'}$ is not needed.

$$\frac{\partial S_1^*}{\partial L''}=-\frac{3}{8}\left\{12\frac{c_{22}}{n_{20}}\frac{\mu_0^4 R_☾^2}{L''^4 G''^3}\sin^2 I''-\frac{\mu_2 L''^3}{n_{20}\mu_0^2 a_☾^3}[(7+3e''^2)\sin^2 I''\right.$$

$$+ 5(1 + e''^2)(1 + \cos^2 I'')\cos 2g']\Big\}\sin(\phi'_{20} - 2h') \tag{108}$$

$$\frac{\partial S_1^*}{\partial G''} = -\frac{3}{8}\Big\{4\frac{c_{22}}{n_{20}}\frac{\mu_0^4 R_☾^2}{L''^3 G''^4}(3 - 5\cos^2 I'') + \frac{\mu_2}{n_{20}\mu_0^2 a_☾^3}\frac{L''^4}{G''}$$
$$\times [3 - 3e''^2 - 5\cos^2 I'' + 5(1 - e''^2 + \cos^2 I'')\cos 2g']\Big\}$$
$$\times \sin(\phi'_{20} - 2h') + \frac{15}{8}\frac{\mu_2}{n_{20}\mu_0^2 a_☾^3}\frac{L''^4 H''}{G''^2}$$
$$\times (2 - e''^2)\sin 2g' \cos(\phi'_{20} - 2h') \tag{109}$$

$$\frac{\partial S_1^*}{\partial H''} = -\frac{3}{4}\Big[4\frac{c_{22}}{n_{20}}\mu_0^4 R_☾^2 \frac{H''}{L''^3 G''^5} + \frac{\mu_2}{n_{20}\mu_0^2 a_☾^3}\frac{L''^4 H''}{G''^2}$$
$$\times \Big(1 + \frac{3}{2}e''^2 - \frac{5}{2}e''^2\cos 2g'\Big)\Big]\sin(\phi'_{20} - 2h')$$
$$- \frac{15}{8}\frac{\mu_2}{n_{20}\mu_0^2 a_☾^3}\frac{L''^4}{G''}e''^2 \sin 2g' \cos(\phi'_{20} - 2h') \tag{110}$$

$$\frac{\partial S_1^*}{\partial D''} = 0 \tag{111}$$

$$\frac{\partial S_2^*}{\partial l'} = 0 \tag{112}$$

$$\frac{\partial S_2^*}{\partial g'} = -\frac{15}{4}\gamma_☾ \frac{\mu_2 L''^4}{n_{23}\mu_0^2 a_☾^3} e''^2 \sin I'' [\cos I'' \sin 2g' \sin(\phi'_{23} - h')$$
$$+ \cos 2g' \cos(\phi'_{23} - h')]$$
$$- \frac{15}{8} e_☾ \frac{\mu_2 L''^4}{\mu_0^2 a_☾^3} e''^2 \Big\{\frac{3}{n_3}\sin^2 I'' \sin 2g' \sin\phi'_3 + \frac{1}{2}(1 + \cos^2 I'')$$
$$\times \sin 2g' \Big[\frac{7}{n_{31}}\sin(\phi'_{31} - 2h') - \frac{1}{n_2}\sin(\phi'_2 - 2h')\Big]$$
$$+ \cos I'' \cos 2g' \Big[\frac{7}{n_{31}}\cos(\phi'_{31} - 2h') - \frac{1}{n_2}\cos(\phi'_2 - 2h')\Big]\Big\}$$
$$- \frac{15}{16}\gamma_☾^2 \frac{\mu_2 L''^4}{\mu_0^2 a_☾^3} e''^2 \Big[\frac{3}{n_{25}}\sin^2 I'' \sin 2g' \sin\phi'_{25} - \frac{1}{n_{20}}(1 + \cos^2 I'')$$
$$\times \sin 2g' \sin(\phi'_{20} - 2h') - \frac{2}{n_{20}}\cos I'' \cos 2g' \cos(\phi'_{20} - 2h')\Big]$$
$$- \frac{15}{8} e_☾ \gamma_☾ \frac{\mu_2 L''^4}{\mu_0^2 a_☾^3} \sin I'' e''^2 \Big\{\cos I'' \sin 2g' \Big[\frac{7}{n_{33}}\sin(\phi'_{33} - h')$$
$$- \frac{1}{n_6}\sin(\phi'_6 - h') - \frac{3}{n_7}\sin(\phi'_7 - h') - \frac{3}{n_8}\sin(\phi'_8 + h')\Big]$$

$$
\begin{aligned}
&+ \cos 2g' \left[\frac{7}{n_{33}} \cos(\phi'_{33} - h') - \frac{1}{n_6} \cos(\phi'_6 - h') \right. \\
&\left.\left. - \frac{3}{n_7} \cos(\phi'_7 - h') + \frac{3}{n_8} \cos(\phi'_8 + h') \right] \right\} \\
- \frac{15}{16} e_{\text{☾}}^2 \frac{\mu_2 L''^4}{\mu_0^2 a_{\text{☾}}^3} e''^2 &\left\{ \frac{9}{n_{24}} \sin^2 I'' \sin 2g' \sin \phi'_{24} \right. \\
&+ (1 + \cos^2 I'') \sin 2g' \left[\frac{17}{n_{41}} \sin(\phi'_{41} - 2h') \right. \\
&\left. - \frac{5}{n_{20}} \sin(\phi'_{20} - 2h') \right] + 2 \cos I'' \cos 2g' \\
&\left.\times \left[\frac{17}{n_{41}} \cos(\phi'_{41} - 2h') - \frac{5}{n_{20}} \cos(\phi'_{20} - 2h') \right] \right\} \\
- \frac{15}{64} \frac{\mu_2 L''^6}{\mu_0^3 a_{\text{☾}}^4} &\left\{ \frac{1}{n_1} \left[\left(e'' + \frac{3}{4} e''^3 \right) (4 - 5 \sin^2 I'') \sin g' \right. \right. \\
&\left. + \frac{105}{4} e''^3 \sin^2 I'' \sin 3g' \right] \sin(\phi'_1 - h') \\
&+ \frac{5}{n_{30}} \left[\left(e'' + \frac{3}{4} e''^3 \right) \sin^2 I'' \sin g' \right. \\
&\left. + \frac{7}{4} e''^3 (4 - 3 \sin^2 I'') \sin 3g' \right] \sin(\phi'_{30} - 3h') \\
&+ \frac{1}{n_1} \cos I'' \left[\left(e'' + \frac{3}{4} e''^3 \right) (4 - 15 \sin^2 I'') \cos g' \right. \\
&\left. + \frac{105}{4} e''^3 \sin^2 I'' \cos 3g' \right] \cos(\phi'_1 - h') \\
&+ \frac{5}{n_{30}} \cos I'' \left[\left(e'' + \frac{3}{4} e''^3 \right) \sin^2 I'' \cos g' \right. \\
&\left.\left. + \frac{7}{4} e''^3 (4 - \sin^2 I'') \cos 3g' \right] \cos(\phi'_{30} - 3h') \right\} \\
+ \frac{45}{16} \frac{c_{20}}{n_{20}^2} \frac{\mu_0^2 \mu_2 R_{\text{☾}}^2}{a_{\text{☾}}^3} \frac{L''}{G''^4} &e''^2 [(4 - 10 \sin^2 I'' + 5 \sin^4 I'') \cos 2g' \\
&\times \cos(\phi'_{20} - 2h') + 4 \cos I'' (1 - 2 \sin^2 I'') \\
&\times \sin 2g' \sin(\phi'_{20} - 2h')] \\
- \frac{45}{16} \frac{c_{22}}{n_{20}^2} \frac{\mu_0^2 \mu_2 R_{\text{☾}}^2}{a_{\text{☾}}^3} \frac{L''}{G''^4} &e''^2 \sin^2 I'' \left\{ \cos 2g' \left[(4 - 10 \sin^2 I'') \right. \right. \\
&\left. \times \cos(\phi'_{20} - 2h') + \frac{1}{2} (6 - 5 \sin^2 I'') \cos(2\phi'_{20} - 4h') \right]
\end{aligned}
$$

$$+ \cos I'' \sin 2g' [4 \sin(\phi'_{20} - 2h') + 3 \sin(2\phi'_{20} - 4h')] \Big\}$$

$$- \frac{45}{16} \frac{\mu_2^2}{n_{20}^2 \mu_0^4 a_☾^6} L''^6 G'' e''^2 [2(1 - 2\sin^2 I'') \cos 2g' \cos(\phi'_{20} - 2h')$$

$$+ \cos I''(2 - 3\sin^2 I'') \sin 2g' \sin(\phi'_{20} - 2h')]$$

$$- \frac{45}{64} \frac{\mu_2^2}{n_{20}^2 \mu_0^4 a_☾^6} \frac{L''^8}{G''} e''^2 \Big\{ \Big[(3 - 3e''^2 - 5\sin^2 I'') \sin^2 I'' \cos 2g'$$

$$- (20e''^2 - 20\cos^2 I'' - \frac{5}{2} \sin^4 I'' - 15e''^2 \sin^2 I'') \cos 4g' \Big]$$

$$\times \cos(2\phi'_{20} - 4h') + \cos I'' [3(1 - e''^2) \sin^2 I'' \sin 2g'$$

$$+ 5(4 - 4e''^2 - 2\sin^2 I'' + e''^2 \sin^2 I'') \sin 4g']$$

$$\times \sin(2\phi'_{20} - 4h') \Big\} \tag{113}$$

$$\frac{\partial S_2^*}{\partial h'} = -\frac{3}{2} I_☾ \frac{c_{22}}{n_{23}} \frac{\mu_0^4 R_☾^2}{L''^3 G''^3} \sin 2I'' \cos(\phi'_{23} - h')$$

$$- \frac{3}{8} \varepsilon_1 \frac{c_{20}}{n_7} \frac{\mu_0^4 R_☾^2}{L''^3 G''^3} \sin 2I'' \cos(\phi'_7 - h')$$

$$+ \frac{3}{8} \gamma_☾ \frac{\mu_2 L''^4}{n_{23} \mu_0^2 a_☾^3} \Big[\sin 2I'' \Big(1 + \frac{3}{2} e''^2 - \frac{5}{2} e''^2 \cos 2g'\Big)$$

$$\times \cos(\phi'_{23} - h') - 5e''^2 \sin I'' \sin 2g' \sin(\phi'_{23} - h') \Big]$$

$$- \frac{3}{8} e_☾ \frac{\mu_2 L''^4}{\mu_0^2 a_☾^3} \Big\{ \Big[\Big(1 + \frac{3}{2} e''^2\Big) \sin^2 I'' + \frac{5}{2} e''^2 (1 + \cos^2 I'') \cos 2g' \Big]$$

$$\times \Big[\frac{7}{n_{31}} \cos(\phi'_{31} - 2h') - \frac{1}{n_2} \cos(\phi'_2 - 2h') \Big]$$

$$+ 5e''^2 \cos I'' \sin 2g'$$

$$\times \Big[\frac{7}{n_{31}} \sin(\phi'_{31} - 2h') - \frac{1}{n_2} \sin(\phi'_2 - 2h') \Big] \Big\}$$

$$+ \frac{3}{8} \gamma_☾^2 \frac{\mu_2 L''^4}{n_{20} \mu_0^2 a_☾^3} \Big\{ \Big[\Big(1 + \frac{3}{2} e''^2\Big) \sin^2 I'' + \frac{5}{2} e''^2 (1 + \cos^2 I'') \cos 2g' \Big]$$

$$\times \cos(\phi'_{20} - 2h') + 5e''^2 \cos I'' \sin 2g' \sin(\phi'_{20} - 2h') \Big\}$$

$$+ \frac{3}{16} e_☾ \gamma_☾ \frac{\mu_2 L''^4}{\mu_0^2 a_☾^3} \Big\{ \sin 2I'' \Big(1 + \frac{3}{2} e''^2 - \frac{5}{2} e''^2 \cos 2g'\Big)$$

$$\times \Big[\frac{7}{n_{33}} \cos(\phi'_{33} - h') - \frac{1}{n_6} \cos(\phi'_6 - h')$$

$$- \frac{3}{n_7} \cos(\phi_7' - h') + \frac{3}{n_8} \cos(\phi_8' + h')\Bigg]$$

$$- 5e''^2 \sin I'' \sin 2g' \left[\frac{7}{n_{33}} \sin(\phi_{33}' - h') - \frac{1}{n_6} \sin(\phi_6' - h') \right.$$

$$\left. - \frac{3}{n_7} \sin(\phi_7' - h') - \frac{3}{n_8} \sin(\phi_8' + h') \right]\Bigg\}$$

$$- \frac{3}{8} e_{☾}^2 \frac{\mu_2 L''^4}{\mu_0^2 a_{☾}^3} \left\{ \left[\left(1 + \frac{3}{2} e''^2\right) \sin^2 I'' + \frac{5}{2} e''^2 (1 + \cos^2 I'') \cos 2g' \right] \right.$$

$$\times \left[\frac{17}{n_{41}} \cos(\phi_{41}' - 2h') - \frac{5}{n_{20}} \cos(\phi_{20}' - 2h') \right]$$

$$+ 5e''^2 \cos I'' \sin 2g' \left[\frac{17}{n_{41}} \sin(\phi_{41}' - 2h') \right.$$

$$\left. \left. - \frac{5}{n_{20}} \sin(\phi_{20}' - 2h') \right] \right\}$$

$$- \frac{5}{64} \frac{\mu_2 L''^6}{\mu_0^3 a_{☾}^4} \left\{ \frac{3}{n_1} \left[\left(e'' + \frac{3}{4} e''^3\right) (4 - 5 \sin^2 I'') \cos g' \right. \right.$$

$$\left. + \frac{35}{4} e''^3 \sin^2 I'' \cos 3g' \right] \cos(\phi_1' - h')$$

$$+ \frac{15}{n_{30}} \left[3\left(e'' + \frac{3}{4} e''^3\right) \sin^2 I'' \cos g' \right.$$

$$\left. + \frac{7}{4} e''^3 (4 - 3 \sin^2 I'') \cos 3g' \right] \cos(\phi_{30}' - 3h')$$

$$+ \frac{3}{n_1} \cos I'' \left[\left(e'' + \frac{3}{4} e''^3\right) (4 - 15 \sin^2 I'') \sin g' \right.$$

$$\left. + \frac{35}{4} e''^3 \sin^2 I'' \sin 3g' \right] \sin(\phi_1' - h')$$

$$+ \frac{15}{n_{30}} \cos I'' \left[3\left(e'' + \frac{3}{4} e''^3\right) \sin^2 I'' \sin g' \right.$$

$$\left. \left. + \frac{7}{4} e''^3 (4 - \sin^2 I'') \sin 3g' \right] \sin(\phi_{30}' - 3h') \right\}$$

$$- 9 \frac{c_{22}}{n_{20}^2} \mu_0^8 R_{☾}^4 \frac{H''}{L''^6 G''^8} \sin^2 I'' [c_{20} \cos(\phi_{20}' - 2h')$$

$$+ c_{22} \cos(2\phi_{20}' - 4h')]$$

$$+ \frac{45}{16} \frac{c_{20}}{n_{20}^2} \frac{\mu_0^2 \mu_2 R_{☾}^2}{a_{☾}^3} \frac{L''}{G''^4} e''^2 (4 - 10 \sin^2 I'' + 5 \sin^4 I'')$$

$$\times \sin 2g' \sin(\phi_{20}' - 2h')$$

$$-\frac{9}{8}\frac{c_{20}}{n_{20}^2}\frac{\mu_0^2\mu_2R_☾^2}{a_☾^3}\frac{L''H''}{G''^5}e''^2\left[\left(\frac{2}{e''^2}+3\right)\sin^2 I''\right.$$

$$\left.-10(1-2\sin^2 I'')\cos 2g'\right]\cos(\phi'_{20}-2h')$$

$$-\frac{45}{16}\frac{c_{22}}{n_{20}^2}\frac{\mu_0^2\mu_2R_☾^2}{a_☾^3}\frac{L''}{G''^4}e''^2\sin^2 I''\sin 2g'[(4-10\sin^2 I'')$$

$$\times\sin(\phi'_{20}-2h')+(6-5\sin^2 I'')\sin(2\phi'_{20}-4h')]$$

$$+\frac{9}{8}\frac{c_{22}}{n_{20}^2}\frac{\mu_0^2\mu_2R_☾^2}{a_☾^3}\frac{L''H''}{G''^5}e''^2\sin^2 I''\left[\left(\frac{4}{e''^2}+6-10\cos 2g'\right)\right.$$

$$\times\cos(\phi'_{20}-2h')-\left(\frac{4}{e''^2}+6+15\cos 2g'\right)$$

$$\left.\times\cos(2\phi'_{20}-4h')\right]$$

$$-\frac{45}{8}\frac{\mu_2^2}{n_{20}^2\mu_0^4a_☾^6}L''^6G''e''^2(1-2\sin^2 I'')\sin 2g'\sin(\phi'_{20}-2h')$$

$$+\frac{9}{16}\frac{\mu_2^2}{n_{20}^2\mu_0^4a_☾^6}L''^6G''e''^2\left[\left(\frac{2}{e''^2}-17\right)\sin^2 I''\right.$$

$$\left.-5(2-3\sin^2 I'')\cos 2g'\right]\cos(\phi'_{20}-2h')$$

$$-\frac{45}{64}\frac{\mu_2^2}{n_{20}^2\mu_0^4a_☾^6}\frac{L''^8}{G''}e''^2\left[2(3-3e''^2-5\sin^2 I'')\sin^2 I''\sin 2g'\right.$$

$$\left.-\left(20e''^2-20\cos^2 I''-\frac{5}{2}\sin^4 I''-15e''^2\sin^2 I''\right)\sin 4g'\right]$$

$$\times\sin(2\phi'_{20}-4h')$$

$$-\frac{9}{64}\frac{\mu_2^2}{n_{20}^2\mu_0^4a_☾^6}\frac{L''^8H''}{G''^2}e''^2\left[\left(\frac{4}{e''^2}+12+9e''^2\right)\sin^2 I''\right.$$

$$+30(1-e''^2)\sin^2 I''\cos 2g'+25(4-4e''^2-2\sin^2 I''$$

$$\left.+e''^2\sin^2 I'')\cos 4g'\right]\cos(2\phi'_{20}-4h') \tag{114}$$

$\dfrac{\partial S_2^*}{\partial d'}$ is not needed.

$$\frac{\partial S_2^*}{\partial L''}=-\frac{9}{2}I_☾\frac{c_{22}}{n_{23}}\frac{\mu_0^4R_☾^2}{L''^4G''^3}\sin 2I''\sin(\phi'_{23}-h')$$

$$-\frac{9}{8}\varepsilon_1\frac{c_{20}}{n_7}\frac{\mu_0^4R_☾^2}{L''^4G''^3}\sin 2I''\sin(\phi'_7-h')$$

$$-\frac{3}{8}\gamma_☾\frac{\mu_2L''^3}{n_{23}\mu_0^2a_☾^3}\{\sin 2I''[(7+3e''^2)-5(1+e''^2)\cos 2g']$$

$$\times \sin(\phi'_{23} - h') + 10(1 + e''^2)\sin I'' \sin 2g' \cos(\phi'_{23} - h')\}$$

$$- \frac{3}{8} e_{☾} \frac{\mu_2 L''^3}{\mu_0^2 a_{☾}^3} \left\{ \frac{1}{n_3} [(7 + 3e''^2)(1 - 3\cos^2 I'') \right.$$

$$- 15(1 + e''^2)\sin^2 I'' \cos 2g'] \sin \phi'_3$$

$$- \frac{1}{2}[(7 + 3e''^2)\sin^2 I'' + 5(1 + e''^2)(1 + \cos^2 I'')\cos 2g']$$

$$\times \left[\frac{7}{n_{31}} \sin(\phi'_{31} - 2h') - \frac{1}{n_2}\sin(\phi'_2 - 2h') \right]$$

$$+ 5(1 + e''^2)\cos I'' \sin 2g'$$

$$\left. \times \left[\frac{7}{n_{31}} \cos(\phi'_{31} - 2h') - \frac{1}{n_2}\cos(\phi'_2 - 2h') \right] \right\}$$

$$- \frac{3}{16} \gamma_{☾}^2 \frac{\mu_2 L''^3}{\mu_0^2 a_{☾}^3} \left\{ \frac{1}{n_{25}} [(7 + 3e''^2)(1 - 3\cos^2 I'') - 15(1 + e''^2) \right.$$

$$\times \sin^2 I'' \cos 2g'] \sin \phi'_{25} + \frac{1}{n_{20}} [(7 + 3e''^2)\sin^2 I''$$

$$+ 5(1 + e''^2)(1 + \cos^2 I'')\cos 2g'] \sin(\phi'_{20} - 2h')$$

$$\left. - \frac{10}{n_{20}} (1 + e''^2)\cos I'' \sin 2g' \cos(\phi'_{20} - 2h') \right\}$$

$$- \frac{3}{16} e_{☾}\gamma_{☾} \frac{\mu_2 L''^3}{\mu_0^2 a_{☾}^3} \left\{ \sin 2I'' [(7 + 3e''^2) - 5(1 + e''^2)\cos 2g'] \right.$$

$$\times \left[\frac{7}{n_{33}} \sin(\phi'_{33} - h') - \frac{1}{n_6}\sin(\phi'_6 - h') - \frac{3}{n_7}\sin(\phi'_7 - h') \right.$$

$$\left. - \frac{3}{n_8}\sin(\phi'_8 + h') \right] + 10(1 + e''^2)\sin I'' \sin 2g'$$

$$\times \left[\frac{7}{n_{33}} \cos(\phi'_{33} - h') - \frac{1}{n_6}\cos(\phi'_6 - h') \right.$$

$$\left. \left. - \frac{3}{n_7}\cos(\phi'_7 - h') + \frac{3}{n_8}\cos(\phi'_8 + h') \right] \right\}$$

$$- \frac{3}{16} e_{☾}^2 \frac{\mu_2 L''^3}{\mu_0^2 a_{☾}^3} \left\{ \frac{3}{n_{24}} [(7 + 3e''^2)(1 - 3\cos^2 I'') \right.$$

$$- 15(1 + e''^2)\sin^2 I'' \cos 2g'] \sin \phi'_{24}$$

$$- [(7 + 3e''^2)\sin^2 I'' + 5(1 + e''^2)(1 + \cos^2 I'')\cos 2g']$$

$$\times \left[\frac{17}{n_{41}} \sin(\phi'_{41} - 2h') - \frac{5}{n_{20}}\sin(\phi'_{20} - 2h') \right]$$

$$+ 10(1 + e''^2)\cos I'' \sin 2g'$$

$$\times\left[\frac{17}{n_{41}}\cos(\phi'_{41}-2h')-\frac{5}{n_{20}}\cos(\phi'_{20}-2h')\right]\bigg\}$$

$$+\frac{15}{64}\frac{\mu_2}{\mu_0^3 a_{\zeta}^4}\frac{L''^5}{e''}\bigg\{\frac{1}{n_1}\bigg[\left(1+\frac{29}{4}e''^2+\frac{9}{4}e''^4\right)(4-5\sin^2 I'')\cos g'$$

$$+\frac{105}{4}(e''^2+e''^4)\sin^2 I''\cos 3g'\bigg]\sin(\phi'_1-h')$$

$$+\frac{5}{n_{30}}\bigg[\left(1+\frac{29}{4}e''^2+\frac{9}{4}e''^4\right)\sin^2 I''\cos g'$$

$$+\frac{7}{4}(e''^2+e''^4)(4-3\sin^2 I'')\cos 3g'\bigg]\sin(\phi'_{30}-3h')$$

$$-\frac{1}{n_1}\cos I''\bigg[\left(1+\frac{29}{4}e''^2+\frac{9}{4}e''^4\right)(4-15\sin^2 I'')\sin g'$$

$$+\frac{105}{4}(e''^2+e''^4)\sin^2 I''\sin 3g'\bigg]\cos(\phi'_1-h')$$

$$-\frac{5}{n_{30}}\cos I''\bigg[\left(1+\frac{29}{4}e''^2+\frac{9}{4}e''^4\right)\sin^2 I''\sin g'$$

$$+\frac{7}{4}(e''^2+e''^4)(4-\sin^2 I'')\sin 3g'\bigg]\cos(\phi'_{30}-3h')\bigg\}$$

$$-27\frac{c_{22}}{n_{20}^2}\mu_0^8 R_{\zeta}^4\frac{H''}{L''^7 G''^8}\sin^2 I''$$

$$\times\left[c_{20}\sin(\phi'_{20}-2h')+\frac{1}{2}c_{22}\sin(2\phi'_{20}-4h')\right]$$

$$+\frac{45}{32}\frac{c_{20}}{n_{20}^2}\frac{\mu_0^2\mu_2 R_{\zeta}^2}{a_{\zeta}^3 G''^4}(2-e''^2)(4-10\sin^2 I''+5\sin^4 I'')$$

$$\times\sin 2g'\cos(\phi'_{20}-2h')$$

$$+\frac{9}{16}\frac{c_{20}}{n_{20}^2}\frac{\mu_0^2\mu_2 R_{\zeta}^2}{a_{\zeta}^3}\frac{H''}{G''^5}[(8-3e''^2)\sin^2 I''-10(2-e''^2)$$

$$\times(1-2\sin^2 I'')\cos 2g']\sin(\phi'_{20}-2h')$$

$$-\frac{45}{32}\frac{c_{22}}{n_{20}^2}\frac{\mu_0^2\mu_2 R_{\zeta}^2}{a_{\zeta}^3 G''^4}(2-e''^2)\sin^2 I''\sin 2g'$$

$$\times\bigg[(4-10\sin^2 I'')\cos(\phi'_{20}-2h')$$

$$+\frac{1}{2}(6-5\sin^2 I'')\cos(2\phi'_{20}-4h')\bigg]$$

$$-\frac{9}{16}\frac{c_{22}}{n_{20}^2}\frac{\mu_0^2\mu_2 R_{\zeta}^2}{a_{\zeta}^3}\frac{H''}{G''^5}\sin^2 I''$$

$$\times \Big\{2\,[(8 - 3e''^2) - 5(2 - e''^2)\cos 2g']\sin(\phi'_{20} - 2h')$$

$$-\left[(8 - 3e''^2) + \frac{15}{2}(2 - e''^2)\cos 2g'\right]\sin(2\phi'_{20} - 4h')\Big\}$$

$$-\frac{45}{8}\frac{\mu_2^2}{n_{20}^2\mu_0^4 a_{☾}^6} L''^5 G''(1 + 2e''^2)(1 - 2\sin^2 I'')\sin 2g'\cos(\phi'_{20} - 2h')$$

$$+\frac{9}{16}\frac{\mu_2^2}{n_{20}^2\mu_0^4 a_{☾}^6} L''^5 H''[(11 + 34e''^2)\sin^2 I''$$

$$+ 5(1 + 2e''^2)(2 - 3\sin^2 I'')\cos 2g']\sin(\phi'_{20} - 2h')$$

$$-\frac{45}{64}\frac{\mu_2^2}{n_{20}^2\mu_0^4 a_{☾}^6}\frac{L''^7}{G''}\Big\{[3(1 + e''^2 - 2e''^4) - 5(1 + 3e''^2)\sin^2 I'']$$

$$\times \sin^2 I''\sin 2g' + 5\Big[2(1 + e''^2 - 2e''^4) - (2 + 3e''^2 - 2e''^4)$$

$$\times \sin^2 I'' + \frac{1}{4}(1 + 3e''^2)\sin^4 I''\Big]\sin 4g'\Big\}\cos(2\phi'_{20} - 4h')$$

$$-\frac{45}{64}\frac{\mu_2^2}{n_{20}^2\mu_0^4 a_{☾}^6}\frac{L''^7 H''}{G''^2}\Big\{\frac{1}{5}(14 + 27e''^2 + 9e''^4)\sin^2 I''$$

$$+ 3(1 + e''^2 - 2e''^4)\sin^2 I''\cos 2g' + 5[2(1 + e''^2 - 2e''^4)$$

$$-(1 + 2e''^2 - e''^4)\sin^2 I'']\cos 4g'\Big\}\sin(2\phi'_{20} - 4h') \qquad (115)$$

$$\frac{\partial S_2^*}{\partial G''} = 3I_{☾}\frac{c_{22}}{n_{23}}\frac{\mu_0^4 R_{☾}^2}{L''^3 G''^4}\cot I''(1 - 5\sin^2 I'')\sin(\phi'_{23} - h')$$

$$+\frac{3}{4}\varepsilon_1\frac{c_{20}}{n_7}\frac{\mu_0^4 R_{☾}^2}{L''^3 G''^4}\cot I''(1 - 5\sin^2 I'')\sin(\phi'_7 - h')$$

$$-\frac{3}{8}\gamma_{☾}\frac{\mu_2}{n_{23}\mu_0^2 a_{☾}^3}\frac{L''^4}{G''\sin I''}\Big\{2\cos I''\Big[1 + \frac{3}{2}e''^2 - 5\sin^2 I''$$

$$-5\left(\frac{1}{2}e''^2 - \sin^2 I''\right)\cos 2g'\Big]\sin(\phi'_{23} - h')$$

$$+ 5(e''^2 - 2\sin^2 I'' + e''^2\sin^2 I'')\sin 2g'\cos(\phi'_{23} - h')\Big\}$$

$$+\frac{3}{8}e_{☾}\frac{\mu_2}{\mu_0^2 a_{☾}^3}\frac{L''^4}{G''}\Big\{\frac{3}{n_3}[1 - e''^2 - 5\cos^2 I'' + 5(e''^2 - \sin^2 I'')\cos 2g']$$

$$\times \sin\phi'_3 + \frac{1}{2}[2 + 3e''^2 - 5\sin^2 I'' - 5(1 - e''^2 + \cos^2 I'')$$

$$\times \cos 2g']\left[\frac{7}{n_{31}}\sin(\phi'_{31} - 2h') - \frac{1}{n_2}\sin(\phi'_2 - 2h')\right]$$

$$+\frac{5}{2}\cos I''(2-e''^2)\sin 2g'\left[\frac{7}{n_{31}}\cos(\phi'_{31}-2h')\right.$$

$$\left.\left.-\frac{1}{n_2}\cos(\phi'_2-2h')\right]\right\}$$

$$+\frac{3}{16}\gamma_{☾}^2\frac{\mu_2}{\mu_0^2 a_{☾}^3}\frac{L''^4}{G''}\left\{\frac{3}{n_{25}}[1-e''^2-5\cos^2 I''+5(e''^2-\sin^2 I'')\right.$$

$$\times\cos 2g']\sin\phi'_{25}-\frac{1}{n_{20}}[2+3e''^2-5\sin^2 I''$$

$$-5(1-e''^2+\cos^2 I'')\cos 2g']\sin(\phi'_{20}-2h')$$

$$\left.-\frac{5}{n_{20}}\cos I''(2-e''^2)\sin 2g'\cos(\phi'_{20}-2h')\right\}$$

$$-\frac{3}{16}e_{☾}\gamma_{☾}\frac{\mu_2}{\mu_0^2 a_{☾}^3}\frac{L''^4}{G''\sin I''}\left\{2\cos I''\left[1+\frac{3}{2}e''^2-5\sin^2 I''\right.\right.$$

$$\left.-5\left(\frac{1}{2}e''^2-\sin^2 I''\right)\cos 2g'\right]\left[\frac{7}{n_{33}}\sin(\phi'_{33}-h')\right.$$

$$\left.-\frac{1}{n_6}\sin(\phi'_6-h')-\frac{3}{n_7}\sin(\phi'_7-h')-\frac{3}{n_8}\sin(\phi'_8+h')\right]$$

$$+5(e''^2-2\sin^2 I''+e''^2\sin^2 I'')\sin 2g'$$

$$\times\left[\frac{7}{n_{33}}\cos(\phi'_{33}-h')-\frac{1}{n_6}\cos(\phi'_6-h')\right.$$

$$\left.\left.-\frac{3}{n_7}\cos(\phi'_7-h')+\frac{3}{n_8}\cos(\phi'_8+h')\right]\right\}$$

$$+\frac{3}{16}e_{☾}^2\frac{\mu_2}{\mu_0^2 a_{☾}^3}\frac{L''^4}{G''}\left\{\frac{9}{n_{24}}[1-e''^2-5\cos^2 I''\right.$$

$$+5(e''^2-\sin^2 I'')\cos 2g']\sin\phi'_{24}+[2+3e''^2-5\sin^2 I''$$

$$-5(1-e''^2+\cos^2 I'')\cos 2g']\left[\frac{17}{n_{41}}\sin(\phi'_{41}-2h')\right.$$

$$\left.-\frac{5}{n_{20}}\sin(\phi'_{20}-2h')\right]+5\cos I''(2-e''^2)\sin 2g'$$

$$\left.\times\left[\frac{17}{n_{41}}\cos(\phi'_{41}-2h')-\frac{5}{n_{20}}\cos(\phi'_{20}-2h')\right]\right\}$$

$$-\frac{15}{64}\frac{\mu_2}{\mu_0^3 a_{☾}^4}\frac{L''^6}{G''e''}\left\{\frac{1}{n_1}\left[\left(4+15e''^2-\frac{3}{2}e''^4-5\sin^2 I''\right.\right.\right.$$

$$\left.-\frac{65}{4}e''^2\sin^2 I''+\frac{15}{4}e''^4\sin^2 I''\right)\cos g'$$

$$- \frac{35}{4} e''^2 (2e''^2 - 3\sin^2 I'' + e''^2 \sin^2 I'') \cos 3g' \Big] \sin(\phi_1' - h')$$

$$- \frac{5}{n_{30}} \Big[\Big(2e''^2 + \frac{3}{2} e''^4 - \sin^2 I'' - \frac{13}{4} e''^2 \sin^2 I''$$

$$+ \frac{3}{4} e''^4 \sin^2 I'' \Big) \cos g' - \frac{7}{4} e''^2 (4 - 2e''^2 - 3\sin^2 I''$$

$$+ e''^2 \sin^2 I'') \cos 3g' \Big] \sin(\phi_{30}' - 3h') - \frac{1}{n_1} \cos I''$$

$$\times \Big[\Big(4 + 39e''^2 + \frac{33}{2} e''^4 - 15\sin^2 I'' - \frac{255}{4} e''^2 \sin^2 I'' \Big)$$

$$\times \sin g' - \frac{35}{4} e''^2 (2e''^2 - 3\sin^2 I'') \sin 3g' \Big] \cos(\phi_1' - h')$$

$$+ \frac{5}{n_{30}} \cos I'' \Big[\Big(2e''^2 + \frac{3}{2} e''^4 - \sin^2 I'' - \frac{17}{4} e''^2 \sin^2 I'' \Big) \sin g'$$

$$- \frac{7}{4} e''^2 (4 - 2e''^2 - \sin^2 I'') \sin 3g' \Big] \cos(\phi_{30}' - 3h') \Big\}$$

$$+ 9 \frac{c_{22}}{n_{20}^2} \mu_0^8 R_☾^4 \frac{H''}{L''^6 G''^9} (1 - 5\sin^2 I'')$$

$$\times \Big[c_{20} \sin(\phi_{20}' - 2h') + \frac{1}{2} c_{22} \sin(2\phi_{20}' - 4h') \Big]$$

$$- \frac{45}{16} \frac{c_{20}}{n_{20}^2} \frac{\mu_0^2 \mu_2 R_☾^2}{a_☾^3} \frac{L''}{G''^5} [4 - 14e''^2 - 5(1 + 3e''^2)$$

$$\times \sin^2 I'' (2 - \sin^2 I'')] \sin 2g' \cos(\phi_{20}' - 2h')$$

$$+ \frac{9}{16} \frac{c_{20}}{n_{20}^2} \frac{\mu_0^2 \mu_2 R_☾^2}{a_☾^3} \frac{L'' H''}{G''^6} [4 + 6e''^2 - 20\sin^2 I'' - 15e''^2 \sin^2 I''$$

$$+ 10(2 + 7e''^2 - 4\sin^2 I'' - 10e''^2 \sin^2 I'')$$

$$\times \cos 2g'] \sin(\phi_{20}' - 2h')$$

$$- \frac{45}{16} \frac{c_{22}}{n_{20}^2} \frac{\mu_0^2 \mu_2 R_☾^2}{a_☾^3} \frac{L''}{G''^5} \sin 2g' \Big[2(2e''^2 - 2\sin^2 I'' + 5\sin^4 I''$$

$$- 14e''^2 \sin^2 I'' + 15e''^2 \sin^4 I'') \cos(\phi_{20}' - 2h')$$

$$+ \frac{1}{2} (6e''^2 - 6\sin^2 I'' + 5\sin^4 I'' - 22e''^2 \sin^2 I''$$

$$+ 15e''^2 \sin^4 I'') \cos(2\phi_{20}' - 4h') \Big]$$

$$- \frac{9}{16} \frac{c_{22}}{n_{20}^2} \frac{\mu_0^2 \mu_2 R_☾^2}{a_☾^3} \frac{L'' H''}{G''^8} \Big\{ (4 + 6e''^2 - 20\sin^2 I'' - 15e''^2 \sin^2 I'')$$

$$
\begin{aligned}
&\times [2 \sin(\phi'_{20} - 2h') - \sin(2\phi'_{20} - 4h')] \\
&- 5\left(e''^2 - \sin^2 I'' - \frac{5}{2} e''^2 \sin^2 I''\right) \\
&\times \cos 2g' [4 \sin(\phi'_{20} - 2h') + 3 \sin(2\phi'_{20} - 4h')]\Bigg\} \\
&+ \frac{45}{16} \frac{\mu_2^2}{n_{20}^2 \mu_0^4 a_☾^6} L''^6 (2 + e''^2 - 4 \sin^2 I'' + 2e''^2 \sin^2 I'') \\
&\times \sin 2g' \cos(\phi'_{20} - 2h') \\
&- \frac{9}{16} \frac{\mu_2^2}{n_{20}^2 \mu_0^4 a_☾^6} L''^6 \cos I'' [2 - 17e''^2 + 15 \sin^2 I'' \\
&+ 5(2 + e''^2 - 3 \sin^2 I'') \cos 2g'] \sin(\phi'_{20} - 2h') \\
&- \frac{45}{256} \frac{\mu_2^2}{n_{20}^2 \mu_0^4 a_☾^6} \frac{L''^8}{G''^2} \Big\{ 2 [6e''^2 (1 - e''^2) - (6 + 11e''^2 + 3e''^4) \sin^2 I'' \\
&+ 5(2 + 3e''^2) \sin^4 I''] \sin 2g' \\
&- 5 \Bigg[2(4 - 6e''^2 + 3e''^4) - (8 - 6e''^2 + 3e''^4) \sin^2 I'' \\
&+ \left(1 + \frac{3}{2} e''^2\right) \sin^4 I'' \Bigg] \sin 4g' \Big\} \cos(2\phi'_{20} - 4h') \\
&+ \frac{45}{128} \frac{\mu_2^2}{n_{20}^2 \mu_0^4 a_☾^6} \frac{L''^8 H''}{G''^3} \Bigg[\frac{1}{5} (4 + 12e''^2 + 9e''^4) - 2(2 + 3e''^2) \sin^2 I'' \\
&+ 6(1 - e''^2)(e''^2 - \sin^2 I'') \cos 2g' - 5(4 - 6e''^2 + 3e''^4 \\
&- 2 \sin^2 I'') \cos 4g' \Bigg] \sin(2\phi'_{20} - 4h') \qquad (116)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial S_2^*}{\partial H''} &= - 3 I_☾ \frac{c_{22}}{n_{23}} \frac{\mu_0^4 R_☾^2}{L''^3 G''^4} \frac{\cos 2I''}{\sin I''} \sin(\phi'_{23} - h') \\
&- \frac{3}{4} \varepsilon_1 \frac{c_{20}}{n_7} \frac{\mu_0^4 R_☾^2}{L''^3 G''^4} \frac{\cos 2I''}{\sin I''} \sin(\phi'_7 - h') + \frac{3}{8} \gamma_☾ \frac{\mu_2}{n_{23} \mu_0^2 a_☾^3} \frac{L''^4}{G'' \sin I''} \\
&\times \left[2 \cos 2I'' \left(1 + \frac{3}{2} e''^2 - \frac{5}{2} e''^2 \cos 2g'\right) \sin(\phi'_{23} - h') \right. \\
&\left. + 5e''^2 \cos I'' \sin 2g' \cos(\phi'_{23} - h') \right] \\
&+ \frac{3}{8} e_☾ \frac{\mu_2 L''^4}{\mu_0^2 a_☾^3} \frac{H''}{G''^2} \Big\{ \frac{3}{n_3} (2 + 3e''^2 - 5e''^2 \cos 2g') \sin \phi'_3 \\
&- \frac{1}{2} (2 + 3e''^2 - 5e''^2 \cos 2g') \Bigg[\frac{7}{n_{31}} \sin(\phi'_{31} - 2h')
\end{aligned}
$$

$$- \frac{1}{n_2} \sin(\phi_2' - 2h') \Big] - \frac{5}{2} \frac{e''^2}{\cos I''} \sin 2g'$$

$$\times \left[\frac{7}{n_{31}} \cos(\phi_{31}' - 2h') - \frac{1}{n_2} \cos(\phi_2' - 2h') \right] \Big\}$$

$$+ \frac{3}{16} \gamma_{☾}^2 \frac{\mu_2}{\mu_0^2 a_{☾}^3} \frac{L''^4 H''}{G''^2} \Big\{ (2 + 3e''^2 - 5e''^2 \cos 2g')$$

$$\times \left[\frac{3}{n_{25}} \sin \phi_{25}' + \frac{1}{n_{20}} \sin(\phi_{20}' - 2h') \right]$$

$$+ \frac{5}{n_{20}} \frac{e''^2}{\cos I''} \sin 2g' \cos(\phi_{20}' - 2h') \Big\}$$

$$+ \frac{3}{16} e_{☾} \gamma_{☾} \frac{\mu_2}{\mu_0^2 a_{☾}^3} \frac{L''^4}{G'' \sin I''} \Big\{ \cos 2I'' (2 + 3e''^2 - 5e''^2 \cos 2g')$$

$$\times \Big[\frac{7}{n_{33}} \sin(\phi_{33}' - h') - \frac{1}{n_6} \sin(\phi_6' - h')$$

$$- \frac{3}{n_7} \sin(\phi_7' - h') - \frac{3}{n_8} \sin(\phi_8' + h') \Big]$$

$$+ 5e''^2 \cos I'' \sin 2g' \Big[\frac{7}{n_{33}} \cos(\phi_{33}' - h') - \frac{1}{n_6} \cos(\phi_6' - h')$$

$$- \frac{3}{n_7} \cos(\phi_7' - h') + \frac{3}{n_8} \cos(\phi_8' + h') \Big] \Big\}$$

$$+ \frac{3}{16} e_{☾}^2 \frac{\mu_2}{\mu_0^2 a_{☾}^3} \frac{L''^4 H''}{G''^2} \Big\{ \frac{9}{n_{24}} (2 + 3e''^2 - 5e''^2 \cos 2g') \sin \phi_{24}'$$

$$- (2 + 3e''^2 - 5e''^2 \cos 2g') \Big[\frac{17}{n_{41}} \sin(\phi_{41}' - 2h')$$

$$- \frac{5}{n_{20}} \sin(\phi_{20}' - 2h') \Big] - 5 \frac{e''^2}{\cos I''}$$

$$\times \sin 2g' \left[\frac{17}{n_{41}} \cos(\phi_{41}' - 2h') - \frac{5}{n_{20}} \cos(\phi_{20}' - 2h') \right] \Big\}$$

$$+ \frac{5}{64} \frac{\mu_2}{\mu_0^3 a_{☾}^4} \frac{L''^6 H''}{G''^2} \Big\{ 30 \left[\left(e'' + \frac{3}{4} e''^3 \right) \cos g' - \frac{7}{4} e''^3 \cos 3g' \right]$$

$$\times \left[\frac{1}{n_1} \sin(\phi_1' - h') - \frac{1}{n_{30}} \sin(\phi_{30}' - 3h') \right]$$

$$- 10 \cos I'' \left[3 \left(e'' + \frac{3}{4} e''^3 \right) \sin g' - \frac{7}{4} e''^3 \sin 3g' \right]$$

$$\times \left[\frac{3}{n_1} \cos(\phi_1' - h') - \frac{1}{n_{30}} \cos(\phi_{30}' - 3h') \right]$$

$$- \frac{3}{n_1} \frac{1}{\cos I''} \left[\left(e'' + \frac{3}{4} e''^3 \right) (4 - 15 \sin^2 I'') \sin g' \right.$$

$$\left. + \frac{35}{4} e''^3 \sin^2 I'' \sin 3g' \right] \cos(\phi'_1 - h') - \frac{5}{n_{30}} \frac{1}{\cos I''}$$

$$\times \left[3 \left(e'' + \frac{3}{4} e''^3 \right) \sin^2 I'' \sin g' + \frac{7}{4} e''^3 (4 - \sin^2 I'') \sin 3g' \right]$$

$$\left. \times \cos(\phi'_{30} - 3h') \right\}$$

$$+ \frac{9}{2} \frac{c_{22}}{n_{20}^2} \frac{\mu_0^8 R_☾^4}{L''^6 G''^8} (1 - 3 \cos^2 I'')$$

$$\times \left[c_{20} \sin(\phi'_{20} - 2h') + \frac{1}{2} c_{22} \sin(2\phi'_{20} - 4h') \right]$$

$$+ \frac{225}{8} \frac{c_{20}}{n_{20}^2} \frac{\mu_0^2 \mu_2 R_☾^2}{a_☾^3} \frac{L'' H''^3}{G''^8} e''^2 \sin 2g' \cos(\phi'_{20} - 2h')$$

$$+ \frac{9}{16} \frac{c_{20}}{n_{20}^2} \frac{\mu_0^2 \mu_2 R_☾^2}{a_☾^3} \frac{L''}{G''^5} e''^2 \left[\left(\frac{2}{e''^2} + 3 \right) (1 - 3 \cos^2 I'') \right.$$

$$\left. + 10 (1 - 6 \cos^2 I'') \cos 2g' \right] \sin(\phi'_{20} - 2h')$$

$$+ \frac{45}{16} \frac{c_{22}}{n_{20}^2} \frac{\mu_0^2 \mu_2 R_☾^2}{a_☾^3} \frac{L'' H''}{G''^6} e''^2 \sin 2g' [4(1 - 5 \sin^2 I'')$$

$$\times \cos(\phi'_{20} - 2h') + (3 - 5 \sin^2 I'') \cos(2\phi'_{20} - 4h')]$$

$$- \frac{9}{16} \frac{c_{22}}{n_{20}^2} \frac{\mu_0^2 \mu_2 R^2}{a_☾^3} \frac{L''}{G''^5} (1 - 3 \cos^2 I'') e''^2$$

$$\times \left[\left(\frac{4}{e''^2} + 6 - 10 \cos 2g' \right) \sin(\phi'_{20} - 2h') \right.$$

$$\left. - \frac{1}{2} \left(\frac{4}{e''^2} + 6 + 15 \cos 2g' \right) \sin(2\phi'_{20} - 4h') \right]$$

$$- \frac{45}{4} \frac{\mu_2^2}{n_{20}^2 \mu_0^4 a_☾^6} L''^6 e''^2 \cos I'' \sin 2g' \cos(\phi'_{20} - 2h')$$

$$- \frac{9}{32} \frac{\mu_2^2}{n_{20}^2 \mu_0^4 a_☾^6} L''^6 e''^2 \left[\left(\frac{2}{e''^2} - 17 \right) (1 - 3 \cos^2 I'') \right.$$

$$\left. + 5 (1 - 9 \cos^2 I'') \cos 2g' \right] \sin(\phi'_{20} - 2h')$$

$$+ \frac{45}{128} \frac{\mu_2^2}{n_{20}^2 \mu_0^4 a_☾^6} \frac{L''^8 H''}{G''^3} e''^2 [2(3 - 3e''^2 - 10 \sin^2 I'') \sin 2g'$$

$$- 5(4 - 3e''^2 - \sin^2 I'')\sin 4g'] \cos(2\phi' - 4h')$$

$$+ \frac{9}{256} \frac{\mu_2^2}{n_{20}^2 \mu_0^4 a_☾^6} \frac{L''^8}{G''^2} e''^2 \left\{ (1 - 3\cos^2 I'') \times \left[\frac{4}{e''^2} + 12 + 9e''^2 + 30(1 - e''^2)\cos 2g' \right] + 25(2 - 3e''^2 + 6\cos^2 I'' - 3e''^2 \cos^2 I'')\cos 4g' \right\} \times \sin(2\phi'_{20} - 4h'). \tag{117}$$

$$\frac{\partial S_2^*}{\partial D''} = 0. \tag{118}$$

A remark must be made on the apparent appearance of potential zero divisors in many of the above expressions. In numerous places quantities like e^2 and $\cos I$ have been factored out for reasons of brevity. This causes the apparent zero divisors inside of brackets. However, the appearance of e or $\sin I$ as denominators in coefficients serves as a signal for the indeterminacy of l and g for zero eccentricity and that of g and h for zero inclination. The computation of the satellite's position is nevertheless always determinate as it can be made with the aid of such nonsingular elements as $l+g$ if $e=0$ and $g+h$ if $I=0$. It is easily verified that $l+g$ is free of the divisor e, explicitly or implicitly, and that $g+h$ is free of the divisor $\sin I$.

B. THE SEQUENCE OF COMPUTATIONS

Numerical integration of the set (93) is straightforward. The orbit is described at epoch by L''_0, G''_0, H''_0, l''_0, g''_0, and h''_0. All arguments of trigonometric functions are clearly defined with the aid of (37). Note that

$$t \equiv d \equiv d' \equiv d'',$$

and that t is always reckoned from epoch. Resulting from this step are

$$L'', G'', H'', l'', g'', h''.$$

These quantities are related to the single-primed variables through (80). A bit of care is required here since the perturbations are expressed in terms of the g' and h' which are yet to be found. Several routes can be taken, but the following is suggested. Calculate g' and h' as indicated by (80), but restricted to the short first-order terms only, namely (109) and (110). The resulting approximate values of g' and h' are in error by quantities of order two. Now employ the complete set (80) using the approximate g' and h' in the right-hand members. This step renders the results correct through order two. If the calculations are restricted to first-order terms only, this device is unnecessary. The variables g'' and h'' may be used in place of g' and h' in the right-hand members immediately.

At this stage one has

$$L', G', H', l', g', h'.$$

There remains only the addition of short-period terms which is accomplished through (51). In place of l, g, and h, the available l', g', and h' may be used in the right-hand members. Solve Kepler's equation in the form

$$E - e' \sin E = l' .$$

This is the E to be used in (99) to (103). Next compute

$$\frac{r}{a'} = 1 - e' \cos E$$

and f from

$$\tan \frac{1}{2} f = \sqrt{\frac{1 + e'}{1 - e'}} \tan \frac{1}{2} E .$$

Since there are only short-period terms of order two, it would have been permissible to write all variables in the last three formulae as well as the right-hand members of (98) through (103) in terms of primed variables only. However, the mixed notation emphasizes the fact that, strictly speaking,

$$\begin{aligned} f &= f(L', G', l), \\ E &= E(L', G', l), \\ r &= r(L', G', l). \end{aligned}$$

One now has the desired osculating elements

$$L, G, H, l, g, h .$$

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