CELESTIAL COORDINATE REFERENCE SYSTEMS IN CURVED SPACE-TIME

S. M. KOPEJKIN
Sternberg State Astronomical Institute, 119899, Moscow, USSR.

Received: 5 February 1988; Accepted: 5 February 1988)

Abstract. In the weak-field and slow-motion approximation of the General Relativity theory the relativistic theory of construction of nonrotating harmonic coordinate reference sysems is developed. The general case of an isolated astronomical N-body system is considered. The bodies are assumed to consist of a perfect fluid and to possess any number of the internal mass and current multipole moments characterizing the internal structure and own gravitational field of the bodies. The description of the coordinate reference systems and gravitational field is realized by the specific forms of the metric tensor. A metric is determined by the method of the Post-Newtonian Approximations (PNA) from the inhomogeneous Einstein equations under the harmonic coordinate condition. We have obtained two different specific forms of a metric, which are related to the inertial and quasiinertial coordinate reference systems.

The metric in inertial coordinates is a near-zone solution of the Einstein equations for an N-body system. The metric in quasi-inertial coordinates is a solution of Einstein equations in the body's neighborhood, which is a world tube surrounding the body under consideration and extending up to another nearest body. The coordinate transformation between the inertial and quasiinertial reference systems is derived by a matching of both solutions in the body's neighborhood. A new method is proposed for construction of the body's proper reference system. This coordinate system has an origin which moves along a nongeodesic (in the general case) worldline of the body's center of inertia. The proper reference system is used for derivation of the Newtonian equations of translational and rotational motion of the body. The equations give us exhaustive information about the nonlinear Newtonian interaction between gravitational fields of the bodies in terms of internal mass multipole moments. Finally, the coordinate transformation between the inertial and proper reference systems is discussed in the first PNA.

0. Notation

In this paper the notations are as in (Misner *et al.*, 1973; Thorne and Hartle, 1985; Blanchet and Damour, 1986). In particular, greek letters $\alpha, \beta, \gamma, \ldots$ run from 0 to 3 and small italic letters a, b, $c, \ldots$ run from 1 to 3; the italic capitals A, B, C number the bodies and run from 1 to N; a comma denotes a usual derivative and semicolon denotes a covariant derivative; repeated indices imply an Einstein summation; round brackets surrounding indices denote symmetrization and square brackets denote anti-symmetrization, for example, $T_{(ij)} = \frac{1}{2}(T_{ij} + T_{ji})$ and $T_{[ij]} = \frac{1}{2}(T_{ij} - T_{ji})$. We designate: G – universal gravitational constant; c – velocity of light; ε_{ijk} – the fully antisymmetric Levi-Civita symbol ($\varepsilon_{123} = 1$) δ_{ij} – the Euclidean metric = diag(1, 1, 1); $\eta_{\alpha\beta} = \eta^{\alpha\beta}$ – the flat metric = diag(-1, 1, 1, 1); $g_{\alpha\beta}$ – a metric of curved space-time; $g = \det(g_{\alpha\beta})$. The spatial indices are raised and lowered by means of the metric δ_{ij}. In order to deal conveniently with sequences of many spatial indices we shall use an abbreviated notation for "multi-indices", where an upper-case letter denotes a multi-index, while the corresponding lower-case letter denotes its number of indices, for example, $L := i_1 i_2 \ldots i_1$; $P := i_1 i_2 \ldots i_p$; $Q_L = Q_{i_1 i_2 \ldots i_l}$. When needed we also use $L - 1$: $i_1 i_2 \ldots i_{l-1}$, so that the tensor $T_{aL-1} = T_{a i_1 i_2 \ldots i_{l-1}}$ has l indices. We also denote $\xi^L = \xi_L =$

$\xi_{i_1}\xi_{i_2}\ldots\xi_{i_l}$ and $\partial^L/\partial x^L = \partial^l/\partial x^{i_1}\partial x^{i_2}\cdots\partial x^{i_l}$. A multi-summation is always understood for repeated multi-indicies: $S_P T^P = S_P T_P = S_{i_1 i_2\ldots i_p} T_{i_1 i_2\ldots i_p}$. The symmetric and trace-free (STF) part of T_P is denoted by $T_{\langle P\rangle} = T_{\langle i_1 i_2\ldots i_p\rangle}$, for instance, $T_{\langle ij\rangle} = T_{(ij)} - \frac{1}{3}\delta_{ij}T_{kk}$. The explicit expression of the STF part of T_P is given in (Thorne, 1980; Blanchet and Damour, 1986). For any positive integer l we shall denote $l! = l(l-1)\ldots 2\cdot 1$; $l!! = l(l-2)\cdot\ldots\cdot(2 \text{ or } 1)$. A dot over any function means a differentiation with respect to time.

1. Introduction and Overview

The consistent and well-founded theory of the astronomical reference frames (RF) has a great principal and practical significance for modern celestial mechanics and astrometry (Mueller, 1981; Kovalevsky and Mueller, 1981; Podobed and Nesterov, 1982; Seidelman, 1985; Guinot, 1986; Soffel *et al.*, 1986a). Construction of the astronomical RF includes (Kovalevsky and Mueller, 1981; Kovalevsky, 1985): (1) a choice of the coordinate reference system (RS); (2) a choice of the basic astronomical objects (repers), for which values of coordinates and a time change of the coordinates are considered as being known; (3) an establishment of the relations between RF's consructed on the base of other RS's and (or) astronomical repers.

In celestial mechanics and astrometry the galactic, solar barycentric, planetocentric, topocentric and satellite RS's are used most often. The origins of these RS's are placed in the center of inertia of the correspoding astronomical system (Galaxy, Solar system etc.). The coordinates of repers are fixed by the initial conditions in the epoch in any, but only one, aforementioned RS. A time change of the coordinates of repers is predicted (Podobed and Nesterov, 1982) either by kinematical methods for objects outside the Solar system (proper motions and radial velocities of the quasars, galaxies and stars) or with the help of the dynamical theories of motion for objects inside the Solar system. The coordinates of repers can easily be recalculated from one RS to the another, if the coordinate transformation between the RS's is known.

We also distinguish (Kovalevsky and Mueller, 1981; Kovalevsky, 1985) between the celestial and terrestrial RS's. There is no principal theoretical distinction betwen them! All differences are in their mathematical definition and practical realization. The celestial RS's are used for a description of the motion of celestial bodies and reduction of the astrometric observations. The terrestrial RS's are used mainly for the study of the internal structure and own gravitational field of the Earth, as well as tectonic deformations of the Earth's crust and description of the locus on the Earth's surface of astrometric and gravimetric stations. In the paper we shall discuss only celestial reference systems.

The classical theory of the astronomical RF's is based on the newtonian theory of gravitation (NGT). In the NGT the notions of space and time are not connected. Time, according to the mathematical point of view, is a universal absolute parameter streamlining a sequence of the events, which take place in the 3-dimensional Euclidean absolute space. The description of physical laws in the absolute space is admitted in any RS. However, in the NGT there exist privileged cartesian RS's. The whole absolute space may be covered by means of only one cartesian RS.

But it is more important to note that the absolute time and Cartesian coordinates have a real physical meaning in Newtonian physics and can be directly measured by astronomical observations. That is why the cartesian RS's are widely used in non-relativisic astronomy. A solution of the NGT equations in the Cartesian RS's allows us to remove from the theoretical predictions all unmeasurable and spurious terms caused only by a bad choice of the coordinate system.

The cartesian RS's may admit an arbitrary rotation and translation in the absolute space. A coordinate transformation between them is:

$$y^i = \mathscr{P}^i_k(t)x^k - x^i_B(t) \tag{1}$$

where $\mathscr{P}^i_k(t)$ is the orthogonal rotation matrix and $x^i_B(t)$ is the vector of translation, both depending on the absolute time t.

It is possible to select from all Cartesian RS's the subset of the inertial RS's which have no rotation and move with constant velocity in the absolute space. The coordinate transformation connecting two inertial RS's is the Galilean one:

$$y^i = \mathscr{P}^i_k x^k - V^i t - b^i \tag{2}$$

Here $\mathscr{P}^i_k, V^i, b^i$ are a constant rotation, velocity and translation, respectively. The Galilean transformation is a particular case of the more common formula (1).

Working in the framework of NGT, one often also uses quasiinertial RS's (Mueller, 1981), which move with arbitrary velocities and accelerations in the absolute space, but their coordinate axes have no rotation. The transformation between inertial RS (t, x^i) and quasiinertial one (t, y^i) follows from (1) under condition that the $\mathscr{P}^i_k$ is a constant matrix. One has:

$$y^i = \mathscr{P}^i_k x^k - x^i_B(t) \tag{3}$$

The Newtonian potential $\hat{U}(t, y) = G\int \rho(t, y')|y - y'|\mathrm{d}^3 y'$ (ρ is a rest mass density) in the quasiinertial RS is linked with the potential $U(t,x) = G\int\rho(t,x)|x - x'|^{-1}\mathrm{d}^3 x'$ in the inertial RS by the transformation (Misner *et al.*, 1973):

$$\hat{U}(t, y) = U(t, x) - \mathscr{P}^i_k x^k \frac{\mathrm{d}^2 x^i_B}{\mathrm{d}t^2} - C(t) \tag{4}$$

where $C(t)$ is an arbitary function of time. Note that both potentials satisfy the same Poisson equation of NGT. For an isolated astronomical N-body system the potential $U(t,x)$ tends to zero when distance from the system increases, but $\hat{U}(t, y)$ becomes infinitely large. One can say that it is these properties of the potentials which define the type of the RS.

A theory of the astronomical RF's is inseparable from the problem of determination of the motion of celestial bodies (Earth, planets, satellites etc.). This problem is decomposed in two subproblems – the internal and external ones (Fock, 1955). The external problem consists in the determination of the translational motion of the body's center of inertia. In the NGT this problem is usually solved in the inertial RS. The purpose of the internal problem is to determine the motion of the body's matter and the laws of time change of the body's multipole moments in the proper (Duboshin, 1975) reference system (PRS). The PRS has an origin which coincides with the body's center of inertia for any moment of time. The coordinate transformation between the inertial RS and PRS is described by the formula (3), where $x_B^i(t)$ now denotes a vector connecting the origin of inertial RS and the body's center of inertia. The explicit dependence of x_B^i on time is found by solving the equations of motion (eq.m.) of the bodies in the external problem. It is very important to note that an influence of the external bodies is manifested in the eq.m of the internal problem only in the form of the tidal terms, which are expressed as second space derivatives of the Newtonian potential $U_E(t, x)$ created only by the external bodies.

The eq.m of the external and internal problems cannot be solved separately in the general case (Duboshin, 1975). A translational motion of bodies perturbs their rotation and *vice versa*. The only exception is the motion of the spherically-symmetric bodies (Duboshin, 1975).

At present there is great confidence that a relativistic theory of astronomical RF's must be founded on General Relativity (GR). GR has passed many a serious test both in the weak gravitational field inside the Solar system (Will, 1981, 1986) and in the strong field inside the double pulsar PSR 1913+16 (Damour, 1987; Taylor, 1987).

In GR the space and time are united together and form the 4-dimensional Riemannian manifold – space-time. It is permissible to introduce any coordinate RS in space-time. The properties of the RS are described by the metric tensor $g_{\alpha\beta}$, which, in contrast to NGT, carry information about the gravitational field of the bodies as well. The metric tensor is determined from the Einstein equations. As a rule, before solving these equations, four resrictions (coordinate or gauge conditions) are imposed on the components of the $g_{\alpha\beta}$. They extract some subset from an infinite set of space-time coordinates. Inside this subset the coordinates are linked by smooth differentiable transformations which do not change the coordinate conditions being chosen.

In the GR there exist no absolute time and Euclidean space. Besides, one cannot in the general case introduce some privileged RS in space-time. Contrary to NGT the co-ordinates in the curved space-time have no physical meaning and cannot be measured directly by the astronomical observations.

Nevertheless, there are special cases, when one can speak in some sense about privileged coordinates in GR. One of them is a case of the space-time having a weak gravitational field and slow motion of matter. Such space-time may be covered by the coordinates, which differ only a little from the absolute time and Cartesian space coordinates of NGT. We shall call these space-time coordinates quasicartesian reference systems. Quasicartesian RS's are most convenient for a development of the relativistic theory of the astronomical RF's inside the Solar system, as well as in the case of any isolated astronomical system, which consist of N well-separated and extended bodies possessing a weak gravitational field and moving with slow orbital and rotational velocities. In this paper we shall analyze only such astronomical systems.

One can select from a set of the quasicartesian RS's the subset of the 'inertial' RS's, which have no rotation and move with a constant velocity one relative to another. The metric tensor in the inertial RS's tends to the flat metric $\eta^{\alpha\beta}$ the further one gets from the N-body system. The way of constructing the inertial RS's is well known in the GR (Fock, 1955; Brumberg, 1972; Misner *et al.*, 1973; Ehlers, 1980; Will, 1981; Damour, 1987). A more complicated task is to construct, in curved space-time, the analog of the PRS of the NGT.

One way aimed at solving this problem is to use the Newtonian transformations:

$$u = t \tag{5}$$

$$w^i = x^i - x^i_B(t) \tag{6}$$

where the inertial RS and PRS are designated as (t, x^i) and (u, w^i) respectively; x^i_B are the coordinates of the center of inertia of body B and are determined by solving the relativistic external problem.

The construction of PRS with the help of transfomations (5) and (6) looks simple and attractive. Therefore, it is widely accepted for interpretation of data of the radio-interferometric observations (Hellings, 1986; Cannon *et al.*, 1986), for description of the motion of the Moon and Earth's satellites (Brumberg, 1958; Brumberg, 1972; Brumberg and Ivanova, 1982; Lestrade and Chapront-Touze, 1982; Newhall *et al.*, 1983; Martin *et al.*, 1985; Brumberg, 1986; Akim *et al.*, 1986). However, the PRS (u, w^i) cannot be recognized as satisfactory for two reasons.

In the first place, the transformations (5) and (6) are not the relativistic ones. They ignore completely the Lorentzian and gravitational (Einsteinian) contractions, the relativistic geodetic precession and effects of the curvature of space-time. All these kinematical and dynamical effects go to the expressions for the metric tensor and eq.m. of the internal problem, where they are shown as terms depending on: (1) the 'absolute' velocity of the body's center of inertia with respect to the inertial RS (t, x^i), and (2) the absolute value and the first space derivatives of the Newtonian potential $U_E(t, x)$ of external bodies. Thus, the relativistic eq.m. of internal problem differ essentially from the Newtonian ones, which do not depend on the 'absolute' velocity and contain only second space derivatives of $U_E(t, x)$, i.e. tidal terms. One must be very cautious when using the PRS (u, w^i) since one can confuse the true physical effects with the coordinate ones. For example, the term with the amplitude of about one meter in the relativistic theory of motion of the Moon (Brumberg, 1958; Misner *et al.*, 1973; Baierlein, 1967) built on the basis of the PRS (u, w^i) has no real physical meaning. The appearance of this term is caused only by a bad choice of the coordinates and, therefore, the one-meter term can not really be observed (Soffel *et al.*, 1986b). The situation with the transformation (5) and (6) in GR is completely analogous with that in Special Relativity, when for description of physical laws the Galilean transformation is used instead of the Lorentzian one. Such an operation is, of course, admissible, but completely inexpedient (Kadomtzev *et al.*, 1972), since it obscures the true nature of things and delivers extremely complicate calculations.

In the second place, the utilization of PRS (u, w^i) confuses and makes difficult the consideration of the important question concerning the notion of rigidity in GR. This notion is frequently used for derivation of the relativistic eq.m. of extended bodies. In many investigations (see, for example, Fock, 1955; Brumberg, 1972; Ehlers and Rudolph, 1977; Hogan and McCrea, 1974; McCrea and O'Brien, 1978; Spyrou, 1977, 1978; Kopejkin, 1985; Grishchuk and Kopejkin, 1986; Fukushima, 1986) a rigidity was understood in the coordinate sense as conditions of the constancy in some concrete RS either of the body's matter or body's multipole moments. Such a definition of rigidity is quite admissible and convenient since it allows us to simplify considerably the complicated and tedious calculations for obtaining the eq.m. of the bodies. However, the coordinate rigidity is not a covariant notion. Therefore, if one chooses RS in an inappropriate manner, one should introduce inside the bodies an additional internal stress and velocity field of matter, which have to compensate the fictitious deformations and motions of body's matter caused only by the bad choice of RS (Hogan and McCrea, 1974; McCrea and O'Brien, 1978; Kopejkin, 1985; Grishchuk and Kopejkin, 1986). The construction of the 'good' RS in which the fictitious additional stresses and velocities are absent, may greatly simplify a solution of the relativistic internal problem and make this solution reflect the nature of physical laws.

The PRS (u, w^i) constructed with the help of the transformations (5) and (6) is 'bad'. The fictitious stresses and velocities are present in this PRS. They compensate the Lorentzian and Einsteinian contractions, the geodetic precession and effects of the curvature of space-time. If one ignored these fictitious stresses and velocities, one would make mistakes in the calculations of the relativistic eq.m. of bodies in terms of order $c^{-2}(L_B/R)^2$ and higher, where L_B is a body's radius and R is the minimal distance between the bodies. Note once again that the present-day relativistic theories of the interferometric observations and motion of the Moon and satellites are constructed in the 'bad' geocentric PRS (u, w^i) of the Earth. The fictitious deformations and shear velocities for the Earth's crust reach about 6 cm and 50 cm/year, respectively, in this PRS and influence the interpretation of observational data. One can use, of course, the 'bad' PRS (u, w^i) and include all spurious terms in the mathematical data processing. But as a consequence comparison with real observations cannot be done at the level of the coordinate description of the theory and must be done using appropriate invariantly defined obsevables (Martin *et al.*, 1985; Brumberg, 1986; Hellings, 1986; Soffel *et al.*, 1986). Such a method is cumbrous. A more suitable method in all respects is to develop a new theory of observations and motion of celestial bodies based on using of the physically adequate 'good' PRS. We shall denote the 'good' PRS by (τ, ξ^i).

The 'good' PRS must be linked with the inertial RS (t, x^i) by the relativistic coordinate transformations, which should introduce no spurious terms in the metric and eq.m. of the relativistic internal problem. The general theoretical consideration

(Misner *et al.*, 1973) shows that the transformation from the 'good' PRS to the inertial RS must have, in the vicinity of worldline of the origin of the 'good' PRS, the structure of a Taylor expansion in powers of ξ^i:

$$x^\alpha = x^\alpha_B(\tau) + e^\alpha_i(\tau)\xi^i + \tfrac{1}{2}\Gamma^\alpha_{ij}(\tau)\xi^i\xi^j + O(\xi^3)$$
$$(\xi^0 \equiv c\tau; \quad x^0 \equiv ct) \qquad (7)$$

Here the function $x^\alpha_B(\tau)$ represents the worldline's description of the origin PRS (τ, ξ^i) in the RS (t, x^i) and the $e^\alpha_i(\tau)$, $\Gamma^\alpha_{ij}(\tau)$ are coefficients of expansion. The relativistic transformation (7) replaces the Newtonian ones (5) and (6).

In the 'good' PRS the following rather remarkable properties should be satisfied:

(1) the metric tensor $g_{\alpha\beta}$ and the eq.m. of the relativistic internal problem must not depend on 'absolute' velocity of motion of the origin of 'good' PRS relative to the inertial RS, but may admit the dependence only on the relative velocities of the bodies;
(2) the own gravitational field of the body for which the 'good' PRS is constructed must be described outside the body by the set of the mass and current internal multipole moments; including monopole, dipole etc.;
(3) the gravitational field of external bodies must be presented only in the form of tidal terms being described by the electric-type and magnetic-type external multipole moments.

These properties express the guess that the 'good' PRS must resemble a RS, which falls freely in the background gravitational field created only by the external bodies. However, one should not think that the 'good' PRS can be realized as a locally inertial RS for a massless test body (Misner *et al.*, 1973; Manasse and Misner, 1963; Ni and Zimmerman, 1978) as the second property of the 'good' PRS prohibits this. The existence of a 'good' PRS has been more or less explicitly assumed by many authors (see, for example, Misner *et al.*, 1973; Will, 1981; Caporali, 1981; Mashhoon, 1985). But these authors have not proposed any exact mathematical procedure for its construction.

Note that there exist many possible ways of constructing a 'good' PRS. One of them was pointed out by Bertotti (1954) and has been recently developed by Ashby and Bertotti (1984, 1986) and Bertotti (1986). An equivalent method has been proposed and developed to the extent of practical applications by Fukushima *et al.* (1986). In these works the 'good' PRS is constructed within the first PNA of GR for the specific form of the EIH metric (Einstein, Infeld and Hoffman, 1938). The EIH metric was obtained (according our terminology) in the inertial RS's. It describes the gravitational field only outside the bodies, which may be regarded as massive point particles (black holes) or spherically-symmetric and nonrotating extended bodies (Fock, 1955).

In the Bertotti–Fukushima method the construction of a 'good' PRS is begun by building up the background external metric for the body under consideration. The external metric is obtained from the complete EIH metric by dropping all the divergent or undefined terms on the body's center of inertia worldline. Then a local Fermi frame (Fermi, 1922; Manasse and Misner, 1963; Misner, *et al.*, 1973) is defined in the body's vicinity using the background metric with respect to which the body moves along a geodesic. After that, the coordinate transformation between the Fermi frame and background metric is obtained. The transformation is applied to the complete EIH metric and, thus, the 'good' PRS is obtained. The body's gravitational field in this PRS is spherically-symmetric (Schwarzschild) and the gravitational field of the distant bodies appears only through the curvature tensor of the background metric, i.e. through the tidal effects.

The Bertotti–Fukushima method is conceptually simple. It confirms our expectation that the 'good' PRS exists and gives an insight into the structure of the transformations (7). However, the method of construction of the Fermi normal coordinates for massive bodies has some disagreable drawbacks. Namely:

(1) the background external metric is not a solution of the Einstein equations;
(2) there are ambiguities in the procedure of constructing the external metric. They are caused by the terms which describe the effects of back-action of the gravitational field of the body under consideration on the external gravitational field of other bodies (Thorne and Hartle, 1985);
(3) the method under review cannot be used for derivation of the eq.m. of bodies, i.e. their worldlines. A choice of the body's center of inertia worldline as a geodesic is justified only *a posteriori* and with the help of a quite different technique (EIH, 1938; Papapetrou, 1951; Fock, 1955; Shirokov and Brodovski, 1956; Infeld and Plebanski, 1960; Brumberg, 1972; Damour, 1983; Thorne and Hartle, 1985; Kopejkin, 1987);
(4) the method has been elaborated only for the special case of spherically-symmetric and nonrotating bodies. It is completely unclear how one can construct the Fermi normal coordinates in real astronomical situations which are, of course, more complicated. Recall that the Earth has oblateness and rotation, which can not be ignored.

Another method of construction of the 'good' PRS has recently been proposed by Thorne and Hartle (1985) (see also Fujimoto and Grafarend (1986)) and developed to some extent by Suen (1986) and Zhang (1986). The method consists in the determination of the metric tensor from the Einstein equations under conditions that one satisfies the properties mentioned above for the 'good' PRS. Thus, the metric in the Thorne–Hartle method is derived at once in the 'good' PRS. The solutions of the Einstein equations is searched for in a vacuum region of space-time under de Donder (harmonic) gauge conditions in the body's neighborhood, where the gravitational field is weak. The metric tensor is represented in the form of an expansion in powers of the small parameters $M_B/\hat{r}$, $\hat{r}/R$ etc., where M_B is the body's mass, $\hat{r}$ is a distance from the body and R is an inhomogeneity scale (distance between the bodies). The coefficients of the expansion are the internal and external multipole moments of the gravitational fields created by the body under consideration, and external ones, respectively. The multipole moments contain a description of the gravitational field as well as complete information about the RS being chosen.

The Thorne–Hartle method is mathematically elegant and appropriate for the derivation of the laws of motion both of compact astrophysical sources (black holes and neutron stars) and extended bodies with a weak gravitational field. However, it is not complete and cannot immediately be used in the ephemeris astronomy for the following reasons:

(1) Thorne and Hartle have constructed only the 'instanteneous' PRS which coincides with the body's center of inertia at some fixed moment of time. As time goes on the origin of the 'instanteneous' PRS propagates along a geodesic, but the body's center of inertia worldline does not do so in the general case. The deviation from the geodesic is caused by the interaction of the body's internal multipole moments to the external multipole moments of the gravitational field of other bodies. The 'instanteneous' PRS is not appropriate for ephemeris astronomy since this science prefers to operate with the RS's whose origins coincide with the body's (or N-bodys') center of inertia for all times.
(2) The Thorne–Hartle solutions of the Einstein equations are formal, since the matching with solutions of the inhomogeneous equations for extended sources has not been performed. Thus, the internal multipole moments are not presented as integrals over volumes of the sources and therefore have no clear physical meaning.
(3) Thorne and Hartle have not derived a relativistic coordinate transformation between the 'good' PRS and the inertial RS, which plays a very important role in the relativistic theory of astronomical RF's (Japanese Ephemeris, 1985; Fukushima *et al.*, 1986; Brumberg and Kopejkin, 1988).

The beautiful method of construction of the 'good' PRS was proposed by D'Eath (1975a,b) (see also works of Kates (1980a,b) and Damour (1983)). These remarkable papers are devoted to the derivation of the eq.m. of compact astrophysical objects – the neutron stars and black holes. However, the authors applied specific mathematical methods which are very interesting, but cannot be used directly for development of the relativistic theory of RF's in the space-time with a weak gravitational field.

Let us also mention the works in which construction of the 'good' PRS has been accomplished either with the help of the infinitesimal (Murray, 1983; Martin *et al.*, 1985; Hellings, 1986; Vincent, 1986) or linear (Pavlov, 1984) transformations. The methods used in these works cannot be considered to be satisfactory since they are based to a considerable extent on heuristic principles, but not on the exact theory.

The crucial step in the problem concerning construction of the exact and appropriate (for astronomical practice) relativistic theory of RF's has recently been done by Brumberg and Kopejkin (1988). The relativistic theory developed by Brumberg and Kopejkin (1987) combines the basic ideas of Fock (1955) – the PNA scheme; Thorne (1980) and Thorne and Hartle (1985) – multipole formalism, and D'Eath (1975a, b)

– asymptotic matching. The theory completely overcomes conceptual and/or technical drawbacks of the works mentioned above and has been founded on two conditions: (1) all nonrotating coordinate RS's have been constructed in the de Donder (harmonic) gauge; (2) the metric tensor in any RS represents the physically adequate solution of the Einstein equations either for the external or internal problems of relativistic celestial mechanics. In such a way, the solar barycentric RS, the geocentric RS, the topocentric RS and the satellite RS have been constructed. The three latter RS's are the 'good' PRS's for the corresponding problem. The relativistic coordinate transformation between different RS's has been derived by a matching of the components of metric tensors in the region of space-time where the RS's overlap.

In the present paper I am developing the ideas put forward in the works (Kopejkin, 1987; Brumberg and Kopejkin, 1988). In Section 2 below the basic conceptions and formulas are given. Section 3 presents a discussion of the harmonic inertial RS's. Section 4 is devoted to a construction of the harmonic quasiinertial RS's. In the Section 5 the relativistic coordinate transformation between inertial and quasiinertial RS's is deduced. Section 6 describes the construction of the 'good' PRS for a massive self-gravitating body. In Section 7 the preceding results of Sections 2–6 are used as a tool for deducing the coordinate transformation between inertial and 'good' PRS.

2. Basic Conceptions and Formulas

We shall investigate a structure of space-time for the case of gravitationally bound and isolated astronomical N-body system. Let us assume that the nongravitational forces are absent, the bodies are well separated and the body's matter consists of a perfect fluid with the energy-momentum tensor $T^{\alpha\beta}$:

$$T^{\alpha\beta} = (\mu c^2 + p)u^\alpha u^\beta + pg^{\alpha\beta} \tag{8}$$

$$\mu = \rho(1 + c^{-2}\Pi) \tag{9}$$

Here ρ denotes the rest mass density in the comoving RS, p is the isotropic pressure connected with ρ by an equation of state $p = p(\rho)$, $u^\alpha = \mathrm{d}x^\alpha/\mathrm{d}s$ is the 4-velocity of a fluid element, Π is the specific internal energy density satisfying the thermodynamic equation:

$$u^\alpha\left(\Pi_{,\alpha} + p\left(\frac{1}{\rho}\right)_{,\alpha}\right) = 0 \tag{10}$$

The N-body system in question is characterized by the following parameters: (1) L_B – size of the body; (2) R – minimal distance from the body under consideration to the nearest companion; (3) M_B – the body's mass; (4) M – total mass of N-body system; (5) v^i – the body's orbital velocity in the barycentric inertial RS; (6) ν^i – internal (rotational ν_{rot} plus oscillatory ν_{osc}) velocity of an element of the body's matter in the 'good' PRS. Since one considers the gravitationally bound N-body system there exist the relations (linked with the virial theorem) $v^2 \sim GM/R$ and $\nu_{\mathrm{osc}}^2 \sim GM_B/L_B$.

We shall restrict our attention only to the N-body systems which have a slow motion of matter and weak gravitational field everywhere both outside and inside the bodies. Thus, one presumes that the following small parameters exist: (1) $\varepsilon \sim v/c \ll 1$; (2) $\eta \sim GM/Rc^2 \ll 1$; (3) $\varepsilon_B \sim v/c \ll 1$; (4) $\eta_B \sim GM_B/L_B c^2 \ll 1$. The parameters ε and η are equivalent, but ε_B and η_B are different. There is also a small parameter α_B, which characterizes a dimensionless measure of the deviation of the distribution of the body's matter from the spherically-symmetric one ($\alpha_B \sim$ (quadrupole moment of the body)$/M_B L_B^2$).

Each body studied in this paper will be supposed to be isolated, i.e. its immediate vicinity will be supposed to be devoid of matter and nongravitational fields and the distance R is large compared to the body's size L_B ($\delta_B \sim L_B/R \ll 1$.) For an isolated body one can split space-time up into three regions as measured in the body's 'instanteneous' PRS (Misner *et al.*, 1973; Thorne and Hartle, 1985): the internal region, which is a world tube surrounding the body and extending out to some radius $\hat{r}_I > L_B$; the buffer region extending from radius $\hat{r}_I$ to some larger radius $\hat{r}_0 < R$; and the external region located outside radius $\hat{r}_0$. In the internal region the body's own gravitational field dominates; but in the external one gravitational fields of other bodies are more important. The buffer region is placed in the vicinity of the distance $\hat{r}^* \sim R(M_B/M)^{1/3}$ from the body. The distance $\hat{r}^*$ is defined from the condition that the body's gravitational influence is approximately equal to the gravitational influence of the external masses. The buffer region plays the role of an asymptotically flat space-time region for the body in question (Misner *et al.*, 1973; Thorne, 1980; Thorne and Hartle, 1985).

We shall characterize gravity by the contravariant metric density $\sqrt{-g}\, g^{\alpha\beta}$ and denote by $h^{\alpha\beta}$ the small deviations of $\sqrt{-g}\, g^{\alpha\beta}$ from the flat metric $\eta^{\alpha\beta}$:

$$h^{\alpha\beta} = \eta^{\alpha\beta} - \sqrt{-g}\, g^{\alpha\beta} \tag{11}$$

We assume the $h^{\alpha\beta}$ to obey the de Donder (harmonic) coordinate conditions, which may be written as

$$(\sqrt{-g}\, g^{\alpha\beta})_{,\beta} = -h^{\alpha\beta}{}_{,\beta} = 0 \tag{12}$$

The Einstein equations under conditions (12) are read (Anderson and Decanio, 1975):

$$\eta^{\alpha\beta} h^{\mu\nu}{}_{,\alpha\beta} = -\frac{16\pi G}{c^4} W^{\mu\nu} \tag{13}$$

where

$$W^{\alpha\beta} \equiv \theta^{\alpha\beta} + \frac{c^4}{16\pi G}(h^{\alpha\mu}{}_{,\nu}\, h^{\beta\nu}{}_{,\mu} - h^{\alpha\beta}{}_{,\mu\nu}\, h^{\mu\nu}) \tag{14}$$

$$\theta^{\alpha\beta} \equiv (-g)(T^{\alpha\beta} + t^{\alpha\beta}) \tag{15}$$

and $t^{\alpha\beta}$ is the Landau and Lifshitz (1967) pseudotensor, which is quadratic in $h^{\alpha\beta}$ and its first partial derivatives.

The aims of this paper are as follows: (1) construct quasicartesian harmonic inertial

RS's, which must cover the space-time manifold as far as possible and move along the straight line with a constant velocity relative to the Lorentzian RS's of the asymptotically flat space-time for the N-body system; (2) construct nonrotating quasiinertial harmonic RS's, whose origins may move along arbitrary timelike worldlines; (3) construct nonrotating 'good' PRS, whose origin has to move along the worldline of the body's center of inertia; (4) derive relativistic coordinate transformations between all reference systems.

3. Harmonic Inertial Reference Systems

The set of the harmonic inertial RS's (t, x^i) is singled out by two boundary conditions imposed on the $h^{\alpha\beta}$ and $h^{\alpha\beta}{}_{,\nu}$ at the past null infinity (Fock, 1955; Damour, 1983):

$$\lim_{\substack{r\to\infty \\ t + r/c = \text{const}}} h^{\alpha\beta} = 0 \tag{16}$$

$$\lim_{\substack{r\to\infty \\ t + r/c = \text{const.}}} \left(r\left(\partial h^{\alpha\beta}/\partial r + \frac{1}{c}\partial h^{\alpha\beta}/\partial t \right)\right) = 0 \tag{17}$$

where $r = (x_i x_i)^{1/2}$. These conditions generalize the boundary condition for the Newtonian potential $U(t, x^i)$ in the NGT:

$$\lim_{\substack{r\to\infty \\ t = \text{const.}}} U(t, x^i) = 0 \tag{18}$$

which singles out from the infinite set of cartesian RS's in the absolute space a more restricted subset of the inertial RS's.

The conditions (16) and (17) mean the absence of the flux of gravitational radiation falling on the N-body system from the external universe (Damour, 1983, 1987). Each astronomical N-body system is termed isolated, if the conditions (16) and (17) may be fulfiled in any inertial RS.

The Einstein equations (13) can be transformed under boundary conditions (16) and (17) into integro-differential equations (Anderson and Decanio, 1975; Damour, 1983):

$$h^{\alpha\beta}(t, x^i) = \frac{4G}{c^4} \int_{R^3} \frac{W^{\alpha\beta}(t - |x - x'|/c, x')}{|x - x'|} \, \mathrm{d}^3 x' \tag{19}$$

where R^3 means integration over the whole spacelike hypersurface of constant time t.

The construction of the global inertial RS's is realized by means of the computation of $h^{\alpha\beta}$ from the equations (19) with the help of the successive iterations in which η and η_B are small parameters. Damour (1983) suggested calling such a method for computation of $h^{\alpha\beta}$ a Post-Minkowskian Approximation scheme (PMA). Before Damour's suggestion this method was called a fast-motion approximation scheme. But this name is not correct, since magnitudes of velocities of the bodies can be arbitrary, both large and small.

Recently Blanchet and Damour (1986) have shown by an elegant mathematical technique how one can compute $h^{\alpha\beta}$ by the PMA in the vacuum region of space-time at any level of approximation. Then, in a recent paper, Blanchet and Damour (1987) have succeeding in matching of the vacuum solution for $h^{\alpha\beta}$ to the interior of the material system consisting of the perfect fluid. The metric tensor obtained by Blanchet and Damour (1987) is suited for any point of space-time. Thus, Blanchet and Damour (1986, 1987) have constructed the inertial RS's, which are indeed global, i.e. they cover the whole space-time manifold.

However, for solving a lot of problems of modern celestial mechanics and astrometry it is quite sufficient to use an inertial RS which covers only a resticted domain of space-time. Usually this domain occupies the material system and is extended as measured in the barycentric inertial RS out to a radius set equal to the minimal length of gravitational waves radiated by the material system. This region of space-time is called a near zone of the source of gravitational radiation. The metric in the near zone may be obtained either from the weak field expansion of $h^{\alpha\beta}$ by additional slow motion expansion in small parameters ε and ε_B or with the help of PNA scheme applied for the solution of equations (19) (Anderson and Decanio, 1975; Ehlers, 1980; Anderson *et al.*, 1982; Futamase and Schutz, 1983; Damour, 1987).

Blanchet and Damour (1987) criticize the concept of the near zone as they have found that it is impossible to express the near zone metric as a functional of the instantaneous state of the material source. There are the terms which depend on the full past history of the material system. These terms arise at the fourth post-Newtonian level (4PNA) and are caused by the gravitational waves emitted by the system in the past and subsequently scattered off the curvature of space-time back onto the system. Fortunately, in most practical tasks of relativistic celestial mechanics and astrometry it is required to know the metric only at the first post-Newtonian level (1PNA). In the 1PNA the metric tensor has a simple form and can be presented as (Fock, 1955; Brumberg, 1972; Will, 1981):

$$g_{00}(t,x) = -1 + c^{-2}\underset{(2)}{g_{00}}(t,x) + c^{-4}\underset{(4)}{g_{00}}(t,x) + O(c^{-5}) \tag{20}$$

$$g_{0i}(t,x) = c^{-3}\underset{(3)}{g_{0i}}(t,x) + O(c^{-5})$$

$$g_{ij}(t,x) = \delta_{ij} + c^{-2}\underset{(2)}{g_{ij}}(t,x) + O(c^{-4})$$

where

$$\underset{(2)}{g_{00}}(t,x) = 2U(t,x); \quad \underset{(2)}{g_{ij}}(t,x) = 2\delta_{ij}U(t,x) \tag{21}$$

$$\underset{(3)}{g_{0i}}(t,x) = -4U^i(t,x) \tag{22}$$

$$g_{00}(t,x) = 2\Phi(t,x) - 2U^2(t,x) - \chi_{,tt}(t,x) \tag{23}$$
(4)

$$U(t,x) = GI_{-1}(\rho^*); \quad U^i(t,x) = GI_{-1}(\rho^* v^i) \tag{24}$$

$$\chi(t,x) = -GI_1(\rho^*)$$

$$\Phi(t,x) = GI_{-1}(\tfrac{3}{2}\rho^* v^2 - \rho^* U + \rho^*\Pi + 3p) \tag{25}$$

$$I_n(F)(t,x) = \int_{R^3} F(t,x')|x - x'|^n \mathrm{d}^3 x' \tag{26}$$

$$\rho^* = \rho\sqrt{-g}\,u^0; \quad v^i = cu^i/u^0 \tag{27}$$

In these formulas the invariant density ρ^* is that of the perfect fluid in the coordinate element of volume dVin arbitrary RS. It satisfies the ordinary Newtonian-looking equation of continuity, which can be written down in the covariant form as:

$$(\rho u^\alpha)_{;\alpha} = \frac{1}{\sqrt{-g}}(\rho\sqrt{-g}\,u^\alpha)_{,\alpha} = 0 \tag{28}$$

The invariant density is a good mathematical tool which drastically simplifies calculations of the eq.m. of bodies (Fock, 1955; Brumberg, 1972; Will, 1981; Kopejkin, 1985). As an example, let us point out two useful formulas:

$$\frac{\partial}{\partial s}\int \rho^*(s,z) f(s,z)\mathrm{d}^3 z = \int \rho^*(s,z)\frac{\mathrm{d}}{\mathrm{d}s} f(s,z)\mathrm{d}^3 z \tag{29}$$

$$\frac{\mathrm{d}}{\mathrm{d}s} \equiv \frac{\partial}{\partial s} + \frac{\mathrm{d}z^i}{\mathrm{d}s}\frac{\partial}{\partial z^i}$$

$$\mathscr{L}_u(\rho^* \mathrm{d}V) = u^\alpha(\rho^* \mathrm{d}V)_{,\alpha} = 0, \tag{30}$$

which are true in any coordinate system (s, z^i).

In the 1PNA the total mass M and coordinates of the center of inertia of an N-body system X^i are defined in any inertial RS by the formulas (Fock, 1955):

$$M = \int_{R^3} \mathrm{d}^3 x(\rho^* + c^{-2}(\tfrac{1}{2}\rho^* v^2 + \rho^*\Pi - \tfrac{1}{2}\rho^* U)) \tag{31}$$

$$MX^i(t) = \int_{R^3} \mathrm{d}^3 x\rho^* x^i(1 + c^{-2}(\tfrac{1}{2}v^2 + \Pi - \tfrac{1}{2}U)) \tag{32}$$

Fock (1955) proved that in the 1PNA the mass M is conserved and the centre of inertia X^i moves in space with a constant velocity along straight line. Thus, $X^i(t) = P^i t + K^i$, where the constants $P^i = \mathrm{d}X^i/\mathrm{d}t$ and K^i are the N-body system's momentum and center of inertia integrals, respectively.

One can show (Will, 1981) that form of the metric tensor (20) is invariant under constant rotation of space coordinates and post-Newtonian Poincaré transformation:

$$t' = t + c^{-2}(\tfrac{1}{2}V^2 t - V^k x^k) + \quad (33)$$
$$+ c^{-4}(\tfrac{3}{8}V^4 t - \tfrac{1}{2}V^2 V^k x^k) + O(c^{-6})$$
$$x'^i = (\delta^{ik} + c^{-2}\tfrac{1}{2}V^i V^k)(x^k - V^k t - b^k) + O(c^{-4})$$

Here (t, x^i) is one inertial RS and (t', x'^i) is another that, V^i and b^i are the constant velocity and displacement respectively.

The form-invariance of the metric (20) under post Newtonian Poincaré transformation (33) justifies the word 'inertial' for harmonic RS's constructed under boundary conditions (16) and (17). One can choose from the set of inertial RS's the barycentric inertial RS. In this RS the functions X^i must be equal to zero for any moment of time. This condition can be satisfied by the applying to the metric (20) transformation (33), where parameters V^i and b^i are to be selected so that P^i and K^i equal zero (for details see Brumberg (1972) and Will (1981)). Barycentric RS is convenient for the solution of the relativistic external problem. The solar system barycentric RS is used for construction of planet ephemerides (Lestrade and Chapront-Touzé, 1982; Newhall *et al.*, 1983; Akim *et al.*, 1986). The coordinate time of the solar barycentric (harmonic) RS must be considered as a TDB time scale, which is extensively used in modern astronomical practice.

Nevertheless (see introduction), the barycentric RS is unsuitable for solving the relativistic internal problem. A fully consistent relativistic description of this problem is possible only in the 'good' PRS in which the external gravitational effects are greatly reduced, leaving only small tidal ones. The construction of the 'good' PRS must be commenced with an intermediate step – the constuction of the set of the quasiinertial RS's.

4. Harmonic Quasiinertial Reference Systems

The harmonic quasiinertial RS's (τ, ξ^i) are defined in the neighborhood of the self-gravitating body. They are constructed so that solution of reduced Einstein equations (13) is to have in the internal and buffer regions the following structure (Thorne and Hartle, 1985): $h^{\alpha\beta} = h^{\alpha\beta}_B + h^{\alpha\beta}_E + h^{\alpha\beta}_I$. Here $h^{\alpha\beta}_B$ describes the gravitational field of the body and may be represented outside the body by two infinite families of internal mass $\hat{M}_B, \mathscr{I}^i_B, \mathscr{I}^{ij}_B, \ldots$ and current $\hat{S}^i_B, \hat{S}^{ij}_B, \ldots$ of the body's multipole moments. The $h^{\alpha\beta}_E$ describes the homogeneous field of force of inertia and tidal gravitational field of the external bodies. Th $h^{\alpha\beta}_E$ is characterized by two families of external electric-type moments $Q_{a_1 a_2 \ldots a_1}$ and magnetic-type moments $C_{a_1 a_2 \ldots a_1}$. All internal and external multipole moments are symmetric and trace-free (STF) tensors under linear coordinate transformations. In our approach the electric-type moments will begin with dipole order Q_i, but the magnetic-type moments will do so only at quadrupole order C_{ij}. This means that one will consider only nonrotation (no angular velocity term C_i), but admitting an arbitrary acceleration (term Q_i) quasiinertial RS. It is understood that rotation and acceleration of the quasiinertial RS takes place with respect to the asymptotically inertial RS in the buffer region of the body (Thorne and Hartle, 1985). The $h_I^{\alpha\beta}$ describes the gravitational field, which represents the interaction between the fields $h_B^{\alpha\beta}$ and $h_E^{\alpha\beta}$ and appears because of the nonlinearity of the Einstein equations.

We shall show that the described form of $h^{\alpha\beta}(\tau, \xi)$ actually exists. It can be exracted from the Einstein equations (13) with the help of a well-prescribed and unambiguous procedure. This procedure is, in fact, a combination of the PNA scheme and the multipole formalism of Thorne (1980) and Thorne and Hartle (1985), used in the internal and buffer regions of the body (Kopejkin, 1987; Brumberg and Kopejkin, 1988).

In the internal and buffer regions the energy-momentum tensor of matter is that of the body under consideration only. One presents $h^{\alpha\beta}$ as a series in powers of G and $1/c$ and substitutes the expansion in the Einstein equations (13). Then, as the first step of iteration in the linearized order of G the inhomogeneous Poisson equation for $h^{\alpha\beta}$ is obtained. In the right-hand- side of this equation a linearized energy-momentum tensor of matter of the body under consideration is included. In the quasiinertial RS's the solution of such an equation is sought as a sum of the two terms. One term represents the linearized solution for $h_B^{\alpha\beta}$, which is the particular solution of the inhomogeneous Poisson equations that are well-behaved at $\hat{r} \to \infty$. Another term describes the linearized solution for $h_E^{\alpha\beta}$. This is sought as the general solution of the homogeneous Laplace equation that is well-behaved at $\hat{r} = 0$. This solution contains 10 independent families of multipole moments (Thorne and Hartle, 1985; Suen, 1986; Zhang, 1986). By imposing de Donder (harmonic) gauge conditions one gets rid of 4 families of moments. Then by performing gauge tansformations with some specific form of generators (Suen, 1986; Zhang, 1986) that are solutions of the Laplace equation one can get rid of 4 more families of moments. Finally only two independent families of moments survive: electric-type moments $Q_{a_1 a_2 \ldots a_l}$ ($l \geqslant 1$) and magnetic-type moments $C_{a_1 a_2 \ldots a_l}$ ($l \geqslant 2$). In the linearized order of G a solution for $h_I^{\alpha\beta}$ is identically equal to zero.

The linearized solution of the Einstein equations (13) is substituted in the right-hand side of these equations and second iteration is done. One obtains:

$$\hat{g}_{00}(\tau, \xi) = -1 + \frac{2}{c^2}\left(\hat{U}_B(\tau, \xi) + \sum_{l=1}^{\infty} \frac{(2l-1)!!}{l!} Q_L\, \xi_{\langle L\rangle}\right) +$$

$$+ \frac{2}{c^4}\left(\hat{\Phi}_B(\tau, \xi) - \frac{1}{2}\frac{\partial^2}{\partial \tau^2}\hat{\chi}_B(\tau, \xi) - \right.$$

$$- 2\hat{U}_B(\tau, \xi) \sum_{l=1}^{\infty} \frac{(2l-1)!!}{l!} Q_L\, \xi_{\langle L\rangle} -$$

$$- G\sum_{l=1}^{\infty} \frac{(2l-1)!!}{l!} Q_L\, I^{(B)}_{-1}(\rho^* \xi^{\langle L\rangle}) +$$

$$+\frac{1}{2}\sum_{l=1}^{\infty}\frac{(2l-1)!!}{l!}\frac{1}{2l+3}\ddot{Q}_L\,\xi_{\langle L\rangle}\xi^2 -$$

$$-\sum_{l=1}^{\infty}\sum_{p=1}^{\infty}\frac{(2l-1)!!}{l!}\frac{(2p-1)!!}{p!}Q_L\,Q_P\,\xi_{\langle L\rangle}\xi_{\langle P\rangle}\Bigg) + O(c^{-5}); \quad (34)$$

$$\hat{g}_{0i}(\tau,\xi) = -\frac{4\hat{U}^i_B(\tau,\xi)}{c^3} - \frac{4}{c^3}\sum_{l=2}^{\infty}\frac{l(2l-1)!!}{(l+1)!}\varepsilon_{ipq}C_{pL-1}\,\xi_{\langle qL-1\rangle} +$$

$$+\frac{4}{c^3}\sum_{l=1}^{\infty}\frac{(2l-1)!!}{(l+1)!}\frac{2l+1}{2l+3}\Bigg(\dot{Q}_L\,\xi_{\langle iL\rangle} - \quad (35)$$

$$-\frac{l}{2l+1}\dot{Q}_{iL-1}\,\xi_{\langle L-1\rangle}\xi^2\Bigg) + O(c^{-5});$$

$$\hat{g}_{ij}(\tau,\xi) = \delta_{ij}\Bigg(1 + \frac{2}{c^2}(\hat{U}_B(\tau,\xi) + \sum_{l=1}^{\infty}\frac{(2l-1)!!}{l!}Q_L\,\xi_{\langle L\rangle})\Bigg) + O(c^{-4}) \quad (36)$$

$$I_n^{(B)}(f)[\tau,\xi] = \int_{V_B} f(\tau,\xi')|\xi-\xi'|^n\,\mathrm{d}^3\xi' \quad (37)$$

$$\hat{U}_B(\tau,\xi) = GI_{-1}^{(B)}(\rho^*);\quad \hat{\chi}_B(\tau,\xi) = -GI_1^{(B)}(\rho^*) \quad (38)$$

$$\hat{U}^i_B(\tau,\xi) = GI_{-1}^{(B)}(\rho^* v^i);\quad v^i = \mathrm{d}\xi^i/\mathrm{d}\tau \quad (39)$$

$$\hat{\Phi}_B(\tau,\xi) = GI_{-1}^{(B)}(\tfrac{3}{2}\rho^* v^2 - \rho^*\hat{U}_B + \rho^*\Pi + 3p) \quad (40)$$

Here the functions $\hat{U}_B(\tau,\xi)$, $\hat{U}^i_B(\tau,\xi)$, $\hat{\Phi}_B(\tau,\xi)$ and $\hat{\chi}_B(\tau,\xi)$ characterize own gravitational field of body under consideration. These functions can be expanded outside the body in infinite series in small parameters $L_B/\hat{r}$, where $\hat{r} = (\xi_i\xi^i)^{1/2}$

$$\hat{U}_B(\tau,\xi) = \frac{G\hat{M}_B}{\hat{r}} + G\sum_{l=1}^{\infty}\frac{(-1)^l}{l!}\mathscr{I}^L_B\,\partial_L\left(\frac{1}{\hat{r}}\right);\quad \partial_L \equiv \frac{\partial^L}{\partial\xi^L}; \quad (41)$$

$$\hat{U}^i_B(\tau,\xi) = \frac{G\hat{P}^i_B}{\hat{r}} + G\sum_{l=1}^{\infty}(-1)^l\frac{l}{(l+1)!}\varepsilon^i_{pq}\hat{S}^{pL-1}_B\,\partial_{qL-1}\left(\frac{1}{\hat{r}}\right) - \quad (42)$$

$$-G\sum_{l=2}^{\infty}\frac{(-1)^l}{l!}\dot{\hat{I}}^{iL-1}_B\,\partial_{L-1}\left(\frac{1}{\hat{r}}\right);$$

$$\hat{\chi}_B(\tau,\xi) = -G\Bigg(\hat{M}_B\hat{r} + \sum_{l=1}^{\infty}\frac{(-1)^l}{l!}\hat{I}^L_B\,\partial_L\,\hat{r}\Bigg); \quad (43)$$

$$\hat{I}^{a_1a_2\ldots a_l}_B = \int_{V_B}\rho^*\xi^{a_1}\xi^{a_2}\ldots\xi^{a_l}\,\mathrm{d}^3\xi; \quad (44)$$

where functions

$$\hat{M}_B = \int_{V_B}\rho^*(\tau,\xi)\,\mathrm{d}^3\xi; \quad (45)$$

$$\hat{\mathscr{I}}_B^{a_1 a_2 \ldots a_l} = \int\limits_{V_B} \rho^* \xi^{<a_1} \xi^{a_2} \ldots \xi^{a_l>} \, \mathrm{d}\xi \tag{46}$$

are the mass internal multipole moments and functions

$$\hat{S}_B^{a_1 a_2 \ldots a_l} = \int\limits_{V_B} \mathrm{d}^3\xi \rho^* \varepsilon^{pq<a_1} \xi^{a_2} \ldots \xi^{a_l>} \xi^p v^q \tag{47}$$

are the current internal multipole moments. Note that the quadrupole $\hat{\mathscr{I}}_B^{ij}$, octupole $\hat{\mathscr{I}}_B^{ijk}$ etc. moments have an order $\hat{M}_B L_B^l \alpha_B$ $(l \geqslant 2)$. The function $\hat{P}_B^i = \int_{V_B} \rho^* v^i \, \mathrm{d}^3\xi$ is the definition of body's linear momentum in the Newtonian approximation. It can be obtained from the dipole moment $\hat{I}_B^i$ by differentiating it with respect to time τ, i.e. $\hat{P}_B^i(\tau) = \dot{\hat{I}}_B^i(\tau)$.

Functions $Q_{a_1 a_2 \ldots a_l}$ $(l \geqslant 2)$ and $C_{a_1 a_2 \ldots a_l}$ $(l \geqslant 2)$ are external multipole moments and characterize a tidal gravitational field of distant bodies. These functions and acceleration Q_i reside at the origin of the quasiinertial RS and depend only on time τ.

The $\hat{M}_B$ is the body's mass. It is constant because of the fullfilment of equation of continuity (28). Coordinates of the body's center of inertia ξ_B^i are defined according to the identical relation: $\hat{M}_B \xi_B^i = \hat{I}_B^i$. One deduces as well from the exact equation $T^{i\alpha};\alpha = 0$ the Euler (Newtonian) hydrodynamical equations for a perfect fluid:

$$\rho^* \frac{\mathrm{d}v^i}{\mathrm{d}\tau} = \rho^* \frac{\partial \hat{U}_B}{\partial \xi^i} - \frac{\partial p}{\partial \xi^i} + \rho^* \sum_{l=1}^{\infty} \frac{(2l-1)!!}{(l-1)!} Q_{iL-1} \xi^{<L-1>} + O(c^{-2})$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau} = \frac{\partial}{\partial \tau} + v^k \frac{\partial}{\partial \xi^k} \tag{48}$$

Equations of motion of the body's center of inertia are obtained in any quasiinertial RS by means of the differentiating of $\hat{P}_B^i$ once with respect to time τ and using equations (48) in the integrand. One has:

$$\mathrm{d}\hat{P}_B^i/\mathrm{d}\tau = \hat{M}_B Q^i + \sum_{l=2}^{\infty} \frac{(2l-1)!!}{(l-1)!} Q_{iL-1} \hat{\mathscr{I}}_B^{L-1} + O(c^{-2}) \tag{49}$$

This formula is identical to the equations of motion of the body's center of inertia in the quasiinertial RS of NGT. Thus one can conclude that the harmonic quasiinertial RS's generalize the notion of the quasiinertial RS's of NGT.

We shall not investigate here the residual coordinate freedom between the quasiinertial RS's in full detail. Note only that in the Newtonian limit space transformation between two quasiinertial RS's (τ', ξ'^i) and (τ, ξ^i) admits constant rotation and translation depending on time τ:

$$\tau' = \tau + O(c^{-2})$$

$$\xi'^i = \mathscr{P}_k^i \xi^k - b^i(\tau) + O(c^{-2}) \tag{50}$$

Transformation (50) does not change the form of the eq.m. (49).

We have constructed the sets of inertial (t, x^i) and quasiinertial (τ, ξ^i) RS's. These RS's are overlaped in internal and buffer regions of the body. Thus, the coordinate transformation between them should exist. It may be found with the help of the asymptotic matching (AM) of components of metric tensors of two RS's in internal and buffer regions. Aside from this the AM allows us to establish the explicit form of the functions $Q_{a_1 a_2 \ldots a_l}$ and $C_{a_1 a_2 \ldots a_l}$ $(l \geqslant 2)$ and deduce the eq.m. of the origin of quasiinertial RS relative to the inertial RS. Acceleration Q_i cannot be derived by the AM and therefore remains arbitrary. This arbitrariness will be used later for appropriate choice of worldline of the origin of quasiinertial RS for construction of 'good' PRS.

5. Coordinate Transformation Between Inertial and Quasiinertial Reference Systems

Coordinate transformation is sought in the form:

$$\tau = t + c^{-2}(\tfrac{1}{2}v_B^2 t + A - v_B^k x^k) + c^{-4}(\tfrac{3}{8}v_B^4 t - \tfrac{1}{2}v_B^2 v_B^k x^k +$$

$$+ B + \sum_{l=1}^{\infty} B_L (x - x)^L + O(c^{-5}) \tag{51}$$

$$\xi^i = (\delta^{ik} + c^{-2}(\tfrac{1}{2}v_B^i v_B^k + F^{ik} + D^{ik}))(x^k - x_B^k) +$$
$$+ c^{-2} D^{ijp}(x^j - x_B^j)(x^p - x_B^p) + O(c^{-4}) \tag{52}$$

Functions x_B^i, $v_B^i = \mathrm{d}x_B^i/\mathrm{d}t$, A, B, F^{ik}, D^{ik}, D^{ijp} and $B^{a_1 a_2 \ldots a_l}$ in the formulas (51) and (52) depend only on time t and have the following properties:

$$F^{ik} = F^{[ik]}; \quad D^{ik} = D^{(ik)}; \quad D^{ijp} = D^{i(jp)}; \quad B^{a_1 a_2 \ldots a_l} = B^{(a_1 a_2 \ldots a_l)} \tag{53}$$

We have not written out the term of order $O(c^{-2}\hat{r}^2/R^2)$ in the formula (51) and the term of order $O(c^{-2}\hat{r}^3/R^3)$ in the formula (52) since (as will be made clear later) these terms are identically equal to zero. Note also that the point $x^i = x_B^i$ in the inertial RS (t, x^i) is the image of the origin of the quasiinertial RS (τ, ξ^i), i.e. point $\xi^i = 0$.

The asymptotic matching of metric tensors (20) and (34)–(36) is realized in the internal and buffer regions of the body according to the formula:

$$g_{\alpha\beta}(t, x) = \hat{g}_{\mu\nu}(\tau, \xi) \frac{\partial \xi^\mu}{\partial x^\alpha} \frac{\partial \xi^\nu}{\partial x^\beta} \tag{54}$$

In fact, formula (54) is equation, in which the left-hand side is known, but the right-hand side is not. This equation may be solved by a step-by-step method, in which the small parameters are the same as in Section 1. Thus, one can determine all unknown functions in the right-hand side of (54): (1) from matching $g_{oo}(t, x^i)$ with $\hat{g}_{\alpha\beta}(\tau, \xi^i)$ one finds $A(t)$, $B(t)$, $Q_{a_1 a_2 \ldots a_l}$ $(l \geqslant 2)$ as well as ordinary differential equations of second order (equations of motion) for function $x_B^i(t)$; (2) from matching $g_{0i}(t, x^i)$ with $\hat{g}_{\alpha\beta}(\tau, \xi^i)$ one finds $B_{a_1 a_2 \ldots a_l}$ $(l \geqslant 1)$, $F^{ik}(t)$ and $C_{a_1 a_2 \ldots a_l}$ $(l \geqslant 2)$; (3) from matching $g_{ij}(t, x^i)$ with $\hat{g}_{\alpha\beta}(\tau, \xi^i)$ one finds $D^{ik}(t)$ and $D^{ijp}(t)$.

4.1. MATCHING $g_{00}(t, x^i)$ WITH $\hat{g}_{\alpha\beta}(\tau, \xi^i)$

For $g_{00}(t, x^i)$ formula (54) gives:

$$g_{00}(t, x) = \hat{g}_{00}(\tau, \xi)\left(\frac{\partial \tau}{\partial t}\right)^2 + \frac{2}{c}\hat{g}_{0i}(\tau, \xi)\frac{\partial \tau}{\partial t}\frac{\partial \xi^i}{\partial t} + \frac{1}{c^2}\hat{g}_{ij}(\tau, \xi)\frac{\partial \xi^i}{\partial t}\frac{\partial \xi^j}{\partial t} \tag{55}$$

Substituting expressions (20), (34)–(36) in formula (55) and using formulas (51), (52) for calculation of derivatives $\partial\tau/\partial t$ and $\partial\xi^i/\partial t$ one obtains:

$$-1 + \frac{2U(t, x)}{c^2} = -1 + \frac{2}{c^2}\left(\hat{U}_B(\tau, \xi) + Q_i\xi^i + \sum_{e=2}^{\infty}\frac{(2l-1)!!}{l!}Q_L\xi^L - \right.$$

$$\left. - (v_B^k a_B^k)t - \dot{A} + a_B^k x^k\right) + O(c^{-4}) \tag{56}$$

where $a_B^i = \mathrm{d}v_B^i/\mathrm{d}t$.

Potential $U(t, x^i)$ satisfies the linear Poisson equation and therefore may be presented in the form:

$$U(t, x^i) = U_B(t, x^i) + U_E(t, x^i) \tag{57}$$

where $U_B(t, x^i)$ is the Newtonian potential of the body under consideration in the inertial RS:

$$U_B(t, x^i) = G\int_{V_B}\rho^*(t, x')|x - x'|^{-1}\,\mathrm{d}^3x' \tag{58}$$

and $U_E(t, x^i)$ is the Newtonian potential in the inertial RS created by the external bodies:

$$U_E(t, x^i) = \sum_{A \neq B} U_A(t, x^i); \quad U_A(t, x^i) = GI^{(A)}_{-1}(\rho^*) \tag{59}$$

One has also (Will, 1981; Kopejkin, 1987), with an accuracy up to order $O(c^{-2})$:

$$U_B(t, x^i) = \hat{U}_B(\tau, \xi^i) + O(c^{-2}) \tag{60}$$

Potential $U_E(t, x^i)$ must be expanded in Taylor series in powers of $(x^i - x_B^i)$ in the neighborhood of the point $x_B^i(t)$:

$$U_E(t, x) = U_E(x_B) + \frac{\partial U_E(x_B)}{\partial x_B^i}(x^i - x_B^i) + \sum_{l=2}^{\infty}\frac{1}{l!}\frac{\partial^L U_E(x_B)}{\partial x_B^L}(x - x_B)^L \tag{61}$$

Let us now substitute formulas (57)–(61) in formula (56), reduce the similar terms and equate the terms having the similar powers $(x^i - x_B^i)$ in the right-hand side and left-hand side of the residual equation. One obtains:

$$\dot{A}(t) = a_B^k x_B^k - (v_B^k a_B^k)t - U_E(x_B) \tag{62}$$

$$a_B^i = \frac{\partial U_E(x_B)}{\partial x_B^i} - Q^i + O(c^{-2}) \tag{63}$$

$$Q_{a_1a_2\ldots a_l}=\frac{1}{(2l-1)!!}\frac{\partial^l U_E(x_B)}{\partial x_B^{a_1}\ldots\partial x_B^{a_l}}+O(c^{-2}) \tag{64}$$

where $l\geqslant 2$.

Formula (62) represents in the $O(c^{-2})$ approximation a part of the infinitesimal coordinate transformation between coordinate time scales t and τ. Using the identities:

$$a_B^k x_B^k=\frac{\mathrm{d}}{\mathrm{d}t}(v_B^k x_B^k)-v_B^2 \tag{65}$$

$$(a_B^k v_B^k)t=\frac{\mathrm{d}}{\mathrm{d}t}(\tfrac{1}{2}v_B^2 t)-\tfrac{1}{2}v_B^2 \tag{66}$$

in the formulas (51) and (62) it is not difficult to obtain the integral transformation between the aforementioned time scales:

$$\tau=t+c^{-2}\left(\int_{t_0}^{t}(-\tfrac{1}{2}v_B^2-U_E(x_B))\,\mathrm{d}t-v_B^k(x^k-x_B^k)\right)+O(c^{-4}) \tag{67}$$

Formula (67) is in very close analogy to the Thomas (1975) equation (13) and the Moyer (1981) equation (21), which describe transformation from the proper time of the Earth to coordinate time of the solar system barycentric RS. Derivation of equation (67) presented here is, in my opinion, more simple, straightforward and clarifies interpretation of the time scale t and τ. Moreover we have proved that the time transfomation (51) actually does not contain terms of order $O(c^{-2}\hat{r}^2/R^2)$.

Formula (63) represents the Newtonian eq.m. of the origin of quasiinertial RS with respect to inertial RS. By making specific choice of Q_i and solving equations (63) one can calculate the explicit dependence of x_B^i on time t.

Formula (64) gives in the Newtonian approximation the explicit expression for external multipole moments $Q_{a_1a_2\ldots a_l}$ ($l\geqslant 2$). The fact that these quantities are STF tensors follows from the formula (64) and the homogeneous Laplace equation $\Delta U_E=0$, which is true in the internal and buffer regions of the body B.

For obtaining function $B(t)$, equations of motion and expressions for external multipole moments $Q_{a_1a_2\ldots a_l}$ in the 1 PNA it is necessary to match $g_{00}(t,x^i)$ with metric $g_{\alpha\beta}(\tau,\xi)$ in the $O(c^{-4})$ approximation (some details and formulas will be presented in Section 7).

4.2 MATCHING $g_{ij}(t,x^i)$ WITH $\hat{g}_{\alpha\beta}(\tau,\xi^i)$

For $g_{ij}(t,x^i)$ formula (54) gives:

$$\delta_{ij}\left(1+\frac{2U(t,x)}{c^2}\right)=\delta_{ij}\left(1+\frac{2\hat{U}_B(\tau,\xi)}{c^2}+\frac{2}{c^2}Q_i\xi^i+\frac{2}{c^2}\sum_{l=2}^{\infty}\frac{(2l-1)!!}{l!}Q_L\xi^L+\right.$$
$$\left.+\frac{2}{c^2}D^{ij}+\frac{2}{c^2}(D^{ijp}+D^{jip})(x^p-x_B^p)+O(c^{-4})\right) \tag{68}$$

Using formulas (57)–(61), (63), (64) and equating the terms having the similar powers of $(x^i - x^i_B)$ in the formula (68) it is not difficult to find:

$$D^{ij} = \delta^{ij} U_E(x_B) \tag{69}$$

$$D^{ijp} = \tfrac{1}{2}(\delta^{ij} a^p_B + \delta^{ip} a^j_B - \delta^{jp} a^i_B) \tag{70}$$

The function D^{ij} defines magnitude of the gravitational (Einsteinian) contraction of the body's shape. This contraction is isotropic in any harmonic RS (Kopejkin, 1987; Brumberg and Kopejkin, 1988; Damour, 1987).

4.3 MATCHING $g_{0i}(t, x^i)$ WITH $\hat{g}_{\alpha\beta}(\tau, \xi^i)$

Matching $g_{0i}(t, x^i)$ with $\hat{g}_{\alpha\beta}(\tau, \xi^i)$ is performed by analogy with that for g_{00} and g_{ij}. Formula (54) is used again with the g_{0i} in the left-hand side of the equation. The potential $U^i(t, x^i)$ is presented in the form:

$$U^i(t, x^i) = U^i_B(t, x^i) + U^i_E(t, x^i) \tag{71}$$

$$U^i_B(t, x^i) = G \int_{V_B} \rho^*(t, x') v^i(t, x') |x - x'|^{-1} \, d^3x' \tag{72}$$

$$U^i_E(t, x^i) = \sum_{A \neq B} U^i_A(t, x^i); \quad U^i_A(t, x^i) = G I^{(A)}_{-1}(\rho^* v^i) \tag{73}$$

Transformation between the local velocities of the fluid's element in the quasiinertial and inertial RS's is established with the help of the formulas (51) and (52). One has in the Newtonian approximation:

$$v^i(t, x^i) = v^i_B(t) + v^i(\tau, \xi^i) + O(c^{-2}) \tag{74}$$

The substitution of the formula (74) in the expressions (71)–(73) gives:

$$U^i(t, x^i) = v^i_B(t) \hat{U}_B(\tau, \xi) + \hat{U}^i_B(\tau, \xi) + U^i_E(t, x^i) + O(c^{-2}) \tag{75}$$

The potential $U^i_E(t, x^i)$ is expanded in the Taylor series in the neighborhood of the point x^i_B in powers of $(x^i - x^i_B)$. Reducing the similar terms and equating symmetric and antisymmetric coefficients attached to the similar powers of $(x^i - x^i_B)$ in the left-hand side and right-hand side, respectively, in the formula (54). With g_{0i} in the left-hand side of it one obtains:

$$B^i = 4U^i_E(x_B) - 3v^i_B U_E(x_B); \tag{76}$$

$$B^{ip} = 2U^{(i,p)}_E(x_B) - 2v^{(i}_B Q^{p)} - v^{(i}_B a^{p)}_B + \tfrac{1}{2}\dot{D}^{ip}; \tag{77}$$

$$B^{ipq} = \tfrac{2}{3} U^{(i,pq)}_E(x_B) - 2v^{(i}_B Q^{pq)} + \tfrac{1}{3}\dot{D}^{(ipq)} + \tfrac{4}{15}\delta^{(ip}\dot{Q}^{q)}; \tag{78}$$

$$B^{ia_1a_2\ldots a_l} = \frac{4}{(l+1)!}\left(U^{(i,a_1a_2\ldots a_l)}_E(x_B) - (2l-1)!!\, v^{(i}_B Q^{a_1a_2\ldots a_l)} + \right. \\ \left. + \frac{l(2l-3)!!}{2l+1} \delta^{(ia_1}\dot{Q}^{a_2\ldots a_l)} \right); \quad (l \geqslant 3) \tag{79}$$

$$\dot{F}^{ij} = -4U_E^{[i,j]}(x_B) + 3v_B^{[i}U_E^{j]}(x_B) + v_B^{[i}Q^{j]}; \tag{80}$$

$$\varepsilon^{ikp}C^{kq} = \tfrac{2}{3}U_E^{[i,p]q}(x_B) - 2v_B^{[i}Q^{p]q} + \tfrac{1}{3}\dot{D}^{[ip]q} - \tfrac{1}{3}\dot{Q}^{[i}\delta^{p]q}; \tag{81}$$

$$\varepsilon^{ika_1}C^{ka_2\ldots a_l} = \frac{2}{(2l-1)!!}U_E^{[i,a_1]a_2\ldots a_l}(x_B) - 2v_B^{[i}Q^{a_1]a_2\ldots a_l} - \\ - \frac{2}{l(2l-1)}\sum_{j=2}^{l}\dot{Q}^{a_2\ldots a_{j-1}a_{j+1}\ldots a_l[i}\delta^{a_1]a_j} \quad (l \geqslant 3) \tag{82}$$

The matching $g_{0i}(t, x^i)$ and $g_{ij}(t, x^i)$ with $\hat{g}_{\alpha\beta}(\tau, \xi^i)$ proves that the space transformation (52) actually doesn't contain the terms of order $O(c^{-2}\hat{r}^3/R^3)$. Therefore, the formula (52) turns out to be convenient for practical applications in the ephemeris astronomy as well as in the theoretical investigations. In this aspect our space transformation (52) favourably differs from the space transformations found by Ashby and Bertotti (1986) and Fukushima *et al.* (1986) which contain infinite number of terms in the $O(c^{-2})$ approximation.

The formulas (81) and (82) give explicit expressions for external magnetic-type multipole moments $C_{a_1a_2\ldots a_l}$. It is not hard to see that these moments depend only on the relative motion of the body under consideration and external bodies. The fact that magnetic-type moments are STF tensors may be proved by the contraction over any two indices (apart from i) in the left-hand side and right-hand side of the formulas (81)–(82), the use of the harmonic coordinate conditions (12) and homogeneous Laplace equation $\Delta W_E^i = 0$, which is true in the internal and buffer regions of the body.

Formula (80) describes relativistic precession of the space axes of the quasiinertial RS (τ, ξ^i) with respect to the space axes of the inertial RS (t, x^i). It is interesting to note that this precession in the presence of massive self-gravitating body has exactly the same form as relativistic precession of the spin of accelerated rotating test body (gyroscope) (Misner *et al.*, 1973).

6. Harmonic Proper Reference System for a Massive Body

Let us call the quasiinertial RS the proper reference system (PRS), if it has the metric tensor in the form (34)–(36) and the origin, which moves along the worldline of the center of inertia of the body. Thus, coordinates of the body's center of inertia ξ_B^i and all its time derivatives are equal zero identically in the PRS for any moment of time. Hence, the left-hand side of the eq.m. (49) has to be equal zero identically as well and one can conclude that the acceleration Q_i (which has so far been retained unspecified) has to be defined according to the formula:

$$Q_i = -\hat{M}_B^{-1}\sum_{l=3}^{\infty}\frac{(2l-1)!!}{(l-1)!}Q_{iL-1}\hat{\mathscr{I}}_B^{L-1} + O(c^{-2}) \tag{83}$$

Here one has dropped the term $3Q_{ij}\hat{I}_B^j$ since the body's dipole moment $\hat{I}_B^i$ is equal zero identically according to the definition of PRS.

PRS is convenient for solving the internal problem both of classical and relativistic celestial mechanics. It is also acceptable for description of the observer's location on the Earth's surface, construction of the geocentric ephemerides of the Moon and artificial satellites of the Earth, interpretation of the VLBI observations etc. As an example of work with the PRS and its usefulness I give the derivation of the Newtonian equations of translational and rotational motion of the body.

As is well known the Newtonian external potential $U_E(t, x^i)$ may be expanded in the series:

$$U_E(t, x^i) = \sum_{A \neq B} \left(\frac{G\hat{M}_A}{R_A} + \sum_{l=2}^{\infty} \frac{(-1)}{l!}^{L} G\hat{\mathscr{I}}_A \partial_L \left(\frac{1}{R_A} \right) \right) \tag{84}$$

where $R_A = (R_{Ai} R_A^i)^{1/2}$, $R_A^i = x^i - x_A^i$, and $\hat{\mathscr{I}}_A^L$ are the internal mass multipole moments of the external bodies, defined in their PRS's. Using the formula:

$$\partial_L \left(\frac{1}{R_A} \right) = (-1)^1 (2l-1)!! R_A^{<j_1} R_A^{j_2} \cdots R_A^{j_l>} / R_A^{2l+1} \tag{85}$$

and substituting formulas (64), (83) and (84) in the (63) one obtains the Newtonian equations of translational motion of the body's center of inertia:

$$\begin{aligned} \hat{M}_B a_B^i = & \sum_{A \neq B} \frac{G\hat{M}_A \hat{M}_B}{R_{AB}^2} N_{AB}^i + \sum_{A \neq B} \sum_{l=2}^{\infty} \frac{(2l+1)!!}{l!} \left(\frac{G\hat{M}_A \mathscr{I}_B^L}{R_{AB}^{l+2}} N_{AB}^{\langle iL \rangle} + \right. \\ & \left. + (-1)^l \frac{G\hat{M}_B \hat{\mathscr{I}}_A^L}{R_{AB}^{l+2}} N_{AB}^{\langle iL \rangle} \right) + \\ & + \sum_{A \neq B} \sum_{l=2}^{\infty} \sum_{k=2}^{\infty} \frac{(-1)^k}{k!} \frac{(2l+2k+1)!!}{l!} \frac{G\hat{\mathscr{I}}_A^K \hat{\mathscr{I}}_B^L}{R_{AB}^{l+k+2}} N_{AB}^{\langle iKL \rangle} \end{aligned} \tag{86}$$

where $R_{AB} = (R_{AB}^i R_{ABi})^{1/2}$; $R_{AB}^i = x_A^i - x_B^i$; $N_{AB}^i = R_{AB}^i / R_{AB}$. Note that the equations (86) have been obtained by the nonstandart way.

One begins with to the derivation of the equation of motion of the body's dipole current multipole moment $\hat{S}_B^i$ (spin). Let us differentiate with respect to time τ both sides of the definition of $\hat{S}_B^i$ (47). Then, one can prove that the torque of the body's internal forces is equal zero identically, but the torque of the external gravitational forces produces tidal precession of the body's spin:

$$\frac{d\hat{S}_B^i}{d\tau} = \sum_{l=2}^{\infty} \frac{(2l-1)!!}{(l-1)!} \varepsilon_{pq}^i Q_{qL-1} \hat{\mathscr{I}}_B^{pL-1} + O(c^{-2}) \tag{87}$$

Using formulas (64), (84) and (85) in the right-hand side of the formula (87) one obtains the Newtonian equations of rotational motion of the body B in the explicit form:

$$\frac{d\hat{S}_B^i}{d\tau} = \sum_{A \neq B} \sum_{l=2}^{\infty} \frac{(2l-1)!!}{(l-1)!} \varepsilon_{pq}^i \frac{G\hat{M}_A \hat{\mathscr{I}}_B^{pL-1}}{R_{AB}^{l+1}} N_{AB}^{\langle qL-1 \rangle} +$$

$$+ \sum_{A \neq B} \sum_{l=2}^{\infty} \sum_{k=2}^{\infty} \frac{(-1)^k}{k!} \frac{(2l+2k+1)!!}{(l-1)!} \varepsilon^i_{pq} \frac{G \hat{\mathscr{I}}^K_A \mathscr{I}^{pL-1}_B}{R^{k+l+1}_{AB}} N^{\langle qKL-1 \rangle}_{AB} \quad (88)$$

The tidal precession describes the law of a time change of the body's spin $\hat{S}^i_B$ with respect to the space axes of PRS. Recall that the space axes of PRS themselves rotate with respect to the space axes of inertial RS (t, x^i) with angular velocity of the relativistic precession (80). Note that the right-hand side of the formula (80) consists of the three terms. The first term is the Lense Thirring or gravitomagnetic precession, the second one is the de Sitter or geodetic precession and the third term is the Thomas precession. The gravitomagnetic and geodetic precessions are well known (Papapetrou, 1951; Barker and O'Connel, 1975; Misner *et at.*, 1973; Thorne and Hartle, 1985). The Thomas precession is caused by the deviation of the worldline of the body's center of intertia from the geodesic one. This deviation is characterized by the acceleration Q_i, which is defined according to the formula (83). As far as I know the Thomas precession of the body's spin $\hat{S}^i_B$ was overlooked in the previous derivations of the relativistic precession for spin of the massive self-gravitating bodies.

One notes also that the coordinate time τ of the PRS for the Earth may be adopted as definition of the TDT time scale. Such a definition is exact and unambiguous. Transformation between TDT and TDB is given according to the formula (67) (see also formula (4.1) from Brumberg and Kopejkin (1988)), where x^i_B and v^i_B are understood now as coordinates and velocity of the Earth's center of intertia in the harmonic solar barycentric RS. The explicit dependence on time t, of coordinates x^i_B are evaluated in the Newtonian approximation from the eq.m. (86). A link between TDB, TDT and proper time of the atomic clocks placed on the Earth's surface or artificial satellite is realized with the help of the additional transformations derived in (Brumberg and Kopejkin, 1988).

7. Coordinate Transformation Between Inertial and Proper Reference Systems

Coordinate transformation is naturally the same as that between inertial and quasiinertial RS's except that functions $x^i_B(t)$ are now meant to be coordinates of the body's center of inertia. In addition to the results presented in Sections 4 and 5 we give here the functions $B(t)$ and $a^i_B(t)$ in the 1 PNA.

The matching procedure in the 1 PNA is tedious and complicated. For example, it is necessary to take into account that all integrals involved in the metric in the inertial RS are taken over hypersurface $t = \text{const.}$ but integrals involved in the metric in the quasiinertial RS are taken over hypersurface $\tau = \text{const.}$ which does not coincide with that for $t = \text{const.}$ Thus, when matching is done one has to transfer by Lie transportation all quantities defined on the hypersurface $\tau = \text{const.}$ to the hypersurface $t = \text{const.}$ Moreover, one must extensively use to a full degree the mathematical formalism developed by Thorne (1980) and Blanchet and Damour (1986) for operations with the STF tensors. We have done all the necessary calculations. The exhaustive final results of AM in the 1 PNA will be published elsewhere.

Here we give coordinate transformaton between inertial and proper RS's only under assumption that the rotational motion and nonsphericity of the bodies give a negligible correction to the relativistic terms in the equations of translational motion of the bodies. The functions involved in the transformation are defined as follows

$$\dot{A} = a_B^k x_B^k - (a_B^k v_B^k)t - \sum_{A\neq B}\left(\frac{G\hat{m}_A}{R_{AB}} + G\sum_{l=2}^{\infty}(-1)^l\frac{(2l-1)!!}{l!}\frac{\hat{\mathscr{I}}_A^L}{R_{AB}^{l+1}}N_{AB}^{\langle L\rangle}\right) \tag{89}$$

$$\begin{aligned}\dot{B} = {} & \tfrac{1}{2}v_B^2(a_B^k x_B^k) + (a_B^k v_B^k)(v_B^p x_B^p) - \tfrac{3}{2}(a_B^k v_B^k)v_B^2 t - \sum_{A\neq B}\frac{G\hat{m}_A}{R_{AB}}\Bigg(\tfrac{3}{2}v_B^2 + \\ & + 2v_A^2 - 4(v_A^k v_B^k) - \tfrac{1}{2}(N_{AB}^k v_A^k)^2 - \frac{1}{2}\sum_{C\neq B}\frac{G\hat{m}_C}{R_{CB}} - \\ & - \sum_{C\neq A}\frac{G\hat{m}_C}{R_{CA}}\left(1 - \frac{1}{2}\frac{R_{AB}^k R_{CA}^k}{R_{CA}^2}\right)\Bigg);\end{aligned} \tag{90}$$

$$\begin{aligned}a_B^i = {} & \sum_{A\neq B}\Bigg(\frac{G\hat{m}_A}{R_{AB}^2}N_{AB}^i + G\sum_{l=2}^{\infty}\frac{(2l+1)!!}{l!}\left(\frac{\hat{m}_A}{\hat{m}_B}\mathscr{I}_B^L + (-1)^l\hat{\mathscr{I}}_A^L\right)\frac{N_{AB}^{\langle iL\rangle}}{R_{AB}^{l+2}} + \\ & + \frac{G}{\hat{m}_B}\sum_{l=2}^{\infty}\sum_{k=2}^{\infty}\frac{(-1)^k}{k!}\frac{(2l+2k+1)!!}{l!}\frac{\mathscr{I}_A^K\mathscr{I}_B^L}{R_{AB}^{l+k+2}}N_{AB}^{\langle iKL\rangle}\Bigg) + \\ & + \sum_{A\neq B}\frac{G\hat{m}_A}{R_{AB}^2}N_{AB}^i\Bigg(v_B^2 + 2v_A^2 - 4(v_A^k v_B^k) - \tfrac{3}{2}(N_{AB}^k v_A^k)^2 - \\ & - 4\sum_{C\neq B}\frac{G\hat{m}_C}{R_{CB}} - \sum_{C\neq A}\frac{G\hat{m}_C}{R_{CA}}\left(1 - \frac{1}{2}\frac{R_{AB}^k R_{CA}^k}{R_{CA}^2}\right)\Bigg) + \\ & + \sum_{A\neq B}\frac{G\hat{m}_A}{R_{AB}^2}N_{AB}^k(4v_B^k - 3v_A^k)(v_A^i - v_B^i) + \\ & + \frac{7}{2}\sum_{A\neq B}\sum_{C\neq A}\frac{G^2\hat{m}_A\hat{m}_C}{R_{AB}R_{CA}^2}N_{CA}^i;\end{aligned} \tag{91}$$

$$D^{ij} = \delta^{ij}\sum_{A\neq B}\frac{G\hat{m}_A}{R_{AB}}; \tag{92}$$

$$D^{ijp} = \frac{1}{2}\sum_{A\neq B}\frac{G\hat{m}_A}{R_{AB}^2}(\delta^{ij}N_{AB}^p + \delta^{ip}N_{AB}^j - \delta^{jp}N_{AB}^i); \tag{93}$$

$$\dot{F}^{ip} = -2\sum_{A\neq B}\frac{G\hat{m}_A}{R_{AB}^2}(v_A^i N_{AB}^p - v_A^p N_{AB}^i) + \frac{3}{2}\sum_{A\neq B}\frac{G\hat{m}_A}{R_{AB}^2}(v_B^i N_{AB}^p - v_B^p N_{AB}^i); \tag{94}$$

$$B^i = \sum_{A\neq B}\frac{G\hat{m}_A}{R_{AB}}(4v_A^i - 3v_B^i); \tag{95}$$

$$B^{ip} = \sum_{A \neq B} \left(\frac{G\hat{m}_A}{R_{AB}^2}(v_A^i N_{AB}^p + v_A^p N_{AB}^i) - \frac{1}{2}\frac{G\hat{m}_A}{R_{AB}^2}(v_B^i N_{AB}^p + v_B^p N_{AB}^i + \right.$$

$$\left. + \delta^{ip} N_{AB}^k (v_A^k - v_B^k)) \right) \tag{96}$$

It should be noted that the body's masses $\hat{m}_A$ in the formulas (89)–(96) are defined as:

$$\hat{m}_A = \int_{V_A} \rho^*(\tau, \xi)\left(1 + \frac{1}{c^2}(\Pi(\tau, \xi) - \tfrac{1}{2}\hat{U}_B(\tau, \xi))\right) \mathrm{d}^3\xi \tag{97}$$

Thus, the masses contain not only the rest mass (45), but corrections for internal energy of the body's matter and energy of own gravitational field of the body. Formula (91) represents the well known eq.m. of the body's center of inertia in 1 PNA, where Newtonian corrections for nonsphericity of the bodies are taken into account. We have not given formulas for $B_{a_1 a_2 \ldots a_l}$ ($l \geqslant 3$) since their knowledge is not necessary for the data analysis of modern observations.

Acknowledgements

I wish to thank V. A. Brumberg for suggesting this problem to me and for many fruitful and stimulating discussions. I am grate to K. S. Thorne for his useful explanations of some principal questions, concerning construction of coordinate systems in the paper of Thorne and Hartle (1985). L. P. Grishchuk participated in many discussions and made some valuable remarks.

References

Akim Eh. L., Brumberg, V. A., Kislik, M. D., Koljuka, Ju. F., Krasinsky, G. A., Pitjeva, E. V., Shishov, V. A., Stephanianz, V. A., Sveshnikov, M. L., and Tikhonov, V. F.: 1986, in: *Relativity in Celestial Mechanics and Astrometry* (IAU Simposium 114), eds. J. Kovalevsky and V. A. Brumberg, p. 63. Reidel: Dordrecht.

Anderson, J. L. and Decanio, T. C.: 1975, *GRG*, **6**, p. 197.

Anderson, J. L.: Kates, R. E.: Kegeles, L. S. and Madonna, R. G.: 1982, *Phys. Rev.*, **D25**, p. 2038.

Ashby, N. and Bertotti, B.: 1984, *Phys. Rev. Lett.*, **52**, p. 485.

Ashby, N. and Bertotti, B.: 1986, *Phys. Rev.* **D34**, p. 2246.

Baierlein, R., 1967, *Phys. Rev.*, **162**, p. 1275.

Barker, B. M. and O'Connell R. F.: 1975, *Phys. Rev.*, **D12**, p. 329.

Bertotti, B.: 1954, *Nuov. Cim.*, **12**, p. 226.

Bertotti, B.: 1986, in: *Relativity in Celestial Mechanics and Astrometry* (IAU Symposium 114), eds. J. Kovalevsky and V. A. Brumberg., p. 233, Reidel: Dordrecht.

Blanchet, L. and Damour, T.: 1986, *Phil. Trans. Roy. Soc. London*, **A320**, p. 379.

Blanchet, L. and Damour, T.: 1987, 'Tail-transported temporal correlations in the dynamics of a gravitating system.' Preprint.

Brumberg, V. A.: 1958, *Bull. Inst. Theor. Astron. Acad. Sci. USSR*, **6**, N10(83), p. 773. (in Russian).

Brumberg, V. A.: 1972, *Relativistic Celestial Mechanics*. Nauka: Moscow. (in Russian).

Brumberg, V. A., and Ivanova T. V.: 1982, *Cel. Mech.*, **26**, p. 77.

Brumberg, V. A. 1986, in: *Astrometric Techniques*, (IAU Symposium 109), eds. H. K. Eichhorn and R. J. Leacock, p. 19, Reidel: Dordrecht.

Brumberg, V. A. and Kopejkin, S. M.: 1988, in: *Reference Systems*, eds. J. Kovalevsky, I. I. Mueller and B. Kolaczek, Reidel: Dordrecht (in press).
Cannon, W. H., Lisewski, D., Finkelstein, A. M. and Kreinovich, V. Ya.: 1986, in *Relativity in Celestial Mechanics and Astrometry*, (IAU Symposium 114), eds. J. Kovalevsky and V. A. Brumberg, p. 255. Reidel: Dordrecht.
Caporali, A.: 1981, *Nuov. Cim.*, **61B**, p. 181 and p. 205.
Damour, T.: 1983, in: *Gravitational Radiation*, eds. N. Deruelle and T. Piran, (Les Houches 1982), p. 59. North-Holland: Amsterdam.
Damour, T.: 1987, in: *Three Hundred Years of Gravitation*, eds. S. W. Hawking and W. Israel, p. 128. Cambridge University Press.
D'Eath, P. D.: 1975a, *Phys. Rev.*, **D11**, p. 1387.
D'Eath, P. D.: 1975b, *Phys. Rev.*, **D12**, p. 2183.
Duboshin, G. N.: 1975, *Celestial Mechanics*. Nauka: Moscow (in Russian).
Ehlers, J., and Rudolph, E.: 1977, *GRG*, **8**, p. 197.
Ehlers, J.: 1980, *Ann. N. Y. Acad. Sci.*, **336**, p. 279.
Einstein, A., Infeld, L. and Hoffmann, B.: 1938, *Ann. Math.*, **41**, p. 65.
Fermi, E.: 1922, *Atti R. Accad. Lincei Rend. Cl. Sci. Fis. Mat. Nat.*, **31**, p. 21 and p. 51.
Fock, V. A.: 1955, *The Theory of Space, Time and Gravitation*. Gostehteorizdat: Moscow (in Russian).
Fujimoto, M., and Grafarend, E.: 1986, in: *Relativity in Celestial Mechanics and Astrometry* (IAU Symmposium 114), eds. J. Kovalevsky and V. A. Brumberg, p. 269. Reidel: Dordrecht.
Fukushima, T. in: *Relativity in Celestial Mechanics and Astrometry* (IAU Simposium 114), eds. J. Kovalevsky and V. A. Brumberg, p. 35. Reidel: Dordrecht.
Fukushima, T., Fujimoto, M., Kinoshita, H. and Aoki, S.: 1986, in: *Relativity in Celestial Mechanics and Astrometry* (IAU Simposium 114), eds. J. Kovalevsky and V. A. Brumberg, p. 145. Reidel: Dordrecht.
Futamase, T. and Schutz, B. F.: 1983, *Phys. Rev.*, **D28**, p. 2363.
Grishchuk, L. P. and Kopejkin, S. M.: 1986, in: *Relativity in Celestial Mechanics and Astrometry*, (IAU Symposium 114), eds. J. Kovalevsky and V. A. Brumberg, p. 19. Reidel: Dordrecht.
Guinot, B.: 1986, in: *Astrometric Techniques* (IAU Symposium 109), eds. H. K. Eichhorn and R. J. Leacock, p. 1. Reidel: Dordrecht.
Hellings, R. W.: 1986, *Astron. J.*, **91**, p. 650.
Hogan, P. A., and McCrea, J. D.: 1974, *GRG*, **5**, p. 79.
Infeld, L. and Plebanski, J.: 1960, *Motion and Relativity*. Oxford: Pergamon Press.
Japanese Ephemeris: 1985, *Basis of the New Japanese Ephemeris*. Tokyo.
Kadomtzev, B. B., Keldysh, L. V., Kobzarev, I. Yu. and Sagdeev, P. Z.: 1972, *Sov. Phys. Uspekhi*, **106**, p. 660. (in Russian).
Kates, R. E.: 1980a, *Phys. Rev.*, **D22**, p. 1853.
Kates, R. E.: 1980b, *Phys. Rev.*, **D22**, p. 1871.
Kopejkin, S. M.: 1985, *Astron. Zh.*, **62**, p. 889 (in Russian).
Kopejkin, S. M.: 1987, *Trans. Sternberg State Astron. Inst.*, **59**, p. 53 (in Russian).
Kovalevsky, J. and Mueller, I. I.: 1981, in: *Reference Coordinate Systems for Earth Dynaimics* (IAU Colloquium 56), eds., E. M. Gaposchkin and B. Kolaczek, p. 375. Reidel: Dordrecht.
Kovalevsky, J.: 1985, *Bull. Astron. Obs. Roy. Belgique*, **10**, N2, p. 87.
Landau, L. D. and Lifshitz, E. M.: 1967, *The Classical Theory of Fields*. Nauka: Moscow (in Russian).
Lestrade, J. F. and Chapront-Touzé, M.: 1982, *Astron. Astrophys.*, **116**, N1, p. 75.
Manasse, F. K., and Misner, C. W.: 1963, *J. Math. Phys.*, **4**, p. 735.
Martin, C. F., Torrence, M. H. and Misner, C. W.: 1985, *J. Geophys. Res.*, **90**, p. 9403.
Mashhoon, B.: 1985, *Found. Phys.*, **15**, p. 497.
McCrea, J. D. and O'Brien, G.: 1978, *GRG*, **9**, p. 1101.
Misner, C. W., Thorne, K. S. and Wheeler, J. A.: 1973, *Gravitation*. San Francisco: W. H. Freeman and Company.
Moyer, T. D., 1981, *Cel. Mech.*, **23**, N1, p. 33.
Mueller, I. I.: 1981, in: *Reference Coordinate Systems for Earth Dynamics* (IAU Colloquium 56), eds. E. M. Gaposchkin and B. Kolaczek, p. 1. Reidel: Dordrecht.
Murray, C. A.: 1983, *Vectorial Astrometry*. Bristol: Adam Hilger LTD.

Newhall, X. X., Standish, E. M. and Williams, J. G., 1983, *Astron. Astrophys.*, **125**, N1, p. 150.
Ni W.-T. and Zimmermann, M.: 1978, *Phys. Rev.*, **D17**, p. 1473.
Papapetrou, A.: 1951, *Proc. Roy. Soc. London*, **A209**, p. 248.
Pavlov, N. V.: 1984, *Astron. Zh.*, **61**, p. 385 and p. 600. (in Russian).
Podobed, V. V. and Nesterov, V. V.: 1982, *General Astrometry*. Nauka: Moscow (in Russian).
Seidelman, P. K.: 1985, *Cel. Mech.*, **37**, N3, p. 199.
Shirokov, M. F. and Brodovski, V. B.: 1956, *Sov. Phys. JETP*, **31**, p. 1027 (in Russian).
Soffel, M., Herold, H., Ruder, H. and Schneider, M.: 1986a, *Veröff. Bayer. Kommis. Int. Erdmess. Bayer. Akad. Wiss. Astron. Geod. Arb.*, **48**, p. 247.
Soffel, M., Ruder, H. and Schneider, M.: 1986, *Astron. Astrophys.*, **157**, p. 357.
Spyrou, N.: 1977, *GRG*, **8**, p. 463.
Spyrou, N.: 1978, *GRG*, **9**, p. 519.
Suen, W. M.: 1986, *Phys. Rev.*, **D34**, p. 3617.
Taylor, J. H.: 1987, in: *General Relativity and Gravitation* (Proceedings of the 11th International Conference on General Relativity and Gravitation, Stockholm, July 6–12, 1986), ed. M. A. H. MacCallum, p. 209. Cambridge University Press.
Thomas, J.: 1975, *Astron. J.*, **80**, p. 405.
Thorne, K. S.: 1980, *Rev. Mod. Phys.*, **52**, p. 299.
Thorne, K. S. and Hartle, J. B.: 1985, *Phys. Rev.*, **D31**, p. 1815.
Vincent, M. A.: 1986, *Cel. Mech.*, **39**, No. 1, p. 15.
Will, C. M.: 1981, *Theory and Experiment in Gravitational Physics*. Cambridge University Press.
Will, C. M.: 1986, *Can. J. Phys.*, **64**, p. 149.
Zhang, X. H.: 1986, *Phys. Rev.*, **D34**, p. 991.