CENTRAL CONFIGURATIONS OF FOUR BODIES WITH ONE INFERIOR MASS*

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Abstract. The number of equivalence classes of central configurations (abbr. c.c.) in the planar 4-body problem with three arbitrary and a fourth small mass is investigated. These c.c. are derived according to their generic origin in the 3-body problem. It is shown that each 3-body collinear c.c. generates exactly 2 non-collinear c.c. (besides 4 collinear ones) of 4 bodies with small $m_4 \geqslant 0$; and that any 3-body equilateral triangle c.c. generates exactly 8 or 9 or 10 (depending on m_1, m_2, m_3) planar 4-body c.c. with $m_4 = 0$. Further, every one of these c.c. can be continued uniquely to sufficiently small $m_4 > 0$ except when there are just 9; then exactly one of them is degenerate, and we conjecture that it is not continuable to $m_4 > 0$.

1. Introduction

Consider N mass points with masses $m_k > 0$ and position vectors q_k, $(k = 1, \ldots, N)$ in the euclidean plane E^2. They form a central configuration $\{q_1, \ldots, q_N; m_1, \ldots, m_N\}$, or abbreviated: a c.c. $\{q; m\}_N$, belonging to the masses m_k, if the vectors q_k satisfy the system of algebraic equations

$$f_1 = \cdots = f_N = 0, \quad \text{where} \quad f_k := \sum_{\substack{j=1 \\ j \neq k}}^{N} m_j r_{jk}^{-3}(q_k - q_j) - q_k, \, r_{jk} := |q_j - q_k| > 0. \tag{1}$$

These configurations describe the ultimate geometry of motion near a collision singularity or for expansion to infinity in the N-body problem, and they furnish homographic solutions of this problem through

$$z_k(t) = q_k z(t), (k = 1, \ldots, N), \quad \text{where } z = z(t) \text{ satisfies } \ddot{z} = -cz|z|^{-3};$$

i.e. $z(t)$ is any solution of a Kepler-problem ($c > 0$) and q_k and z are considered as complex numbers. When $N = 3$ and m_1, m_2, m_3 arbitrary, all solutions (q_1, q_2, q_3) of (1) are known: They are the 3 collinear configurations of Euler and the 2 equilateral triangle configurations of Lagrange. For $N \geqslant 4$ and given masses the solutions $(q_1, \ldots, q_N)$ of (1) are not yet all known, nor have their equivalence classes been completely enumerated. We shall make here a small contribution to the latter task when $N = 4$, which we assume from now on. Then (1) constitutes 8 scalar equations for 8 unknowns (the vector components of the q_k) with 4 parameters m_k. At first we derive an algebraic reduction of (1) to 3 equations in 3 unknowns, and then we reduce the problem to the case $m_4 = 0$. We can here only outline the ideas and techniques being used and refer for a more detailed treatment and complete proofs to a forthcoming paper in *Celestial Mechanics*.

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2. Algebraic Reduction

Applying in (1) the scaling (and then dropping the primes)

$$m_k = m_k' m,\ q_k = q_k' m^{1/3},\ m := \sum_1^N m_k, \quad \text{yields} \quad \sum_1^4 m_k = 1. \tag{2}$$

Now (1) with $N = 4$ implies

$$\sum_1^4 m_k q_k = -\sum_1^4 m_k f_k = 0, \text{ and with } z_k := q_k - q_4 \text{ then } q_k = z_k - \sum_1^3 m_j z_j. \tag{3}$$

Setting

$$R_{jk} := m_j m_k (r_{jk}^{-3} - 1),\ (j \neq k),\ R_k := \sum_{\substack{j=1 \\ j \neq k}}^{4} R_{jk},\ (k = 1, \ldots, 4) \tag{4}$$

yields

$$m_k f_k = \sum_{\substack{j=1 \\ j \neq k}}^{4} R_{jk}(z_k - z_j) = R_k z_k - \sum_{\substack{j=1 \\ j \neq k}}^{3} R_{jk} z_j,\ (z_4 = 0,\ r_{jk} = |z_j - z_k|)$$

and it becomes visible that (1) is equivalent to $Mz = 0$, where

$$M := \begin{pmatrix} R_1 & -R_{12} & -R_{13} \\ -R_{21} & R_2 & -R_{23} \\ -R_{31} & -R_{32} & R_3 \end{pmatrix},\ z := \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix};\ \text{since} \begin{pmatrix} m_1 f_1 \\ m_2 f_2 \\ m_3 f_3 \end{pmatrix} = Mz.$$

Since $|z_k| = r_{k4} > 0$; i.e. $z \neq 0$ in $\mathscr{C}^3$ (considering q_k, z_k as complex numbers), det $M = 0$. Hence rank $M \leqslant 2$. We can show that rank $M = 2$ implies z_1, z_2, z_3 are parallel, and thus the endpoints of the position vectors $q_1, \ldots, q_4$ from the origin are collinear. Clearly rank $M > 0$, since otherwise all $R_{jk} = 0$; i.e. all $r_{jk} = 1$, $(1 \leqslant j < k \leqslant 4)$, which is geometrically impossible. Hence

$$\text{rank } M = 1 \text{ on every non-collinear c.c. } \{q; m\}_4 \text{ in } E^2 = \mathscr{C}.$$

In this case every 2-by-2 subdeterminant of M vanishes, which yields

$$R_{12}R_{34} = R_{13}R_{24} = R_{14}R_{23} =: R^*,\ R_4 R^* = -R_{14}R_{24}R_{34},\ R_j R_4 = R_{j4}^2; \tag{5}$$

and it can be shown that $R^* \neq 0$ on any non-collinear c.c. Then

$$R_4 R_{ij} = -R_{i4}R_{j4} \neq 0,\ (1 \leqslant i < j \leqslant 3),\ \sum_{k=1}^{3} R_{k4} z_k = 0 \tag{6}$$

the last from $Mz = 0$. Conversely, we can show that (6) implies rank $M = 1$ and $Mz = 0$, and that the resulting q_k given in (3) determine a non-collinear c.c. $\{q; m\}_4$.

The proof of the latter fact is difficult. (6) represents 5 scalar equations for 6 unknowns (the real and imaginary parts of z_1, z_2, z_3), but we may choose $Imz_3 = 0$, which amounts to a rotation of the c.c.

We shall consider only non-collinear c.c.; i.e. (6), and achieve for this case a further reduction by introducing barycentric coordinates for q_4 with respect to q_1, q_2, q_3; i.e. if $q_1 - q_3$ and $q_2 - q_3$ are linearly independent over R, we write

$$q_4 = \sum_1^3 b_j q_j, \sum_1^3 b_j = 1; (q_4 \Rightarrow b_1, b_2, b_3 \text{ uniquely}). \tag{7}$$

Indeed the required independence follows from the last equation in (6) with $z_k = q_k - q_4$, since $\{q; m\}_4$ is non-collinear. That same equation and (7) then imply

$$b_k = R_{k4}/R_4, (k = 1, 2, 3), R_4 \neq 0. \tag{8}$$

Setting

$$\rho_k := r_{ij} \text{ for } (i, j, k) = (1, 2, 3), (2, 3, 1), (3, 1, 2); \; r_k := r_{k4}, \quad (k = 1, 2, 3) \tag{9}$$

we derive using (7)

$$r_k^2 = b_i^2 \rho_j^2 + b_j^2 \rho_i^2 + b_i b_j (\rho_i^2 + \rho_j^2 - \rho_k^2), \;\; (k = 1, 2, 3; i, j \text{ as before}). \tag{10}$$

Considering this as a system of linear equations for $\rho_1^2, \rho_2^2, \rho_3^2$ we obtain

$$\rho_k^2 = b_i^{-1} b_j^{-1} [(b_i + b_j) b_i r_i^2 + (b_i + b_j) b_j r_j^2 - b_k^2 r_k^2], \quad (k = 1, 2, 3; \text{etc.}). \tag{11}$$

Setting

$$m_4 F_k := R_4 R_{ij} + R_{i4} R_{j4}, S_k := r_k^{-3} - 1, \quad (k = 1, 2, 3; \text{etc.}) \tag{12}$$

we obtain from (6), (4) and (9) the conditions

$$F_1 = F_2 = F_3 = 0 \text{ with } F_k = (\rho_k^{-3} - 1) \sum_1^3 m_l S_l + m_4 S_i S_j \tag{13}$$

and

$$\rho_k^2 = r_i^2 (1 + m_i S_i / m_j S_j) + r_j^2 (1 + m_j S_j / m_i S_i) - r_k^2 m_k^2 S_k^2 / m_i m_j S_i S_j \tag{14}$$

by (11) and (8). Now (13) constitutes 3 equations in the 3 unknowns r_1, r_2, r_3. The steps leading from (6) to (13) can actually be reversed, and thus any solution (r_1, r_2, r_3) of (13) with $r_k > 0, 0 < \rho_k < \rho_i + \rho_j$ and $r_k \neq 1$ $(k = 1, 2, 3)$ so that all $R_{ij} \neq 0$, yields a non-collinear c.c. $\{q; m\}_4$ and every such c.c. can be so obtained.

To find solutions of (13) with given masses satisfying (2) we first determine all solutions of (13) when $m_4 = 0$ and then try to continue them to small positive m_4 by the implicit function theorem. Therefore it is required that the associated Jacobian functional determinant

$$D_3 = \frac{\partial(F_1, F_2, F_3)}{\partial(r_1, r_2, r_3)} \text{ at } \rho_1 = \rho_2 = \rho_3 = 1, m_4 = 0 \tag{15}$$

does not vanish on the considered solution. A tricky calculation using (14) and (12) yields

$$D_3 = -27R^3\frac{r_1r_2r_3}{b_1b_2b_3}\left[1 - 3\sum_1^3 b_k r_k^{-3} S_k^{-1} + \frac{27}{4}\frac{b_1b_2b_3}{S_1S_2S_3}(r_1r_2r_3)^{-5}\sum_1^3 b_k r_k^5 S_k\right] \tag{16}$$

with

$$r_k^2 = b_i^2 + b_j^2 + b_ib_j, b_k = m_kS_kR^{-1}, \quad (k = 1, 2, 3), \quad R := \sum_1^3 m_lS_l \neq 0 \tag{17}$$

by (10), (8) and (12). The fact that this calculation of D_3 becomes possible at all provides additonal motivation for our reduction of (1) to (13).

3. Central Configurations with $m_4 = 0$ and their Continuation

Again we consider only non-collinear c.c. and obtain for these from (13) as $m_4 \to 0$ the conditions

$$E: \sum_1^3 m_kS_k = 0, \quad \text{or} \quad L: \rho_1 = \rho_2 = \rho_3 = 1. \tag{18}$$

In the first (Eulerian) case $R = m_4^{-1}R_4 = 0$ and the use of barycentric coordinates is impossible. In fact we can show that in this case the endpoints of q_1, q_2, q_3 drawn from the origin lie on a straight line and thus $\{q;m\}_3$ is an Eulerian collinear c.c. Furthermore in this case, using (1) with $m_4 = 0$ we can show by direct calculations that there are exactly 2 solutions for q_4 such that $\{q, m\}_4$ with $m_4 = 0$ is non-collinear. Clearly these c.c. are symmetric to the Euler-configuration. Also, the relevant Jacobian determinant does not vanish, so that exactly two mirror symmetric non-collinear $\{q;m\}_4$ for small $m_4 > 0$ arise from each of the 3 collinear c.c. $\{q;m\}_3$. Palmore (1982) has shown (with a much more elegant proof than ours) that, more generally, every collinear c.c. $\{q;m\}_N$ for $N \geqslant 2$ generates for small $m_{N+1} > 0$ exactly 2 non-collinear c.c. $\{q;m\}_{N+1}$ (besides the $N+1$ collinear c.c. of Moulton).

In the second (Lagrangian) case q_1, q_2, q_3 point to the corners of an equilateral triangle with side 1, by (9) and (1). Thus $\{q;m\}_3$ is a Lagrangian c.c., and q_4 with (7) is to be determined so that (8), (10) and (18) case L, hold; i.e. so that (17) holds. This requires 2 equations for 2 barycentric coordinates, say, or

$$0 = Rq_4 - \sum_1^3 m_kS_kq_k = \sum_1^3 m_kS_kz_k = \sum_1^3 m_kr_k^{-3}z_k + q_4, r_k = |z_k|, \tag{19}$$

since

$$z_k = q_k - q_4, \sum_1^3 m_kq_k = 0, \sum_1^3 m_k = 1, m_4 = 0. \tag{20}$$

Setting

$$z := q_4 = x + iy,\ q_k = 3^{-1/2}e^{2\pi ik/3} - q,\ q = 3^{-1/2}\sum_1^3 m_k e^{2\pi ik/3} \tag{21}$$

then (19) is equivalent to

$$g_1 = g_2 = 0, \quad \text{where} \quad g_1 + ig_2 = g(z) := z + \sum_1^3 m_k r_k^{-3}(q_k - z). \tag{22}$$

This constitutes 2 equations for 2 unknowns, x and y, with 3 parameters m_1, m_2, m_3. These equations also follow from (1) with $m_4 = 0$ directly. Any solution z of (22) on which D_3 in (16) is not zero, generates a non-collinear c.c. $\{q; m\}_4$ for small $m_4 > 0$, as seen in Section 2.

We interprete m_1, m_2, m_3 as barycentric coordinates of a point m in the mass triangle $T_m: m_k > 0, m_1 + m_2 + m_3 = 1$. Any solution $z = z(\tilde{m})$ of (22) for given $m = \tilde{m}$ can be uniquely continued as a solution of (22) to nearby m, if the Jacobian determinant

$$\begin{aligned} D_2 := g_{1x}g_{2y} - g_{2x}g_{1y} = D_2(z, m) = R^2 - 3R\sum_1^3 m_k r_k^{-3} + \\ + \frac{27}{4}\sum_{k=1}^3 m_i m_j b_k^2 r_i^{-5} r_j^{-5} \end{aligned} \tag{23}$$

does not vanish at $z = z(\tilde{m}), m = \tilde{m}$. Now from (22) follows (19); i.e. (17) using (7), and especially $m_k = Rb_k S_k^{-1}$. Hence $D_2(z(m), m) = 0$, iff.

$$D^* := 1 - 3\sum_1^3 b_k S_k^{-1} r_k^{-3} + \frac{27}{4} b_1 b_2 b_3 \sum_{k=1}^3 b_k S_i^{-1} S_j^{-1} r_i^{-5} r_j^{-5} = 0. \tag{24}$$

Furthermore

$$m_k = S_i S_j b_k S^{-1},\ S := \sum_{k=1}^3 S_i S_j b_k;\ (S_k = r_k^{-3} - 1); \tag{25}$$

i.e. every solution $z = z(m)$ of (22) determines $m = m(z)$ in T_m uniquely. Now it turns out that the solutions (b_1, b_2, b_3) of (24) constitute a 1-dim. manifold on which q_4 of (7) describes a simple closed curve C^* contained entirely in the equilateral triangle T determined by q_1, q_2, q_3. The proof of this fact is complicated and is based on suitable local power series expansions of the related function

$$F(z) := \sum_1^3 b_k S_k^{-1} r_k^{-3} - \left|\sum_1^3 b_k S_k^{-1} r_k^{-3} e^{2i\psi_k}\right|,\ e^{i\psi_k} := z_k/r_k \tag{26}$$

near the midpoint and the corners of T, since (24) is equivalent to $F(z) = 2/3$.

By (25) the image $C = m(C^*)$ is a simple closed curve in T_m, which divides T_m into two connected regions, say U containing the midpoint of T_m and $V = T_m - U - C$. If m is at the midpoint; i.e. $m_1 = m_2 = m_3 = 1/3$, we can show that every solution z of (22) lies on one of the 3 straight lines $r_1 = r_2, r_2 = r_3, r_3 = r_1$, and that each of these lines carries exactly 4 solutions including $z = 0$, which they have in common.

Hence there are exactly 10 solutions of (22) for this m. Since $D^* \neq 0$ in U, it follows that for every m in U there exist exactly 10 solutions z of (22), and exactly 4 of these are in T. Further, since $D^* \neq 0$ in the connected region V, (22) has one and the same number of solutions z for all m in V. To determine these solutions let $m_2 = m_3$ and $m_1 > 0$ very small, so that m is in V. Then a detailed analysis of the algebraic curves C_{ij} in the z-plane with equations

$$m_i(r_i^{-3} - 1)b_j = m_j(r_j^{-3} - 1)b_i, \quad (i, j) = (1, 2), (2, 3), (3, 1) \tag{27}$$

by (17) and (7) with (21), reveals that these curves intersect in exactly 8 points z, of which 2 are in T. Hence it follows that (22) has for m on the singular curve C exactly 9 solutions z, one of those is on C^*, which is a manifold of double points. Summarizing, it follows the *theorem*: Any equilateral triangle c.c. $\{q\,; m\}_3$ generates exactly 8, 9 or 10 planar c.c. $\{q;m\}_4$ with $m_4 = 0$ depending on whether m in T_m lies outside, on or inside the curve C.

We now ask for the continuability of the corresponding solutions of (13) with $m_4 = 0$; i.e. of (18) L, to positive m_4. We choose differentiable functions $m_k(\mu) > 0$ such that

$$\sum_1^4 m_k(\mu) = 1, m_4(\mu) = \mu \geqslant 0 \quad \text{and} \quad \sum_1^3 \tilde{m}_j := 1, \tilde{m}_j := m_j(0) > 0 \tag{28}$$

and consider (13) and (14) with $m_k = m_k(\mu)$. From (16) and (24) follows the important fact that D^* divides D_3, hence $D_3 = 0$ on $\{q;m(0)\}_4$, iff. $q_4 = z = z(\tilde{m})$ is on the critical curve C^*. It follows that every $\{q\,; m\}_4$ with $m_4 = 0$ (given in the above theorem) and with q_4 not on C^*, can be continued uniquely to a c.c. $\{q, m\}_4$ for small $m_4 > 0$, by Section 2. However, if q_4 is on C^* (and thus $\tilde{m}$ on C), the continuability is in doubt. In this case one can expand the F_k in (13) into power series in $r_i - r_i^*$ $(i = 1, 2, 3)$ and μ about the corresponding solution $r_i = r_i^*$, $\mu = 0$ of (13) with (28). We find that the matrix $((\partial F_k/\partial r_i))^*$ at this solution has rank 2. Hence 2 of the equations in (13) can be solved by linear elimination and the resulting third equation can, with the help of the Weierstrass preparation theorem, be converted into a polynomial equation in one unknown (say $x = r_3 - r_3^*$) with coefficients depending on μ and of degree at least two, since $D_3^* = 0$. The explicit computation of this polynomial is quite laborious and has not yet been accomplished. We conjecture that its roots x for small $\mu > 0$ are not real, for all q_4 on C^*. If this is true, then we may conclude that every equilateral c.c. $\{q, \tilde{m}\}_3$ generates for sufficiently small $m_4 > 0$ either 8 or 10 plannar c.c. $\{q, m\}_4$ with (28), depending on whether m lies outside or inside (including on) C.

4. Concluding Remarks

By the foregoing there is a $\delta = \delta(m_1, m_2, m_3) > 0$ such that for any choice of $m_1, m_2, m_3 > 0$ and m_4 with $0 < m_4 < \delta$ the number of equivalence classes of c.c.

$\{q\,;m\}_4$ is either 34 or 38, namely $3(4+2)=18$ arising from Eulerian c.c. and, if our conjecture holds, 2(8 or 10) arising from Lagrangian c.c. For the case $m_4=0$ exhaustive numerical computations of the positions of m_4 in the Lagrangian case have been performed by Pedersen (1944). There also the singular curves C^* and C are being numerically exhibited. Further extensive numerical computations of $\{q\,;m\}_4$ have been carried out by Simó (1978), who traces their evolution as m moves from the midpoint of the mass tetrahedron to one of its faces (say $m_4=0$). The numerical findings are consistent with the analytical results described in this paper. The careful mathematical analysis and rigorous calculations required to prove these results are contained in the Ph.D. dissertation (1981) of my student J. R. Gannaway.

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