A NOTE CONCERNING THE TR-TRANSFORMATION

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Abstract. The canonical transformation which Scheifele (1970) proposes to make a coordinate of the true anomaly is the product of a Whittaker transformation by an extension to space-time of the one-parameter family of canonical transformations that Hill (1913) defined for the same purpose.

While exploring ways of shortening the literal developments involved by Delaunay's method in lunar and planetary theories, Hill (1913) offered a new canonical transformation $(R, \Theta, N, r, \theta, \nu) \to (F, G, H, f, g, h)$ depending on a parameter. In the Hill maps, the true anomaly f plays the role held by the mean anomaly l in Delaunay's transformation.

In dealing with the perturbations caused by the planet's oblateness on the motion of a close satellite, Mr. Scheifele (1970, 1972, 1973, 1974) has developed a canonical transformation $(R, \Theta, N, T, r, \theta, \nu, t) \to (F, G, H, U, f, g, h, u)$ which he calls a TR-mapping.

It is shown here that the TR-mapping is an extension of Hill's transformation from the 6-dimensional phase space $(R, \Theta, N, r, \theta, \nu)$ to an 8-dimensional manifold $(R, \Theta, N, T, r, \theta, \nu, t)$.

1. Hill's One-Parameter Family

According to Andoyer (1913), one may present Hill's transformation in the following manner. It is defined by the implicit equations

$$\begin{aligned} R &= \frac{\partial S}{\partial r}, \quad \Theta = \frac{\partial S}{\partial \theta}, \quad N = \frac{\partial S}{\partial \nu}, \\ f &= \frac{\partial S}{\partial F}, \quad g = \frac{\partial S}{\partial G}, \quad h = \frac{\partial S}{\partial H}, \end{aligned} \tag{1}$$

derived from the generator

$$S \equiv S(F, G, H, r, \theta, \nu; U) = H\nu + G\theta + \int_P^r Q^{1/2}\, \mathrm{d}r. \tag{2}$$

The integrand is given by the relation

$$Q \equiv Q(F, G, r; U) = -2U + 2\frac{\mu}{r} - \frac{(G-F)^2}{r^2} \tag{3}$$

and the lower limit $P \equiv P(F, G; U)$ is a root of the equation

$$Q(F, G, P; U) = 0\,. \tag{4}$$

The quantity U is a parameter of the transformation or what Hill calls an 'absolute constant.'

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In view of (2), the transformation Equations (1) are

$$R = Q^{1/2}, \quad \Theta = G, \quad N = H, \qquad f = \int_P^r \frac{(G-F)\,\mathrm{d}w}{w^2 Q^{1/2}}, \quad g = \theta - f, \quad h = \nu. \tag{5}$$

The simplest way of performing the quadrature in (5_4) is by factoring the integrand Q. To this effect, introduce the auxiliary functions

$$a \equiv a(U) = \mu/2U, \tag{6_1}$$

$$p \equiv p(F, G) = (G-F)^2/\mu, \tag{6_2}$$

$$\eta \equiv \eta(F, G; U) = \sqrt{p/a}, \tag{6_3}$$

$$e \equiv e(F, G; U) = \sqrt{1-\eta^2}. \tag{6_4}$$

In these notations, the integrand may be written as the product

$$Q = \frac{\mu}{a}\left(\frac{A}{r} - 1\right)\left(1 - \frac{P}{r}\right). \tag{7}$$

the roots being

$$A \equiv A(F, G; U) = a(1+e), \quad P \equiv P(F, G; U) = a(1-e). \tag{8}$$

Then, in the quadrature (5_4), substitute for r an angle $\phi = \phi(F, G, r; U)$ such that

$$p = r(1 + e\cos\phi). \tag{9}$$

It is readily seen that

$$\frac{A}{r} - 1 = \frac{e}{1-e}(1+\cos\phi), \qquad 1 - \frac{P}{r} = \frac{e}{1+e}(1-\cos\phi), \qquad Q = \frac{\mu}{p}e^2\sin^2\phi, \qquad \frac{\mathrm{d}r}{r^2} = \frac{1}{p}e\sin\phi\,\mathrm{d}\phi. \tag{10}$$

Hence (5_4) in closed form becomes the equation

$$f = \phi. \tag{11}$$

Notice that, on account of (10) and (11), Equation (5_1) may be written as the relation

$$R = \sqrt{\frac{\mu}{p}}\, e\sin f, \tag{12}$$

Applications of Hill's transformation to dynamic systems involve the differential quantity $r^2\,\mathrm{d}f$. We show now how, on account of the transformation alone irrespective of the dynamic system to which the transformation is applied, it can be expressed in a form well suited to the perturbation theories of celestial mechanics.

Introduce an angle $\psi \equiv \psi(F, G, r; U)$ such that

$$r = a(1 - e\cos\psi)\,. \tag{13}$$

On account of (9), this definition implies that

$$r\cos f = a(\cos\psi - e)\,. \tag{14}$$

There follows that

$$r(1+\cos f) = a(1-e)(1+\cos\psi)\,,$$
$$r(1-\cos f) = a(1+e)(1-\cos\psi)\,.$$

and finally that

$$\tan\frac{f}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{\psi}{2}\,. \tag{15}$$

Observe also that substituting r as a function of ψ in the integrand Q defined by (7) yields that

$$Q = \frac{\mu a}{r^2}e^2\sin^2\psi$$

Hence, on account of (5_1),

$$R = \frac{\sqrt{\mu a}}{r}e\sin\psi \tag{16}$$

which implies through (12) that

$$r\sin f = a\eta\sin\psi\,. \tag{17}$$

Taking the logarithmic derivative of (15) produces the Pfaffian

$$\frac{\mathrm{d}f}{\sin f} = \frac{\mathrm{d}e}{\eta^2} + \frac{\mathrm{d}\psi}{\sin\psi}\,.$$

Then, on account of (17), it is found that

$$r^2\,\mathrm{d}f = a^2\eta\left[\mathrm{d}(\psi - e\sin\psi) + \left(1+\frac{r}{p}\right)\sin\psi\,\mathrm{d}e\right]. \tag{18}$$

In the manner of illustration, Hill's transformation is applied to Keplerian systems

$$\mathscr{H} = \frac{1}{2}\left(R^2 + \frac{\Theta^2}{r^2}\right) - \frac{\mu}{r}\,. \tag{19}$$

It converts the Hamiltonian into the function

$$\mathcal{H} = -U + \frac{F}{r^2}\left(G - \tfrac{1}{2}F\right). \tag{20}$$

The parameter U is chosen to be minus the energy. Then a new independent variable s is chosen so that

$$r^2\,\mathrm{d}s = (G - \tfrac{1}{2}F)\,\mathrm{d}t. \tag{21}$$

Therefore the phase flow determined by (20) on the manifold $\mathcal{H} - U = 0$ in the chart (F, G, H, f, g, h, t) is in $1-1$ correspondence with the phase flow determined by $\mathcal{K} = F$ on the manifold $\mathcal{K} = 0$ in the chart (F, G, H, f, g, h, s). In view of the differential equation

$$\frac{\mathrm{d}f}{\mathrm{d}s} = \frac{\partial K}{\partial F} = 1,$$

the new independent variable s turns out to be the true anomaly itself (provided s is counted from a passage at perigee). Since $F = 0$, the functions a, p and e defined by (6_1), (6_2) and (6_4) are respectively the semi-major, the semi-latus rectum and the eccentricity of a Keplerian ellipse. Moreover, because $\mathrm{d}e = 0$ in this system, the Pfaffian (21) becomes

$$d(\psi - e\sin\psi) = d\left(\sqrt{\frac{\mu}{a}}\,t\right)$$

which is the differential form of Kepler's equation in a Keplerian system.

2. The TR-Transformation

We extend Hill's canonical transformation to an 8-dimensional phase space. The extension $(R, \Theta, N, T, r, \theta, \nu, t) \to (F, G, H, U, f, g, h, u)$ is defined by the implicit equations

$$\begin{aligned} R &= \frac{\partial S}{\partial r}, \quad \Theta = \frac{\partial S}{\partial \theta}, \quad N = \frac{\partial S}{\partial \nu}, \quad T = \frac{\partial S}{\partial t}, \\ f &= \frac{\partial S}{\partial F}, \quad g = \frac{\partial S}{\partial G}, \quad h = \frac{\partial S}{\partial H}, \quad u = \frac{\partial S}{\partial U} \end{aligned} \tag{22}$$

derived from the generator

$$S \equiv S(F, G, H, U, r, \theta, \nu, t) = Ut + H\nu + G\theta + \int_P^r Q^{1/2}\,\mathrm{d}r \tag{23}$$

with the integrand Q and the lower limit P the same as for Hill's transformation.

There follows that

$$T = U, \quad u = t - \int_P^r \frac{\mathrm{d}r}{Q^{1/2}}. \tag{24}$$

To perform the quadrature in (24), substitute for r an angle $\psi = \psi(F, G, r; U)$ such that

$$r = a(1 - e\cos\psi). \tag{25}$$

Accordingly one obtains that

$$A - r = ae(1 + \cos\psi),$$

$$r - P = ae(1 - \cos\psi),$$

$$Q = \frac{\mu a}{r^2} e^2 \sin^2\psi,$$

$$\mathrm{d}r = ae \sin\psi \, \mathrm{d}\psi.$$

So that (24) in closed form becomes the equation

$$t - u = \sqrt{\frac{a^3}{\mu}}(\psi - e\sin\psi). \tag{26}$$

It means that u may be interpreted as an *instant of passage* at perigee.

The extension of Hill's transformation defined by the implicit equations (22) is the TR-transformation presented by Scheifele.

Indeed Scheifele starts from a set of elements $(R, B, \Lambda, T, r, \beta, \lambda, t)$ related to spherical coordinates. But, as was done by Hill, one would perform a Whittaker transformation $(B, \Lambda, \beta, \lambda) \to (\Theta, N, \theta, \nu)$ to pass from the set $(R, B, \Lambda, T, r, \beta, \lambda, t)$ to the set $(R, \Theta, N, T, r, \theta, \nu, t)$. Such a transformation (Whittaker 1904, Hill 1913) is defined by the implicit equations

$$B = \frac{\partial W}{\partial \beta} = \Theta, \quad \theta = \frac{\partial S}{\partial \Theta} = \int_0^\beta \frac{\Theta \, \mathrm{d}\beta}{\sqrt{\Theta^2 - (N^2/\cos^2\beta)}},$$

$$\Lambda = \frac{\partial W}{\partial \lambda} = N, \quad \nu = \frac{\partial S}{\partial \Lambda} = \lambda - \int_0^\beta \frac{N \, \mathrm{d}\beta}{\cos^2\beta \sqrt{\Theta^2 - (N^2/\cos^2\beta)}}$$

derived from the generator

$$S \equiv S(\Theta, N, \beta, \lambda) = N\lambda + \int_0^\beta \sqrt{\Theta^2 - (N^2/\cos^2\beta)} \, \mathrm{d}\beta.$$

On introducing an angle $I \equiv I(\theta, N)$ such that

$$\cos I = \frac{N}{\Theta}, \quad \sin I = \sqrt{1 - \frac{N^2}{\Theta^2}}$$

one readily obtains that

$$\sin \beta = \sin I \sin \theta,$$

$$\tan(\lambda - \nu) = \tan I \tan \theta.$$

In the coordinate set $(R, \Theta, N, T, r, \theta, \nu, t)$, Scheifele considers a Hamiltonian

$$\mathcal{H}' = r^2 \left[\frac{1}{2}\left(R^2 + \frac{\Theta^2}{r^2}\right) - \frac{\mu}{r} + T\right]$$

to obtain a complete integral of its Hamilton–Jacobi equation, which he then uses as the generator of a canonical transformation $(R, \Theta, N, T, r, \theta, \nu, t) \to (F', G', H', U', f', g', h', u')$. The particular generator retained by Scheifele is the function

$$S \equiv S(F', G', H', U', r, \theta, \nu, t) = U't + G'\theta + H'\nu + \int_P^r Q'^{1/2}\, \mathrm{d}r \tag{27}$$

with the integrand such that

$$Q' = -2U' + 2\frac{\mu}{r} - \frac{G'^2 - 2F'}{r^2}. \tag{28}$$

But, in the transformation generated by S,

$$f' = \frac{\phi}{\sqrt{G'^2 - 2F'}}, \quad \theta = g' - \frac{G'}{\sqrt{G'^2 - 2F'}}\phi;$$

hence the new coordinate f' is not the true anomaly, not even a (dimensionless) angle. Scheifele remedies this deficiency by performing a second canonical transformation $(F', G', H', U', f', g'. h', u') \to (F, G, H, U, f, g, h, u)$. It may be defined by the explicit equations

$$u = u', \qquad U = U',$$

$$f = f'\sqrt{G'^2 - 2F'}. \qquad F = G' - \sqrt{G'^2 - 2F'},$$

$$g = g' + f'(G' - \sqrt{G'^2 - 2F'}), \qquad G = G',$$

$$h = h', \qquad H = H'.$$

But replacing the moments (F', G', H', U') in (27) by their expressions in terms of (F, G, H, U) proves that the complete integral (27) is identical to the generator (22), or that the Tr-transformation is nothing but an extension of Hill's transformation to the extended phase space $(R, \Theta, N, T, r, \theta, \nu, t)$.

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