# ON THE RESTRICTED THREE-BODY PROBLEM WHEN THE MASS PARAMETER IS SMALL*

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**Abstract.** We study some aspects of the restricted three-body problem when the mass parameter $\mu$ is sufficiently small. First, we describe the global flow of the two-body rotating problem, $\mu = 0$, and we use it for the analysis of the collision and parabolic orbits when $\mu \gtrsim 0$. Also we show that for any fixed value of the Jacobian constant and for any $\varepsilon > 0$, there exists a $\mu_0 > 0$ such that if the mass parameter $\mu \in [0, \mu_0]$, then the set of bounded orbits which are not contained in the closure of the set of symmetric periodic orbits has Lebesgue measure less than $\varepsilon$.

## 1. Introduction

We consider the circular planar restricted three-body problem (usually, the restricted three-body problem) in a rotating coordinate system $q = (q_1, q_2)$ of rotational frequency equal to 1. In this frame we put the larger primary $m_1$ of mass $1 - \mu$ at the origin and the smaller primary $m_2$ of mass $\mu$ at the position $e_2 = (-1, 0)$. The Hamiltonian which governs the motion of the zero mass particle $m_3$ is given by

$$H = \|p\|^2/2 + q_2 p_1 - q_1 p_2 - \|q\|^{-1} + \mu(\|q\|^{-1} - \|q - e_2\|^{-1} - p_2) \qquad (1.1)$$

where $p = (p_1, p_2)$ are the momentum variables conjugate to the $q$. It is clear that $C = -2H$ is a first integral of the Hamiltonian system associated with $H$. This integral is called the Jacobi integral. Note that our Jacobian constant differs from the usual in the constant $\mu(1 - \mu)$ (see [12]).

The goal of this paper is to study some aspects of the restricted three-body problem as the mass parameter $\mu$ is sufficiently small. First, in Section 2 we describe the global flow of the two-body rotating problem, $\mu = 0$, and use it, in Section 3, for the analysis of the collision and parabolic orbits when $\mu$ is small enough.

A solution of the restricted three-body problem has a collision with $m_1$ (resp. $m_2$) in the instant $t_0$ if the distance between $m_3$ and $m_1$ (resp. $m_2$) tends to zero as $t \to t_0$.

Our main results about the collision orbits are the following two theorems.

THEOREM A. *For the restricted three-body problem and for each value of the Jacobian constant, the set of orbits which end or begin at collision with $m_1$ or $m_2$ is topologically homeomorphic to a cylinder.*

THEOREM B. *For values of the mass parameter $\mu$ sufficiently small the following statements hold for the restricted three-body problem.*

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(*i*) *For each value of the Jacobian constant* $C > 0$, *there is a value of the mass parameter* $\mu_0 = \mu_0(C)$ *such that, for every* $\mu \in (0, \mu_0]$, *at least there are two orbits which leave the collision with the mass* $1 - \mu$, *cross one time the surface* $\dot{r} = 0$ *and go to collision with the mass* $1 - \mu$, *again.*

(*ii*) *For each value of the Jacobian constant* $C < 2$, *there exists a value of the mass parameter* $\mu_0 = \mu_0(C)$ *such that, for every* $\mu \in (0, \mu_0]$, *there is one and only one orbit such that it leaves the collision with the mass* $1 - \mu$ *and it goes to collision with the mass* $\mu$ *without to cross the surface* $\dot{r} = 0$.

Since the set of orbits of the restricted three-body problem is invariant under the symmetry $(q_1, q_2, p_1, p_2, t) \to (q_1, -q_2, -p_1, p_2, -t)$, we have, from (ii) of Theorem B and in the same hypotheses, that there is one and only one orbit such that it leaves the collision with the mass $\mu$ and it goes to collision with the mass $1 - \mu$ without to cross the surface $\dot{r} = 0$.

A solution of the restricted three-body problem is a parabolic solution as $t \to +\infty$ (resp. $t \to -\infty$) if the body $m_3$ reaches infinity with zero radial velocity as the time tends to $+\infty$ (resp. $-\infty$).

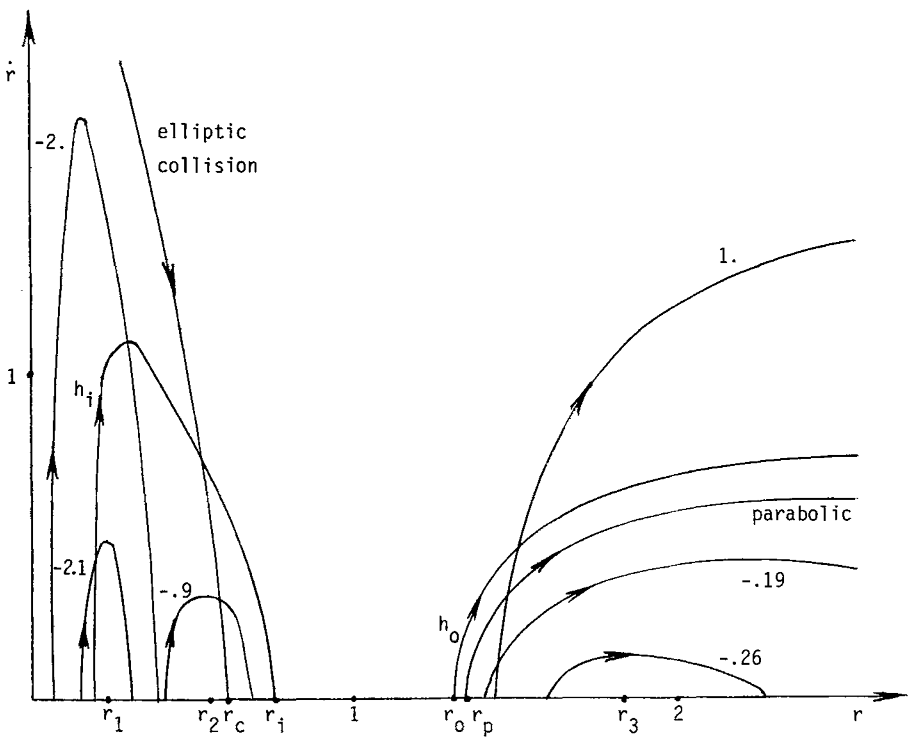


Fig. 1a. $C = 3.25$; $r_1 = 0.236\,786\,5\ldots$, $r_2 = 0.578\,375\,9\ldots$, $r_3 = 1.825\,462\,7\ldots$, $r_i = 0.740\,139\,4\ldots$, $r_0 = 1.314\,906\,6\ldots$, $h_i = -1.077\,193\,7\ldots$, $h_0 = 0.103\,979\,5\ldots$, $(0, r_1)$ elliptic retrograde orbits, $(0, r_2)$ elliptic direct orbits, $(r_p, r_3)$ elliptic direct orbits.

McGehee [9] proved a theorem similar to Theorem A for the parabolic solutions; i.e., for the restricted three-body problem and for each value of the Jacobian constant, the set of parabolic orbits as $t \to +\infty$ or $t \to -\infty$ is topologically homeomorphic to a cylinder.

The following theorem summarizes our results about parabolic orbits.

THEOREM C. *For values of the mass parameter $\mu$ sufficiently small the following statements hold for the restricted three-body problem.*

(*i*) *For each $\varepsilon > 0$ and for all $C \notin (-\varepsilon, \varepsilon)$, there is a value of the mass parameter $\mu_0 = \mu_0(C)$ such that, for every $\mu \in (0, \mu_0]$, at least there are two parabolic orbits such that they leave the infinity, cross only one time the surface $\dot{r} = 0$ and reach the infinity.*

(*ii*) *For each $\varepsilon > 0$ sufficiently small and for all $C \in (-8^{1/2} + \varepsilon, 8^{1/2} - \varepsilon)$, there is a value of the mass parameter $\mu_0 = \mu_0(C)$ such that, for every $\mu \in (0, \mu_0]$, we have one and only one parabolic orbit such that it leaves the infinity and goes to collision with $m_2 = \mu$ without to cross the surface $\dot{r} = 0$.*

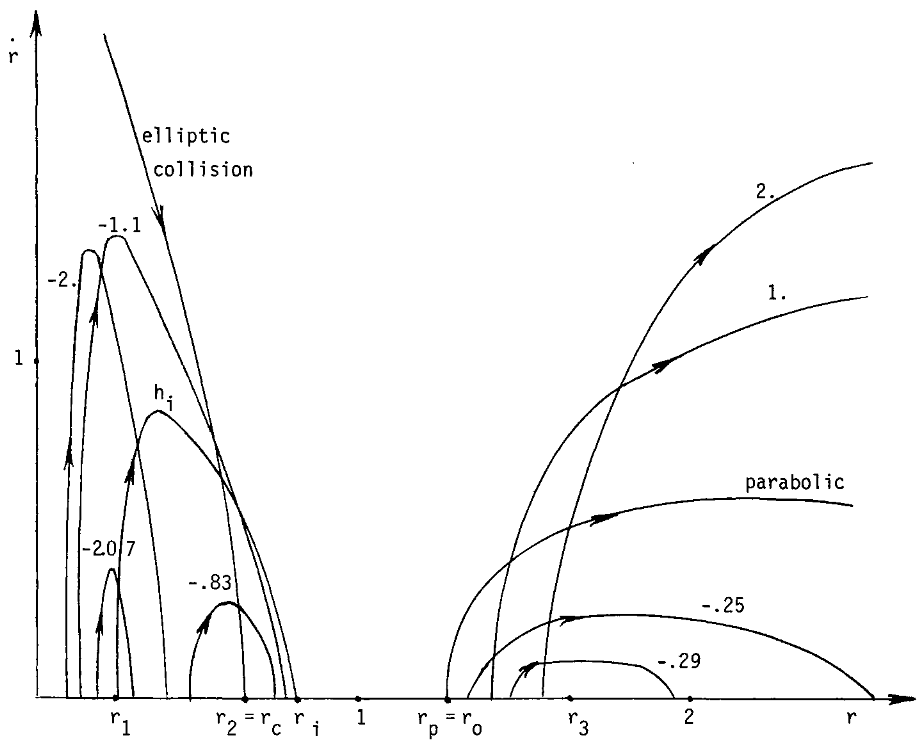


Fig. 1b. $C = 32^{1/3}$, $r_1 = 2^{-5/3}(3 - 5^{1/2})$, $r_2 = 2^{-2/3}$, $r_3 = 2^{-5/3}(3 + 5^{1/2})$, $r_i = 2^{-2/3}(5^{1/2} - 1)$, $r_0 = 2^{1/3}$, $h_i = 2^{-1/3}(1 - 5^{1/2})$, $h_0 = 2^{2/3}$, $(0, r_1)$ elliptic retrograde orbits, $(0, r_2)$ elliptic direct orbits, $(r_p, r_3)$ elliptic direct orbits.

By using the symmetry $(q_1, q_2, p_1, p_2, t) \to (q_1, -q_2, -p_1, p_2, -t)$, we have, from (ii) of Theorem C and in the same hypotheses, that there is one and only one parabolic orbit such that it leaves the collision with $m_2 = \mu$ and goes to infinity without to cross the surface $\dot{r} = 0$.

The proofs of Theorems A, B, and C are given in Section 3.

In the restricted three-body problem a periodic orbit symmetric with respect to the $q_1$-axis crosses the $q_1$-axis twice and only twice at right angles. The crossings at right angles may be divided into four classes, lower and higher passages at opposition and at conjunction, respectively (see Section 4 for definitions). Then there are ten possible combinations of pairs of the four classes of right-angle crossings. We prove the following result about the 'density' of the symmetric periodic orbits.

THEOREM D. *For any fixed value of the Jacobian constant and for any $\varepsilon > 0$, there exists a $\mu_0 > 0$ such that if the mass parameter $\mu \in [0, \mu_0]$, then the set of bounded orbits which are not contained in the closure of one of the ten classes of symmetric periodic orbits has Lebesgue measure less than $\varepsilon$.*

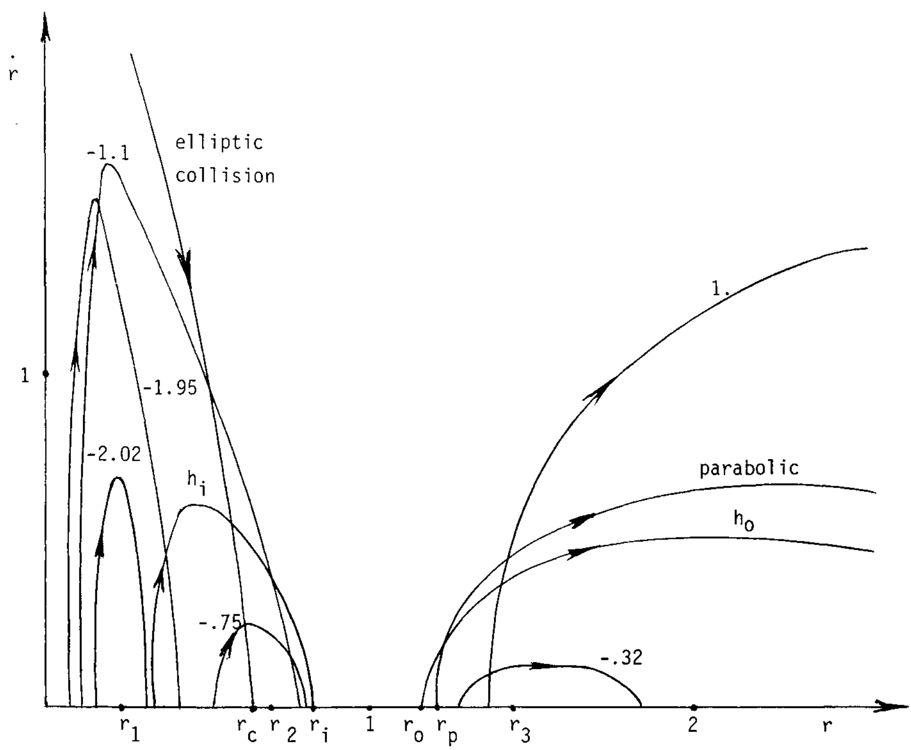


Fig. 1c. $C = 3.1$, $r_1 = 0.244\,555\,5\ldots$, $r_2 = 0.702\,251\,7\ldots$, $r_3 = 1.455\,692\,7\ldots$, $r_i = 0.828\,829\,4\ldots$, $r_0 = 1.193\,310\,8\ldots$, $h_i = -0.863\,041\,8\ldots$, $h_0 = -0.126\,009\,2\ldots$, $(0, r_1)$ elliptic retrograde orbits, $(0, r_2)$ elliptic direct orbits, $(r_0, r_p)$ elliptic direct orbits, $(r_0, r_3)$ elliptic direct orbits.

The topology with respect to which we take the closure is the usual topology in a surface of section induced by the topology of $(R^2 - \{0, e_2\}) \times R^2$ when we have fixed a value of the Jacobian constant.

Theorem D improves a theorem about the density of the periodic orbits in the set of bounded orbits for the restricted three-body problem given in [5].

The proof of Theorem D is given in Section 4.

## 2. The Global Flow of the Two-body Rotating Problem

When the mass parameter $\mu$ is zero the restricted three-body problem is called the two-body rotating problem. This problem is defined by the Hamiltonian

$$H = \|p\|^2/2 + q_2 p_1 - q_1 p_2 - \|q\|^{-1},$$

where $\| \ \|$ is the Euclidean norm in $R^2$. The two-body rotating problem as Hamiltonian system is integrable, because other integrals are the sidereal energy

$$h = \|p\|^2/2 - \|q\|^{-1},$$

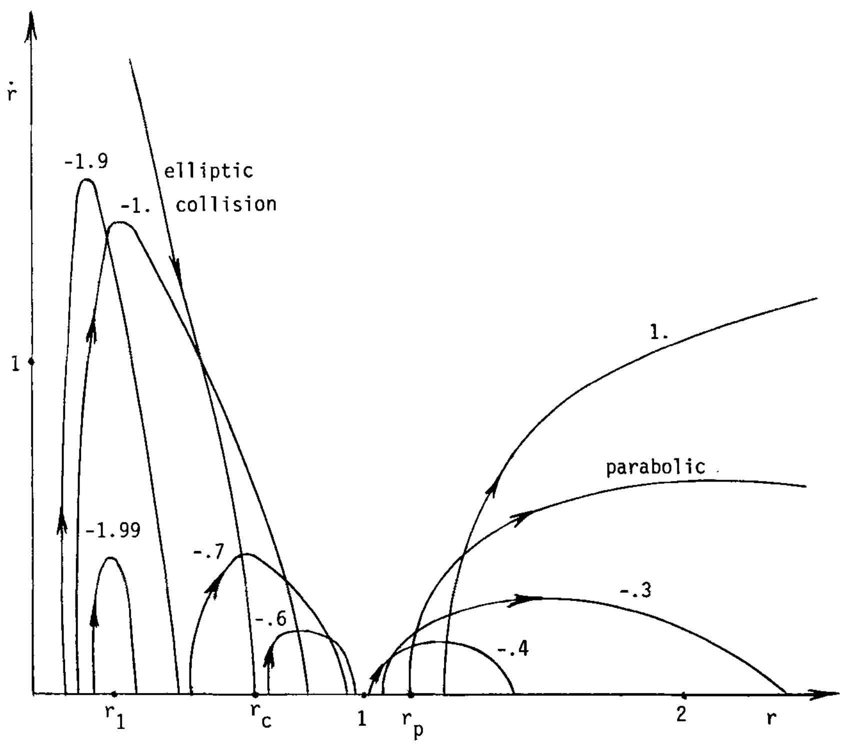


Fig. 1d. $C = 3$, $r_1 = 0.25$, $r_2 = r_3 = r_i = r_0 = 1$, $h_i = h_0 = -0.5$, $(0, r_1)$ elliptic retrograde orbits, $(0, 1)$ elliptic direct orbits, $(1, r_p)$ elliptic direct orbits.

and the sidereal angular momentum

$$M = q_1 p_2 - q_2 p_1 .$$

Of course, since $H = h - M = -C/2$, only two of these three integrals have linearly independent gradients.

Let $I_C$ be the set of points of the phase space with Jacobian constant equals $C$. In a similar way we define the set $I_h$, and denote by $I_{hC}$ the set $I_h \cap I_C$.

Now we study the sets $I_{hC}$. In what follows we fix the value of the Jacobian constant $C$. We introduce polar coordinates $(r, \theta)$, i.e. $q_1 + \mathbf{i}q_2 = r \exp(\mathbf{i}\theta)$. It is known (see [6]) that if $M \neq 0$, then every solution of the two-body rotating problem satisfies

$$\begin{aligned} r &= M^2[1 + (1 + 2hM^2)^{1/2} \cos(\theta - \theta')]^{-1}, \\ \dot{r} &= (2r^{-1} - M^2 r^{-2} + 2h)^{1/2}, \end{aligned} \tag{2.1}$$

where $\theta'$ is an adequate constant. When $M = 0$ we have a collision orbit which satisfies

$$\dot{r}^2 = 2r^{-1} + 2h. \tag{2.2}$$

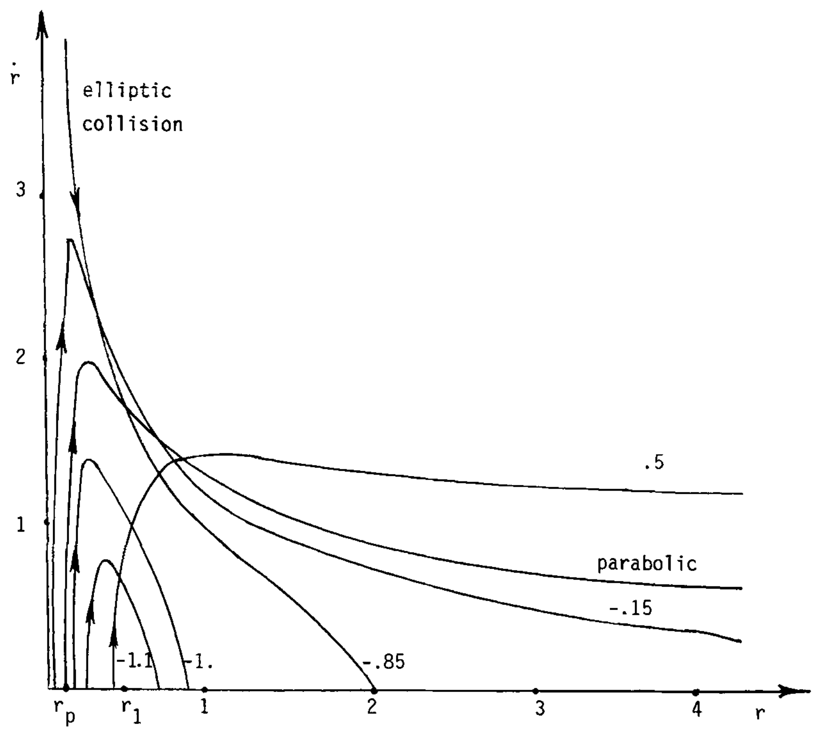


Fig. 1e. $C = 1$, $r_1 = 0.432\,040\,8\ldots$, $(0, r_1)$ elliptic retrograde orbits, $(0, r_2)$ elliptic direct orbits.

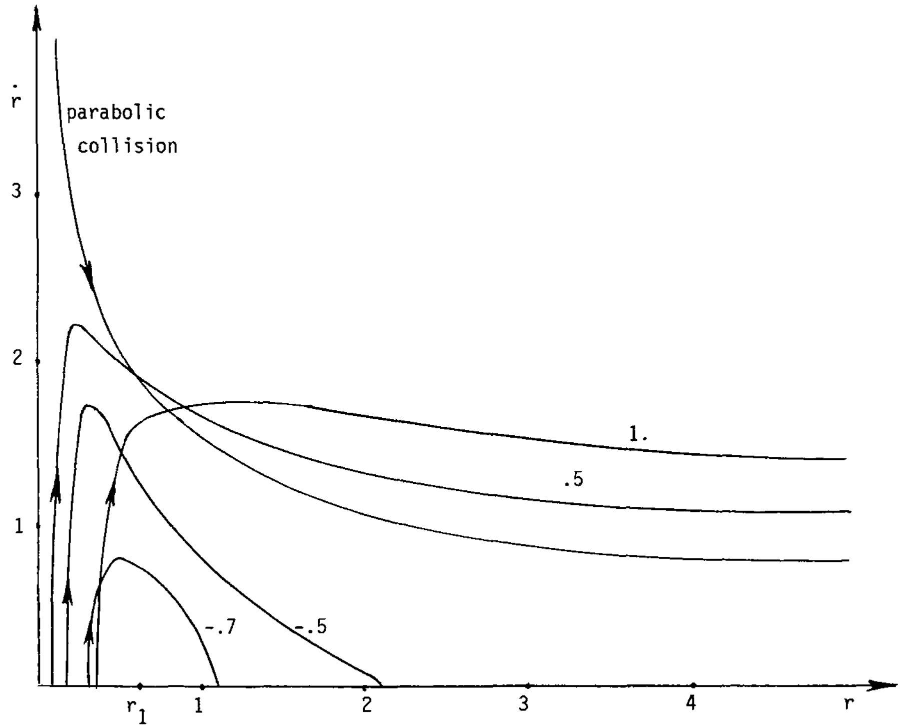


Fig. 1f. $C=0$, $r_1 = 2^{-2/3}$, $(0, r_1)$ elliptic retrograde orbits.

For a fixed value of $\theta$ we plot on the half-plane $(r, \dot{r})$ the sidereal energy curves $h$ equals constant. In Figures 1a – g we have drawn these curves for seven values of the Jacobian constant $C$ according as the value of $C$ satisfies $C > 32^{1/3}$, $C = 32^{1/3}$, $3 < C < 32^{1/3}$, $C = 3$, $0 < C < 3$, $C = 0$ and $C < 0$. Equations (2.1) and (2.2) are useful in the plot of Figure 1. Since the sidereal energy curves are symmetric with respect the $r$-axis on the half-plane $(r, \dot{r})$, we only have drawn on the quadrant $\dot{r} > 0$. The full picture of the sets $I_{hC}$ is obtained by rotating the cross-section about the $\dot{r}$-axis. Then we only obtain five different qualitative pictures of the flow on $I_C$ according as the value of $C$ satisfies $C > 3$, $C = 3$, $0 < C < 3$, $C = 0$ and $C < 0$. We describe these five possibilities.

Case $C > 3$

The set $I_C = \bigcup_h I_{hC}$ has two components. First component, $I_C^1$, correspond with values of the sidereal energy $h \in [h_1, h_2]$ and the second one, $I_C^2$, with values $h \in [h_3, +\infty)$, where $h_i = -(2r_i)^{-1}$ and $r_i$, for $i = 1, 2, 3$, are the three positive roots of the polynomial

$$4r^3 - C^2 r^2 + 2Cr - 1, \tag{2.3}$$

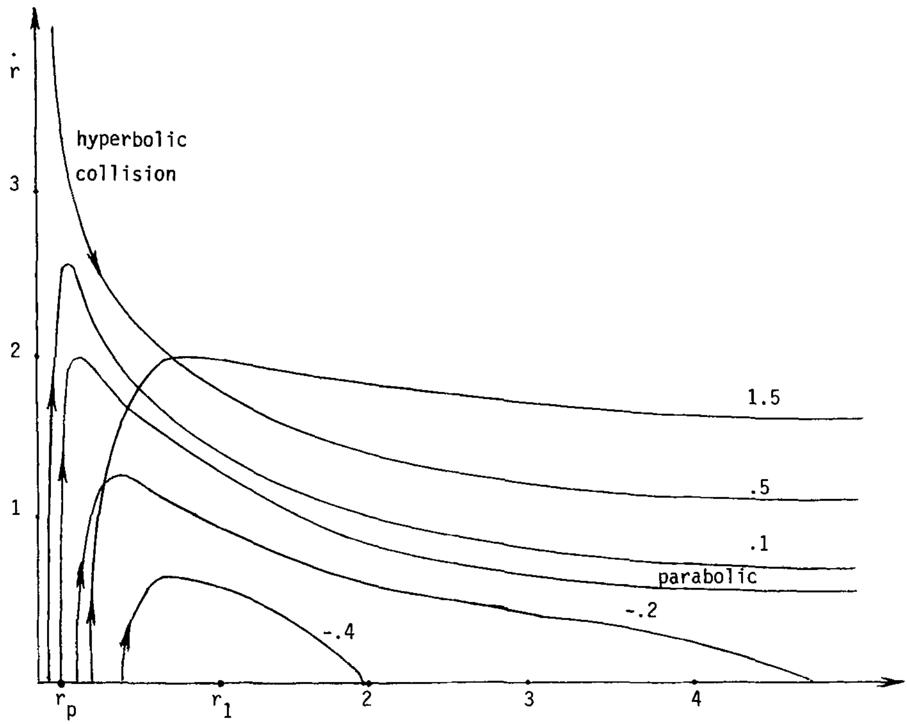


Fig. 1g. $C = -1$, $r_1 = 1$, $(r_p, r_1)$ elliptic retrograde orbits.

(see [3] p. 706). Also, we can say that first component correspond with values of $r \in [0, r_i]$ and the second one with values $r \in [r_0, +\infty)$, where $a$ and $b$ are the two positive roots of the polynomial $r^3 - Cr + 2$.

The description of the sets $I_{hC}$ in $I_C^1$ is given in Table I. This table follows essentially from Figure 1. Ever, we have that $h_C = -C/2$ and $r_C = -h_C^{-1}$ if $h_C < 0$, and that $r_p = C^2/8$.

A direct orbit is an orbit such that its projection on the $q$-plane turns around the origin counterclockwise. If the particle is moving in the opposite direction, we speak of a retrograde orbit (for more details, see [3], p. 706). It is known that the bounded orbits of the two-body rotating problem come from the elliptic orbits of the two-body problem. For this reason, we called elliptic orbits to the bounded orbits of the two-body rotating problem.

Now, we give a qualitative picture of the flow into $I_C^1$. First we consider a configuration, as in Figure 2, formed by two solid cones. This configuration is the union of a family of cylinders, one for each value of $h \in (h_1, h_2)$. For $h = h_1$ or $h = h_2$, we have a circle or a segment, respectively. In this configuration we identify (i.e. we consider as the same) two points on the boundary having the same projection on the plane $P$ with one exception. The exception occurs for the cylinder $h = h_C$. For this cylinder we do not identify the points on its boundary. From Table I, the result is

TABLE I

The component $I_C^1$ when $C > 3$

| Sidereal energy | Topology of $I_{hC}$ | $I_{hC}$ is formed by |
|---|---|---|
| $h = h_1$ | $S^1$ | one circular retrograde orbit |
| $h_1 < h < h_C$ | $S^1 \times S^1$ | a set of $S^1$ elliptic retrograde orbits |
| $h = h_C$ | $S^1 \times R$ | a set of $S^1$ elliptic collision retrograde orbits |
| $h_C < h < h_2$ | $S^1 \times S^1$ | a set of $S^1$ elliptic direct orbits |
| $h = h_2$ | $S^1$ | one circular direct orbit |

$I_C^1$. Furthermore, it is not difficult to see that $I_C^1$ is topologically an open solid torus and that the flow on $I_C^1$ is that of Figure 2. The Levi–Civita regularization of the binary collisions (see [11]) means that we may identify the boundary points of the cylinder $h = h_C$. Then $I_C^1$ is topologically a 3-sphere.

We remark that the direct (resp. retrograde) circular orbit is encircled by a family of tori formed by elliptic direct (resp. retrograde) orbits. These two families are separated by the cylinder of the elliptic collision orbits. If the rotation number of an elliptic orbit of a torus $I_{hC}$ is rational (resp. irrational), then all the orbits of this torus are periodic (resp. quasiperiodic) orbits, for more details see [2]. If an orbit is quasi-periodic then it is dense on its torus.

In a similar way, Table II gives us the sets $I_{hC}$ of $I_C^2$. We recall that an orbit is called parabolic (resp. hyperbolic) if it reaches the infinity with zero (resp. positive) radial velocity as the time tends to $+\infty$ or $-\infty$.

We consider a configuration, as in Figure 3, formed by the union of a family of cylinders, one for each value of $h \in (h_3, +\infty)$. For $h = h_3$ we have a segment. In this configuration we identify two points on the boundary having the same projection on

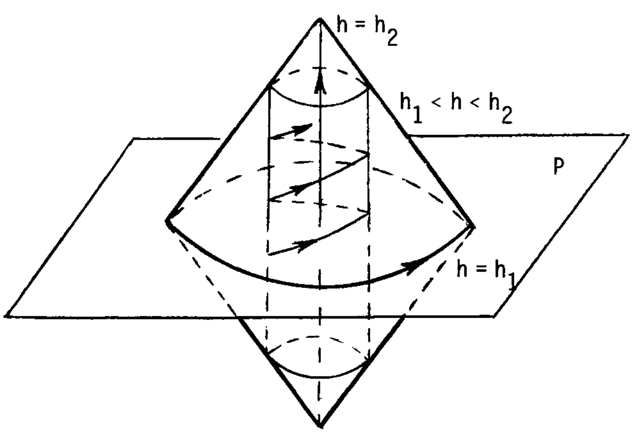


Fig. 2. The component $I_C^1$.

TABLE II
The component $I_C^2$ when $C > 3$

| Sidereal energy | Topology of $I_{hC}$ | $I_{hC}$ is formed by |
|---|---|---|
| $h = h_3$ | $S^1$ | one circular direct orbit |
| $h_3 < h < 0$ | $S^1 \times S^1$ | a set of $S^1$ elliptic direct orbits |
| $h = 0$ | $S^1 \times R$ | a set of $S^1$ parabolic orbits |
| $h > 0$ | $S^1 \times R$ | a set of $S^1$ hyperbolic orbits |

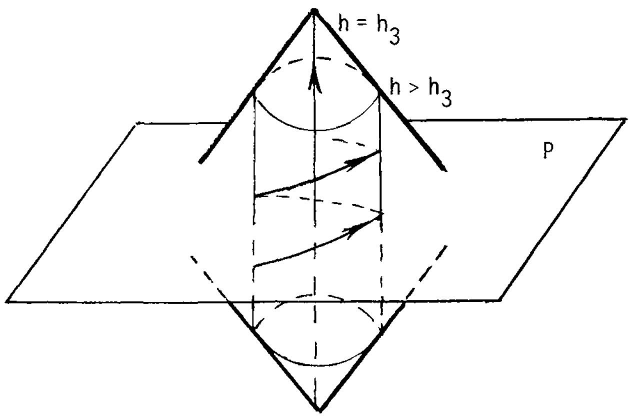


Fig. 3. The component $I_C^2$.

the plane $P$ if they are in the cylinders $h \in [h_3, 0)$. Now, the result is $I_C^2$, an open solid torus, and the flow on $I_C^2$ is that of Figure 3. This qualitative picture of the flow on $I_C^2$ was already given in [8], by using another coordinate system.

## Case $C = 3$

This case is obtained from the above if we identify the two circular direct orbit. Therefore $I_C$ has only one component.

TABLE III
The set $I_C$ when $0 < C < 3$

| Sidereal energy | Topology of $I_{hC}$ | $I_{hC}$ is formed by |
|---|---|---|
| $h = h_1$ | $S^1$ | one circular retrograde orbit |
| $h_1 < h < h_C$ | $S^1 \times S^1$ | a set of $S^1$ elliptic retrograde orbits |
| $h = h_C$ | $S^1 \times R$ | a set of $S^1$ elliptic collisions orbits |
| $h_C < h < 0$ | $S^1 \times S^1$ | a set of $S^1$ elliptic direct orbits |
| $h = 0$ | $S^1 \times R$ | a set of $S^1$ parabolic orbits |
| $h > 0$ | $S^1 \times R$ | a set of $S^1$ hyperbolic orbits |

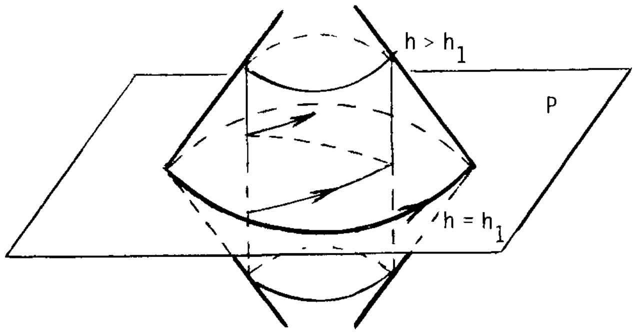


Fig. 4. The set $I_C$ as $C < 3$.

CASE $0 < C < 3$

For values of the Jacobian constant $C < 3$ the polynomial (2.3) has only one positive root $r_1$. Then the set $I_C = \bigcup_h I_{hC}$ has only one component, where $h \in [h_1, +\infty)$. Table III gives us the sets $I_{hC}$ of $I_C$. Now, we consider a configuration as in Figure 4, formed by the union of a family of cylinders one for each value of $h \in (h_1, +\infty)$ with a circle for $h = h_1$. In this configuration we identify two points on the boundary having the same projection on the plane $P$ if they are in the cylinders $h \in (h_1, h_C) \cup (h_C, 0)$. The result is $I_C$, an open solid torus minus its axis, and the flow on $I_C$ is given in Figure 4. Again, the Levi–Civita's regularization of the binary collision means that we may identify the boundary points of the cylinder $h = h_C$. Then $I_C$ is topologically an open solid torus.

The cases $C = 0$ and $C < 0$ are studied in Tables IV and V.

TABLE IV
The set $I_C$ when $C = 0$

| Sidereal energy | Topology of $I_{hC}$ | $I_{hC}$ is formed by |
|---|---|---|
| $h = h_1$ | $S^1$ | one circular retrograde orbit |
| $h_1 < h < h_C$ | $S^1 \times S^1$ | a set of $S^1$ elliptic retrograde orbits |
| $h = h_C = 0$ | 2 copies of $S^1 \times R$ | a set of $S^1 \cup S^1$ parabolic collision orbits |
| $h > 0$ | $S^1 \times R$ | a set of $S^1$ hyperbolic orbits |

TABLE V
The set $I_C$ when $C < 0$

| Sidereal energy | Topology of $I_{hC}$ | $I_{hC}$ is formed by |
|---|---|---|
| $h = h_1$ | $S^1$ | one circular retrograde orbit |
| $h_1 < h < 0$ | $S^1 \times S^1$ | a set of $S^1$ elliptic retrograde orbits |
| $h = 0$ | $S^1 \times R$ | a set of $S^1$ parabolic orbits |
| $0 < h < h_C$ | $S^1 \times R$ | a set of $S^1$ hyperbolic orbits |
| $h = h_C$ | 2 copies of $S^1 \times R$ | a set of $S^1 \cup S^1$ hyperbolic collision orbits |
| $h > h_C$ | $S^1 \times R$ | a set of $S^1$ hyperbolic orbits |

## 3. Collision Orbits and Parabolic Orbits

Let $W_c^s$ (resp. $W_c^u$) be the set of orbits which end (resp. begin) at collision with one of the primaries for the restricted three-body problem. Note that for the two-body rotating problem there exists only one primary. Furthermore, as we have seen in Section 2, if $C > 0$ then $W_c^s$ coincides with $W_c^u$ and it is topologically a cylinder, i.e. the cylinder of elliptic collision orbits. If $C \leqslant 0$ then $W_c^s$ does not coincide with $W_c^u$, and both are topologically a cylinder. Furthermore, $W_c^s$ and $W_c^u$ are formed by parabolic (resp. hyperbolic) collision orbits when $C = 0$ (resp. $C < 0$).

We shall show for the two-body rotating problem and for the restricted three-body problem, that $W_c^s$ and $W_c^u$ are respectively the stable and unstable invariant manifold associated to a convenient invariant set. By using this important fact we shall prove, for the restricted three-body problem, that $W_c^s$ and $W_c^u$ are topologically a cylinder.

We need a complete picture of the local behavior of the solutions near a collision. The binary collision of the third body with the primary of mass $1 - \mu$ can be regularized by using the variables of McGehee. We 'blow up' the collision point, the origin, and replace it with an invariant boundary called the collision manifold. The dynamical system extends smoothly (after a scaling of time) over this collision manifold, and so we get a new flow on an augmented phase space. It turns out that this new flow on and near the collision manifold is extremely simple. It is this fact that enables us to readily understand the behavior of the solutions near a binary collision.

For the restricted three-body problem the usual variables of McGehee (see [4]) do not work since its Hamiltonian cannot be written in the form $H(q, p) = \|p\|^2/2 + V(q)$ (furthermore, it is necessary that $V(q)$ is a homogeneous function). Nevertheless, the ideas of McGehee work and, now, we use them.

We introduce the usual canonical transformation to polar coordinates

$$\begin{aligned}
q_1 &= Q_1 \cos Q_2,\\
q_2 &= Q_1 \sin Q_2,\\
p_1 &= P_1 \cos Q_2 - P_2 Q_1^{-1} \sin Q_2,\\
p_2 &= P_1 \sin Q_2 + P_2 Q_1^{-1} \cos Q_2.
\end{aligned}$$

The Hamiltonian (1.1) becomes

$$\begin{aligned}
H = {} & (P_1^2 + P_2^2 Q_1^{-2})/2 - P_2 - Q_1^{-1} + \mu[Q_1^{-1} - (Q_1^2 + 1 + 2Q_1 \cos Q_2)^{-1/2} -\\
& - P_1 \sin Q_2 - P_2 Q_1^{-1} \cos Q_2],
\end{aligned}$$

and the equations of motion are

$$\begin{aligned}
\dot{Q}_1 &= P_1 - \mu \sin Q_2,\\
\dot{Q}_2 &= P_2 Q_1^{-2} - 1 - \mu Q_1^{-1} \cos Q_2,
\end{aligned}$$

$$\dot{P}_1 = \dot{P}_2^2 Q_1^{-3} - Q_1^{-2} + \mu[Q_1^{-2} - (Q_1 + \cos Q_2)(Q_1^2 + 1 + 2Q_1 \cos Q_2)^{-3/2} - - P_2 Q_1^{-2} \cos Q_2] \tag{3.1}$$

$$\dot{P}_2 = \mu[Q_1 \sin Q_2 (Q_1^2 + 1 + 2Q_1 \cos Q_2)^{-3/2} + P_1 \cos Q_2 - P_2 Q_1^{-1} \sin Q_2].$$

The radial coordinate $Q_1 = r$ is the distance between the larger primary $1 - \mu$ at the origin and the third body $m_3$, while the angular coordinate $Q_2 = \theta$ is the angle between the $q_1$-axis and the radius vector. Then $y = \dot{r}$ and $x = r\dot{\theta}$ are the components of the velocity in polar coordinates. In coordinates $(r, y, \theta, x)$, the equations of motion (3.1) becomes

$$\begin{aligned} \dot{r} &= y, \\ \dot{y} &= x^2 r^{-1} + 2x + r - r^{-2} + \mu[\cos\theta + r^{-2} - \\ &\quad - (r + \cos\theta)(r^2 + 1 + 2r\cos\theta)^{-3/2}], \\ \dot{\theta} &= r^{-1} x, \\ x &= -xyr^{-1} - 2y + \mu \sin\theta[(r^2 + 1 + 2r\cos\theta)^{-3/2} - 1] \end{aligned} \tag{3.2}$$

and the energy relation gives

$$H = (x^2 + y^2 - r^2)/2 - r^{-1} - \mu[\mu/2 + r\cos\theta + (r^2 + 1 + 2r\cos\theta)^{-1/2} - r^{-1}], \tag{3.3}$$

The system (3.2) is no longer Hamiltonian, but (3.3) defines a codimension one invariant set which we continue to call the energy level. Introduce

$$\begin{aligned} u &= r^{1/2} x, \\ v &= r^{1/2} y. \end{aligned}$$

The system (3.2) becomes

$$\begin{aligned} \dot{r} &= r^{-1/2} v, \\ \dot{v} &= r^{-3/2}(v^2/2 + u^2 - 1) + 2u + r^{3/2} + \mu r^{1/2}[\cos\theta + r^{-2} - \\ &\quad - (r + \cos\theta)(r^2 + 1 + 2r\cos\theta)^{-3/2}], \\ \dot{\theta} &= r^{-3/2} u, \\ \dot{u} &= -r^{-3/2} uv/2 - 2v + \mu r^{1/2} \sin\theta[(r^2 + 1 + 2r\cos\theta)^{-3/2} - 1]. \end{aligned} \tag{3.4}$$

This system still has singularities at $r = 0$, but now they can be removed by a change of time scale. Introduce a new time variable $\tau$ via

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = r^{3/2}.$$

Then (3.4) becomes

$$r' = rv,$$

$$\begin{aligned} v' &= v^2/2 + u^2 - 1 + \mu + 2ur^{3/2} + r^3 + \\ &\quad + \mu r^2[\cos\theta - (r+\cos\theta)(r^2+1+2r\cos\theta)^{-3/2}], \\ \theta' &= u, \\ u' &= -uv/2 - 2r^{3/2}v + \mu r^2 \sin\theta[(r^2+1+2r\cos\theta)^{-3/2} - 1], \end{aligned} \tag{3.5}$$

where the prime indicates differentiation with respect to $\tau$. The energy relation goes over to

$$\begin{aligned} (u^2+v^2)/2 - 1 + \mu &= rH + r^3/2 + \\ &\quad + \mu r[\mu/2 + r\cos\theta + (r^2+1+2r\cos\theta)^{-1/2}], \end{aligned} \tag{3.6}$$

When $r = 0$, the energy relation shows that the collision manifold is a two dimensional torus in $r = 0$ defined by

$$\begin{aligned} &(u^2+v^2)/2 = 1 - \mu, \\ &\theta \text{ arbitrary.} \end{aligned} \tag{3.7}$$

The vectorfield on the collision manifold is then given by

$$\begin{aligned} v' &= u^2/2, \\ \theta' &= u, \\ u' &= -uv/2, \end{aligned} \tag{3.8}$$

where we have used the energy relation to simplify $v'$.

From (3.5), (3.6) and (3.8), the collision manifold and the flow on and near this manifold are qualitatively the same that in the Kepler problem, see [4] p. 224. The collision manifold for the Kepler problem has been studied by Devaney [4] and the flow on and near it is sketched in Figure 5. That is, there are two circles of equilibrium points on the torus, its points are $u = 0$, $v = \pm[2(1-\mu)]^{1/2}$ and $\theta$ is arbitrary. All

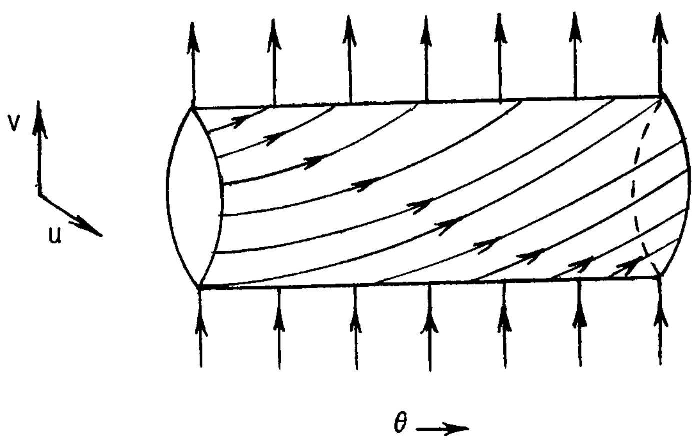


Fig. 5. The flow on and near the collision manifold.

other solutions curves on the torus move from the lower circle to the upper one. We see that the set of orbits which end, $W_c^s$, (resp. begin, $W_c^u$) at collision forms a cylinder in a neighbourhood of the collision. Then we have proved Theorem A.

In fact we have studied the collision orbits with the primary $m_1 = 1 - \mu$. But, since $\mu \in (0, 1)$ the same result follows for the collision orbits with the primary $m_2 = \mu$.

Now, we study the collision orbits for the two-body rotating problem. Thus, Equations (3.5) become

$$\begin{aligned} r' &= rv, \\ v' &= v^2/2 + u^2 - 1 + 2ur^{3/2} + r^3, \\ \theta' &= u, \\ u' &= -uv/2 - 2vr^{3/2}. \end{aligned}$$

when we take $\mu = 0$, and the energy relation (3.6) goes over to $(u^2 + v^2)/2 - 1 = rH + r^3/2$.

For the two-body rotating problem, the collision orbits are characterized by the fact that its sidereal angular momentum $M$ is zero. This implies that the energy relation is reduced to

$$v^2/2 - 1 = rH, \tag{3.9}$$

on the collision orbits. Note that for these orbits

$$u = -r^{3/2},$$

where the minus sign is due to the fact that all the collision orbits are retrograde orbits.

In cylindrical coordinates $(r, \theta, v)$, the cylinder of collision orbits for the two-body rotating problem is plotted in Figure 6 for negative values of $H$. This picture follows from (3.9) and (3.10). Of course, $W_c^u$ coincides with $W_c^s$.

For the restricted three-body problem we have studied in [7] the first cut, $\gamma^u$ and $\gamma^s$, of $W_c^u$ and $W_c^s$ with the plane $v = 0$, respectively. First we fix the energy $H < 0$. Then we know that if $\mu = 0$, in the space $(r, \theta, v)$, $\gamma^u = \gamma^s$ is a circle of radius $-H^{-1}$. If $\mu$ is small enough and positive then $\gamma^u$ and $\gamma^s$ are real analytic simple curves, $\gamma^u \neq \gamma^s$ and $\gamma^u$ and $\gamma^s$ intersect at the points $\theta = 0$ and $\theta = \pi$, nontangentially. Then we have put the Bernoulli shift as a subsystem of the Poincaré map defined by the surface of section $v = 0$, and we have given a geometrical interpretation of the orbits associated with the shift (see [7] for details).

The well-known symmetry $S$ of the restricted three-body problem (see Section 1) in coordinates $(r, v, \theta, u, \tau)$ becomes

$$S : (r, v, \theta, u, \tau) \to (r, -v, -\theta, u, -\tau).$$

Since $S(\gamma^u) = \gamma^s$, it follows immediately, in the above hypotheses, that $\gamma^u$ and $\gamma^s$ intersect at the points $\theta = 0$ and $\theta = \pi$.

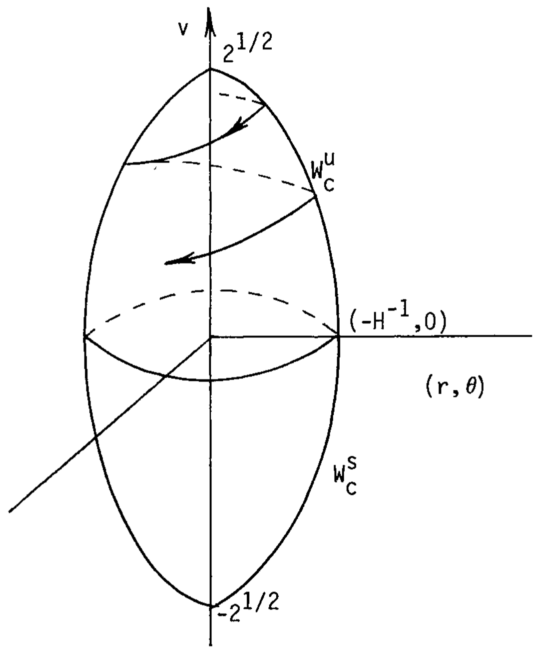


Fig. 6. The flow on the cylinder of collision orbits as $H < 0$ and $\mu = 0$.

*Proof of Theorem B.*

(i) follows immediately from the fact that the two curves $\gamma^u$ and $\gamma^s$ intersect at the points $\theta = 0$ and $\theta = \pi$. We recall that $H = -C/2$. Then $C > 0$ implies $H < 0$.

(ii) If $C < 2$ then $-H^{-1} > 1$. For values of $\mu$ sufficiently small the curves $\gamma^u$ and $\gamma^s$ are near the circle of radius $r = -H^{-1}$ if $0 < C$. Since the primary $m_2 = \mu$ is on the circle of radius one in the $q$-plane, (ii) follows for values of $C$ such that $0 < C < 2$. If $C \leqslant 0$ then $W_c^u$ goes to infinity without crossing the surface $\dot{r} = 0$ when $\mu = 0$. Then for $\mu$ sufficiently small the projection of $W_c^u$ on the position plane crosses the circle $r = 1$ and (ii) follows.

Let $W_p^s$ (resp. $W_p^u$) be the set of parabolic orbits for $t \to +\infty$ (resp. $-\infty$). For the two-body rotating problem we have seen in Section 2 that if $C \neq 0$ then $W_p^s$ coincides with $W_p^u$ and it is topologically a cylinder. If $C = 0$ then $W_p^s$ does not coincide with $W_p^u$, and both are topologically a cylinder, the cylinders of parabolic collisions.

For the restricted three-body problem, McGehee in [9] proved that $W_p^s$ and $W_p^u$ are the stable and unstable invariant manifolds of a periodic orbit at the infinity, respectively. Also, he showed that $W_p^s$ and $W_p^u$ are topologically a cylinder. In [8] we have studied the first cut $\gamma^s$ and $\gamma^u$ of $W_p^s$ and $W_p^u$ with the surface $\dot{r} = 0$, respectively. If $\mu$ is sufficiently small and $C$ sufficiently large then $\gamma^s$ and $\gamma^u$ intersect only in two points, nontangentially (see Theorem 5.1 of [8]).

Now, the proof of Theorem C follows in a similar way to the proof of Theorem B by using Section 2 and the symmetry of the problem. Note that for values of $C \in (-8^{1/2}, 8^{1/2})$ the parabolic cylinder of the two-body rotating problem cuts the surface $\dot{r} = 0$ in a circle of radius $r < 1$.

## 4. Symmetric Periodic Orbits

From now on we consider the restricted three-body problem in a rotating coordinate system $q = (q_1, q_2)$ of rotational frequency equal to 1. In this frame we put the larger primary $m_1 = 1 - \mu$ at the position $e_1 = (\mu, 0)$ and the smaller primary $m_2 = \mu$ at the position $e_2 = (\mu - 1, 0)$. The Hamiltonian which governs the motion of the zero mass particle $m_3$ is given by

$$H = \|p\|^2 + q_2 p_1 - q_1 p_2 - (1-\mu)\|q - e_1\|^{-1} - \mu\|q - e_2\|^{-1},$$

where $p = (p_1, p_2)$ are the momentum variables conjugate to the $q$, and $\| \quad \|$ is the Euclidean norm of $R^2$.

We have introduced polar coordinates through $q_1 + iq_2 = r\exp(i\theta)$. A periodic orbit which is symmetric with respect to the $q_1$-axis crosses the $q_1$-axis twice and only twice at right angles. The crossing at right angles may be divided into four classes. The first class, $C_1$, corresponds with the states of motion $x > \mu$, $y = 0$, $\dot{r} = 0$ and $\ddot{r} > 0$. We call these states of motion lower passage at opposition. The second class, $C_2$, corresponds with crossing at right angles for $x > \mu$, $y = 0$, $\dot{r} = 0$ and $\ddot{r} < 0$, this state of motion may be called higher passage at opposition. The third $C_3$ and fourth $C_4$ classes, $x < \mu$, $y = 0$, $\dot{r} = 0$, $\ddot{r} > 0$ and $\ddot{r} < 0$, respectively, may be called lower and higher passage at conjunction. There are ten possible combinations $C_{ij}$ of pairs of the four classes of right-angle crossings, where $C_{ij} = C_i \cap C_j$ with $i, j \in \{1, 2, 3, 4\}$.

The main result of this section is the following theorem.

THEOREM D. *For any fixed value of the Jacobian constant and for any $\varepsilon > 0$, there exists a $\mu_0 > 0$ such that if the mass parameter $\mu \in [0, \mu_0]$, then the set of bounded orbits which are not contained in the closure of the class $C_{ij}$ of symmetric periodic orbits has Lebesgue measure less than $\varepsilon$.*

From now on we fix the value $C$ of the Jacobian constant. From Figures 2, 3 and 4 it is clear, for the two-body rotating problem, that the Poincaré map associated with the the surface of section given by the plane $P$, are twist mappings. But for the study of symmetric periodic orbits we consider the Poincaré map associated to the surface of section $\Sigma$ defined by $\dot{r} = 0$, $\ddot{r} > 0$. From Figure 1 or from [5], we obtain Table VI. This table gives us the different component rings of the surface of section $\Sigma$ for each value of the Jacobian constant. Actually $\Sigma$ has only one connected component if $C \leqslant 3$ and two connected components if $C > 3$. From Section 2 we recall that $r_1, r_2, r_3$ are the radii of the circular orbits, $r_p = C^2/8$ is the radius of the circle of the parabolic cylinder when it cuts the surface $\dot{r} = 0$, and $r = r_0$ is the boundary circle of $I_C^2$ when

TABLE VI

| Orbits | $C$ | $r_i$ inner radius | $r_0$ outer radius |
|---|---|---|---|
| Retrograde | $(-\infty, 0)$ | $C^2/8$ parabolic | $r_1$ circular |
| | 0 | 0 parabolic collision | $r_1$ circular |
| | $(0, \infty)$ | 0 elliptic collision | $r_1$ circular |
| Direct | $(0, 3)$ | 0 elliptic collision | $C^2/8$ parabolic |
| | 3 | 0 elliptic collision | $r_2 = r_3 = 1$ circular |
| | | $r_2 = r_3 = 1$ circular | $C^2/8$ parabolic |
| | $(3, 32^{1/3})$ | 0 elliptic collision | $r_2$ circular |
| | | $r_0$ elliptic | $C^2/8$ parabolic |
| | | $r_0$ elliptic | $r_3$ circular |
| | $[32^{1/3}, \infty)$ | 0 elliptic collision | $r_2$ circular |
| | | $C^2/8$ parabolic | $r_3$ circular |

$C > 3$. All these curves in the $(C, r)$-plane are plotted qualitatively in Figure 7.

Let $R$ be one of the rings of Table VI, i.e.

$$R = \{(r, \theta) : r_i < r < r_0, 0 \leqslant \theta \leqslant 2\pi, \dot{r} = 0, \ddot{r} > 0\}.$$

For any point $P$ of $R$ we denote by $P'$ the point at which the orbit that passes through $P$ first meets $R$. The transformation which takes $P$ to $P'$ defines a Poincaré map $T : R \to R$. The equations of $T$ are:

$$\begin{aligned} r' &= r, \\ \theta' &= \theta + f(r), \end{aligned} \tag{4.1}$$

where $f(r) = -2\pi a^{3/2}$ and from

$$r = a\{1 - [1 - (1 - Ca)^2 (4a^3)^{-1}]^{1/2}\}, \tag{4.2}$$

we have $a = a(r)$, for more details see [3] p. 709.

The mapping (4.1) is a twist mapping, That is, every circle is invariant by $T$ and $T$ rotates each circle $r = c$ an angle $f(c)$. Furthermore, $|df/dr| > 0$ for any $r_i < r < r_0$.

Let $\psi$ be the position of the particle in its ellipse (true anomaly), and $\phi$ the longitude of the apsidal line with reference to the rotating $q_1$-axis. Then the angular polar synodic coordinate $\theta$ equals $\phi + \psi$.

A lower crossing at right angles corresponds to $\phi = 0$, $\psi = 0$ or $\phi = \pi$, $\psi = 0$ according as the crossing is of $C_1$ or $C_3$ type. A higher crossing at right angles is represented by $\phi = \pi$, $\psi = \pi$ or $\phi = 0$, $\psi = \pi$ according as the crossing is of $C_2$ or $C_4$ type.

The lower crossings at right angles are represented on the ring $R$ by the segments:

$$\gamma_1 = \{(r, \theta) \in R : \theta = 0\},$$

$$\gamma_3 = \{(r, \theta) \in R : \theta = \pi\},$$

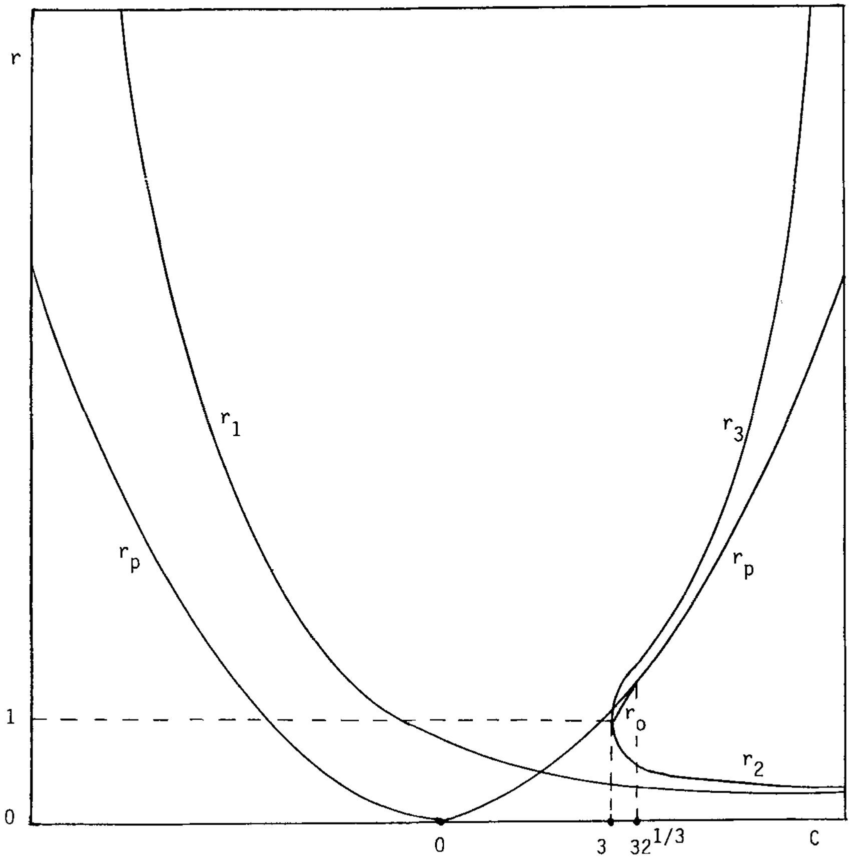


Fig. 7. The curves $r_1(C)$, $r_2(C)$, $r_3(C)$, $r_p(C) = C^2/8$ and $r_0(C)$.

according to the crossing being of $C_1$ or $C_3$ type. In the higher crossings of $C_2$ and $C_4$ we have $\dot{r} < 0$, and hence they are not represented in the ring $R$. However, we may associate with each such crossing the first state of motion along the orbit which lies on the ring. These form two continuous arcs:

$$\gamma_2 = \{(r, \theta) \in R : \theta = \pi(1 - a^{3/2}) \quad \text{where } a = a(r) \text{ from (4.2)}\},$$

$$\gamma_4 = \{(r, \theta) \in R : \theta = -\pi a^{3/2} \quad \text{where } a = a(r) \text{ from (4.2)}\},$$

according as the crossing is of $C_2$ or $C_4$ type (see [3], p. 746).

A symmetric periodic orbit is completely characterized by the fact that it has two and only two states of motion represented by points on the arcs $\gamma_1, \gamma_2, \gamma_3, \gamma_4$. Therefore, if a point $P$ of one of these arcs is carried into a point $P'$ of such an arc by the $k$-fold iteration of $T$, then the corresponding orbit is symmetric and periodic. Hence the class $C_{ij}$ of symmetric periodic orbits is represented on the ring $R$ by the points of

$$\Gamma_{ij} = \bigcup_{k=-\infty}^{+\infty} T^k(\gamma_{ij}),$$

where $\gamma_{ij} = \gamma_i \cap \Gamma_j$, with

$$\Gamma_j = \bigcup_{k=-\infty}^{+\infty} T^k(\gamma_j).$$

We shall use the following lemma, see [2], p. 98, for a proof.

LEMMA 4.1. (Jacobi's Theorem). *Let $S^1 = R/Z$ the circle, and let $F$ be the map $x \to x + \omega$ (mod. 1), $\omega \in R$. Then an orbit of $F$ is dense if and only if $\omega$ is irrational.*

LEMMA 4.2 *Let $\gamma : r = r(\lambda)$, $\theta = \theta(\lambda)$ $(0 \leqslant \lambda \leqslant 1)$ be a continuous arc on $R$, such that $\gamma$ meets $r = r_i$ and $r = r_0$ in the endpoints $\lambda = 0$ and $\lambda = 1$, respectively. Then the set*

$$\bigcup_{k=-\infty}^{+\infty} T^k(\gamma),$$

*is dense in $R$.*

Since $T$ is a twist mapping, by Lemma 4.1, the proof of Lemma 4.2 follows.

Let $Y$ be a given subset of a topological space X. The closure of $Y$ in $X$ will be denoted by $\mathrm{cl}_X(Y)$. If $Z$ is a subset of $Y$ we denote by $\mathrm{cl}_{Y,X}(Z)$, the closure in $Y$ of $Z$ in the relative topology of $Y$ with respect to the topology of $X$.

By Lemma 4.2 and since $T$ is a twist mapping, we have that the closure of $\gamma_{ij}$, $\mathrm{cl}_{\gamma_i, R}(\gamma_{ij})$, is $\gamma_i$. Then, by continuity:

$$T^k(\gamma_i) = \mathrm{cl}_R(T^k(\gamma_{ij})). \tag{4.3}$$

THEOREM 4.3 *For any fixed value $C$ of the Jacobian constant and for $\mu = 0$, the set of bounded orbits is contained in the closure of the class $C_{ij}$ of symmetric periodic orbits; i.e., $\mathrm{cl}_R(\Gamma_{ij}) = R$, where $R$ is any ring of Table VI for the given value of $C$.*

*Proof.* From (4.3) it follows that

$$\Gamma_i = \bigcup_{k=-\infty}^{+\infty} \mathrm{cl}_R(T^k(\gamma_{ij})) \subset \mathrm{cl}_R(\Gamma_{ij}).$$

Hence, by Lemma 4.2, we obtain

$$R = \mathrm{cl}_R(\Gamma_i) \subset \mathrm{cl}_R(\Gamma_{ij}) \subset R.$$

Then Theorem D is proved for $\mu = 0$; i.e. for the two-body rotating problem.

For a fixed value of the Jacobian constant $C$, let $\Sigma_\mu$ be the surface of section for the restricted three-body problem with mass parameter $\mu$ defined by $\dot{r}=0$, $\ddot{r}>0$ for the bounded orbits, and let $T_\mu$ be the Poincaré map defined on $\Sigma_\mu$.

Let $R$ be the ring $\{(r,\theta)\in\Sigma_\mu : a<r<b\}$ with $r_i<a<b<r_0$, where $r_i$ and $r_0$ are the inner and outer radius, respectively, of the rings of Table VI. For the fixed value of $C$, if $\mu$ is sufficiently small we can take the different rings $R$ so that the surface of section $\Sigma_\mu$ not contained in the rings $R$ has measure of Lebesgue arbitrarily small.

The Poincaré map $T_\mu$ on $R$ takes Equations (4.1) into

$$r' = r + \mu g_1(r,\theta,\mu),$$
$$\theta' = \theta + f(r) + \mu g_2(r,\theta,\mu)(\text{mod. } 2\pi),$$

where $g_1$ and $g_2$ are real, analytic and $2\pi$-periodic in $\theta$. Since the polar synodic coordinates $r, \theta$ are the position coordinates of a canonical transformation of the restricted three-body problem, we have that $T_\mu$ is an area-preserving mapping.

For a function $g\in C^1(R)$ we consider the norm

$$|g|_s = \sup_{m+n\leqslant s}\left|\frac{\partial^{m+n}g}{\partial^m r\,\partial^n\theta}\right|.$$

We take $s=5$. If $\mu$ is small enough by the Kolmogorov–Arnold–Moser–Rüssman Theorem (see [10], p. 52) there exists a $\delta$ depending on $\varepsilon, s, f(r)$ and $\mu$ such that if $T_\mu$ satisfies $\mu(|g_1|_s+|g_2|_s)<\nu\delta$, where $|\mathrm{d}f/\mathrm{d}r|\geqslant\nu>0$ in $R$. Then $T_\mu$ has an invariant curve of the form $r=c+u(\xi)$, $\theta=\xi+v(\xi)$, in $R$, where $u$ and $v$ are continuously differentiable of period $2\pi$ and satisfy $|u|_1+|v|_1<\varepsilon$, and $c$ is a constant in $(a,b)$. Furthermore, the induced mapping on this curve is given by $\xi\to\xi+\omega$, where $\omega$ is incommensurable with $2\pi$, and satisfies infinitely many conditions $|\omega/(2\pi)-j/k|\geqslant$ $\geqslant\gamma k^{-\tau}$, with some positive $\gamma$, $\tau$ for all integers $k>0, j$. Furthermore, we have the following theorem.

THEOREM 4.4 (Arnold, for a proof see [1]). *The set of all invariant curves of $T_\mu$ leaves out only a set of measure of Lebesgue arbitrarily small in R, if $\delta$ (and so $\mu$) is sufficiently small.*

Without loss of generality we can assume that the Poincaré map $T_\mu$ is such that $\partial\theta'/\partial r>0$.

LEMMA 4.5 (see Lemma 1 of [5] p. 340). *Let $(r,\theta)$ be any point of an invariant curve $\Gamma$ of $T_\mu$ and let $T^n_\mu(r,\theta)=(r_n(r,\theta),\theta_n(r,\theta))$. Let $K>0$ be the minimum value of the continuous map $(\partial\theta_1/\partial r)$ on $\Gamma$. Then we have $\partial\theta_n/\partial r\geqslant K$, for each positive integer $n>1$.*

Again, the lower crossings at right angles are represented on the ring $R$ by the segments:

$$\gamma_1=\gamma_1(\mu)=\{(r,\theta)\in R:\theta=0\},$$

$$\gamma_3=\gamma_3(\mu)=\{(r,\theta)\in R:\theta=\pi\},$$

according as the crossing is of $C_1$ or $C_3$ type. The arcs $\gamma_2 = \gamma_2(\mu)$ and $\gamma_4 = \gamma_4(\mu)$ change only slightly from $\gamma_2$ and $\gamma_4$, respectively, if $\mu$ is small. Hence the class $C_{ij}$ of symmetric periodic orbits is represented in the ring $R$ by the points of

$$\Gamma_{ij} = \Gamma_{ij}(\mu) = \bigcup_{k=-\infty}^{+\infty} T_\mu^k(\gamma_{ij}(\mu)),$$

where $\gamma_{ij}(\mu) = \gamma_i(\mu) \cap \Gamma_j(\mu)$, with

$$\Gamma_j = \Gamma_j(\mu) = \bigcup_{k=-\infty}^{+\infty} T_\mu^k(\gamma_j(\mu)).$$

LEMMA 4.6. *Let $\Gamma$ be an invariant curve of $T_\mu$. Then for any $\varepsilon > 0$ there exists a symmetric periodic point $p$ in $\Gamma_{ij}$ such that $d(p, \Gamma) < \varepsilon$.*

*Proof*. Let $R_0$ be the outside component of the curve $\Gamma$ into the ring $R$. Let $A = \{x \in R_0 : d(x, \Gamma) < \varepsilon\}$. Set $q_i = \gamma_i \cap \Gamma$, $q_j = \gamma_j \cap \Gamma$ and $\gamma_i' = \gamma_i \cap A$.

One of the endpoints of the arc $\gamma_i'$ is $q_i$, let $q_i'$ be the other. If $\varepsilon$ is sufficiently small, then the line through the point $q_i'$ with slope $K$ (defined in Lemma 4.5) intersects the curve $\Gamma$ in a point $q_i''$, see Figure 8.

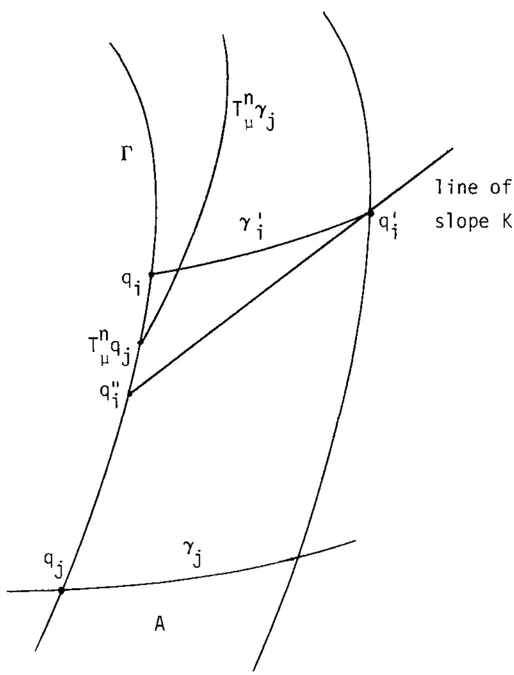


Fig. 8. The region $A$ near the segment $\gamma_i'$.

From Moser's Theorem we have that the orbit of $q_j$, $\{q_j, T_\mu q_j, T_\mu^2 q_j, \cdots\}$ is dense in $\Gamma$. Then there is a positive integer $n = n(q_j)$ such that the point $T_\mu^n q_j$ is between the points $q_i''$ and $q_i$ when we go from $q_i''$ to $q_i$ counterclockwise. By Lemma 4.5, $T_\mu^n \gamma_j$ cuts the arc $\gamma_i'$. Then there are points $p$ in $\Gamma_{ij}$ such that $d(p, \Gamma) < \varepsilon$.

THEOREM 4.7. *The closure of the symmetric periodic orbits* $\Gamma_{ij}$ *of* $T_\mu$ *is a set which contains the invariant curves of* $T_\mu$.

*Proof*. Suppose that there is an invariant curve $\Gamma$ and a neighborhood $U$ of a point of $\Gamma$ such that $U$ does not contain any symmetric periodic point of $\Gamma_{ij}$. By Moser's Theorem, the set $\bigcup_{n=0}^{\infty} T_\mu^n U$ contains the compact $\Gamma$. Then, there is a set $\bigcup_{i=1}^{k} T_\mu^{n(i)} U$ which contains also to $\Gamma$. Since $T_\mu$ is a diffeomorphism, there exists $\varepsilon$ and an annulus $A = \{x \in R : d(x, \Gamma) < \varepsilon\}$ contained in $\bigcup_{i=1}^{k} T_\mu^{n(i)} U$, without symmetric periodic points of $\Gamma_{ij}$. This is a contradiction with Lemma 4.6, and the theorem follows.

Theorem D follows from Theorems 4.4 and 4.7.

## References


[1] V. I. Arnold, 'Proof of a Theorem of A. N. Kolmogorov on the Invariance of Quasi-periodic Motions Under Small Perturbations of the Hamiltonian', *Russian Math. Surveys* **18** (1963), 9–36.

[2] V. I. Arnold and A. Avez, *Problemes ergodiques de la Mécanique Classique*, Gauthier-Villars, Paris, 1967.

[3] G. D. Birkhoff, *Collected Mathematical Papers*, Vol. 1, Proc. Amer. Academy of Arts and Sciences, 1923.

[4] R. L. Devaney, 'Singularities in Classical Mechanical Systems', in A. Katok (ed.), *Ergodic Theory and Dynamical Systems I*, Proceedings Special Year, Maryland 1979–80, Birkhäuser, Basel, 1981, pp. 211–333.

[5] G. Gomez and J. Llibre, 'A Note on a Conjecture of Poincaré', *Celes. Mech.* **24** (1981), 335–343.

[6] H. Golstein, *Classical Mechanics*, Addison-Wesley, London, 1971.

[7] J. Llibre and C. Pinol, *Collision Orbits in the Planar Circular Restricted Three-body Problem*, to appear.

[8] J. Llibre and C. Simó, 'Oscillatory Solutions in the Planar Restricted Three-body Problem', *Math. Ann.* **248** (1980), 153–184.

[9] R. McGehee, 'A Stable Manifold Theorem for Degenerate Fixed Points with Applications to Celestial Mechanics', *J. Differential Equations* **14** (1973), 70–88.

[10] J. Moser, *Stable and Random Motions in Dynamical Systems*, Princeton Univ. Press, 1973.

[11] E. L. Stiefel and G. Scheifele, *Linear and Regular Celestial Mechanics*, Springer-Verlag, Berlin, 1971.

[12] V. Szebehely, *Theory of Orbits*, Academic Press, New York, 1967.