HOW CRITICAL IS THE CRITICAL INCLINATION?

ANTHONY G. LUBOWE
Bell Telephone Laboratories Inc., Whippany, N.J., U.S.A.

(Received 23 August, 1968)

Abstract. It has been claimed that the representation of satellite motion in the vicinity of the critical inclination is a matter of practical, as well as theoretical interest, since "the perturbations in the coordinates are of the order of 25 times greater near the critical inclination than away from it" (Message *et al.*, 1962). In this paper we show, using Encke's method of numerical integration for satellites which are at, near, and away from the critical inclination, that there are no discernible features in the coordinate perturbations which distinguish the critical inclination from any other.

1. Introduction

Since the advent of the space age the calculation of the orbits of artificial satellites about an oblate spheroid has been a problem of great interest, both theoretical and practical. Another problem, of some theoretical interest, is the representation of the satellite motion when the orbit plane is in the vicinity of the so-called 'critical' inclination, and, to first order, the motion of perigee vanishes. Some authors attach an intrinsic physical significance to the critical inclination, even going so far as to suggest that numerical integration would disclose its extraordinary character (Message *et al.*, 1962).

In the following discussion we show, using Encke's method of numerical integration for satellites which are at, near, and away from the critical inclination, that the critical inclination presents only a theoretical problem, since there are no discernible features in the coordinate perturbations which distinguish the critical inclination from any other.

2. Discussion

Some analytical treatments of the artificial satellite problem are not valid when the satellite inclination, i, is near the critical inclination, i.e., when i is a solution of the equation

$$5 \cos^2 i - 1 = 0 .$$

A partial list of such treatments includes Garfinkel (1959), Kozai (1959), and Brouwer (1959). A common feature of these methods which are not valid near the critical inclination, is the desire to obtain satellite elements at some arbitrary future time, given elements at an initial time, without obtaining elements at intermediate times. Thus the long-periodic components of the satellite perturbations must be obtained explicitly, and it is here that the difficulties arise, unless the critical inclination is treated as a special case (Garfinkel, 1960; Hori, 1960).

In orbit prediction for communication satellite applications, as discussed by Claus and Lubowe (1963), the requirement for high accuracy prediction exists but the requirement that there be no intermediate steps does not. Indeed, since one must predict all passes visible to all ground stations, an integration step of one satellite period is as desirable as an integration step of several weeks or months. A method which obtains the secular perturbations, to order J_2^2, for example (where J_2 is the coefficient of the second harmonic of the geopotential), and updates the orbital elements every period (using the secular second-order perturbations) will be as accurate as the above three treatments, since it can be shown that it obtains the long-periodic perturbations implicitly by the updating procedure. Also, it avoids any difficulties at the critical inclination. As an illustration, a 2000 period prediction (8 months) was made for a satellite orbit similar to that of TELSTAR I, except that the orbit was initially at the critical inclination, in the reference by Claus and Lubowe (1963), and the perturbations were in no way exceptional.

An interesting study by Kikuchi (1967) indicates that with proper treatment of the initial conditions even the mathematical methods which usually yield singular results around the critical inclination can be made uniformly valid. His heuristic explanation that then "the long-periodic term continuously transfers to the secular term" seems reminiscent of the way the method of Claus and Lubowe (1963) obtains the long-periodic perturbations by rectification of the secular perturbations.

However, the claim has been made (Message *et al.*, 1962) that the critical inclination problem is indeed a physical one since the coordinate perturbations may be 25 times greater near the critical inclination than away from it. They base this claim on an order of magnitude argument, stating that perturbations away from the critical inclination are of order J_2, while those near the critical inclination are of order $\sqrt{J_2}$ (Garfinkel, 1960) and the ratio of these quantities is about 25. This statement has been repeated in at least two survey articles (Brouwer, 1963, p. 230; Cook, 1963, p. 411).

Unfortunately, the statement is not correct. This is indicated by the result of Claus and Lubowe (1963) described above but even more directly by the results presented here. Message *et al.* (1962) suggest "Perhaps numerical integration in coordinates, including cases both near to, and far from, the critical case, could provide additional verification". We follow this suggestion in this paper.

3. Numerical Results

Encke's method of numerical integration for satellites acted upon by the 2nd (J_2) and 4th (J_4) harmonics of the geopotential was used on satellites initially at, near, and away from the critical inclination. (Specifically, values of $i=60.0°$, $63.0°$, $63.4°$, $\cos^{-1}(1/\sqrt{5})$, $63.5°$, $64.0°$, and $67.0°$, were used.)* Also, since there is even no theoretical difficulty when the eccentricity is zero or when $J_2^2+J_4=0$ (Garfinkel, 1959, 1960), these quantities were also treated as parameters. Values of $e=0.0$ and $e=0.9$, and $J_4=0$, $J_4=-J_2^2$, and $J_4=-1.532755259J_2^2$ (the Apollo mission values) were used. The perturbations in spherical coordinates were examined for all these cases. The

* Strictly speaking, these are osculating elements and the mean elements should be used. However the mean critical inclination is certainly well within the range of i considered.

TABLE I

Initial Conditions

Series I	$(J_2=0, J_4=0)$
Series II	$(J_2=1082.3\times10^{-6}, J_4=0)$
Series III	$(J_2=1082.3\times10^{-6}, J_4=-1.8\times10^{-6}\doteq -1.5J_2^2)$
	$a_0=4545.4546$ miles
	$e_0=0.0$
	$\omega_0=\Omega_0=\tau_0=0$
Series IV	$(J_2=0, J_4=0)$
Series V	$(J_2=1082.3\times10^{-6}, J_4=0)$
Series VI	$(J_2=1082.3\times10^{-6}, J_4=-1.17\times10^{-6}\doteq -J_2^2)$
	$a_0=41\,000.000$ miles
	$e_0=0.9$
	$\omega_0=\Omega_0=\tau_0=0$
Series I–VI:	Run 1: $i_0=60.0°$
	2 63.0°
	3 63.4°
	4 63.43...°
	5 63.5°
	6 64.0°
	7 67.0°

TABLE II

Perturbations for $e_0=0.0$

	$-\Delta r$(miles)	$\Delta\alpha(10^{-4}$ radians)	$\Delta\delta(10^{-4}$ radians)	$\Delta\tilde{\theta}(10^{-4}$ radians)
$t=1$ hour:				
$i=60°$	5.6	14.1	−46.3	35.8
63	5.3	10.4	−43.6	32.6
63.4	5.3	10.0	−43.2	32.2
63.43...	5.3	10.0	−43.2	32.2
63.5	5.3	10.0	−43.1	32.1
64	5.2	9.3	−42.7	31.6
67	4.9	6.4	−39.9	29.0
$t=2$ hours:				
$i=60°$	2.2	55.2	59.9	47.9
63	2.2	47.2	57.5	42.8
63.4	2.2	46.1	57.2	42.2
63.43...	2.2	46.0	57.2	42.1
63.5	2.2	45.8	57.1	42.0
64	2.2	44.5	56.7	41.2
67	2.2	36.9	54.0	37.1
$t=24$ hours:				
$i=60°$	1.1	361.3	970.3	683.8
63	1.2	274.5	912.5	609.7
63.4	1.2	263.4	904.2	600.7
63.43...	1.1	262.5	903.5	599.9
63.5	1.1	260.9	902.4	598.8
64	1.2	247.7	892.3	588.3
67	1.2	175.5	831.6	532.2

perturbations in the longitude in the orbit (the sum of the longitude of the node, the argument of perigee, and the true anomaly), were also examined.

Six series of numerical integrations, each consisting of 7 runs, were made. The initial conditions are tabulated in Table I. (For simplicity these are given in terms of orbital elements.) The numerical integration method used is known to produce errors of less than 0.1 mile in coordinates after 24 hours by comparison with exact conic routines and with double precision integration routines using Cowell's Method. (Thus the results in Tables II and III are correct to within at least 1 digit in the last figure given.)

TABLE III

Perturbations for $e_0 = 0.9$

	$-\Delta r$(miles)	$\Delta\alpha$(10^{-4} radians)	$\Delta\delta$(10^{-4} radians)	$\Delta\tilde{\theta}$(10^{-4} radians)
$t = 1$ hour:				
$i = 60°$	9.1	1.6	7.2	2.4
63	9.0	0.7	6.7	1.9
63.4	8.9	0.6	6.6	1.8
63.43...	8.9	0.6	6.6	1.8
63.5	8.9	0.5	6.6	1.8
64	8.9	0.4	6.5	1.7
67	8.8	−0.4	5.9	1.4
$t = 2$ hours:				
$i = 60°$	21.7	1.34	10.6	5.7
63	21.5	0.49	9.9	5.1
63.4	21.5	0.39	9.8	5.0
63.43...	21.5	0.38	9.8	5.0
63.5	21.5	0.36	9.8	5.0
64	21.5	0.34	9.6	4.9
67	21.4	−0.40	8.9	4.4
$t = 24$ hours:				
$i = 60°$	869.6	24.3	53.4	55.1
63	869.5	21.4	53.6	54.2
63.4	869.4	21.0	53.7	54.1
63.43...	869.4	21.0	53.7	54.1
63.5	869.4	20.9	53.7	54.1
64	869.4	20.4	53.7	54.0
67	869.2	17.7	53.8	53.3

The perturbations in radius (Δr), right ascension ($\Delta\alpha$), declination ($\Delta\delta$), and longitude in the orbit ($\Delta\tilde{\theta}$), are given at 1, 2, and 24 hours after the start of the integration in Tables II and III. Table II gives the perturbations for the runs with $e_0 = 0.0$ (Series I and II in the notation of Table I) and Table III gives the perturbations for the runs with $e_0 = 0.9$ (Series IV and V). The effect of different values of J_4 (Series III and VI) is not visible in the figures retained as significant in the tables. (Order of magnitude estimates, unencumbered by the critical inclination mystique, could have indicated this from the start.)

4. Conclusion

The numerical results presented here show that for zero or non-zero eccentricity and for zero or non-zero values of $J_2^2+J_4$ there are no discernible features in the perturbations in the coordinates of a satellite of an oblate spheroid which distinguish the so-called critical inclination from any other.

Acknowledgments

I am grateful to Dr. J. M. Gormally and Mrs. E. Meyers, my colleagues at the Bell Telephone Laboratories, for obtaining the numerical integration results described herein and to the reviewer, Dr. A. Deprit, of the Boeing Scientific Research Laboratories, for his comments on the manuscript.

References

Brouwer, D.: 1959, 'Solution of the Problem of Artificial Satellite Theory Without Drag', *Astronom. J.* **64**, 378–397.
Brouwer, D.: 1963, 'Review of Celestial Mechanics', *Ann. Rev. Astron. Astrophys.* **1**, 219–234.
Claus, A. J. and Lubowe, A. G.: 1963, 'A High Accuracy Perturbation Method with Direct Application to Communication Satellite Orbit Prediction', *Astronaut. Acta*, **IX**, 275–301.
Cook, A. H.: 1963, 'The Contribution of Observations of Satellites to the Determination of the Earth's Gravitational Potential', *Space Sc. Rev.* **2**, 355–437.
Garfinkel, B.: 1959, 'The Orbit of a Satellite of an Oblate Planet', *Astronom. J.* **64**, 353–367.
Garfinkel, B.: 1960, 'On the Motion of a Satellite in the Vicinity of the Critical Inclination', *Astronom. J.* **65**, 624–627.
Hori, G.: 1960, 'The Motion of an Artificial Satellite in the Vicinity of the Critical Inclination', *Astronom. J.* **65**, 291–300.
Kikuchi, S.: 1967, 'Commensurability and Critical Inclination', *Astron. Nachr.* **289**, 241–245.
Kozai, Y.: 1959, 'The Motion of a Close Earth Satellite', *Astronom. J.* **64**, 367–377.
Message, P. J., Hori, G., and Garfinkel, B.: 1962, 'On the Critical Case in the Motion of an Artificial Satellite', *Observatory* No. 929, pp. 168–170.