ANALYSIS OF SOME DEGENERATE QUADRUPLE COLLISIONS* **

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Abstract. We consider the trapezoidal problem of four bodies. This is a special problem where only three degrees of freedom are involved. The blow up method of McGehee can be used to deal with the quadruple collision. Two degenerate cases are studied in this paper: the rectangular and the collinear problems. They have only two degrees of freedom and the analysis of total collapse can be done in a way similar to the one used for the collinear and isosceles problems of three bodies. We fully analyze the flow on the total collision manifold, reducing the problem of finding heteroclinic connections to the study of a single ordinary differential equation. For the collinear case, from which arises a one parameter family of equations, the analysis for extreme values of the parameter is done and numerical computations fill up the gap for the intermediate values. Dynamical consequences for possible motions near total collision as well as for regularization are obtained.

1. Introduction

The trapezoidal problem of four bodies consists in the description of the motion of four particles of mases $m_1, m_2 = m_1, m_3, m_4 = m_3$ with initial coordinates (a, b), $(-a, b), (c, d)$ and $(-c, d)$, respectively and velocities such that the symmetry of coordinates is kept for all time. We can suppose that the center of masses remains at the origin, i.e., $m_1 b + m_3 d = 0$. New variables $x = 2a, y = 2c, z = b - d$ can be introduced (see Figure 1). The motions near quadruple collision for that problem have been partially described in [5]. In order to give a complete picture of the flow on the total collision manifold we restrict ourselves to two degenerate cases: the rectangular and the collinear. In the first the four masses are equal and $a = c, b = d$. In the second one has $b = d = 0$ but we still have one parameter: the mass ratio $\alpha = m_2/m_1$. Then the total collision manifold is two dimensional (see [6] and [1]) and the invariant manifolds associated to the critical points are one dimensional. The study can be done on the same lines as the one found in [6] and [7] for the collinear three-body problem, or in [1], [8] and [2] for the isosceles problem. However, the analysis of the behavior of the invariant manifolds is done using a single ordinary differential equation. A similar method was formulated in [4] and [3].

2. The Rectangular Case

First we set the masses equal to one for the bodies. We write down the Lagrangian

$$L = \tfrac{1}{2}(\dot{x}^2 + \dot{y}^2) + \frac{2}{x} + \frac{2}{y} + \frac{2}{(x^2 + y^2)^{1/2}}$$

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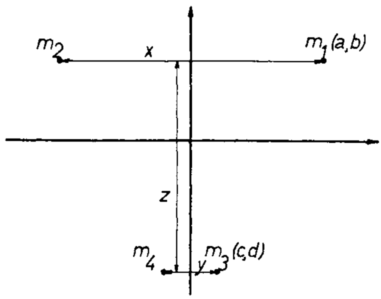

Fig. 1.

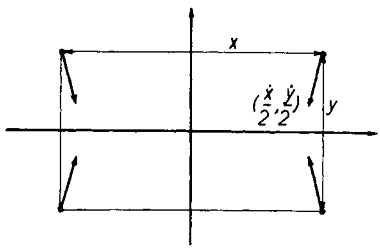

Fig. 2.

and the corresponding Hamiltonian

$$H = \tfrac{1}{2}(p_x^2 + p_y^2) - \frac{2}{x} - \frac{2}{y} - \frac{2}{(x^2 + y^2)^{1/2}}$$

where the coordinates are described in Figure 2.

The resulting Hamilton equations are

$$\dot{x} = p_x, \quad \dot{p}_x = -\frac{2}{x^2} - \frac{2x}{\zeta^3},$$

$$\dot{y} = p_y, \quad \dot{p}_y = -\frac{2}{y^2} - \frac{2y}{\zeta^3},$$

where $\zeta = (x^2 + y^2)^{1/2}$.

Let us introduce the change of variables (see [6]):

$$X = x\zeta^{-1}, \quad Y = y\zeta^{-1}, \quad P_X = p_X\zeta^{1/2}, \quad P_Y = p_Y\zeta^{1/2}, \quad ' = \frac{\mathrm{d}}{\mathrm{d}\tau} = \zeta^{3/2}\frac{\mathrm{d}}{\mathrm{d}t}.$$

Then we have $X^2 + Y^2 = 1$ and $\dot{\zeta} = \zeta^{-1}(xp_x + yp_y)$. Introducing $V = XP_X + YP_Y$ we get the blown up equations:

$$X' = P_X - XV, \quad P'_X = -\frac{2}{X^2} - 2X + \tfrac{1}{2}VP_X,$$

$$Y' = P_Y - YV, \quad P'_Y = -\frac{2}{Y^2} - 2Y + \tfrac{1}{2}VP_Y.$$

On $\zeta = 0$ (total collision manifold C) the equations are still regular and we shall use the description of the flow on C to get information about passage near total collision. As we know the change of variables is a diffeomorphism for $\zeta > 0$. The change has blown up the point $x = y = 0$ to the manifold C. This has no physical meaning neither the fact that the new time τ on C is obtained by an infinite slowing down of the physical time. However, the regularity of the equations on C gives information for small positive values of ζ and this has a clear physical importance.

On C the equation of the energy is $\frac{1}{2}(P_X^2 + P_Y^2) - U = 0$, where $U = (2/X) + (2/Y) + 2$ and we get $V' = U - (V^2/2)$.

The equilibrium points are $U_c = U_{\min} = U(1/\sqrt{2}, 1/\sqrt{2}) = 2 + 4\sqrt{2}$ and $V_c = \pm\sqrt{(8\sqrt{2} + 4)}$.

We now introduce a new change of coordinates: $X = \cos\theta$, $Y = \sin\theta$ and therefore

$$X' = -\sin\theta \cdot \theta', \quad \theta' = -\frac{P_X - XV}{\sin\theta}.$$

But $P^2 = P_X^2 + P_Y^2 = 2U$ and $\operatorname{Arg}\mathbf{P} = \gamma$ allow us to write $P\cos(\gamma - \theta) = V$. Therefore

$$P_X = P\cos\gamma = V\cos\theta - \sqrt{(P^2 - V^2)}\sin\theta.$$

After substitution we get

$$\theta' = \pm\sqrt{2V'}\ , \quad V' = U(\theta) - \frac{V^2}{2}, \quad U(\theta) = \frac{2}{\cos\theta} + \frac{2}{\sin\theta} + 2$$

which we integrate from $\theta = \pi/4$, $V = -\sqrt{(8\sqrt{2} + 4)}$ to obtain the unstable manifold W_A^u of the lower equilibrium point A (see Figure 3). Now we have several possibilities for studying the equations of the manifold. We can obtain $d\theta/dV$ (see Section 3) or

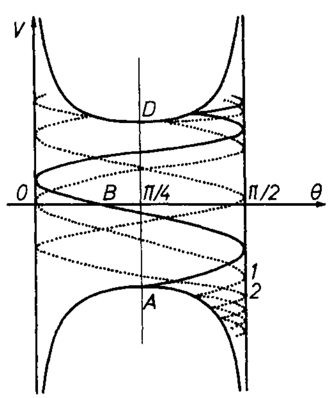

Fig. 3.

we can use the arc parameter σ along W_A^u as independent variable. The new equations become

$$\frac{dV}{d\sigma} = (1 + 2/V')^{-1/2}, \quad \frac{d\theta}{d\sigma} = \pm(1 + V'/2)^{-1/2},$$

avoiding all the singularities. The change of sign in $d\theta/d\sigma$ is produced when $\theta = 0$ or $\pi/2$.

3. Numerical Computations and Analytical Estimations for the Rectangular Case

The last equations have been integrated starting at A up to $V = 0$ (point B). The values obtained are $\theta(B) = 0.5877$, $\sigma(B) = 4.459$. Using the symmetry with respect to $\theta = \pi/4$ and $V = 0$ it is clear that, to have a connection between lower, A, and upper, D, equilibrium points requires $\theta(B)$ to be a multiple of $\pi/4$. The value 0.5877 is quite different from 0 and $\pi/4$. However, for people who dislike results obtained through numerical integration, we offer a proof of the fact that $W_D^s \neq W_A^u$ that involves only inequalities and a few evaluations of trigonometric and hyperbolic functions.

Dividing θ' by V' we get

$$\left|\frac{d\theta}{dV}\right| = \sqrt{2/V'} = \left(\sec\theta + \operatorname{cosec}\theta + 1 - \frac{V^2}{4}\right)^{-1/2}.$$

We intend to show that starting at B_1 and going backwards we reach the curve $V = \sqrt{(2U(\theta))}$ to the right of point A and starting at B_2 we reach $\theta = \pi/4$ above point A (see Figure 4).

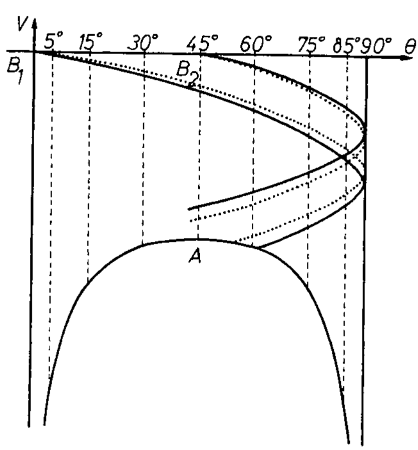

Fig. 4.

To prove the first assertion we show that this is true for a vectorfield F such that

$$\left|\frac{d\theta}{dV}\right| \leqslant F$$

and for the second, that it is true for a G such that

$$\left|\frac{d\theta}{dV}\right| \geqslant G.$$

Let $k_i = \min_{\theta\in[\theta_i,\theta_{i+1}]}$ $(\sec\theta + \operatorname{cosec}\theta)$. In this range of θ we take $F = 1/\sqrt{k_i + 1 - (V^2/4)}$. If we set $d\theta/dV = F$ one has $\Delta\theta = \int_{V_i}^{V_{i+1}} dV/\sqrt{k_i + 1 - (V^2/4)}$ where V_i, V_{i+1} are the values at extreme points. Letting $V = 2\sqrt{k_i + 1}\sin\alpha$ we obtain $\Delta\theta = 2\Delta\alpha$. Now we split the range of θ in the following set of intervals (in degrees) $[0°, 5°]$, $[5°, 15°]$, $[15°, 30°]$, $[30°, 60°]$, $[60°, 75°]$, $[75°, 85°]$, $[85°, 95°]$, $[95°, 105°]$ $[105°, 120°]$, $[120°, 135°]$. For angles greater than 90° we take the symmetrical with respect to 90°. The partition points separating intervals are $\theta_0 = 0°$, $\theta_1 = 5°, \dots, \theta_9 = 120°$, $\theta_{10} = 135°$. At each one of such points we shall compute V_i. Note that for each V_i we have two values of α, α_{2i-1}, α_{2i}, depending on the value of k_i used, the one related to the left or the right interval. Using symmetry and convexity $k_0 = \sec 5° + \operatorname{cosec} 5° = k_6$, $k_1 = 2\sqrt{6} = k_5 = k_7$, $k_2 = 2 + 2/\sqrt{3} = k_4 = k_8$, $k_3 = 2\sqrt{2} = k_9$.

We set up the recurrence $\alpha_{2i+1} = \alpha_{2i} + (\theta_{i+1} - \theta_i)/2$, $\sqrt{(k_i + 1)}\sin\alpha_{2i+1} = \sqrt{(k_{i+1} + 1)}\sin\alpha_{2i+2}$, $i = 0$–8, starting with $\alpha_0 = 0$. A few computations of trigonometric functions give the values $\alpha_1 = \pi/72, \alpha_3 = 0.153\,246\,335, \alpha_5 = 0.313\,807\,297$, $\alpha_7 = 0.589\,183\,175$, $\alpha_9 = 0.693\,534\,686$, $\alpha_{11} = 0.653\,534\,211$, $\alpha_{13} = 0.501\,227\,454$, $\alpha_{15} = 0.900\,181\,094$, $\alpha_{17} = 1.335\,010\,689$ and then we obtain $\sin\alpha_{18} > 1$ showing that under F we reach the value $V = V_c$ to the right of point A.

Now we proceed to study the solutions of $d\theta/dV = G$ starting at $V = 0$, $\theta = \pi/4$. Consider the interval $[a, b] \subset [0, \pi/4]$. Suppose that $V(a) < V(b)$. Then we take as $1/G$ the function

$$\sqrt{\frac{d}{\theta} + \sec(a) + 1 - \frac{V(a)^2}{4}}$$

where $d = b/\sin b$. We have $\Delta V = \int_a^b 1/G(\theta)\,d\theta$. Let $\varphi(m, \theta) = \sqrt{\theta + m\theta^2} + (1/\sqrt{m})$ argtanh$\sqrt{(m\theta/(1 + m\theta))}$ if $m > 0$ and $\sqrt{\theta + m\theta^2} + (1/\sqrt{-m})\operatorname{arctg}\sqrt{(-m\theta/(1 + m\theta))}$ if $m < 0$. Define $g = \sec(a) + 1 - (V(a)^2/4)$. Then $\Delta V = \sqrt{d}(\varphi(g/d, b) - \varphi(g/d, a))$. When θ goes from $\pi/4$ to $\pi/2$ and again to $\pi/4$ and V decreases, the variation ΔV is equal to the variation obtained going from $\pi/4$ to 0 and again to $\pi/4$. Using twice the partition $[\pi/6, \pi/4]$, $[\pi/12, \pi/6]$, $[\pi/36, \pi/12]$, $[0, \pi/36]$ (the same partition used for F) we get

$$\Delta V = \sum_{i=1}^{8} \sqrt{di}\left(\varphi\left(\frac{g_i}{di}, \theta_{i+1}\right) - \varphi\left(\frac{g_i}{di}, \theta_i\right)\right),$$

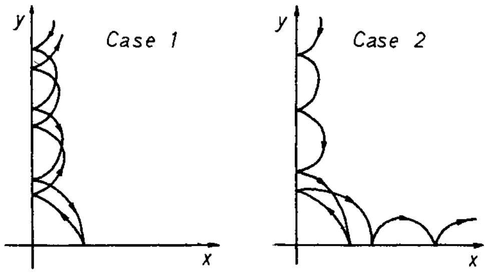

Fig. 5.

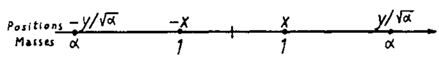

Fig. 6.

where in g_i the value V is taken as $\sum_{j=1}^{i-1}$. The values of θ_i are $\pi/4$, $\pi/6$, $\pi/12$, etc. The evaluation of the required inverse trigonometric and hyperbolic functions gives $\Delta V = 3.856\,090\,805 < \sqrt{(4 + 8\sqrt{2})} = |V_c|$, proving that under G we reach $\theta = \pi/4$ at a point above A, as desired. In conclusion we proved the following result.

THEOREM 2.1. *The right side of the invariant unstable manifold of the lower equilibrium point A reaches the value $V = 0$ for $\theta \in (0, \pi/4)$.*

The conclusions regarding the remaining side of W_A^u can be obtained by symmetry. After a sequence of binary collisions (couples of simultaneous double collisions, of course) of types 1 and 2 (see Figure 5), slightly below or above the quadruple collision point A, the bodies escape as shown in Figure 5. A similar behavior is obtained for left hand side collisions.

4. The Collinear Case

Let $m_1 = m_2 = 1, m_3 = m_4 = \alpha$ be the masses of the four bodies and x, $-x$, $y/\sqrt{\alpha}$, $-y/\sqrt{\alpha}$ the coordinates (see Figure 6). We again write down the Lagrangian

$$L = \dot{x}^2 + \dot{y}^2 + \frac{1}{2x} + \frac{\alpha^{5/2}}{2y} + \frac{2\alpha^{3/2}}{y - x\sqrt{\alpha}} + \frac{2\alpha^{3/2}}{y + x\sqrt{\alpha}}$$

and the corresponding Hamiltonian, setting, $p_x = 2\dot{x}, p_y = 2\dot{y}$,

$$H = \frac{p_x^2}{4} + \frac{p_y^2}{4} - \frac{1}{2x} - \frac{\alpha^{5/2}}{2y} - \frac{2\alpha^{3/2}}{y - x\sqrt{\alpha}} - \frac{2\alpha^{3/2}}{y + x\sqrt{\alpha}} = \frac{p_x^2}{4} + \frac{p_y^2}{4} - U(x, y).$$

Introducing $\zeta = (2x^2 + 2y^2)^{1/2}$ and the same change of coordinates as in Section 2 we get again

$$X' = \frac{P_X}{2} - XV, \quad P'_X = -\frac{1}{2X^2} - \frac{2\alpha}{\left(\frac{Y}{\sqrt{\alpha}} + X\right)^2} + \frac{2\alpha}{\left(\frac{Y}{\sqrt{\alpha}} - X\right)^2} + \tfrac{1}{2} V P_X,$$

$$Y' = \frac{P_Y}{2} - YV, \quad P'_Y = -\frac{\alpha^{5/2}}{2Y^2} - \frac{2\alpha^{1/2}}{\left(\frac{Y}{\sqrt{\alpha}} - X\right)^2} - \frac{2\alpha^{1/2}}{\left(\frac{Y}{\sqrt{\alpha}} + X\right)^2} + \tfrac{1}{2} V P_Y,$$

where $V = XP_X + YP_Y$ as before. We get again $V' = U - \frac{1}{2}V^2$ on $\zeta = 0$.

Introducing $X = (1/\sqrt{2}) \cos \theta$, $Y = (1/\sqrt{2}) \sin \theta$, the equation $\theta' = \pm\sqrt{2V'}$ is obtained.

The equilibrium points are obtained in the following way: let $z = y/\sqrt{\alpha}$. From $\ddot{x}/x = \ddot{z}/z$ and letting $z = \mu x$ we get

$$\alpha = \frac{\mu^3(\mu^2 - 1)^2 - 8\mu^2(\mu^2 + 1)}{17\mu^4 - 2\mu^2 + 1}.$$

When α ranges from 0 to ∞, the parameter μ ranges between μ_0 and ∞, where μ_0 is the zero of $\mu(\mu^2 - 1)^2 = 8(\mu^2 + 1)$ (approximately $\mu_0 = 2.396\,812\,289$). The minimum value of θ is given by $\theta_0 = \operatorname{arctg}\sqrt{\alpha}$ and the critical one by $\theta_c = \operatorname{arctg}(\mu\sqrt{\alpha})$.

In order to study the connection of the invariant manifolds starting at points $(\theta_c, \pm\sqrt{2U(\theta_c)})$, we introduce a new change of coordinates (only useful for this purpose). Let

$$a = \frac{\pi/2 - \theta_0}{2}, \quad b = \frac{\pi/2 + \theta_0}{2} \quad \text{and} \quad \theta = b + a \sin \gamma.$$

Then we get

$$\frac{dV}{d\gamma} = \pm\frac{a}{\sqrt{2}}\sqrt{V' \cos^2 \gamma}$$

and

$$V' \cos^2 \gamma = \left(\frac{\alpha^{5/2}}{\sqrt{2}\sin\theta} + \frac{(2\alpha)^{3/2} \cos \theta_0}{\sin(\theta + \theta_0)} - \frac{V^2}{2}\right) \cos^2 \gamma + $$
$$+ \frac{1/\sqrt{2}\cos^2 \gamma}{\sin(a(1 - \sin \gamma))} + \frac{(2\alpha)^{3/2} \cos \theta_0 \cos^2 \gamma}{\sin(a(1 + \sin \gamma))}.$$

The term

$$\frac{\cos^2 \gamma}{\sin(a(1 - \sin \gamma))}$$

(and the one with the $+$ sign in a similar way) has an avoidable singularity. If $\gamma = \pi/2 + \varepsilon$, for instance, we merely write

$$\frac{\cos^2 \gamma}{\sin(a(1 - \sin\gamma))} = \frac{4\cos^2 \varepsilon/2}{\dfrac{\sin(2a\psi)}{\psi}},$$

(where $\psi = \sin^2 \varepsilon/2$) and compute $(\sin(2a\psi))/\psi$ as $2a - \frac{4}{3}a^3\psi^2 + \frac{4}{15}a^5\psi^4 - \frac{8}{315}a^7\psi^6 + + \dots$ The computation must be started with $\gamma = \gamma_c = \arcsin((\theta_c - b)/a) \in \in (-\pi/2, \pi/2)$, $V_c = -\sqrt{(2U(\theta_c))}$. In $\mathrm{d}V/\mathrm{d}\gamma$ the $+$ sign is used for the unstable manifold (right branch) and the $-$ sign (with γ decreasing) for the left branch.

5. Numerical Computations for the Collinear Case

Using the equation numerically regularized as described in Section 4 we have computed the point $\gamma_+(\gamma_-)$, where the right (left) branch of the unstable manifold of the lower equilibrium point reaches the value $V = 0$.

The independent parameter has been the parameter μ. Table I shows some results. Figure 7 offers a rough representation of $\gamma_\pm$ as a function of α, including the region of small values of α. The computations have been done using a RK routine of fourth order with a step equal to 0.02. Some errors can be introduced for this value of the step for large values of α.

In order to study possible motions on the total collision manifold as a function of α we need the connections between the equilibrium points. For $\gamma_+ = (2k - 1)\pi/2$, $k \in \mathbb{N}$ or $-\gamma_- = (2k + 1)\pi/2$, one of the branches of W^u_A coincides with one of the branches of W^s_0. For $\gamma_+ - \gamma_- = 2k\pi, k \in \mathbb{N}$, both branches meet on account of

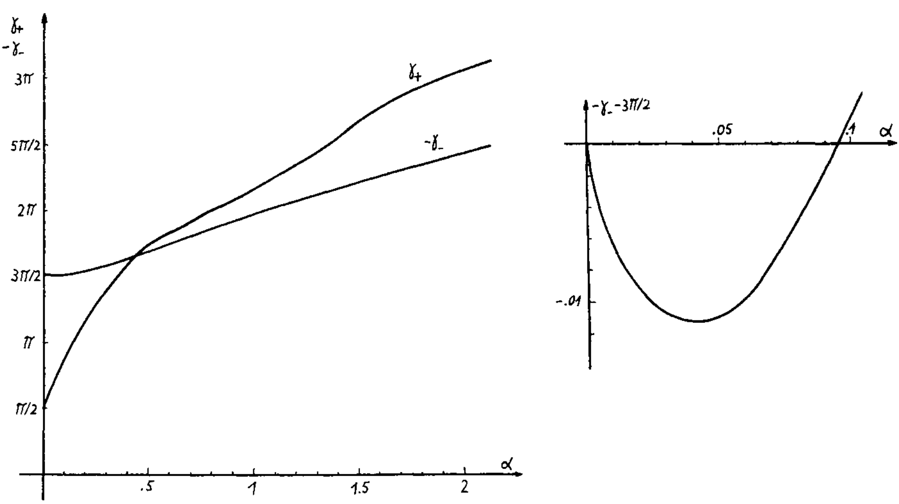

Fig. 7.

TABLE I

μ	α	γ_c	V_c	γ_+	γ_-	μ	α	γ_c	V_c	γ_+	γ_-
2.396 82	0.000 008	− 1.471 8	− 1.189 2	1.571 2	− 4.712 0	3.3	1.242 54	0.277 5	− 5.078 7	7.464 0	− 6.564 5
2.396 85	0.000 036	− 1.424 4	− 1.189 4	1.572 9	− 4.711 9	3.4	1.431 95	0.312 6	− 5.599 3	8.195 7	− 6.833 9
2.396 9	0.000 086	− 1.388 6	− 1.189 6	1.575 2	− 4.711 8	3.5	1.633 14	0.344 2	− 6.1455	8.792 9	− 7.105 7
2.397	0.000 183	− 1.350 1	− 1.190 0	1.578 9	− 4.711 6	3.6	1.846 50	0.372 9	− 6.719 1	9.246 5	− 7.384 1
2.398	0.001 155	− 1.217 9	− 1.194 0	1.604 1	− 4.710 7	3.7	2.072 39	0.399 2	− 7.321 9	9.642 4	− 7.671 7
2.400	0.003 103	− 1.115 3	− 1.201 9	1.641 6	− 4.709 3	3.8	2.311 19	0.423 3	− 7.955 6	10.016	− 7.969 0
2.402	0.005 055	− 1.053 4	− 1.209 9	1.673 3	− 4.708 2	3.9	2.563 26	0.445 7	− 8.622 0	10.385	− 8.274 1
2.406	0.008 966	− 0.969 2	− 1.225 9	1.729 7	− 4.706 4	4.0	2.828 98	0.466 6	− 9.322 9	10.761	− 8.584 5
2.412	0.014 856	− 0.883 3	− 1.249 8	1.805 4	− 4.704 4	4.5	4.374 92	0.553 1	− 13.410	12.715	− 10.578
2.420	0.022 752	− 0.801 1	− 1.281 5	1.897 7	− 4.702 6	5.0	6.316 19	0.619 5	− 18.648	15.986	− 13.400
2.430	0.032 692	− 0.723 7	− 1.321 2	2.005 5	− 4.701 3	5.5	8.697 52	0.673 0	− 25.294	18.383	− 15.688
2.438	0.040 698	− 0.673 6	− 1.352 8	2.088 2	− 4.701 1	6.0	11.563 4	0.717 6	− 33.628	22.057	− 19.281
2.45	0.052 802	− 0.610 6	− 1.400 2	2.208 7	− 4.701 8	6.5	14.958 2	0.755 6	− 43.946	24.966	− 22.134
2.46	0.062 974	− 0.566 0	− 1.439 6	2.307 0	− 4.703 4	7.0	18.926 1	0.788 7	− 56.564	28.993	− 26.150
2.47	0.073 226	− 0.526 6	− 1.478 9	2.404 3	− 4.705 7	7.5	23.511 4	0.817 8	− 71.813	32.716	− 30.459
2.48	0.083 555	− 0.491 3	− 1.518 2	2.500 8	− 4.708 9	8.0	28.758 3	0.843 8	− 90.042	37.139	− 34.200
2.49	0.093 965	− 0.459 2	− 1.557 5	2.596 9	− 4.712 8	8.5	34.710 9	0.867 2	− 111.616	41.624	− 38.698
2.5	0.104 456	− 0.429 7	− 1.596 7	2.693 0	− 4.717 4	9.0	41.413 4	0.888 3	− 136.917	46.709	− 43.730
2.6	0.213 869	− 0.221 5	− 1.989 6	3.646 3	− 4.802 8	9.5	48.910 0	0.907 7	− 166.343	50.925	− 47.973
2.7	0.331 810	− 0.090 4	− 2.387 7	4.529 9	− 4.956 4	10.0	57.244 6	0.925 4	− 200.308	56.344	− 53.323
2.8	0.458 723	0.005 1	− 2.795 5	5.195 6	− 5.174 6	11.99	99.594 8	0.983 8	− 389.219	78.643	− 75.591
2.9	0.595 040	0.080 0	− 3.216 3	5.687 8	− 5.441 5	12.00	99.847 6	0.984 1	− 390.416	78.726	− 75.671
3.0	0.741 177	0.141 4	− 3.652 8	6.107 2	− 5.728 9	12.01	100.100 8	0.984 3	− 391.616	78.808	− 75.755
3.1	0.897 541	0.193 3	− 4.107 4	6.508 0	− 6.015 0	20.0	467.9	1.119 6	− 2 610.1	191.0	− 187.9
3.2	1.064 53	0.238 1	− 4.582 1	6.934 5	− 6.293 2	30.0	1 584.4	1.203 4	− 11 917.9	366.9	− 363.8

TABLE II

k	α_k	Type	k	α_k	Type
1	0.092 97	$-\gamma_- = 3\pi/2$	16	10.323 0	$\gamma_+ = 13\pi/2$
2	0.361 53	$\gamma_+ = 3\pi/2$	17	12.868 8	$-\gamma_- = 13\pi/2$
3	0.907 88	$\gamma_+ - \gamma_- = 4\pi$	18	13.088 0	$\gamma_+ - \gamma_- = 14\pi$
4	1.345 2	$\gamma_+ = 5\pi/2$	19	13.303 5	$\gamma_+ = 15\pi/2$
5	2.218 1	$-\gamma_- = 5\pi/2$	20	16.107 2	$-\gamma_- = 15\pi/2$
6	2.636 2	$\gamma_+ - \gamma_- = 6\pi$	21	16.310 5	$\gamma_+ - \gamma_- = 16\pi$
7	2.998 6	$\gamma_+ = 7\pi/2$	22	16.511 5	$\gamma_+ = 17\pi/2$
8	4.498 4	$-\gamma_- = 7\pi/2$	23	19.557 2	$-\gamma_- = 17\pi/2$
9	4.821 0	$\gamma_+ - \gamma_- = 8\pi$	24	19.748 4	$\gamma_+ - \gamma_- = 18\pi$
10	5.122 9	$\gamma_+ = 9\pi/2$	25	19.937 9	$\gamma_+ = 19\pi/2$
11	7.051 5	$-\gamma_- = 9\pi/2$	26	23.215	$-\gamma_- = 19\pi/2$
12	7.323 7	$\gamma_+ - \gamma_- = 10\pi$	27	23.397	$\gamma_+ - \gamma_- = 20\pi$
13	7.585 9	$\gamma_+ = 11\pi/2$	28	23.578	$\gamma_+ = 21\pi/2$
14	9.846 9	$-\gamma_- = 11\pi/2$	29	27.080	$-\gamma_- = 21\pi/2$
15	10.087 8	$\gamma_+ - \gamma_- = 12\pi$	30	27.254	$\gamma_+ - \gamma_- = 22\pi$
			31	27.427	$\gamma_+ = 23\pi/2$

symmetry. Table II offers some values of α for which such connections are established.

6. Analytical Study of the Limiting Cases

We study the behavior of γ_+, γ_- and, incidentally, γ_c, V_c for $\alpha \downarrow 0, \alpha \uparrow \infty$.

For $\alpha = 0$ we have $\theta_0 = \theta_c = 0$, $V_c = -2^{1/4}$, $a = b = \pi/4$, $\gamma_c = -\pi/2$. The differential equation is $dV/d\theta = \frac{1}{2}\sqrt{(\sqrt{2}\sec\theta - V^2)}$, and scaling $V = 2^{1/4}\bar{V}$, we get $\bar{V}^2 + (d\bar{V}/d\theta)^2 \cdot 4 = \sec\theta$ with $\bar{V} = -1$ for $\theta = 0$.

LEMMA. *The solution of* $\bar{V}^2 + 4(d\bar{V}/d\theta)^2 = \sec\theta$ *such that* $\bar{V}(0) = -1$ *reaches* $\bar{V} = 0$ *for* $\theta = \pi/2$.

Proof. It is enough to check that the solution is given by $\bar{V}(\theta) = -\sqrt{\cos\theta}$.

COROLLARY. *For* $\alpha = 0$ *we have* $\gamma_+ = \pi/2$, $-\gamma_- = 3\pi/2$.

Now we study what happens with $\alpha > 0$ sufficiently small. First of all we have, approximately, $\theta_0 = \sqrt{\alpha}$, $\theta_c = \mu_0\sqrt{\alpha}$. Therefore

$$U(\theta_c) = \frac{1}{\sqrt{2}} + \alpha\left(\frac{1}{\sqrt{2}}\frac{\mu_0^2}{2} + \frac{2^{3/2}}{\mu_0 + 1} + \frac{2^{3/2}}{\mu_0 - 1}\right)\alpha + O(\alpha^2)$$

and

$$V_c = -\sqrt[4]{2} - \sqrt[4]{2}\left(\frac{\mu_0^2}{4} + \frac{2}{\mu_0 + 1} + \frac{2}{\mu_0 - 1}\right)\alpha + O(\alpha^2)$$

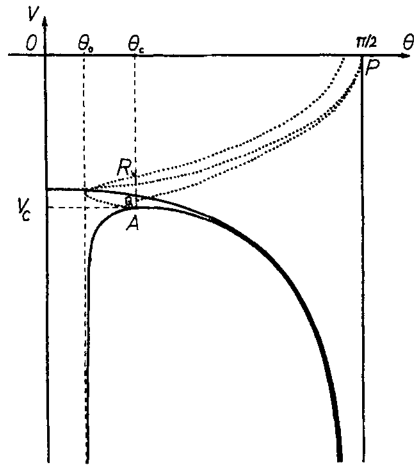

Fig. 8.

In order to check the behavior observed in Section 5 we have to prove two things: $\gamma_+ > \pi/2$, $-\gamma_- < 3\pi/2$.

We start at $P(V = 0, \theta = \pi/2)$ and follow the differential equation

$$\frac{\mathrm{d}V}{\mathrm{d}\theta} = \sqrt{\frac{U(\theta)}{2} - \frac{V^2}{4}}$$

backwards.

Writing down $V = -2^{1/4}\sqrt{\cos\theta} + w$, $w(\pi/2) = 0$ and retaining first order terms we get

$$\frac{\mathrm{d}w}{\mathrm{d}\theta} = \frac{w}{2^{5/4}}\frac{\cos\theta}{\sin\theta} + \frac{2\alpha^{3/2}\cos^{1/2}\theta}{\sin\theta}\left(\frac{1}{\sin(\theta+\theta_0)} + \frac{1}{\sin(\theta-\theta_0)}\right).$$

The solution of the homogeneous equation is $w = c(\sin\theta)^{-1/2^{5/4}}$ and the method of variation of the constants gives us

$$\frac{\mathrm{d}c}{\mathrm{d}\theta} = \frac{2\alpha^{3/2}\cos^{1/2}\theta}{(\sin\theta)^{(1-1/2^{5/4})}}\left(\frac{1}{\sin(\theta+\theta_0)} + \frac{1}{\sin(\theta-\theta_0)}\right).$$

Therefore $w(\theta_c) = -(\mu_0\sqrt{\alpha})^{-1/2^{5/4}}\Delta c$, where $\Delta c = \int_{\theta_c}^{\pi/2} \mathrm{d}c/\mathrm{d}\theta$. The value Δc can be estimated in the following way $\int_{\theta}^{\pi/2} = \int_{\theta_c}^{z} + \int_{z}^{\pi/2}$, where z is a small but finite quantity and so

$$\int_{z}^{\pi/2} = O(\alpha^{3/2}).$$

It remains to compute the main contribution $\int_{\theta_c}^{z}$. We bind $\cos^{1/2}\theta$ by 1, put

$$\frac{1}{\sin(\theta+\theta_0)}+\frac{1}{\sin(\theta-\theta_0)}<\frac{2\sin\theta}{\sin(\theta+\theta_0)\sin(\theta-\theta_0)}$$

and approximate the sines by the angles. We then get

$$\Delta c\simeq\int_{\theta_c}^{z}\frac{4\alpha^{3/2}\theta^{1/2^{5/4}}}{\theta^2-\alpha}<4\alpha^{3/2}\frac{\mu_0^2}{\mu_0^2-1}\int_{\theta_c}^{z}\theta^{-2+1/2^{5/4}}\simeq$$

$$\simeq\frac{4\alpha^{3/2}\mu_0^2}{(1-1/2^{5/4})(\mu_0^2-1)}(\mu_0\sqrt{\alpha})^{-1+1/2^{5/4}}.$$

Then

$$w(\theta_c)\simeq-\frac{4\alpha\mu_0}{(1-1/2^{5/4})(\mu_0^2-1)}.$$

As

$$\frac{4\mu_0}{(1-1/2^{5/4})(\mu_0^2-1)}<2^{1/4}\left(\frac{\mu_0^2}{4}+\frac{2}{\mu_0+1}+\frac{2}{\mu_0-1}\right)$$

the point Q (Figure 8) is above the equilibrium point (θ_c, V_c), showing that $\gamma_+>\pi/2$.

Now let us look for point R (see Figure 8). The first order terms for γ_c give us

$$\gamma_c=-\frac{\pi}{2}+\sqrt{\frac{8(\mu_0-1)}{\pi}}\alpha^{1/4}.$$

On the other hand the main term in $\mathrm{d}V/\mathrm{d}\gamma$ is $-\sqrt{(\pi/4\sqrt{2})}$, near the left hand side collision. Therefore the value of ΔV from the point (θ_c, V_c) to R is $2\sqrt{(\pi/4\sqrt{2})}\times\times(\pi/2+\gamma_c)=\sqrt{(8(\mu_0-1)/\sqrt{2})}\alpha^{1/4}$, showing that $-\gamma_-<3\pi/2$. We now have proved the following result.

PROPOSITION 6.1 *For α small enough $\gamma_+>\pi/2$, $-\gamma_-<3\pi/2$.*

For large α we have

$$\theta_0\simeq\frac{\pi}{2}-\frac{1}{\sqrt{\alpha}},\quad \theta_c\simeq\frac{\pi}{2}-\frac{1}{\sqrt{\alpha\sqrt[3]{17\alpha}}}.$$

Introducing $\bar{V}=V/\alpha^{5/4}$ and retaining the dominant term in the differential equation we get

$$V_c\simeq-\alpha\sqrt[4]{2\alpha},\quad \frac{\mathrm{d}\bar{V}}{\mathrm{d}\gamma}=\frac{a}{\sqrt{2}}\sqrt{1-\frac{\bar{V}^2}{\sqrt{2}}}|\cos\gamma|,$$

where $a = 1/(2\sqrt{\alpha})$. Then

$$\int_{-\sqrt[4]{2}}^{0} \frac{\mathrm{d}\bar{V}}{\sqrt{1-\bar{V}^2/2}} = \frac{a}{\sqrt{2}} T \frac{2}{\pi},$$

where T is the γ interval and $2/\pi$ is the average value of $|\cos\gamma|$. We immediately get $T = 2^{-1/4}\pi^2\alpha^{1/2}$. We then state the result, which shows good agreement with the results in Table I.

PROPOSITION 6.2. *For α sufficiently large $\gamma_+ \simeq \pi^2\, 2^{-1/4}\,\alpha^{1/2}$ and $\gamma_+ + \gamma_- \to \pi$.*

COROLLARY 6.3. *There are infinite values for which the left-hand branch of W_A^u coincides with the left-hand branch of W_D^s and for which the right-hand one coincides with the right-hand one and for which $W_A^u \equiv W_D^s$. In the last case the left-hand branch of W_A^u coincides with the right-hand one of W_D^s and vice versa.*

The first-mentioned values for which these coincidences are obtained were given in Table II.

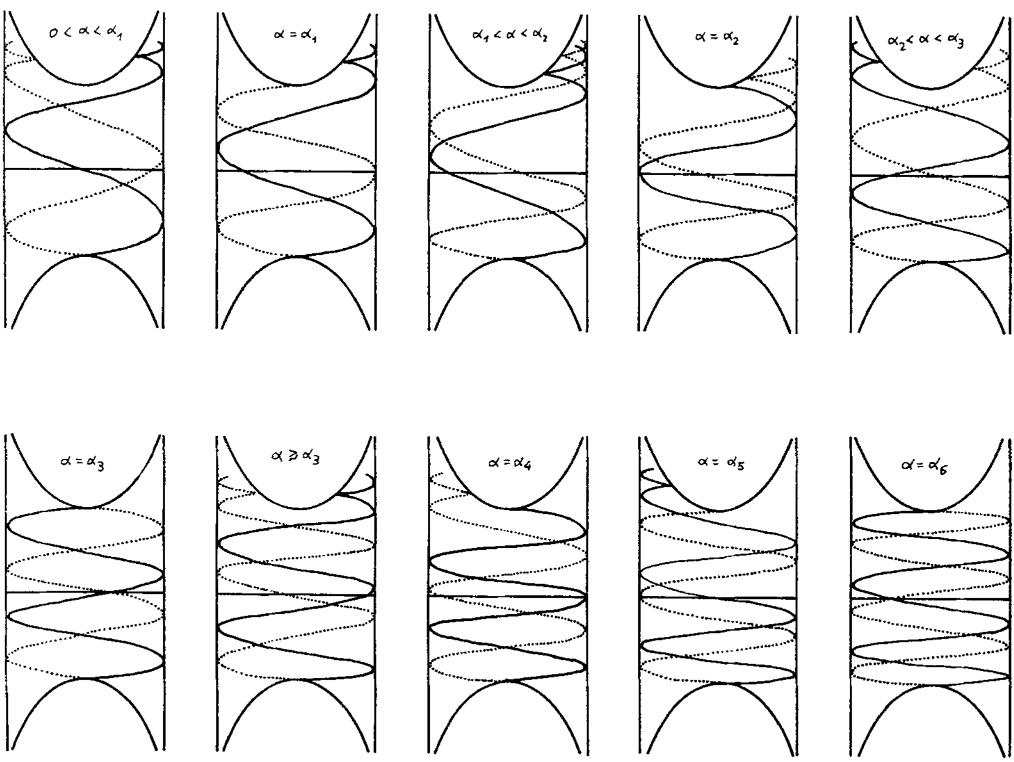

Fig. 9.

7. Some Dynamical Consequences

Let α_1 be the unique value $\alpha > 0$ such that $-\gamma_-(\alpha_1) = 3\pi/2$, α_2 such that $\gamma_+(\alpha_2) = 3\pi/2$, α_3 such that $\gamma_+(\alpha_3) - \gamma_-(\alpha_3) = 4\pi$, etc. Figure 9 shows a qualitative picture of the invariant manifolds of the lower equilibrium point for an initial range of values of α containing those values $(0 < \alpha_1 < \alpha_2 < \alpha_3)$.

The consequences with respect to orbits passing near quadruple collision are now easily obtained in the same way as they where obtained for the rectangular case (see orbits type 1, 2 in Figure 3). We recall that other necessary conditions for regularization can be obtained (for the good values of α, i.e., such that $W_A^u \equiv W_D^s$) in the way introduced in [7]. Sufficient conditions will be given in a forthcoming paper [9].

The way of escaping after approaching a quadruple collision and the number of collisions taking place between central bodies or simultaneous double collisions between external bodies can be predicted from Figure 9.

Picture similar to Figure 9 can be given for the full range of values of α. (Note that, according to Table II, there is α_4 similar to α_2, α_5 similar to α_1 and α_k similar to α_{k-3} for all $k \geqslant 6$).

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