NON-LINEAR STABILITY AROUND THE TRIANGULAR LIBRATION POINTS

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Abstract. The configuration space around the triangular libration points in the Earth-Moon system is partitioned according to the stability of the motion. The regions around L_4 and L_5 are established where particles placed with zero initial velocity will librate. The complexity of the partitioning is revealed.

1. Description of the Dynamical System

The model used is the restricted problem of three bodies. Two point masses, called the primaries, orbit one another in circular Keplerian motion, each influencing, but not influenced by, a third body of infinitesimal mass. The motion of this third body is examined. (For a treatise of the restricted problem see for instance, Szebehely, 1967.)

There are five equilibrium points: three collinear (located along the line of syzygies connecting the primaries), and two triangular, located in the plane of the motion of the primaries at the vertices of the equilateral triangles which have the primaries as the other vertices (see Figure 1). The triangular equilibrium points, denoted L_4 and L_5 are stable, provided the mass ratio, $\mu = M_2/(M_1+M_2)$, is less than a critical value $\mu_{cr} \simeq 0.0385$. The present investigation is concerned with the Earth-Moon system,

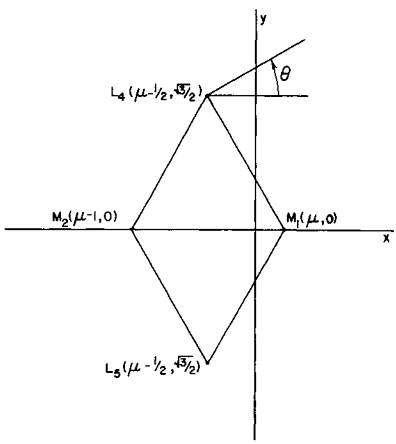

Fig. 1. The synodic coordinate system.

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for which $\mu = M_M/(M_E+M_M) \simeq 0.0121286$; therefore, the criterion for stability is satisfied at L_4 and at L_5.

Because of this stability a smooth transition is expected in the behavior of the orbits of particles initially near L_4 and L_5. Furthermore, when a particle is initially near one of the stable equilibrium points, with zero or small initial velocity, it is expected to remain near the point. A recent study, however, showed considerable complexity of the phase space around the triangular libration points (Szebehely and Cooper, 1979) and the present paper intends to clarify the question of global stability.

The standard non-dimensional units are used: the unit of distance is the Earth-Moon distance, $l = 384\,000$ km and the unit of time is $29^{\mathrm{d}}.5/(2\pi) = 4^{\mathrm{d}}.7$.

2. Definition of Libration and Selection of the Time Interval of Integration

We distinguish between 'librational' motion and 'unstable' motion. When the particle leaves the vicinity of L_4 (or L_5), we refer to global instability and when the particle stays in the vicinity of the equilateral points we speak of librational motion.

It has been the authors' experience that once the orbit of a particle (which begins its motion with zero velocity near L_4 or L_5) crosses the x-axis it will orbit one or both primaries or it will have a close encounter with one of the primaries. That is, the character of the motion changes from libration around the equilibrium point to motion about the primaries after the first crossing of the axis of syzygies. Note that periodic orbits with special initial conditions may cross the axis of syzygies and still maintain their librational character (Rabe, 1961). Such highly special and unstable periodic orbits do not appear in the present investigation. Consequently, the following, admittedly arbitrary, but for practical purposes useful definition of librational motion is accepted as one that does not intersect the axis of syzygies. This use of the word 'libration' is more general than the one used in the literature, nevertheless, it is short and descriptive.

Since the linear theory loses its validity for the large amplitude motions considered in this paper, numerical integration must be used to establish the behaviour of the particle. The selection of the length of the interval of time for which the numerical integration is performed becomes most critical. It is admittedly desirable to use for the time of integration, $t \to \infty$. On the other hand due to the accumulation of errors, very long time intervals may lead to meaningless results.

The selection of the time interval (t_F) is made by examining the effect of t_F on the maximum distance the particle may be started from L_4 along two given lines, before its motion ceases to be librational. The first line points from L_4 away from the origin of the coordinate system ($\theta_1 = 119^\circ.694$). The second line is inclined $\theta_2 = 299^\circ.694$ with respect to the x-axis, pointing towards the origin of the coordinate system. The maximum distances from L_4 along θ_1 and θ_2 for libration are denoted by R_1 and R_2 and are given in Table I for $t_F = 120$, 240, 360, 480, 600, 720, 840, and 960 non-dimensional time units, corresponding to 564, 1128, 1692, 2256, 2820, 3384,

TABLE I

Maximum Radii for Bounded Motion

T_f	R_1 ($\theta = 119\overset{\circ}{.}694$)	R_2 ($\theta = 299\overset{\circ}{.}64$)
120	0.01331	0.01557
240	0.01305	0.01512
360	0.01210	0.01501
480	0.01210	0.01494
600	0.01209	0.01412
720	0.01201	0.01396
840	0.01193	0.01396
960	0.01154	0.01396

3948, and 4512 days, or to approximately 1.54, 3.09, 4.64, 6.18, 7.73, 9.27, 10.82 and 12.36 years. Note that the long period of the solution of the linear equations is about 92 days. (In fact this motion is often called 'libration'.)

As can be seen from the table, the distances, R_1 and R_2 decrease as t increases; that is, as the time of integration increases the maximum distance the particle may be placed initially from L_4 for libration, is decreased. The data shown in Table I is accurate to $\pm 10^{-5}$ concerning R_1 and R_2. These results are also shown in Figure 2. After the first large change in the distance between 120 and 240 time units, the effect of final time of integration on the distance appears to be small. Since as the time of integration increases the accuracy of the numerical integration becomes more and

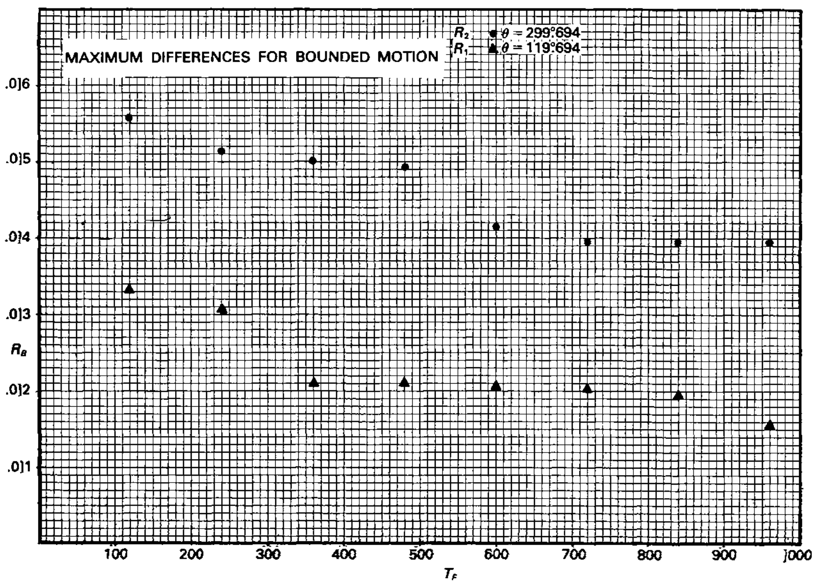

Fig. 2. Maximum distances for bounded motion.

more questionable, a reasonable compromise appears to be $t_F = 480$. Consequently, the results of this paper will refer to integration carried out for $t_F = 480$ or for a little over 6 years, corresponding to approximately 25 long periods.

3. Initial Positions for Motion about L_4

To determine the region about L_4 in which a particle with zero initial velocity (in the synodic system) librates about L_4, the region in the vicinity of L_4 is divided into grids, with a mesh size of 0.005 non-dimensional units (1,920 km). The trajectories of particles placed at each node point with zero initial velocity are then integrated for 480 time units. The integration is performed with a 12th-order, variable mesh, multistep method (Shampine and Gordon, 1975). The progress of the solution is monitored at intervals of one time unit. If the orbit does not touch or cross the x-axis, the initial position is considered to be one which gives libration.

The results are presented in Figure 3. The point (0, 0) denotes L_4. The symbols ξ_4, η_4 are coordinates centered at L_4 and parallel with the x, y axes. The outside line

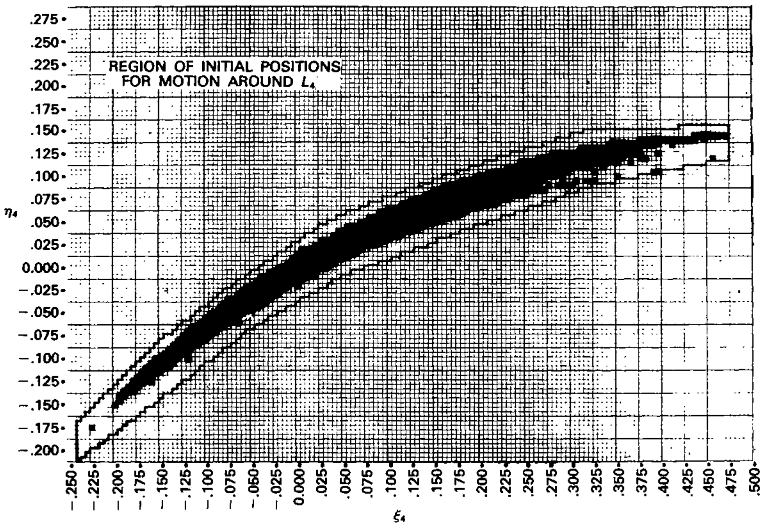

Fig. 3. Region of initial positions for motion around L_4.

indicates the boundary of the region within which initial positions were tested. Initial positions which remained near L_4 (i.e. which librated) are indicated by solid black boxes; those which resulted in crossing the axis of syzgies are white. In the immediate vicinity of L_4, the librational region is well defined, as expected. The interesting aspect of the regions appear farthest from L_4, where there are isolated regions of librational motion. In the region above and to the right of L_4, these islands are most prominent.

Figure 4 shows the same about L_5 and closely approximates a mirror image of the region about L_4.

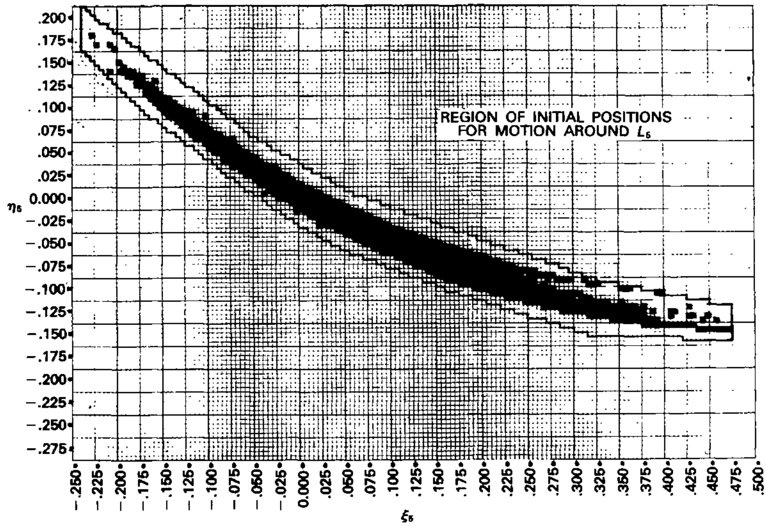

Fig. 4. Region of initial positions for motion around L_5.

4. Notes and Comments

(1) The above described results suggested the investigation of stability when the motion begins at L_4 with non-zero velocity. At present the determination of the velocities which result in librational motions is in progress.

(2) A method for predicting, approximately, the more uniform portion of the regions given in this paper is also under development, using the Gaussian curvature of the restricted problem of three bodies.

The function

$$\Omega = \tfrac{1}{2}[(1-\mu)r_1^2 + \mu r_2^2] + \frac{1-\mu}{r_1} + \frac{\mu}{r_2}$$

is the generalized potential function of the restricted problem and it is used in the description of the regions of possible motion. The Gaussian curvature,

$$K = C_1 C_2 = \frac{\Omega_{xx}\Omega_{yy} - \Omega_{xy}^2}{(\Omega_x^2 + \Omega_y^2 + 1)^2}$$

is the product of the reciprocals of the principle radii of curvature of the surface, $z = \Omega(x, y)$ (Plummer, 1918).

Figure 5 is a portion of the $K = 0$ line in the vicinity of L_4. An examination of Figures 5 and 3 indicates a correlation between the $K = 0$ line and the region of librational motion. In fact most of the librational regions are above the $K = 0$ line and fall into the $K > 0$ area, indicating local stability.

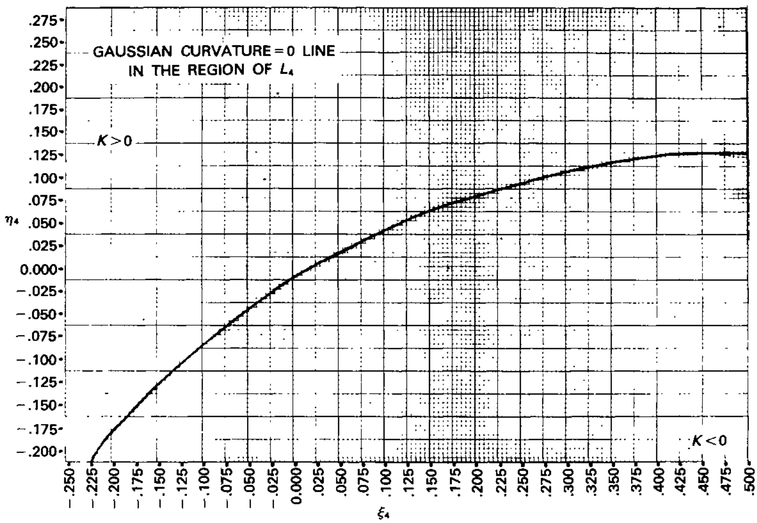

Fig. 5. The line of zero gaussian curvature.

5. Conclusions

An important aspect of these results is the fact that small changes in the initial position (0.005) may result in completely different behavior. Near the equilibrium points there apparently exists a sharp demarcation line between the two types of motion. Even though we are near the equilibrium point no analytical method is available for predicting this region. The complexity of the region near the extremities is evidenced by isolated islands. Whether longer time of integration or higher numerical accuracy would change the picture is unknown. Since the restricted, problem is a non-integrable dynamical system and in addition to the Jacobian integral no other analytic integrals exist with global validity (Poincaré, 1896), we do not find it surprising that discontinuities in the librational regions appear and that Whittaker's (1904) comments on the frequency dependence of the adelphic integrals are applicable. Recent work by Prigogine (1977) questioning the deterministic aspects of classical mechanics seems to be also applicable to the findings of this paper.

Acknowledgement

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