A NEW FAMILY OF PERIODIC ORBITS FOR THE RESTRICTED PROBLEM

E. A. BELBRUNO

Department of Mathematics, Boston University, 264 Bay State Rd., Boston, MA 02215, U.S.A.

(Received February, 1980; Accepted August, 1980)

Abstract. A new family of periodic orbits of the three-dimensional restricted three-body problem which continue off from a consecutive collision orbit are numerically studied. Their behavior for varying energy is unexpected. In particular, associated with our system is a countable set of resonant energy values and each time the energy passes through one of them the periodic orbit forms a loop by self-intersection. Any number of loops can form by this process and the resulting orbits take on an interesting appearance.

1. Introduction

(a) We consider the three-dimensional restricted three-body problem in a rotating coordinate system $q = (q_1, q_2, q_3)$ of rotational frequency equal to 1. In this frame we put the larger primary m_1 of mass $1 - \mu$ at the origin and the smaller primary m_2 of mass μ at the position $e_2 = (0, 1, 0)$ (see Figure 1).

The Hamiltonian which governs the motion of the zero mass particle m_3 is given by

$$H = \tfrac{1}{2}|p|^2 - |q|^{-1} + \omega(q_2 p_1 - p_2 q_1) + \mu G(p, q),$$
$$G = |q|^{-1} - \Delta^{-1} - p_1, \qquad \Delta = |q - e_2|, \tag{1}$$

where $p = (p_1, p_2, p_3)$ are the momentum variables conjugate to the q[†]. For $\mu = 0$ we consider the consecutive collision orbit $\phi^*(t) = (p^*(t), q^*(t))$ which oscillates on the positive q_3-axis. Let ω^* denote the frequency of ϕ^*, then the following result was proven by the author [1].

THEOREM 1. There exists a unique continuation $\phi(t, \mu)$ to $\phi^*(t)$ on $H = -h < 0$ for $|\mu| \neq 0$ sufficiently small of period $T(\mu) = 2\pi/\omega^* + \mathcal{O}(\mu)$ provided

$$\omega^* \neq \frac{1}{j}, \qquad j = 1, 2, 3, \ldots.$$

Moreover $\phi(t, \mu)$ is symmetric with respect to reflection in the $q_2 q_3$-plane in q-space.

The proof of this Theorem was accomplished by a Poincaré continuation argument, however this must be done in regularized coordinates since every time ϕ^* collides with m_1, $q = 0$ which is a singularity of the differential equations $\dot{q} = H_p$, $\dot{p} = -H_q$, $\dot{} \equiv \mathrm{d}/\mathrm{d}t$, defined by H. The map Φ that was used to regularize the flow at $q = 0$ is equivalent to a simple rational map and is valid in n-dimensions. It is also dimension preserving which is not the case, in particular, for the KS-map which

[†] $\omega = 1$

Celestial Mechanics **25** (1981) 195–217. 0008–8714/81/0252–0195$03.45.

increases dimension [1]. This increase of dimension in the KS-map is not advantageous for numerical work and causes unpleasant degeneracies to occur as far as the analysis is concerned [1]. The above theorem was also proven by Guillaume at an earlier time [3]. The regularized coordinates chosen in that proof were the KS-coordinates.

(b) The main goal in the following numerical investigation is to numerically reproduce the continued orbit $\phi(t, \mu)$ and to study the geometric behavior of $\phi(t, \mu)$ for different parameters. $\phi(t, \mu)$ is found by translating the Poincaré continuation argument used in the proof of Theorem A into a numerical scheme whereby a differential correction procedure is used. This is described in Section 2. The results obtained for the behavior of $\phi(t, \mu)$ are now briefly described: Theorem A gives the existence of $\phi(t, \mu)$ on each fixed energy surface $H = H(p, q) = h < 0$ provided $1/\omega^* \neq j, j = 1, 2, 3, \ldots$, for μ sufficiently small. Therefore, Kepler's equation, $T^* = 2\pi(-2h)^{-3/2}$ where T^* is the period of $\phi^*(t)$, rules out a discrete set of energies h_j since $\omega^* = 2\pi/T^*$. We now fix $\mu > 0$ sufficiently small; then for $-\infty < h < h_1$ the orbit in q-space which must be symmetric with respect to the q_2q_3-plane, is inclined with respect to the q_3-axis and is a simply closed curve. (This inclination and the shape of the orbit will be explained fully in Section 2.) As h passes through h_1 this inclination changes in a discontinuous manner, and further on in the open interval $h_1 < h < h_2$ our symmetric orbit develops one self intersection, i.e. two loops. Within the interval (h_1, h_2) the new angle of inclination with the q_3-axis to the orbit is maintained. As h passes through h_2 this angle of inclination of the orbit again reverses itself discontinuously and is then maintained throughout the interval (h_2, h_3). Within this interval the orbit develops another self intersection so that it now has 3 loops, etc. In general, the numerical work suggests that in each open interval $I_j = (h_j, h_{j+1})$ our orbit makes a continuous transition from j to $j + 1$ loops as h increases within I_j where $j = 1, 2, 3, \ldots$, and where there is a discontinuous change of inclination at each value h_j which is then maintained in the subsequent interval I_j. These results will be seen in detail in Section 2.

In this paper we do not perform a stability analysis. This will be presented at a later time in another paper.

We finally mention that if one takes $\mu = 0.001$, a physical example of our problem would be $m_1 = \text{Sun}$, $m_2 = \text{Jupiter}$, $m_3 = \text{comet}$, minor planet or meteor-stream traveling approximately perpendicular to the plane determined by m_1 and m_2. It turns out, [5], that such minor planets are indeed observed traveling in highly eccentric close approach orbits about the Sun approximately perpendicular to the Sun-Jupiter plane. It is also interesting to note the orbits presented in Section 2 need not be elliptical in shape; thus, observationally determining comet orbits about the Sun which are approximately perpendicular to the Sun-Jupiter plane to be elliptical, which is the usual practice, might well be in error.

2. Numerical Investigations

The goals and results for our numerical work were briefly discussed in Section 1(b). It is recalled that our main goal was to graphically reproduce our orbits on the computer and investigate their behavior. This behavior was described in 1(b) in a brief way. We will now make the description more precise; however, before doing this we first describe the numerical procedure used to obtain the symmetric periodic orbits on the computer. The numerical technique is a translation of the proof of Theorem A into a numerical scheme. This scheme can in general be viewed as a Newton–Raphson scheme, or more specifically as the method of differential correction. Before we present the numerical procedure, the method of proof of Theorem A is briefly outlined.

We will let (P, Q) denote the regularized coordinates cooresponding to the coordinates (p, q) under the map Φ referred to above in Section 1(a). This map is fully developed [1]; however, we record it here,

$$P_k = \begin{cases} \dfrac{|p|^2 - 1}{|p + e_1|^2}, & k = 1 \\ \dfrac{2p_k}{|p + e_1|^2}, & k = 2, 3 \end{cases}$$

$$Q_k = \begin{cases} \dfrac{1 - |p|^2}{2} q_1 + (p, q)(p_1 + 1), & k = 1 \\ \dfrac{|p|^2 + 1}{2} q_k + p_1 q_k - p_k q_1 - (p, q) p_k, & k = 2, 3, \end{cases}$$

where $(p, q) = \sum_{k=1}^{3} p_k q_k$, $e_1 = (1, 0, 0)$. As is proven in reference 1 the Hamiltonian flow corresponding to (1) is regular at collision with m_1 if one transforms to the coordinates P, Q above. The collision state $q = 0$, $p = \infty$ on $H = -h < 0$ goes into $P = e_1$, $|Q| = 1 - \mu$ on $\Gamma(P, Q) = 0$ where $\Gamma(P, Q)$ is the transformation of the Hamiltonian H in the new coordinates (P, Q) after a change of the independent variable $t \to s$. We will let $X^*(s)$ denote the special collision orbit $\Phi^*(t)$ in our new coordinates. Now, let L denote the hyperplane $Q_1 = P_2 = P_3 = 0$. One can prove, [1] that if a solution $X(s)$ to the restricted problem intersects L at $s = 0$ and again at $s = S$ then $X(s)$ is a periodic orbit of period $2S$. One verifies that $X^*(s)$ for $\mu = 0$ has such a property, thus we are led to the following system, which clearly has $X^*(s)$ as a solution for $\mu = 0$,

$$\begin{aligned} Q_1(S, Q_2^\circ, Q_3^\circ, P_1^\circ) &= 0 \\ P_2(S, Q_2^\circ, Q_3^\circ, P_1^\circ) &= 0 \\ P_3(S, Q_2^\circ, Q_3^\circ, P_1^\circ) &= 0, \end{aligned} \tag{2}$$

where we assume $X(s) \in L$ at $s = 0$ which implies $Q_1^\circ = P_2^\circ = P_3^\circ = 0$. Now, by virtue of the fact $\Gamma_{Q_3} = -1 \neq 0$ at $X^*(0)$ we can solve for

$$Q_3^\circ = f(P_1^\circ, Q_2^\circ) \tag{3}$$

and therefore use P_1°, Q_2° for independent variables on L for $s = 0$. In fact, we can reduce system (2) even further by noting that $\Gamma_{Q_3^\circ} = -1 \neq 0$ at $X^*(0)$ which implies $dP_3/dS \neq 0$ so that we can determine the time S of the next intersection with L as

$$S = g(Q_2^\circ, f(P_1^\circ, Q_2^\circ), P_1^\circ) \tag{4}$$

in view of (3). This reduces (2) to the 2×2 system

$$\begin{aligned} Q_1(S, Q_2^\circ, Q_3^\circ, P_1^\circ) &= 0 \\ P_2(S, Q_2^\circ, Q_3^\circ, P_1^\circ) &= 0 \end{aligned} \tag{5}$$

for P_1°, Q_2° where S and Q_3° are given by (2), (4) respectively. By Theorem A, System (2) has a unique solution P_1°, Q_2°.

We now translate the above into a numerical scheme: Step 1. – Input $Q_1^\circ = P_2^\circ = P_3^\circ = 0$, $Q_2^\circ = Q_2^*(0)$, $P_1^\circ = P_1^*(0)$ and determine Q_3° from $\Gamma(P^\circ, Q^\circ) = 0$ by a Newton–Raphson scheme. This brings us to (2), (3). Step 2. – Integrate the system of differential equations with P°, Q° determined in step 1 until a time S is reached such that $P_3(S) = 0$ which brings us to (4), (5). Step 3. – With $Q_3^\circ, P_1^\circ, Q_2^\circ, S$ determined above we can't yet expect to satisfy (5), however, we will find corrections $\Delta P_1^\circ, \Delta Q_2^\circ$ to P_1°, Q_2° so that (5) is satisfied. These corrections will be found in an approximate form at first; then the above procedure is iterated by starting at step 1 with P_1°, Q_2° replaced by $P_1^\circ + \Delta P_1^\circ, Q_2^\circ + \Delta Q_2^\circ$ until a desired accuracy is obtained. We require an accuracy of at least 10 significant digits throughout.

To calculate the corrections $\Delta P_1^\circ, \Delta Q_2^\circ$ we proceed as follows: Clearly, one would like to satisfy

$$\begin{aligned} Q_1(Q_2^\circ + \Delta Q_2^\circ, P_1^\circ + \Delta P_1^\circ) &= 0 \\ P_2(Q_2^\circ + \Delta Q_2^\circ, P_1^\circ + \Delta P_1^\circ) &= 0 \end{aligned} \tag{6}$$

where we have suppressed the dependence on S and Q_3°. Now, we form the difference

$$Q_1(Q_2^\circ + \Delta Q_2^\circ, P_1^\circ + \Delta P_1^\circ) - Q_1(Q_2^\circ, P_1^\circ) = A_1 \Delta Q_2^\circ + A_2 \Delta P_1^\circ, \tag{7}$$

and similarly,

$$P_2(Q_2^\circ + \Delta Q_2^\circ, P_1^\circ + \Delta P_1^\circ) - P_2(Q_2^\circ, P_1^\circ) = B_1 \Delta Q_2^\circ + B_2 \Delta P_1^\circ, \tag{8}$$

where

$$A_1 = \frac{\partial Q_1}{\partial Q_2^\circ}, \qquad A_2 = \frac{\partial Q_1}{\partial P_1^\circ}, \qquad B_1 = \frac{\partial P_2}{\partial Q_2^\circ}, \qquad B_2 = \frac{\partial P_2}{\partial P_1^\circ}.$$

When computing these partial derivatives one must be careful, e.g.,

$$Q_1(P_1^\circ, Q_2^\circ) = Q_1(S, Q_2^\circ, Q_3^\circ, P_1^\circ),$$

where $Q_3^\circ = f(P_1^\circ, Q_2^\circ)$, $S = g(Q_2^\circ, Q_3^\circ, P_1^\circ)$; thus, to compute, e.g., $\partial Q_1^\circ / \partial Q_2^\circ$ by a first order forward difference formula

$$\frac{Q_1(Q_2^\circ + \delta Q_2^\circ) - Q_1(Q_2^\circ)}{\delta Q_2^\circ},$$

we have to compute $Q_1(Q_2^\circ + \delta Q_2^\circ)$. This is done by keeping P_1° fixed and with Q_2° replaced by $Q_2^\circ + \delta Q_2^\circ$ we go back to step 1 to recompute Q_3° and S. We then integrate the differential equations up to S and see what value Q_1 takes on – which will be $Q_1(Q_2^\circ + \delta Q_2^\circ)$. With this value, the difference formula is readily computed. The other derivatives are similarly computed. It was found that for the derivative $\partial Q_1^\circ / \partial Q_2^\circ$, an optimum choice of δQ_2° was 10^{-5}. In fact, this was the case with all the increments for the computations of all the other derivatives.

Now, with $A_i, B_i, i = 1, 2$, computed we go back to (7), (8) and set $P_2(Q_2^\circ + \Delta Q_2^\circ, P_1^\circ + \Delta P_1^\circ) = Q_1(Q_2^\circ + \Delta Q_2^\circ, P_1^\circ + \Delta P_1^\circ) = 0$ and uniquely solve the resulting 2×2 linear system

$$\begin{pmatrix} A_1 & A_2 \\ B_1 & B_2 \end{pmatrix} \begin{pmatrix} \Delta Q_2^\circ \\ \Delta P_1^\circ \end{pmatrix} = - \begin{pmatrix} Q_1(Q_2^\circ, P_1^\circ) \\ P_2(Q_2^\circ, P_1^\circ) \end{pmatrix}$$

for the corrections $\Delta Q_2^\circ, \Delta P_1^\circ$.

It was found that to achieve an accuracy of 10^{-10}, approximately 3 or 4 iterations were needed.

The integration of the system of differential equations was accomplished by an implicit Adams method.‡ The above numerical scheme is similar to one used [4].

As described in Section 1(b), the following behavior of our orbit was observed in the q-space: With μ fixed sufficiently small, e.g. $\mu = 0.001$, our symmetric orbit on the energy surface $H(p, q) = -h < 0$ makes a continuous transition from j to $j+1$ loops, i.e. from $j-1$ to j self-intersections, in each open interval $I_j = (h_j, h_{j+1})$ where $h_j < 0$ corresponds to the frequency $(1/j) = \omega^* = (-2h)^{3/2}$ of Φ^*, and where $j = 1, 2, 3, \ldots$. In addition, the orbit changes its inclination to be defined below, in a discontinuous manner with respect to m_1 and m_2 as h passes through h_j. These results are now graphically illustrated.

For the energy values $h = -1.5, -0.38, -0.27$, which lie in the consecutive intervals I_j for $j = 1, 2, 3$ respectively, we observe the corresponding loop states in Figures 2, 3, and 4 respectively where the view is along the q_2-axis, i.e. projected on the $q_1 q_3$-plane. Now, if we look at the orbits, for the above energy values, projected on the $q_1 q_2$-plane we have Figures 5, 6, and 7. Finally, the views projected on the $q_2 q_3$-plane, the symmetry plane, are shown in Figures 8, 9, and 10. In this set of figures

‡ All numerical work was done on a CDC 6600 computer.

we see that the orbits are constrained to lie on their own planes which are folded but not twisted. One also notes the following phenomenon in this set of figures; If we look at the tip of each curve, i.e. the point where the mass m_3 turns around to approach m_1 which is marked, for example, by the point A in Figure 8, then one sees that this point alternates between pointing in the positive and negative q_2-directions in each consecutive figure. This is what we previously referred to as the change of inclination of our orbit as h passes through $h_j, j = 1, 2, \ldots$. In this context we also numerically observe that the loop formation in each interval I_j occurs only after the orbit changed its inclination at h_j.

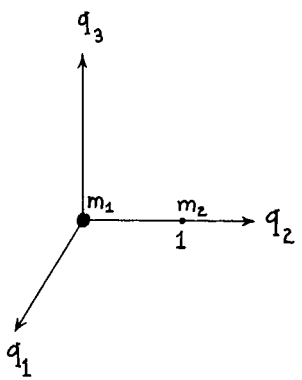

Fig. 1. The rotating coordinate system.

A particularly interesting orbit is illustrated in Figures 11, 12. This orbit corresponds to the case $h = -0.055 \in I_{27}$.

In Figures 13, 14, and 15 we illustrate a loop transition in the interval I_2 from 2 to 3 loops which is projected in the q_1q_3-plane.

We illustrate the discontinuous behavior of the change of inclination in Figures 16, 17 which are the projections on the q_2q_3-plane for $h = -0.52, -0.48$ respectively where one notes that $h_1 = -0.5$ is the first resonance energy value.

The loop formation is clearly a resonance phenomenon corresponding to the additional number of times m_2 completes its period before m_3 completes one period.

Acknowledgements

I would like to express my gratitude to Professor Jurgen Moser for his help and encouragement which was vital for the successful completion of this work.

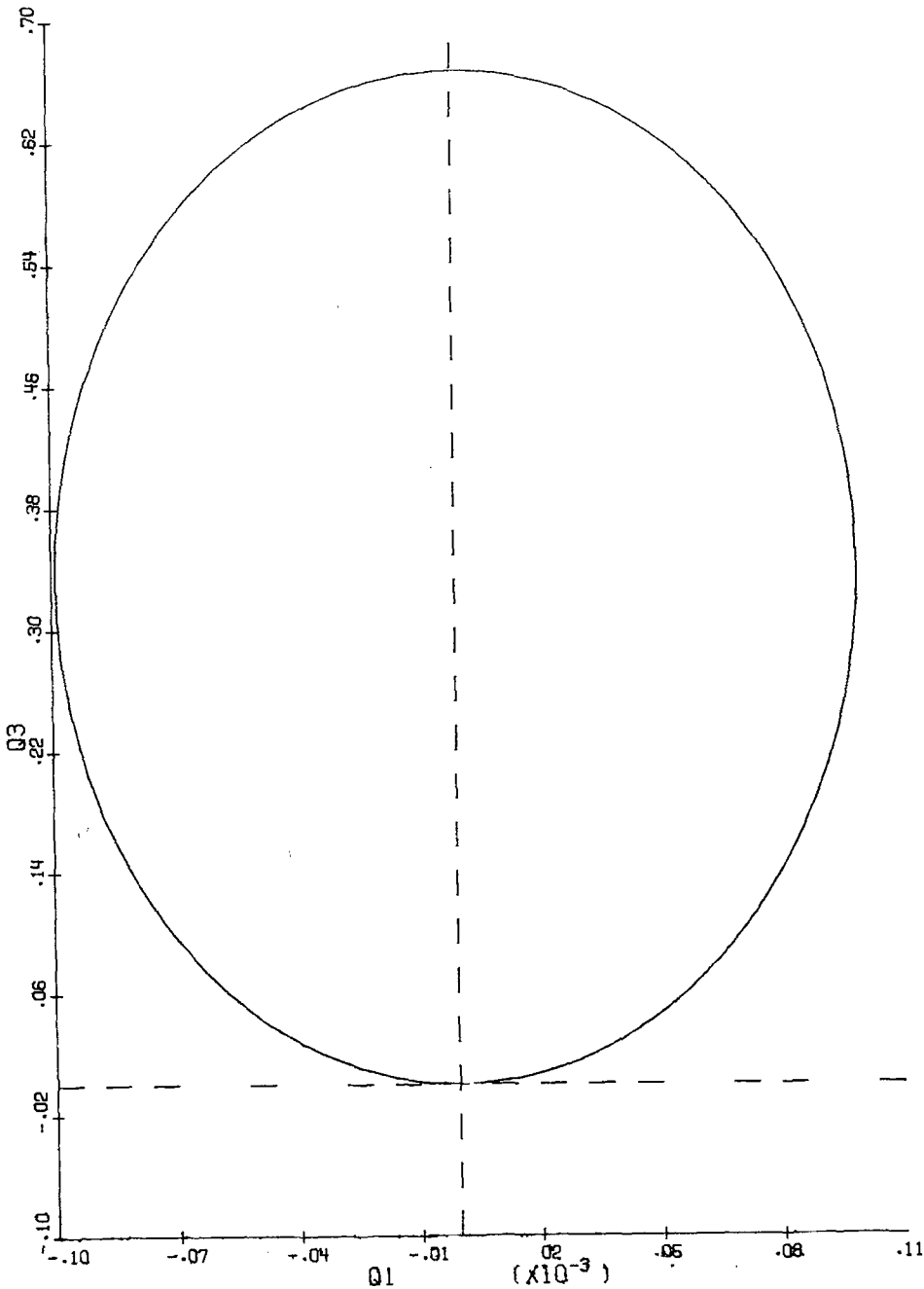

Fig. 2. $H = -1.5$.

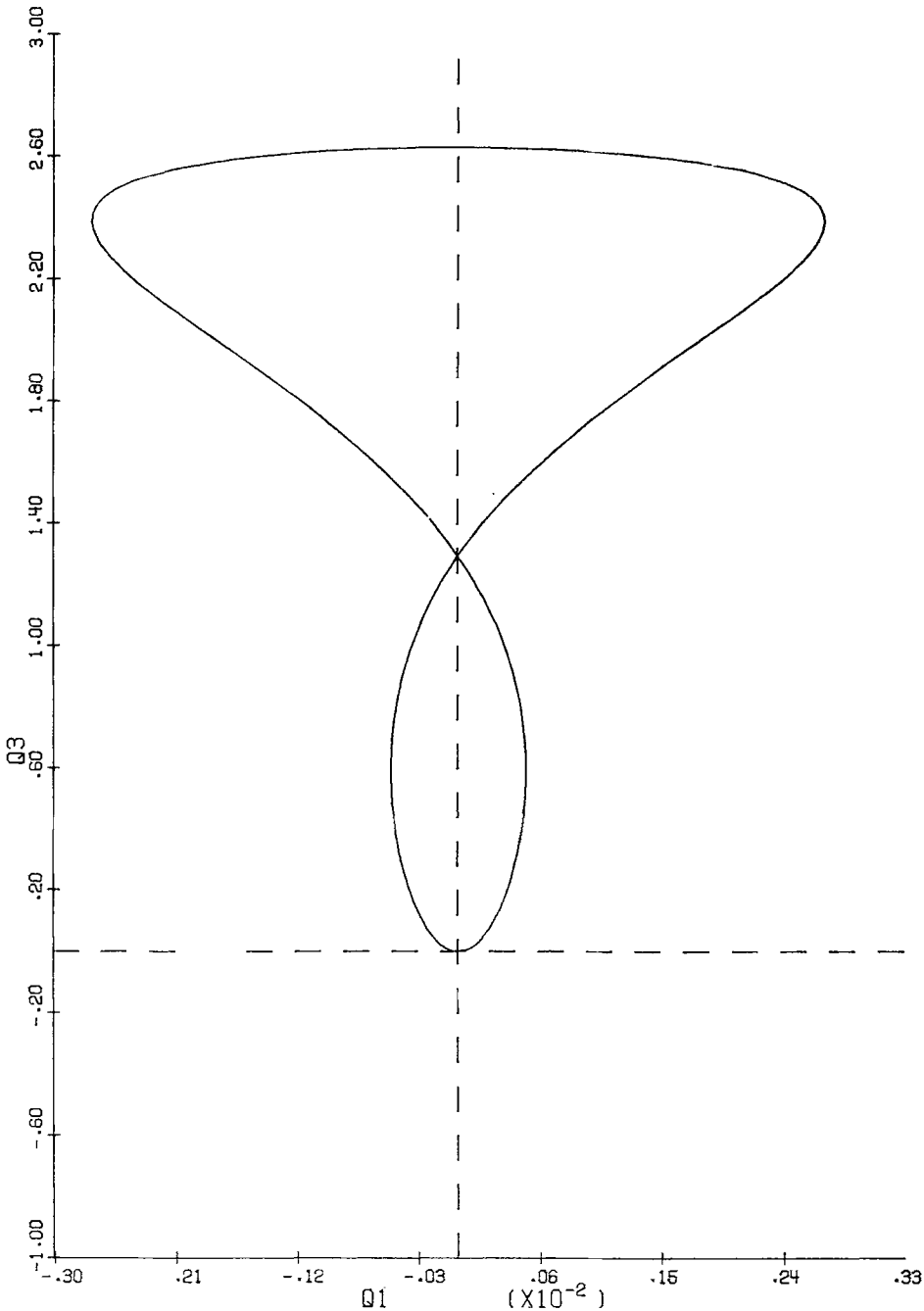
3.00
2.60
2.20
1.80
1.40
1.00
.60
.20
-.20
-.60
-1.00
Q3
-.30
.21
-.12
-.03
.06
.15
.24
.33
Q1
(X10^{-2})

Fig. 3. $H = -0.38$.

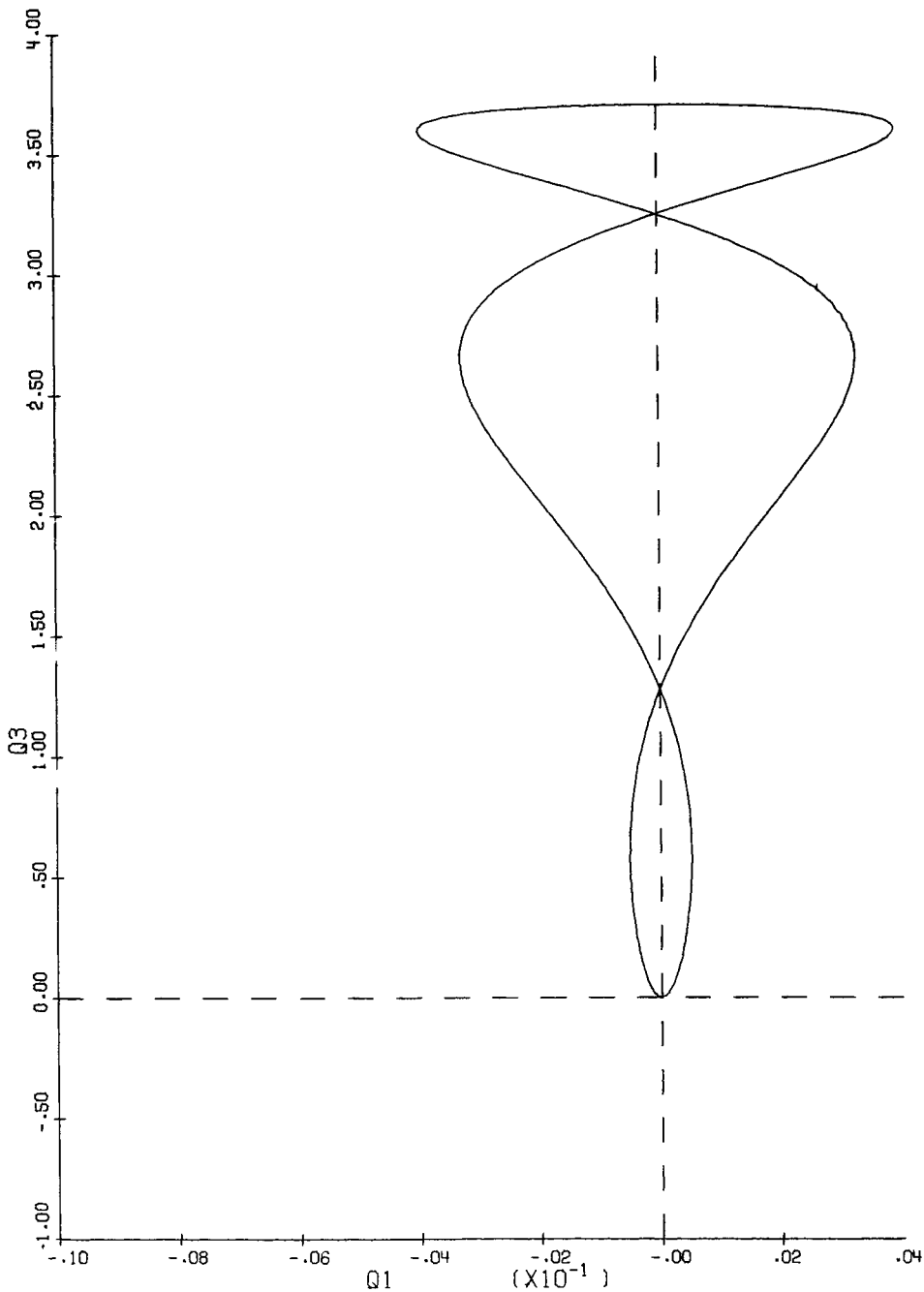
4.00
3.50
3.00
2.50
2.00
1.50
1.00
.50
0.00
-.50
-1.00
Q3
-.10
-.08
-.06
-.04
-.02
-.00
.02
.04
Q1
(X10^{-1})

Fig. 4. $H = -0.27$.

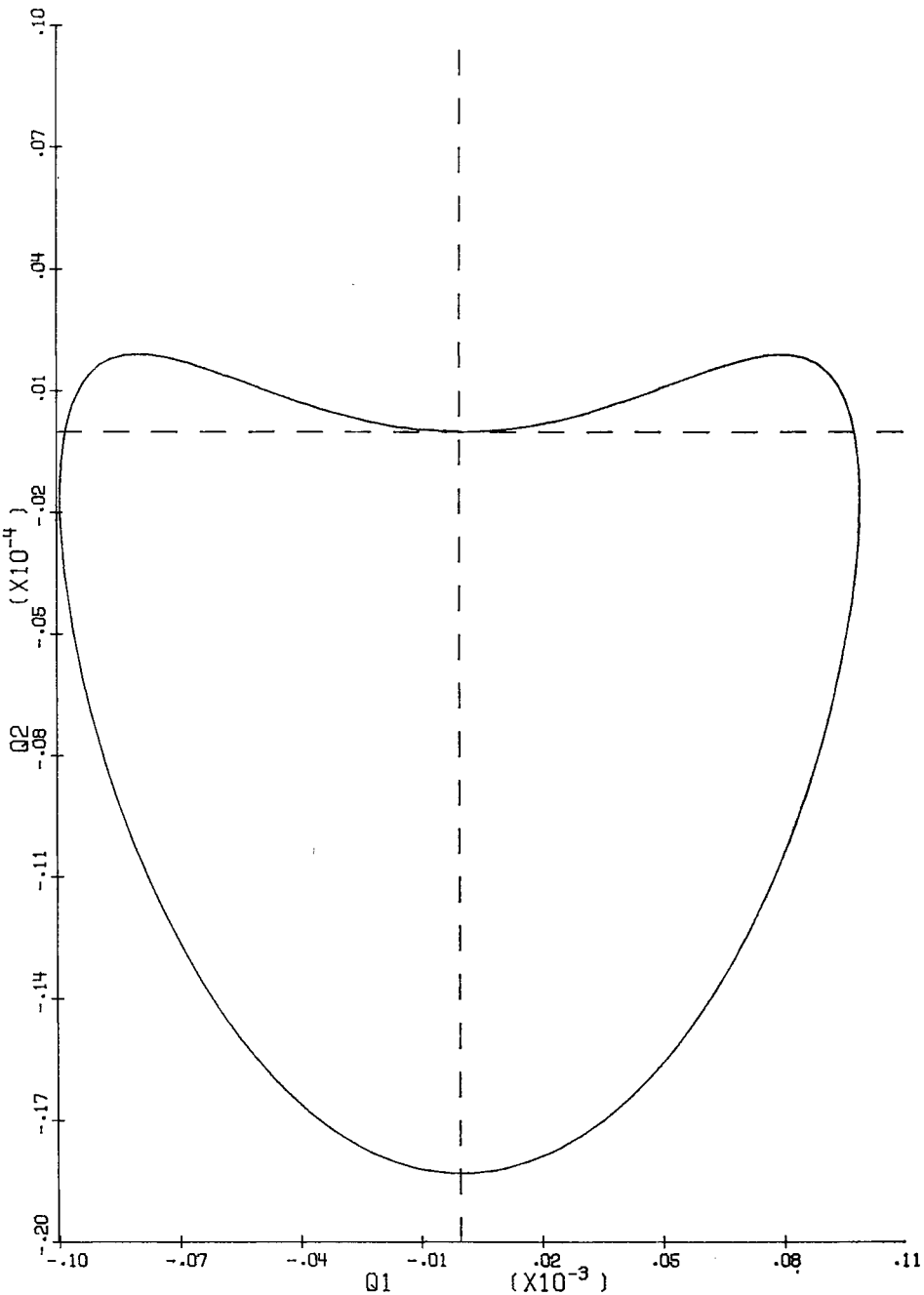
.10
.07
.04
.01
-.02
-.05
-.08
-.11
-.14
-.17
-.20
Q2
(X10^-4)
-.10
-.07
-.04
-.01
.02
.05
.08
.11
Q1
(X10^-3)

Fig. 5. $H = -1.5$.

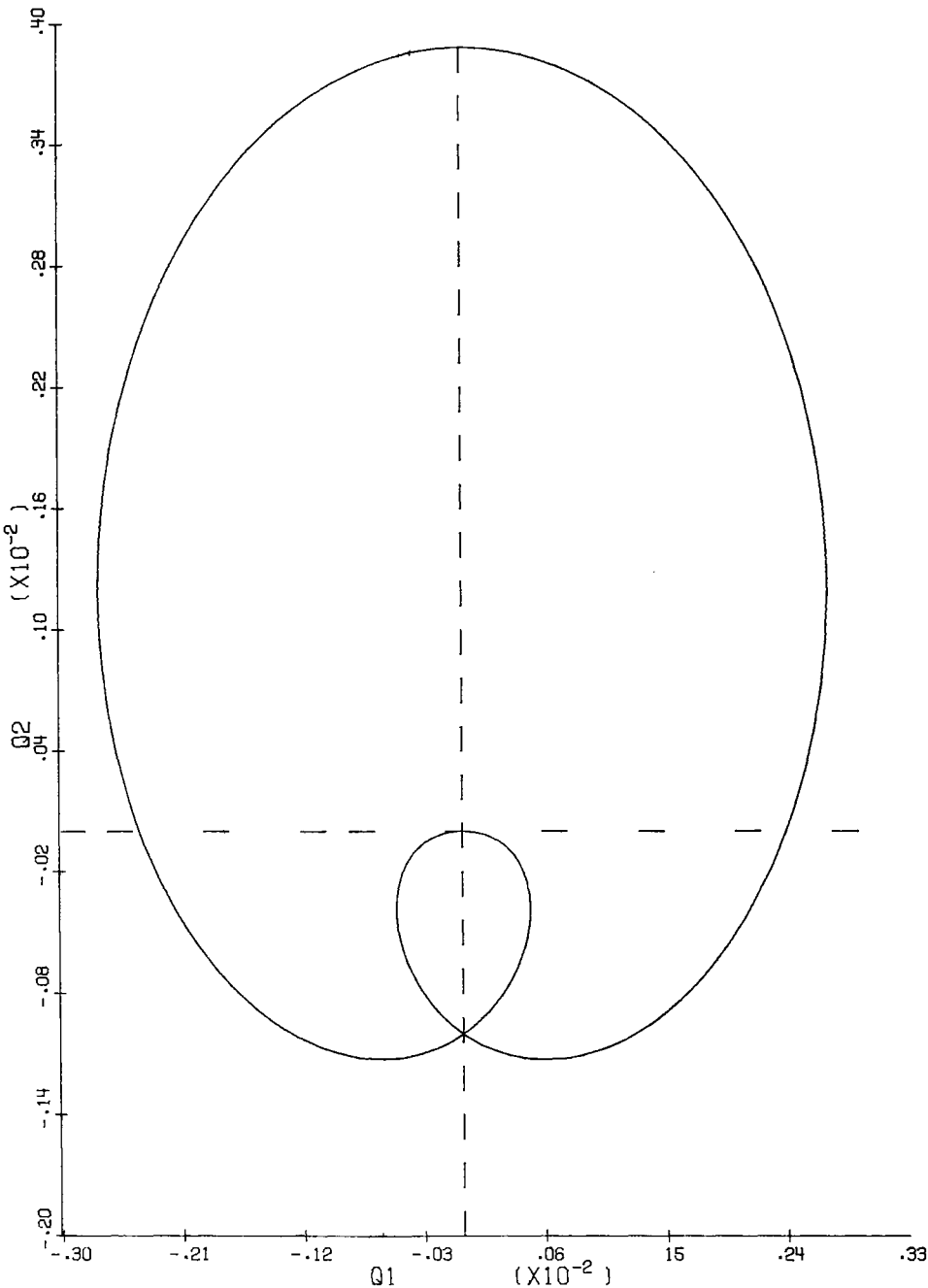
.40
.34
.28
.22
.16
.10
.04
-.02
-.08
-.14
-.20
Q2
(X10^{-2})
-.30
-.21
-.12
-.03
.06
.15
.24
.33
Q1
(X10^{-2})

Fig. 6. $H = -0.38$.

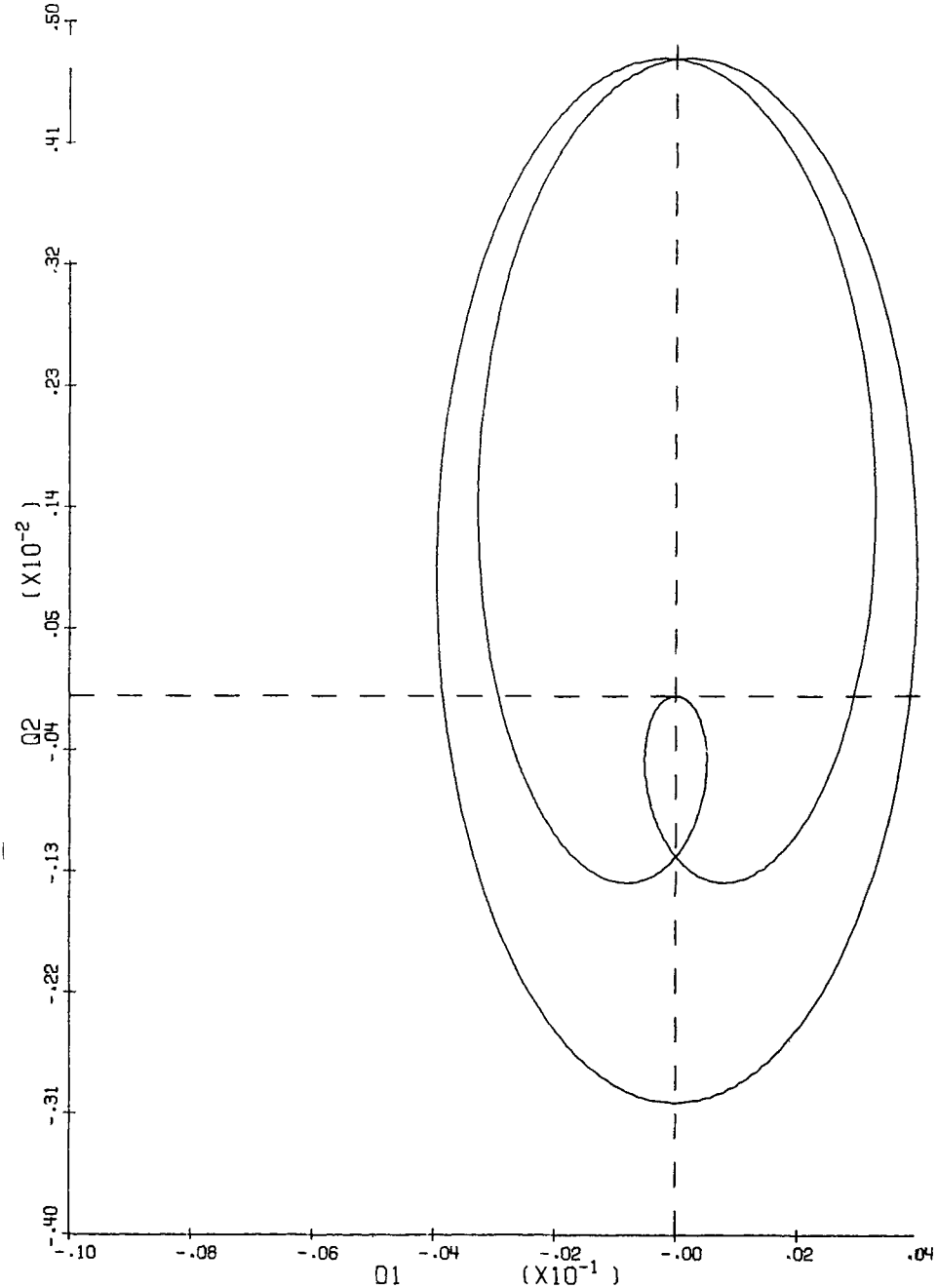
.50
.41
.32
.23
.14
.05
-.04
-.13
-.22
-.31
-.40
Q2
(X10^-2)
-.10
-.08
-.06
-.04
-.02
-.00
.02
.04
Q1
(X10^-1)

Fig. 7. $H = -0.27$.

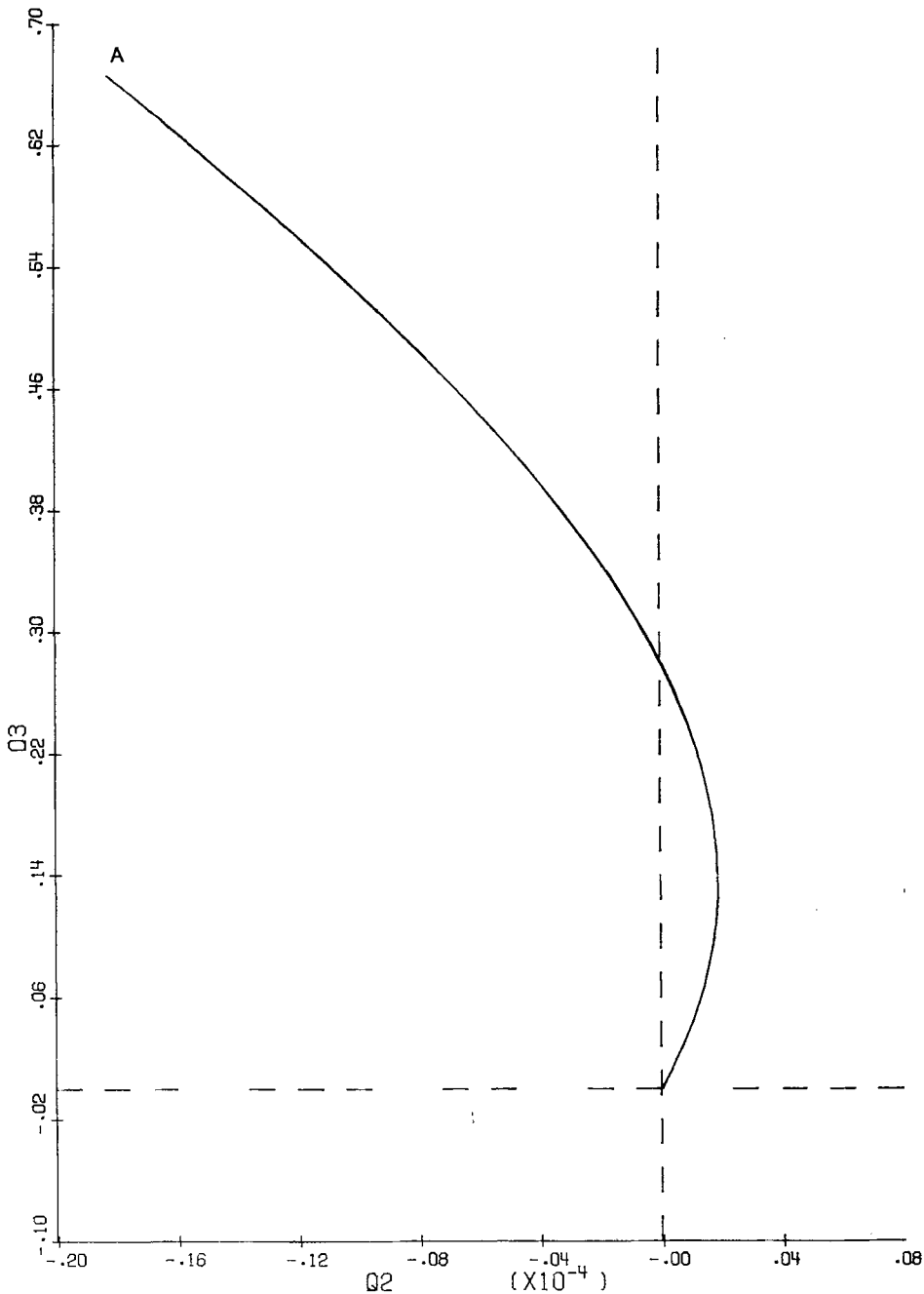

Fig. 8. $H = -1.5$.

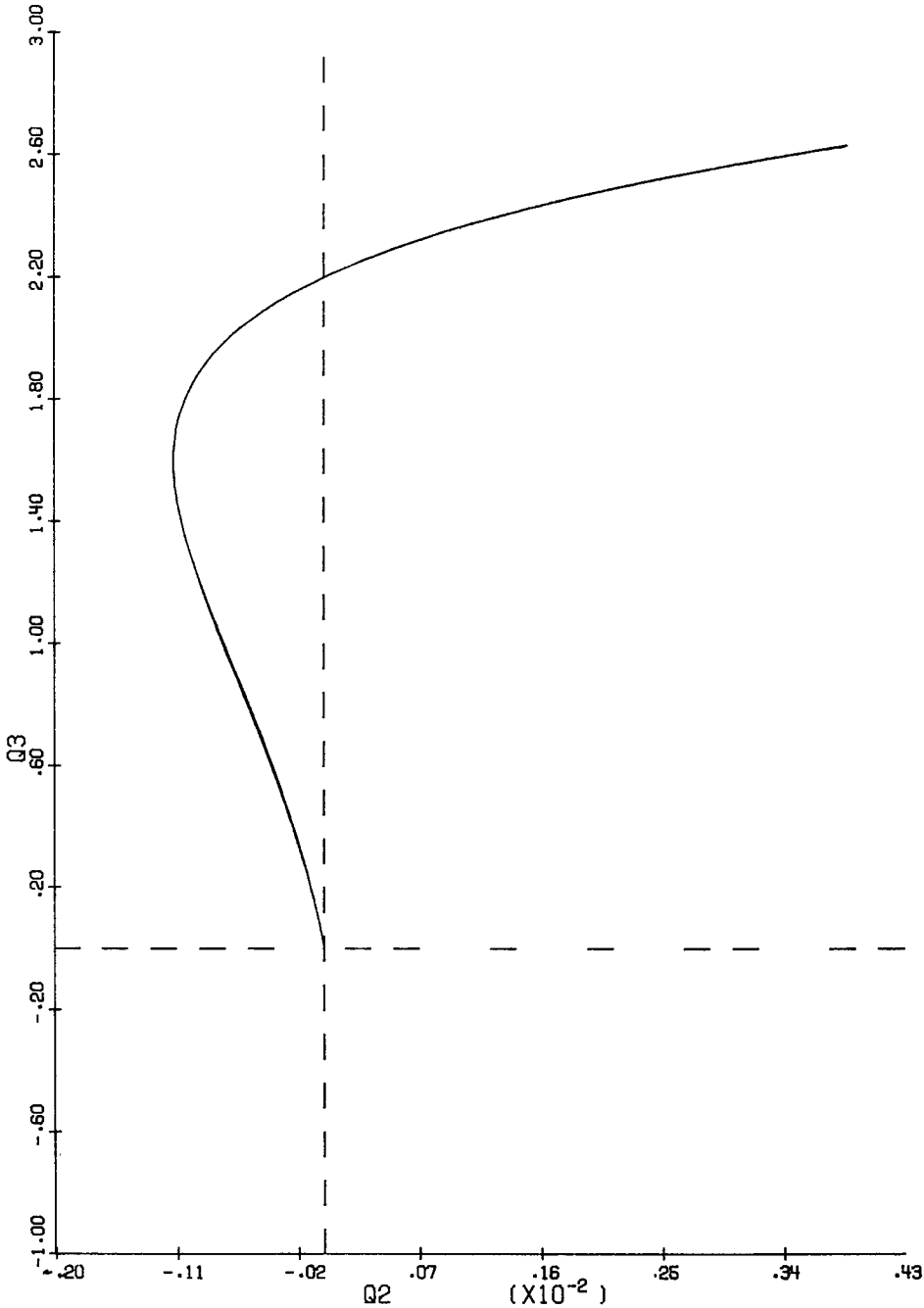
3.00
2.60
2.20
1.80
1.40
1.00
.60
.20
-.20
-.60
-1.00
Q3
-.20
-.11
-.02
.07
.16
.25
.34
.43
Q2
(X10^{-2})

Fig. 9. $H = -0.38$.

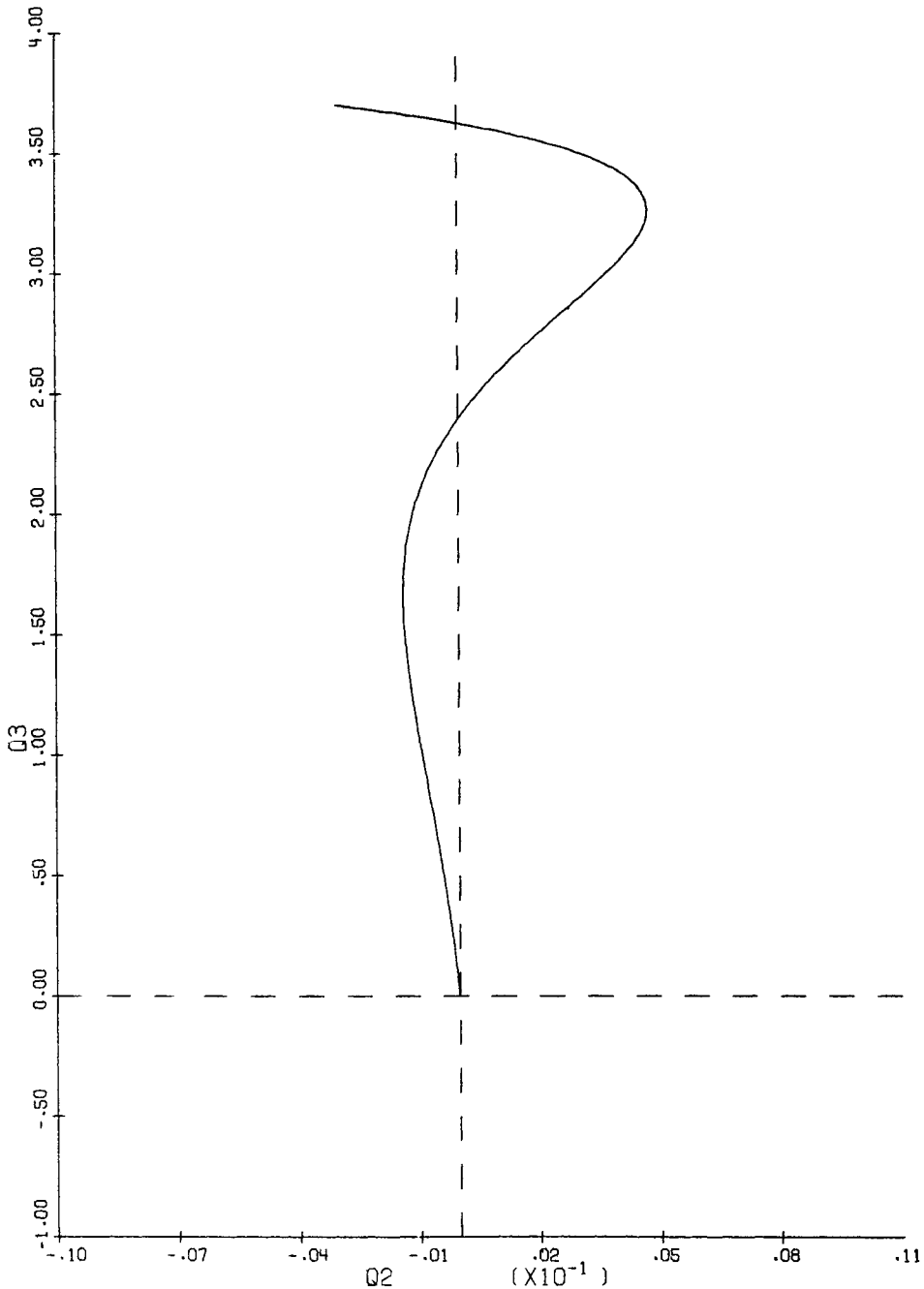
4.00
3.50
3.00
2.50
2.00
1.50
1.00
.50
0.00
-.50
-1.00
Q3
-.10
-.07
-.04
-.01
.02
.05
.08
.11
Q2
(X10^{-1})

Fig. 10. $H = -0.27$.

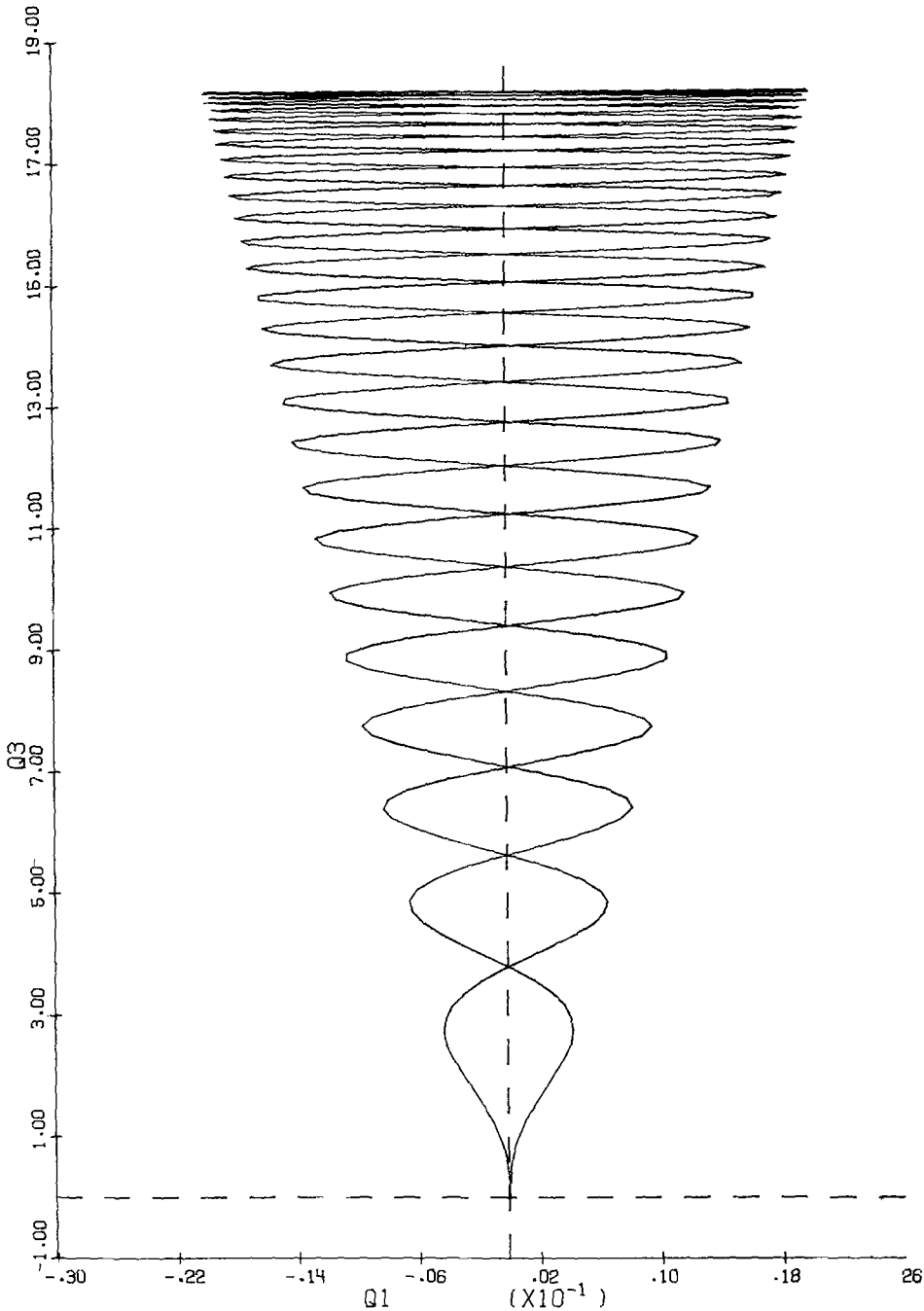
19.00
17.00
15.00
13.00
11.00
9.00
7.00
5.00
3.00
1.00
-1.00
Q3
-.30
-.22
-.14
-.06
.02
.10
.18
26
Q1
(X10^{-1})

Fig. 11. $H = -0.055$.

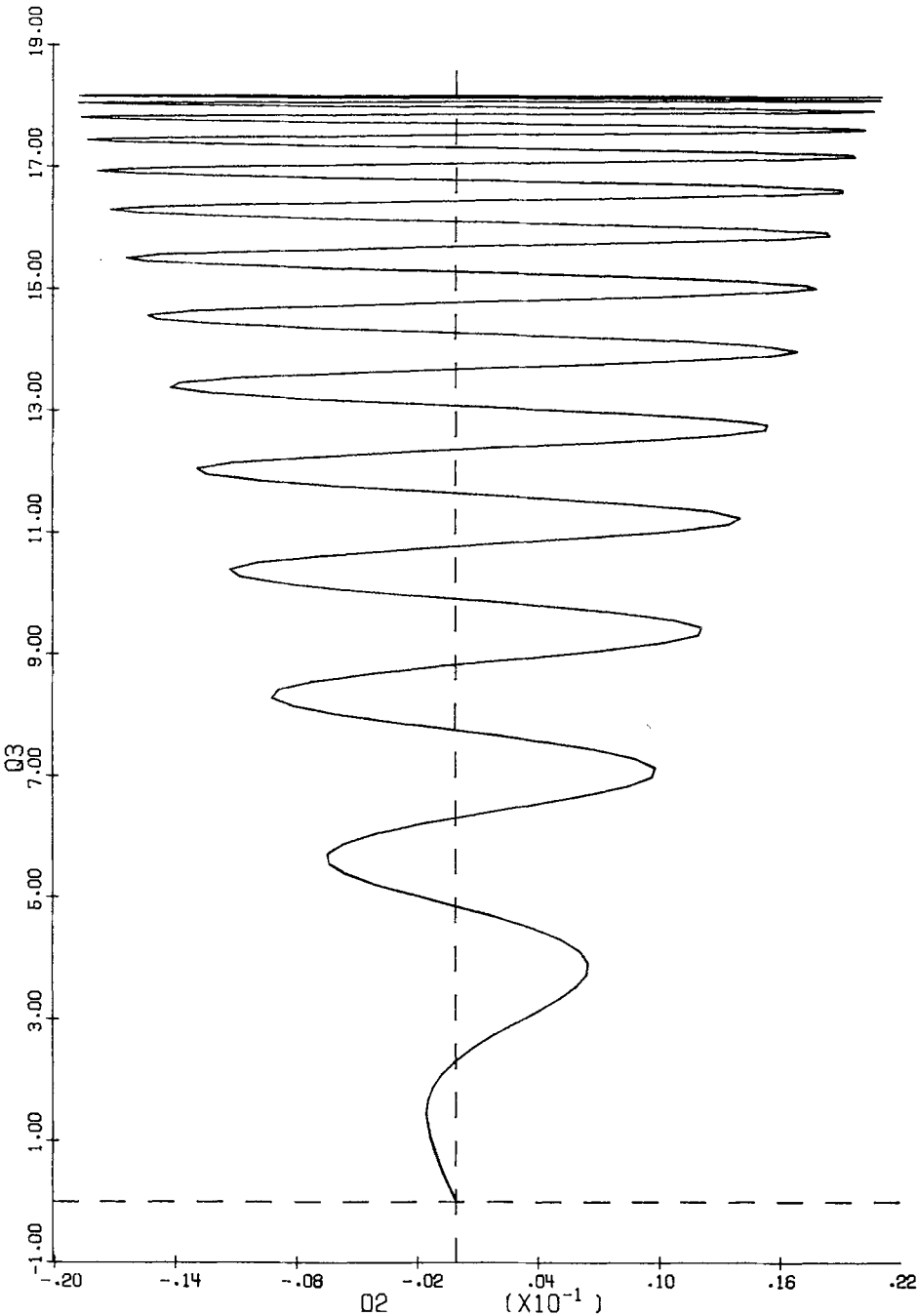
19.00
17.00
15.00
13.00
11.00
9.00
7.00
5.00
3.00
1.00
-1.00
Q3
-.20
-.14
-.08
-.02
.04
.10
.16
.22
Q2
(X10^{-1})

Fig. 12. $H = -0.055$.

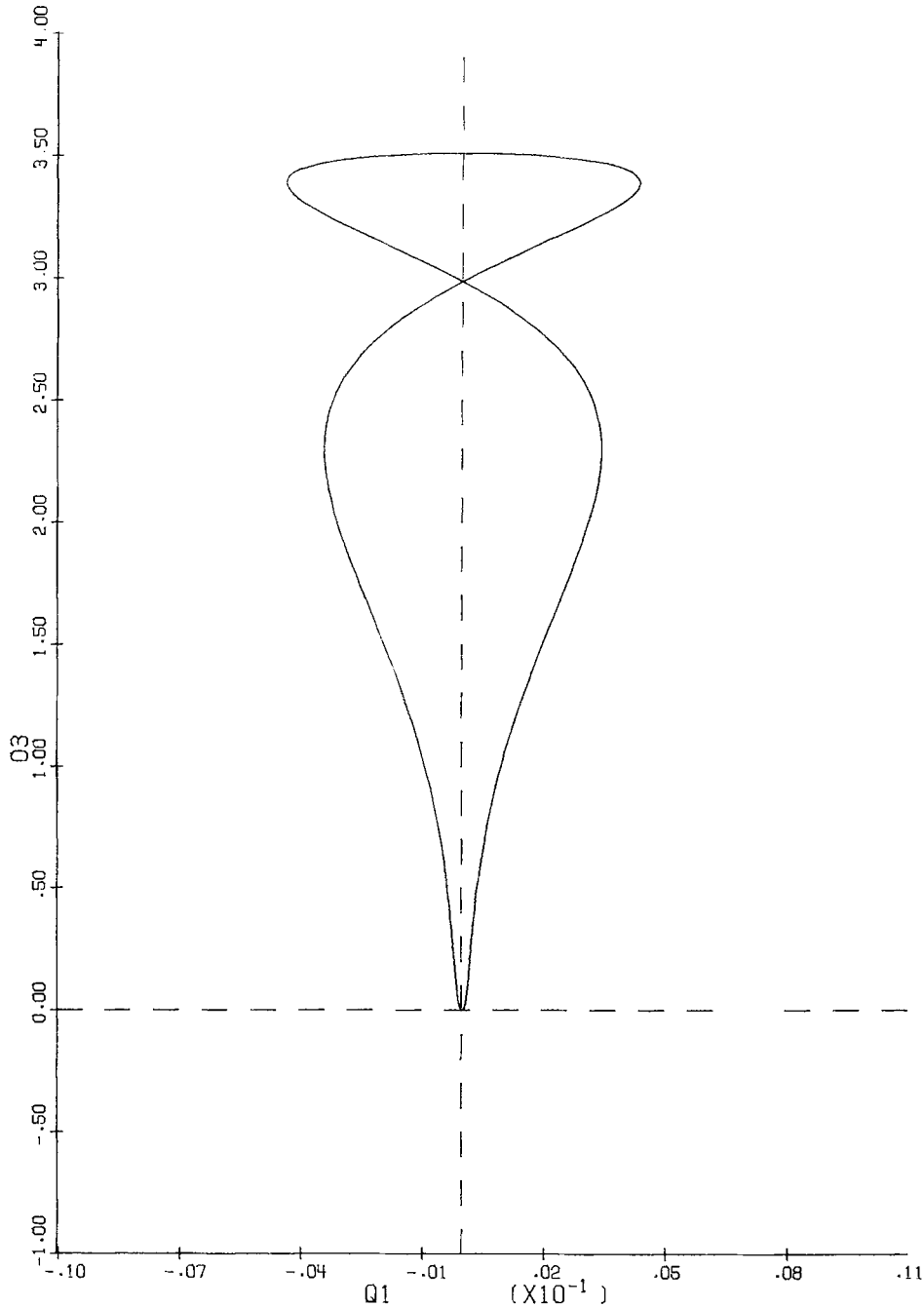
4.00
3.50
3.00
2.50
2.00
1.50
1.00
.50
0.00
-.50
-1.00
Q3
-.10
-.07
-.04
-.01
.02
.05
.08
.11
Q1
(X10⁻¹)

Fig. 13. $H = -0.285$.

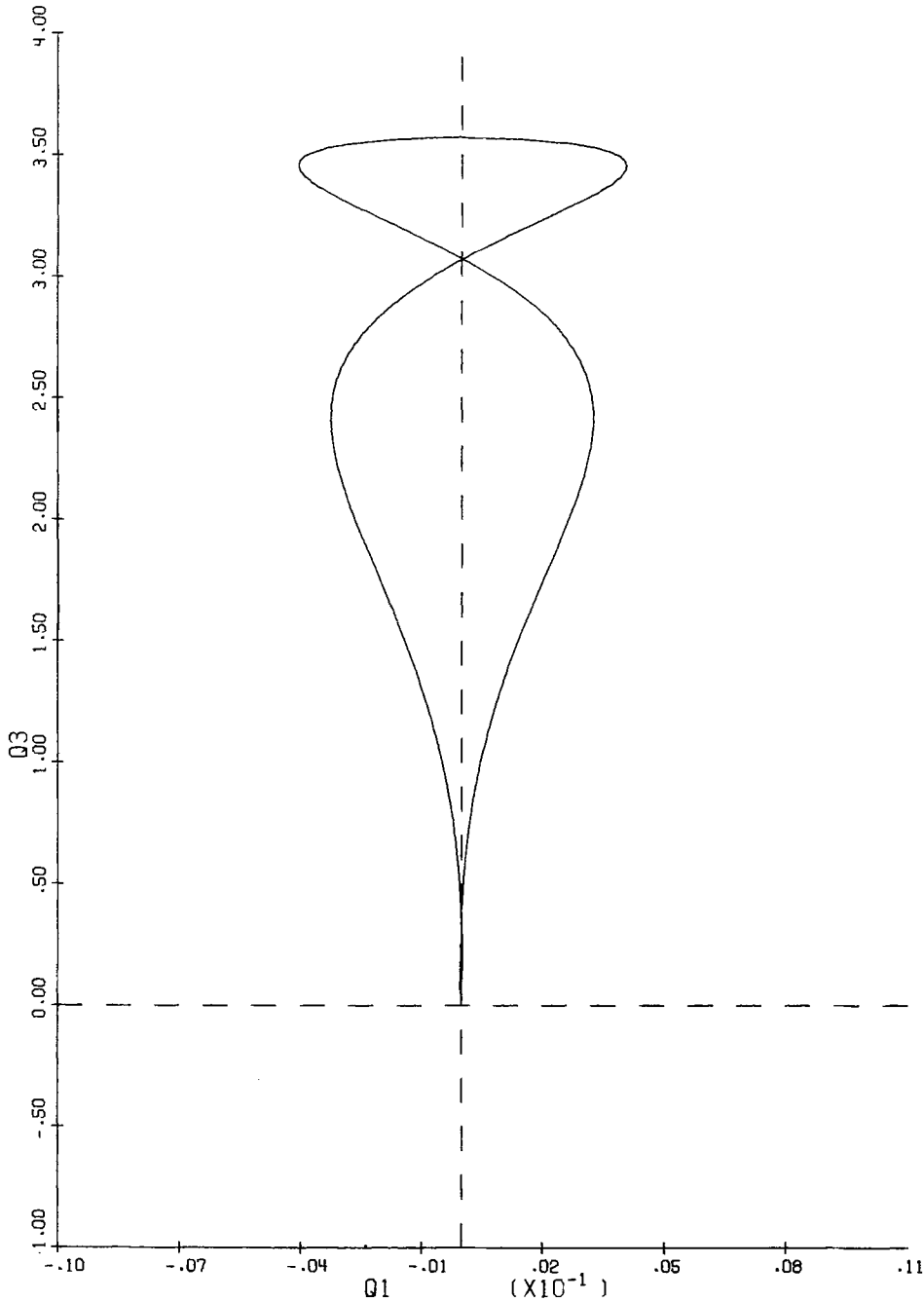
4.00
3.50
3.00
2.50
2.00
1.50
1.00
.50
0.00
-.50
-1.00
Q3
-.10
-.07
-.04
-.01
.02
.05
.08
.11
Q1
(X10^{-1})

Fig. 14. $H = -0.28$.

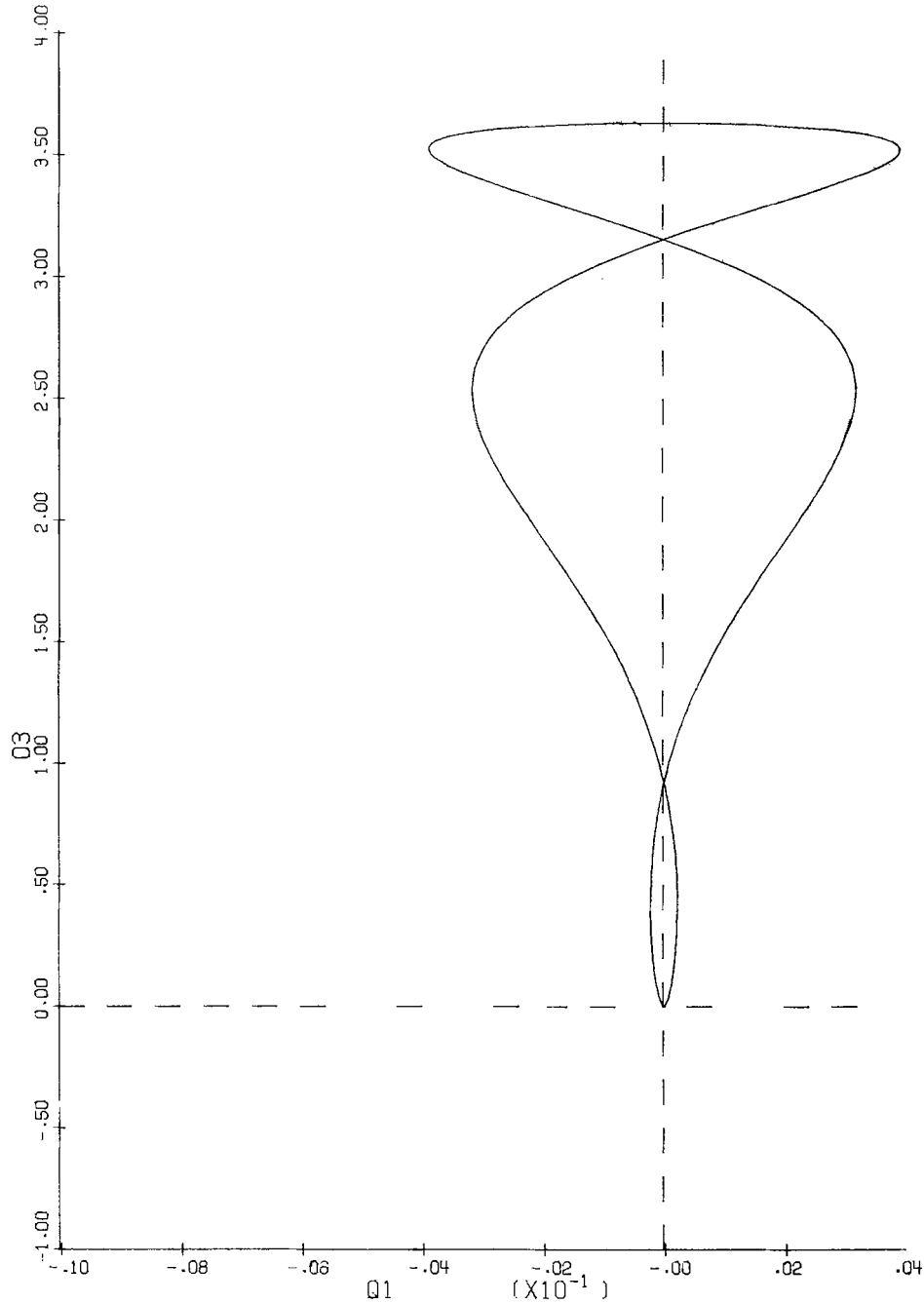
4.00
3.50
3.00
2.50
2.00
1.50
1.00
.50
0.00
-.50
-1.00
Q3
-.10
-.08
-.06
-.04
-.02
-.00
.02
.04
Q1
(X10^{-1})

Fig. 15. $H = -0.275$.

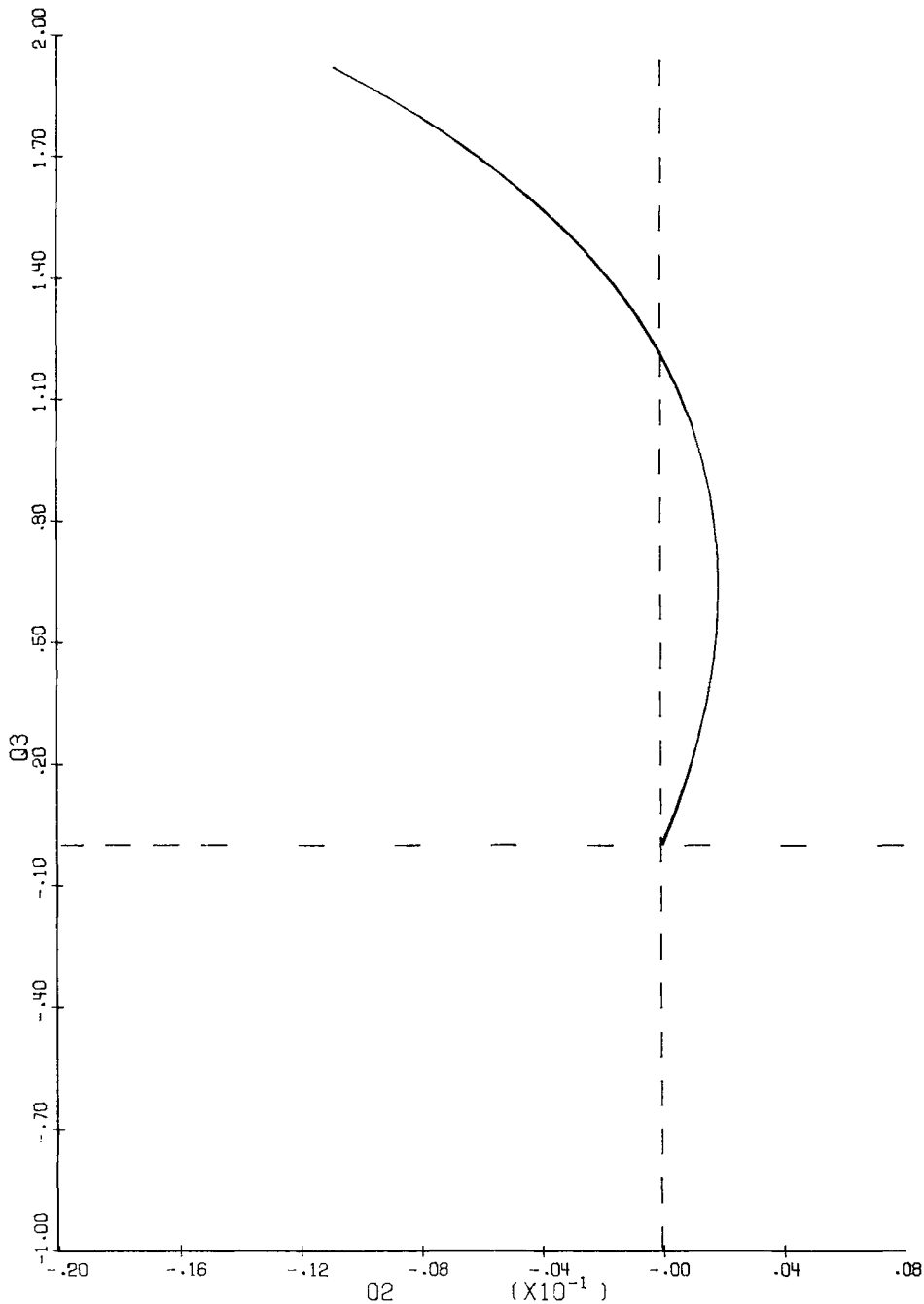
2.00
1.70
1.40
1.10
.80
.50
.20
-.10
-.40
-.70
-1.00
Q3
-.20
-.16
-.12
-.08
-.04
-.00
.04
.08
Q2
(X10^{-1})

Fig. 16. $H = -0.52$.

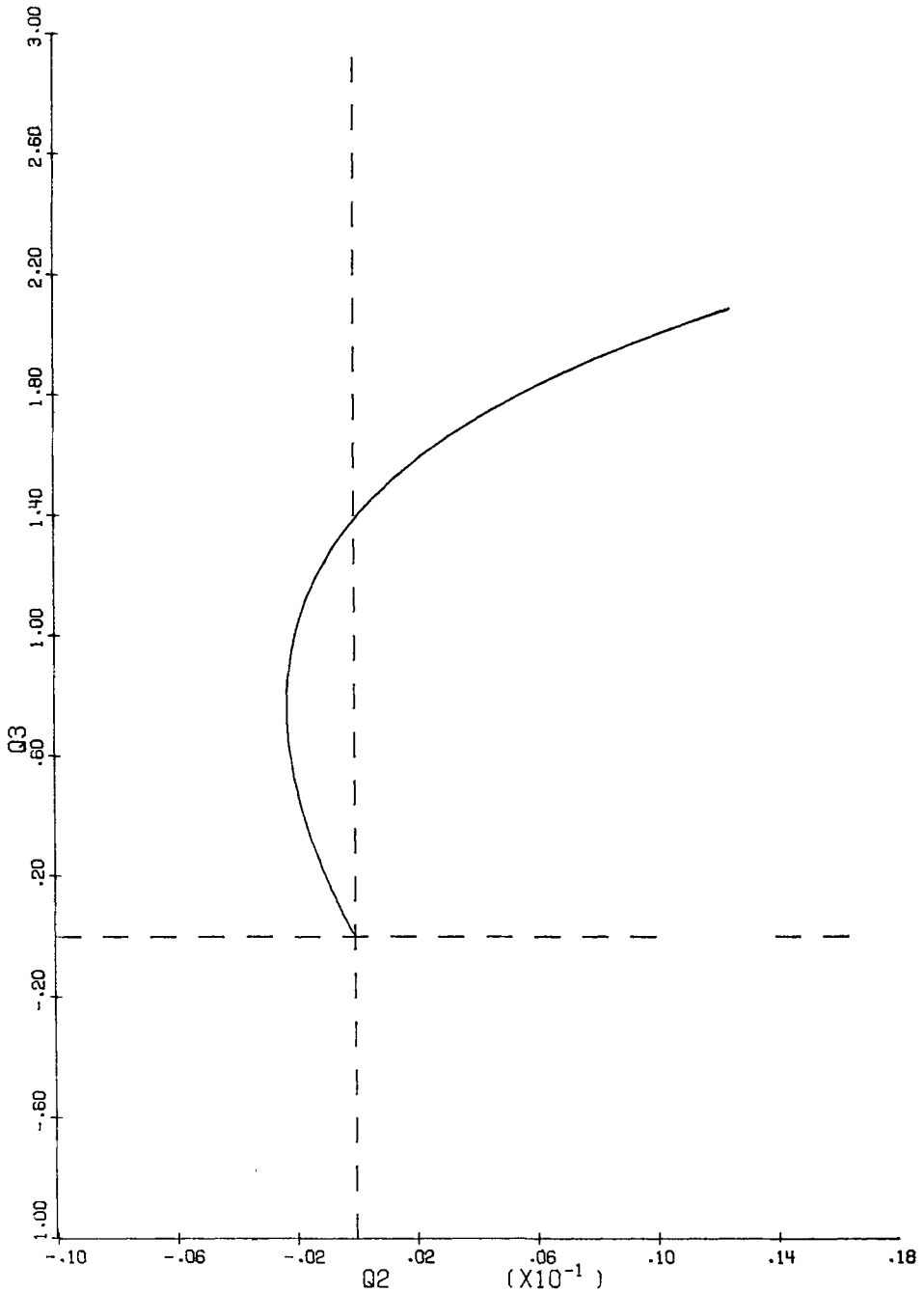
3.00
2.60
2.20
1.80
1.40
1.00
.60
.20
-.20
-.60
-1.00
Q3
-.10
-.06
-.02
.02
.06
.10
.14
.18
Q2
(X10^{-1})

Fig. 17. $H = -0.49$.

References

[1] Belbruno, E. A.: 1981, 'A New Regularization to the Restricted Three-Body Problem and an Application', to appear in *Celes. Mech.*
[2] Brjuno, A. D.: 1978, *Celes. Mech.* **18**, 51.
[3] Guillaume, Pierre: 1974, *Celes. Mech.* **10**, 475.
[4] Hadjidemetriou, John D. and Christides, T. H.: 1975, *Celes. Mech.* **12**, 175.
[5] Richter, N. B.: 1963, *The Nature of Comets*, Mathuen and Company LTD, London.
[6] Szebehely, V.: 1967, *Theory of Orbits*, Academic Press.