GYLDEN–MEŠČERSKII PROBLEM

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To Professor G.N. Duboshin on his 75th Birthday.

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Abstract. Classical non-stationary two-body problem, described by the equation of the form

$$\ddot{\mathbf{r}} = -\mu(t)\frac{\mathbf{r}}{r^3}$$

is investigated using differential equation transformation methods developed by the author.

All laws of mass variation for which Gylden–Meščerskii problem is reduced to autonomous form are stated. The problem symmetry properties are investigated and reviews of integrable cases from the group point of view are made.

1. Introduction

The classical non-stationary two-body Gylden–Meščerskii problem (Gylden, 1889; Meščerskii, 1893) used for describing double-star evolution at the secular mass loss owing to photon and corpuscular activity is one of the most famous in celestial mechanics. Gylden–Meščerskii's problem is also used as a mathematical model for different cases of variable mass body dynamics relative motion of two bodies, when their Newton's interaction greatly exceeds reactive forces or when there is a perturbing force a type of "friction", compensating reactive forces, etc. (see Meščerskii, 1897, 1902, 1949; Duboshin, 1925; Omarov, 1975).

The question is the motion equation of the form

$$\ddot{\mathbf{r}} = -\mu(t)\frac{\mathbf{r}}{r^3}, \tag{1}$$

where $\mathbf{r} = (x; y)$ is the radius vector of motion of a particle with respect to the other one in the orbit plane, $r = |\mathbf{r}| = \sqrt{x^2 + y^2}$, $\mu(t)$ is a time function.

It should be noted that the Dirac cosmogonic hypothesis about secular variation of gravitational constant (Dirac, 1937, 1938) – in the two-body problem leads to the same equation.

The following laws of mass variation are known, remarkable for integrating (1) in quadrature

$$\mu(t) = (\alpha t + \beta)^{-1}, \tag{2}$$

$$\mu(t) = (\alpha t + \beta)^{-1/2}, \tag{3}$$

$$\mu(t) = (\alpha t^2 + \beta t + \gamma)^{-1/2}, \tag{4}$$

called the first, second, joint Meščerskii laws, respectively. Being stated mathematically they are physical based on the theory of interior structure and

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evolution of stars in the form of Edington–Jeans law (Jeans, 1925).

$$\dot{\mu} = -\kappa\mu^{\nu}, \tag{5}$$

where κ is a small positive number and exponent ν is supposed to be equal to 3. According to the theory mass loss is interpreted by electro-magnetic radiation. It is considered at present that corpuscular radiation (Ambartsumjan, 1952; Fessenkov, 1952) is mainly responsible for mass loss and exponent ν satisfies inequality $0.4 < \nu < 4.4$ (see, e.g. Hadjidemetriou, 1967) $\nu = 2$ and $\nu = 3$ corresponds to the Meščerskii's first and second laws, respectively. Duboshin (1930) proposed that mass variation laws differing from those mentioned above, are also realized physically in nature.

Natural generalization of Equation (5) may be obtained when removing a limitation of ν (specifically, provided $\nu = 0$ the mass varies according to the linear law, while at $\nu = 1$, according to the exponential one). The following mass variation laws are also known

$$\mu(t) = (1 + \lambda t)(1 + 2\lambda t)^{-2} \tag{6}$$

(Niţă, 1958 and 1973, carried out approximate analytical integration) and more general law, as well

$$\mu(t) = (\mathrm{d}t + \beta)(\gamma t + \delta)^{-2} \tag{7}$$

(Gelfgat, 1959 and 1968, carried out exact integration by means of solution uniformity through Airy junctions of auxiliary parameter).

All the known integrability cases permit reduction of problem (1) to the stationary form. On the other hand, the above mentioned mass variation laws are such that one can find suitable functions $v(t)$ and $u(t)$, at which problem (1) by variables transformation

$$\mathbf{r} = v(t)\boldsymbol{\rho}, \qquad \mathrm{d}\tau = u(t)\,\mathrm{d}t, \boldsymbol{\rho} = (\xi, \eta), \tag{8}$$

$$v(t), u(t) \in \mathbb{C}_I^2, \qquad u(t)v(t) \neq 0, \qquad \forall t \in I,$$

where I is the open, finite or non-finite time axis interval t, $\mathbb{C}_I^2$ is the twice continuously differentiated functions space over I is reduced to the stationary form (with fictitious time τ)

$$\boldsymbol{\rho}'' \pm b_1 \boldsymbol{\rho}' + b_0 \boldsymbol{\rho} = \frac{-\mu_0 \cdot \boldsymbol{\rho}}{\rho^3}, \qquad (') = \frac{\mathrm{d}}{\mathrm{d}\tau}, \tag{9}$$

b_0, μ_0 are the real constants and b_1 may be either a real or purely imaginary constant.

Note that, according to the well-known Stäkkel–Lie (Stäckel, 1893) theorem, (8) is the most general transformation keeping the equation order, the linearity of its linear part and the structure of the non-linear one.

The present paper, both stating the new and systematizing the known results, sums up the investigations in finding out mathematical mass variation laws and exact integrability of problem (1). It gives the answer to the question: What are all the possible mass variation laws for which problem (1) is reduced to (9) by transformation (8)?

The mass variation laws found are remarkable for their remaining invariant in total in respect to the symmetry group admitted by problem (1).

At present "it has become more natural to deduce the laws of nature and to, test them by means of invariance principles" (Wigner, 1970). The invariance (symmetry) principles play the part of the touchstone when testing the correctness of the supported natural law.

It turned out that Eddington–Jeans law did not stand the test. However, what the mass variation law admitted by problem (1) will be, its integration is reduced to the case of mass variation by Eddington–Jeans law for $1 \leqslant \nu \leqslant 3$.

It follows from the results received in the work that material mass can be neither periodic nor oscillatory. All the linear motions in problem (1) have been found. The review of the integrable cases of problem (1) from the group point of view is made. There are constructed the finite transformation groups, permitting us to divide the problem of integrating (1) into non-crossing classes in accordance with various mass variation laws. The ability to integrate problem (1) for one case of mass variation involves integration of the whole class by means of the known transformations.

Remark 1. In this paper either results are proved or corresponding references are made. The end of the proof is denoted by ■. When citing the literature references are made only to those works which are directly concerned with discussion of the questions set above. As for various aspects of the two-body variable mass problem, the reviews made by Duboshin (1930), Lapin (1944) and especially Mikhailov (1974, 1975) are quite complete and include a period of centenary development of non-stationary problems in celestial mechanics up to World War II. The reviews by Dommanget (1963) and Hadjidemetriou (1967) include the more up-to-date period.

2. Autonomization Method

The works by Omarov (1975), Berković and Gelfgat (1975) are dedicated to the investigation of broad classes of non-stationary problems in celestial mechanics; the latter work includes the general method of transformation of non-stationary problems into stationary ones.

One of the main questions solved in the present work is that of finding all the mass variation laws $\mu(t)$ for which and only for which (1) is reduced to (9) through transformation (8). Since the unknown laws are not only sufficient but also necessary for reduction of (1), (8), and (9), the question can be investigated neither by semireversal method nor by means of heuristic substitutions.

The solution is given on the basis of differential equation autonomization method in which the regular theory of differential equation transformation is developed (Berković, 1971, 1978), see also (Berković and Netchayevskii, 1976, 1979). This method will now be set forth in detail in conformity with the Gylden–Meščerskii problem.

LEMMA 1. *In order to reduce* (1) *by means of transformation* (8) *to* (9) *it is necessary* *and sufficient that the kernel $u(t)$ and the factor $v(t)$ of (8) satisfies the equations*

$$\frac{1}{2}\frac{\ddot{u}}{u}-\frac{3}{4}\left(\frac{\dot{u}}{u}\right)^2-\frac{1}{4}\delta u^2=0, \qquad \delta=b_1^2-4b_0, \tag{10}$$

$$\ddot{v}-b_0v^{-3}=0, \tag{11}$$

$$\ddot{v}-b_0b_1^{-2}v^{-3}\left(\int_{t_0}^{t}v^{-2}\,\mathrm{d}t\right)^{-2}=0, \quad b_1\neq 0, \tag{12}$$

where $v(t)$, $u(t)$ and $\mu(t)$ are connected by relations

$$v(t)=|u|^{-1/2}\exp\left(\pm\frac{1}{2}b_1\int_{t_0}^{t}u\,\mathrm{d}t\right), \tag{13}$$

$$\ddot{v}-b_0u^2(t)v=0, \tag{14}$$

$$\mu(t)=\mu_0u^2(t)v^3(t). \tag{15}$$

In this case problem (1) admits Lie group with infinitesimal operator

$$X=\frac{1}{u}\frac{\partial}{\partial t}+\frac{\dot{v}}{uv}\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right) \tag{16}$$

and has particular solutions

$$\mathbf{r}=v(t)\boldsymbol{\lambda}, \qquad \lambda^3=-\mu_0b_0^{-1}, \quad b_0\neq 0. \tag{17}$$

Proof. Let us note first of all that the initial non-linear non-autonomous vectorial equation represents a sum of two items, one of which is the linear part $\ddot{\mathbf{r}}$, and the other – non-linear one – $|u(t)\mathbf{r}/r^3$.

Necessity. Let us apply transformation (8) to (1) and put on reduction requirement (1) to (9). As a result (9) will become identical to the equation

$$\boldsymbol{\rho}''-\frac{1}{u^2}\left(2\frac{\dot{v}}{v}u+\dot{u}\right)\boldsymbol{\rho}'+\frac{\ddot{v}}{u^2v}\boldsymbol{\rho}=-\frac{\mu(t)}{u^2v^3}\frac{\boldsymbol{\rho}}{\rho^3} \tag{18}$$

whence it immediately follows (14) and (15).

Equating the factor at $\boldsymbol{\rho}'$ we come to the linear equation with respect to v:

$$\dot{v}=\left(-\frac{1}{2}\frac{\dot{u}}{u}\pm\frac{b_1}{2}u\right)v, \tag{19}$$

by integration of which we get (13). Having substituted then (13) into (14) we obtain (10). Having chosen u as a dependent variable we get from (19) Bernoulli equation which has as solutions

$$u=v^{-2} \quad \text{if} \quad b_1=0, \tag{20}$$

$$u = \pm \frac{v^{-2}}{b_1 \int_{t_0}^{t} v^{-2}\,\mathrm{d}t} \quad \text{if } b_1 \neq 0. \tag{20'}$$

The substitution of (20) and (21) to (14) leads to (11) and (12), respectively.

To get Equation (17) note that a solution of Equation (19) is $\boldsymbol{\rho} = \boldsymbol{\lambda}$, where $\lambda = |\boldsymbol{\lambda}|$ satisfies the relation $\lambda^3 = -\mu_0 b_0^{-1}, b_0 \neq 0$. Then, because of (8) we arrive at particular solutions (17) which present linear motions in problem (1).

The statement that (1) admits one parameter Lie-group follows from the requirement of reduction (1) to autonomous form (9); in this case corresponding infinitesimal operator

$$X = \xi(t, x, y)\frac{\partial}{\partial t} + \eta(t, x, y)\frac{\partial}{\partial x} + \zeta(t, x, y)\frac{\partial}{\partial y}$$

takes the form (16) (Berković, 1971, and also Wilczynski, 1906)

Sufficiency. When all the statements of Lemma 1 are fulfilled. Then transformation (8) reduces (1) to (18). Since all the requirements of (13)–(15) are satisfied, (18) is identical to (9).■

Remark 2. Name Equation (10) after Kummer–Schwarz, Equation (11) after Ermakov and transformation

$$\mathbf{r} = |u|^{-1/2} \exp\left(\pm \tfrac{1}{2} b_1 \int_{t_0}^{t} u\,\mathrm{d}t \right) \boldsymbol{\rho}, \qquad \mathrm{d}\tau = u\,\mathrm{d}t \tag{21}$$

after Kummer–Liouville who were concerned with the said equations and transformation (Kummer, 1834; Ermakov, 1880). For $b_1 = 0$ (21) becomes Nechvile transformation (1926)

$$\mathbf{r} = |u|^{-1/2}\boldsymbol{\rho}, \qquad \mathrm{d}\tau = u\,\mathrm{d}t, \tag{22}$$

which has been used earlier by Liouville (1837).

LEMMA 2. *The general solution of Equation* (10) *is*

$$u(t) = (\alpha_1 t + \beta_1)^{-1}(\alpha_2 t + \beta_2)^{-1}, \qquad \delta = (\alpha_1\beta_2 - \alpha_2\beta_1)^2 > 0,$$

$$u(t) = (At^2 + Bt + C)^{-1}, \qquad \delta = B^2 - 4AC < 0,$$

$$u(t) = (\alpha t + \beta)^{-2}, \qquad \delta = 0,$$

the important particular cases of which are represented by formula

$$u(t) = (\alpha t + \beta)^{-1}, \qquad u(t) = 1.$$

The validity of Lemma 2 *may be verified by means of direct test.* ■

LEMMA 3. (a) *The general solution of Equation* (11) *is*

$$v(t)=\sqrt{(\alpha_1 t+\beta_1)(\alpha_2 t+\beta_2)}, \qquad -4b_0=(\alpha_1\beta_2-\alpha_2\beta_1)^2>0,$$

$$v(t)=\sqrt{At^2+Bt+C}, \qquad -4b_0=B^2-4AC<0,$$

$$v(t)=\alpha t+\beta, \qquad b_0=0,$$

the important particular cases of which are

$$v(t)=(\alpha t+\beta)^{1/2}, \qquad v(t)=1.$$

(b) *The general solution of integro-differential Equation* (12) *is*

$$v(t)=(\alpha_1 t+\beta_1)^{1/2\pm b_1/2\sqrt{\delta}}(\alpha_2 t+\beta_2)^{1/2\mp b_1/2\sqrt{\delta}}, \qquad \delta>0, \tag{23}$$

$$v(t)=(At^2+Bt+C)^{1/2}\exp\left(\pm\frac{b_1}{\sqrt{-\delta}}\operatorname{arctg}\frac{2At+B}{\sqrt{-\delta}}\right), \quad \delta<0, \tag{24}$$

$$v(t)=(\alpha t+\beta)\exp\left(\mp\frac{b_1}{2\alpha(\alpha t+\beta)}\right), \quad \delta=0, \tag{25}$$

the limiting cases of which are

$$v(t)=(\alpha t+\beta)^{1/2\pm b_1/2\alpha}, \tag{26}$$

$$v(t)=\exp\left(\pm\tfrac{1}{2}b_1 t\right) \tag{27}$$

(b_1 *may everywhere take both real and purely imaginary values*).

Proof. (a) Since Equations (11) and (10) are interconnected by transformation of $v=u^{-1/2}$, then from Lemma 2 we get the statements of Lemma 3a; special cases of Lemma 2 leads here to special cases of Lemma 3.

(b) Due to (13) we get from Lemma 2 the relations of Lemma 3b. ■

Instead of non-linear Equations (11) and (12) one may consider linear Equation (14) with variable factor $-b_0u^2(t)$, which takes different expressions according to Lemma 2. Then we obtain an equation with rational factors

$$\ddot{v}+\frac{f}{(at^2+bt+c)^2}v=0, a, b, c, f-\text{const}, \tag{28}$$

which arises in various problems of mechanics and mathematics. It was Besge (1844) who first pointed out the principle possibility of integration (28) in the finite form through elementary functions though he did not integrate.

Further complete integration of (28) will be given of various correlations between the parameters included in it, according to Berković (1978).

LEMMA 4. (1) *Let* $\delta=(\alpha_1\beta_2-\alpha_2\beta_1)^2>0$, *Equations* (12) *and* (28) *take the form*

$$\ddot{v}+\tfrac{1}{4}[(\alpha_1\beta_2-\alpha_2\beta_1)^2-b_1^2](\alpha_1 t+\beta_1)^{-2}(\alpha_2 t+\beta_2)^{-2}v=0$$

and has as a basis (21), *if* $b_1 \neq 0$, *or* (*for* $b_1 = 0$)

$$\left\{\sqrt{(\alpha_1 t+\beta_1)(\alpha_2 t+\beta_2)}, \sqrt{(\alpha_1 t+\beta_1)(\alpha_2 t+\beta_2)}\ln\left|\frac{\alpha_1 t+\beta_1}{\alpha_2 t+\beta_2}\right|\right\}.$$

(2) *Let* $\delta = B^2 - 4AC < 0$, *Equations* (14) *and* (28) *take the form*

$$\ddot{v} + \tfrac{1}{4}(B^2 - 4AC - b_1^2)(At^2 + Bt + C)^{-2} v = 0$$

and has as a basis (22) *if* $b_1 \neq 0$, *or* (*for* $b_1 = 0$)

$$\left\{\sqrt{At^2+Bt+C}, \sqrt{At^2+Bt+C}\,\mathrm{arctg}\,\frac{2At+B}{\sqrt{-\delta}}\right\},$$

(3) *Let* $\delta = 0$, *Equations* (14) *and* (28) *take the form* $\ddot{v} - \frac{1}{4}b_1^2(\alpha t + \beta)^{-4} v = 0$ *and has as a basis* (28) *if* $b_1 \neq 0$ *or* (*for* $b_1 = 0$) $\{1, t\}$;

(4) *Let* $\delta = \alpha^2 > 0$ *Equations* (14) *and* (28) *degenerate into Euler–Legendre equation*

$$\ddot{v} + \tfrac{1}{4}(\alpha^2 - b_1^2)(\alpha t + \beta)^{-2} v = 0$$

and has as a basis (24), *if* $b_1 \neq 0$ *or* (*for* $b_1 = 0$),

$$\{\sqrt{\alpha t+\beta}, \sqrt{\alpha t+\beta}\ln|\alpha t+\beta|\}.$$

(5) *Let* $\delta = 0$, $u = 1$, *Equations* (14) *and* (28) *degenerate into the equation with constant coefficient* $\ddot{v} - \frac{1}{4}b_1^2 v = 0$ *and has as a basis* (27) *if* $b_1 \neq 0$, *or* (*for* $b_1 = 0$) *has as a basis* $\{1, t\}$.

Proof. The proof of Lemma 4 is based on Lemmas 2 and 1. Furthermore, the results received may be verified directly. ■

3. The Laws of Mass Variation in Differential and Integro-Differential Form

As has been already mentioned in section 2, autonomization method permits reduction of non-stationary problem (1) to the stationary form (9). All the mass variation laws may be stated in this case.

THEOREM 1. *In order to reduce* (1) *to* (9) *by transformation* (29) (*for* $b_1 = 0$) *it is necessary and sufficient that mass satisfies the differential equation*

$$\ddot{\mu} - \frac{2}{\mu}\dot{\mu}^2 + b_0\mu^5 = 0. \tag{29}$$

Here (22), (16) *and* (17) *take respectively the form*

$$\mathbf{r} = \mu^{-1}\boldsymbol{\rho}, \qquad \mathrm{d}\tau = \mu^2\,\mathrm{d}t, \tag{30}$$

$$X = \mu^{-2}\frac{\partial}{\partial t} - \mu^{-3}\dot{\mu}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right), \tag{31}$$

$$\mathbf{r} = \mu^{-1}\boldsymbol{\lambda}, \qquad \lambda^3 = -\mu_0 b_0^{-1}, \quad b_0 \neq 0. \tag{32}$$

Proof. Necessity. Since by (22) $v = u^{-1/2}$, then (15) takes the form $\mu(t) = \mu_0 u^{1/2}$, hence we get

$$u(t) = \mu^2 \,\mathrm{d}t, \quad v(t) = \mu^{-1}(t), \tag{33}$$

if we take $\mu_0 = 1$.

Having substituted (33) into the equation

$$\frac{1}{2}\frac{\ddot{u}}{u} - \frac{3}{4}\left(\frac{\dot{u}}{u}\right)^2 + b_0 u^2 = 0 \tag{34}$$

and also into (22), (16) and (17) we come to (27)–(32), respectively.

Sufficiency. Let $\mu(t)$ satisfy (29). By means of substitutions $\mu(t) = u^{-2}$, $\mu(t) = v^{-1}$, (29) is reduced to Equations (34) and (11) respectively, which is the criterion of reduction of (1) to (9) (for $b_1 = 0$). ■

THEOREM 2. *In order to reduce problem* (1) *by transformation* (21) *to* (9) ($b_1 \neq 0$) *it is necessary and sufficient that mass $\mu(t)$ satisfies the integro-differential equation*

$$\ddot{\mu} - 2\mu^{-1}\dot{\mu}^2 + \frac{2b_1^2 + b_0}{9b_1^2}\mu^5\left(\int_{t_0}^{t} \mu^2 \,\mathrm{d}t\right)^{-2} = 0, \tag{35}$$

in this case the transformation (21), *admitted by* (1) *infinitesimal operator X, and particular solutions take respectively the form*

$$\mathbf{r} = \mu^{-1}\left(3b_1 \int_{t_0}^{t} \mu^2 \,\mathrm{d}t\right)^{2/3} \mathbf{p}, \qquad \mathrm{d}\tau = \mu^2\left(\pm 3b_1 \int_{t_0}^{t} \mu^2 \,\mathrm{d}t\right)^{-1} \mathrm{d}t, \tag{36}$$

$$X = \mu^{-2}\left(\pm 3b_1 \int_{t_0}^{t} \mu^2 \,\mathrm{d}t\right)\frac{\partial}{\partial t} + \left[\mu^{-3}\left(\mp 3b_1 \int_{t_0}^{t} \mu^2 \,\mathrm{d}t\right)\dot{\mu} \pm 2b_1\right]\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right), \tag{37}$$

$$\mathbf{r} = \mu^{-1}\left(3b_1 \int_{t_0}^{t} \mu^2 \,\mathrm{d}t\right)^{2/3} \boldsymbol{\lambda}, \qquad \lambda^3 = -\mu_0 b_0^{-1}, \quad b_0 \neq 0. \tag{38}$$

Proof. Necessity. Having substituted (13) to (15) we get

$$\mu(t) = \mu_0 u^{1/2} \exp\left(\pm \tfrac{3}{2} b_1 \int_{t_0}^{t} u \,\mathrm{d}t\right), \quad \mu_0 = 1, \tag{39}$$

hence we get Bernoulli equation with respect to u:

$$\dot{u} = 2\frac{\dot{\mu}}{\mu}u \mp 3b_1 u^2, \quad b_1 \neq 0,$$

the solution of which is the function

$$u = \mu^2 \left(\pm 3b_1 \int_{t_0}^{t} \mu^2 \, dt + \kappa \right)^{-1}, \tag{40}$$

where κ is the integration constant. When substituting (40) to (10), (21), (16) and (17) we obtain respectively (35)–(38).

Sufficiency. Let the mass $\mu(t)$ satisfy Equation (35). By substitution (39) and

$$\mu(t) = b_1^{-2} v^{-1} \left(\int_{t_0}^{t} v^{-2} \, dt \right)^{-2}$$

(the latter is received from (15) by (20′)) Equation (35) is reduced to (10) and (12), respectively. But this is the criterion of reduction of (1) to (9) by transformation (21). ■

Remark 3. We shall name Equations (29) and (35) respectively the differential and the integro-differential mass variation laws.

The work by Lapin (1944) contains Equation (29). The united integro-differential mass variation law, containing as particular cases (29) and (35) has the following form

$$\ddot{\mu} - 2\mu^{-1}\dot{\mu}^2 + \frac{(2b_1^2 + b_0)\mu^5}{(\kappa + 9b_1^2)\left(\int_{t_0}^{t} \mu^2 \, dt \right)^2} = 0 \begin{cases} \kappa = 1, & \text{if } b_1 = 0 \\ \kappa = 0, & \text{if } b_1 \neq 0. \end{cases} \tag{41}$$

4. Mass Variation Laws in the Finite Form and Linear Motions in Problem (1)

In Section 3 mass variation laws in differential and integro-differential form were stated. However, in many cases it is more convenient to use the finite form of these. In addition the finite form permits us to compare the known results recently received.

Mass variation laws in the finite form may be derived by integration of Equation (11).

4.1. INDIRECT INTEGRATION METHOD (41)

THEOREM 3. *All the laws of mass variation with time $\mu(t)$ in problems* (1), (8) *and* (9) *are given by means of the following finite equations*

$$\mu(t) = (\alpha_1 t + \beta_1)^{-1/2 \pm 3b_1/2\sqrt{\delta}} (\alpha_2 t + \beta_2)^{1/2 \mp 3b_1/2\sqrt{\delta}}, \quad \delta > 0, \alpha_1 \neq 0, \alpha_2 \neq 0; \tag{42}$$

$$\mu(t) = (At^2 + Bt + C)^{-1/2} \exp\left(\pm \frac{3b_1}{2\sqrt{-\delta}} \operatorname{arctg} \frac{2At + B}{\sqrt{-\delta}} \right), \quad \delta < 0; \tag{43}$$

$$\mu(t) = (\alpha t + \beta)^{-1} \exp\left[\mp \frac{3b_1}{2\alpha(\alpha t + \beta)} \right], \quad \delta = 0; \tag{44}$$

$$\mu(t) = (\alpha t + \beta)^{-1/2 \pm 3b_1/2\alpha}, \quad \delta = \alpha^2; \tag{45}$$

$$\mu(t) = \mu_0 \exp(\pm \tfrac{3}{2} b_1 t), \quad \delta = 0 \tag{46}$$

(b_1 *takes both real and purely imaginary values*; $\alpha, \beta, \gamma, \delta, A, B, C, \mu_0$ *the arbitrary constants.*).

Proof. The proof may be obtained indirectly on the basis of relations between the functions $(u(t), v(t), \mu(t))$ [formula (15)]. The former ones are found by Lemmas 2, and 3 or 2 and 4. ■

Consequence 1. Real mass $\mu(t)$ in problems (1), (8) and (9) cannot vary neither according to periodic nor oscillatory laws.

Consequence 2. To reduce problem (1) by means of transformation (8) to the classical two-body problem

$$\boldsymbol{\rho}'' = -\frac{\boldsymbol{\rho}}{\rho^3}, \tag{47}$$

it is necessary and sufficient that the mass $\mu(t)$ satisfies the equation

$$\ddot{\mu} - 2\mu^{-1}\dot{\mu}^2 = 0, \tag{48}$$

i.e. varies according to the Meščerskii first law (2).

Consequence 3. To reduce problem (1) to the stationary form

$$\boldsymbol{\rho}'' + b_0 \boldsymbol{\rho} = \frac{-\mu_0 \boldsymbol{\rho}}{\rho^3}, \quad b_0 \neq 0 \tag{49}$$

by transformation (6) (motion in gravitation medium) it is necessary and sufficient that the mass $\mu(t)$ satisfies the differential Equation (29) and was described by the following finite equations

$$\mu(t) = (\alpha t + \beta)^{1/2}, \tag{50}$$

$$\mu(t) = [(\alpha_1 t + \beta_1)(\alpha_2 t + \beta_2)]^{-1/2}, \quad \delta > 0 \tag{51}$$

$$\mu(t) = (At^2 + Bt + C)^{-1/2}, \quad \delta < 0. \tag{52}$$

Consequence 4. To reduce problem (12) by transformation (8) to the stationary form

$$\boldsymbol{\rho}'' + b_1 \boldsymbol{\rho}' = \frac{-\mu_0 \boldsymbol{\rho}}{\rho^3}, \tag{53}$$

it is necessary and sufficient that mass $\mu(t)$ satisfies integro-differential equation

$$\ddot{\mu} - 2\mu^{-1}\dot{\mu}^2 + \tfrac{2}{9}\mu^5\left(\int_{t_0}^{t} \mu^2 \, dt \right)^{-2} = 0 \tag{54}$$

and was described by the following finite equations

$$\mu(t) = \alpha t + \beta, \tag{55}$$

$$\mu(t) = (\alpha t + \beta)^{-2}, \tag{56}$$

$$\mu(t) = \begin{cases} (\alpha_1 t + \beta_1)(\alpha_2 t + \beta_2)^{-2} \\ (\alpha_1 t + \beta_1)^{-2}(\alpha_2 t + \beta_2). \end{cases} \tag{57}$$

Note, that for $b_1 > 0$ (53) is the equation of the constant mass body motion affected by the force of Newton's attraction to some centre and the resistance force, proportional to the rate; instead of the latter for $b_1 < 0$ we have tangential accelerating force proportional to the rate.

THEOREM 3.1. *The united mass variation law in the complex-valued form in problems* (1), (8) *and* (9) *may be given as* (42).

Proof. Formula (43) has complex-valued form (42), (44) is the limiting case of (42); (45) is the particular case of (42), and (46) is the limiting case of (45). ■

Remark 4. Lovett (1902), who believed that (1) is reduced to (47) at linear mass variation law (55) but not at (2) criticized Meščerskii's work (1893). In fact, as Meščerskii showed (1902) law (2) is correct. As for (55) in this case, according to consequence 4 (1) is reduced to (53). From (42) – (46) for $b_1 = 0$ Meščerskii laws follow.

Formulas (45) and (46) present the finite form of the general Eddington–Jeans law. Equation (42) in real form was obtained by Berković and Gelfgat (1972, 1975), Equation (42) (in complex-valued form), (43) and (44), and also integro-differential mass variation laws (35) and (41) – by the author.

4.2. DIRECT INTEGRATION TECHNIQUE OF (41). EXACT LINEARIZATION

Because of the importance of mass variation laws it is worth mentioning one more method for obtaining them. Consider now direct integration method of (41) and in particular (29), based on the exact linearization method (Berković, 1979). One may obtain Theorem 3, giving the finite form of mass variation laws, by the following direct way.

LEMMA 5. *Commutative factorization*

$$(\mu^{-2}D - r_2)(\mu^{-2}D - r_1)\mu = 0 \tag{58}$$

corresponds to Equation (29), $r_1 r_2 = b_0$, $r_1 + r_2 = 0$, *it admits particular solutions*

$$\mu = (-2rt + \beta)^{-1/2}, \qquad r^2 + b_0 = 0 \tag{59}$$

(*β-integration constant*), *by transformation*

$$d\tau = \mu^2\, dt \tag{60}$$

is reduced to the linear form

$$\mu'' + b_0 \mu = 0 \tag{61}$$

and has the general solution of type (51) *and* (52).

Proof. If we multiply Equation (29) by μ^{-4}, then it is easy to verify that factorization (58) is valid. Due to commutativity of (58), Equation (29) has particular solutions being first-order equation integrals

$$\mu^{-2}\dot{\mu} - r_\kappa \mu = 0, \quad \kappa = 1, 2 \tag{62}$$

from which (59) follows. It is not difficult to verify that Equation (29) is reduced to (61) by transformation of (60).

To obtain (51) and (52) write the solution of Equation (61) in the following form (for $b_0 > 0$)

$$\mu(r) = \sqrt{\frac{A}{b_0}} \cos \sqrt{b_0}\, \tau, \tag{63}$$

where A is an arbitrary constant (analogous reasoning is given also for the case of $b_0 < 0$). Having substituted (63) to (60) and integrated, we obtain

$$\mathrm{tg} \sqrt{b_0}\, \tau = \frac{A}{\sqrt{b_0}}(t + C)$$

(C is integration constant). As $\sqrt{b_0}\,\tau = \mathrm{arctg}\,(A/\sqrt{b_0})(t + C)$, then due to (63) we get

$$\mu(t) = \left(At^2 + 2ACt + AC^2 + \frac{b_0}{A} \right)^{-1/2}.$$

Having taken $B = 2AC$, $C = B^2/4A + b_0/A$, we obtain (52). Since no restrictions are put on square trinomial discriminant, the formula found includes (51) as well. ■

LEMMA 6. *Commutative factorization*

$$\left(\frac{\pm 3b_1 \int \mu^2\, \mathrm{d}t}{\mu^2} D - r^2 \right) \left(\frac{\pm 3b_1 \int \mu^2\, \mathrm{d}t}{\mu^2} D - r_1 \right) \mu = 0 \tag{64}$$

where $r_{1,2}$ satisfy the algebraic equation

$$r^2 \mp 3b_1 r + b_0 + 2b_1^2 = 0, \tag{65}$$

corresponds to integro-differential Equation (35); *it admits particular solutions*

$$\mu(t) = (\alpha t + \beta)^{-1/2 \pm 3b_1/2\sqrt{\delta}} \tag{66}$$

(α, β *are the arbitrary constants,* $\delta = b_1^2 - 4b_0$): *by transformation* $\mathrm{d}\tau = \mu^2\, \mathrm{d}t/(\pm 3b_1 \int \mu^2 \mathrm{d}t)$ (35) *is reduced to the linear-differential equation*

$$\mu'' \mp 3b_1 \mu' + (2b_1^2 + b_0)\mu = 0 \tag{67}$$

and has general solution of types (42) *and* (43).

Proof. If we multiply *Equation* (35) *by* $9b_1^2(\int\mu^2\,dt)^2\mu^{-4}$, then it is not difficult to verify, that factorization (64) is valid providing (65). Due to commutativity of factorization (64) Equation (35) has particular solutions, satisfying integro-differential equations

$$\frac{\int\mu^2\,dt}{\mu^2}\dot{\mu}-\frac{r_\kappa}{\pm 3b_1}\mu=0,\quad \kappa=1,2. \tag{68}$$

By dividing variables we come to the equations

$$\frac{d\mu}{\mu}=\frac{r_\kappa}{\pm 3b_1}\frac{\mu^2\,dt}{\int\mu^2\,dt}$$

by integration of which we obtain integral equations

$$\ln C\mu=\frac{r_\kappa}{\pm 3b_1}\ln\int\mu^2\,dt \tag{69}$$

(C is the integration constant). By use of potentiation and differentiation we arrive at the first-order differential equation

$$\frac{\pm 3b_1}{r_\kappa}\mu^{\pm 3b_1/r_\kappa-1}\dot{\mu}=C\mu^2 \tag{70}$$

by integration of which we come to the solutions (66) corresponding to $r_{1,2}=(\pm 3b_1\pm\sqrt{\delta})/2$.

To deduce formulas (42) and (43) let us write the solution of Equation (67) as

$$\mu=\frac{2\sqrt{A}}{\sqrt{-\delta}}\exp\left(\pm\frac{3b_1}{2}r\right)\cos\left(\frac{\sqrt{-\delta}}{2}\tau\right) \tag{71}$$

for $\delta<0$, A is the arbitrary constant (analogous reasoning is given for $\delta>0$, as well). Let us write down τ in the form of

$$\tau=\frac{1}{\pm 3b_1}\ln(\pm 3b_1\int\mu^2\,dt), \tag{72}$$

hence, due to (71) we get

$$\operatorname{arc\,cos}\frac{\sqrt{-\delta}}{2\sqrt{A}}\frac{\mu}{\sqrt{\pm 3b_1\int\mu^2\,dt}}=\frac{\sqrt{-\delta}}{\pm 6b_1}\ln(\pm 3b_1\int\mu^2\,dt). \tag{73}$$

Let us differentiate the integral Equation (73). Then we come to the integro-differential equation.

$$\frac{-\sqrt{-\delta}/(2\sqrt{A})(d/dt)(\mu/\sqrt{\pm 3b_1\int\mu^2\,dt})}{\sqrt{1+(\delta/4A)\mu^2/(\pm 3b_1\int u^2\,dt)}}=\frac{\sqrt{-\delta}}{2}\frac{\mu^2}{\pm 3b_1\int\mu^2\,dt}. \tag{74}$$

Having introduced the designation

$$v = \frac{\mu}{\sqrt{\pm 3b_1 \int \mu^2 \, dt}}, \tag{75}$$

we write (74) as

$$-\frac{1}{\sqrt{A}} \frac{\dot{v}}{\sqrt{1 + \delta/4A}} = v^2. \tag{76}$$

Integrating (76), we obtain

$$v(t) = (At^2 + Bt + C)^{-1/2}, \quad \delta = B^2 - 4AC. \tag{77}$$

Now let us solve the integral equation

$$(At^2 + Bt + C)^{-1/2} = \mu(\pm 3b_i \int \mu^2 \, dt)^{-1/2}. \tag{78}$$

From (78) we get the relation

$$\pm 3b_1 \int \mu^2 \, dt = \exp\left(\pm \frac{6b_1}{\sqrt{-\delta}} \operatorname{arctg} \frac{2At + B}{\sqrt{-\delta}} \right), \tag{79}$$

from which (43) immediately follows. As has been mentioned earlier, (42) may be obtained by the analogous way, the function $v = (\alpha_1 t + \beta_1)^{-1/2}(\alpha_2 t + \beta_2)^{-1/2}$ being the solution of (76) for $\delta > 0$. ■

From Lemmas 5 and 6 follow Theorem 3.

4.3. LINEAR MOTIONS IN PROBLEM (1).

In conclusion let us consider the question about linear motions in problem (1).

THEOREM 4. (a) *In order that problem* (1) *admits linear motions* (17) *it is necessary and sufficient that the mass* $\mu(t)$ *satisfies Equation* (35) *and consequently to one of the laws* (42)–(46) (*for* $b_0 \neq 0$).

(b) *If* $\mu(t)$ *satisfies* (42)–(46), *then as* $v(t)$ *one should choose respectively* (23)–(27).

Proof. The proof of the theorem follows immediately from Lemmas 2 and 3 and Theorem 3. ■

5. About Gylden–Meščerskii Problem Transformation Into Itself

As is known, the most characteristic properties of the physical problem are those which remain invariant when transforming it into itself by variable substitutions "... mathematical laws, governing the nature, are the source of symmetry in nature" (Weyl, 1952).

Find the transformation of Kummer–Liouville type (8), transforming Gylden–Meščerskii problem (1) into itself, i.e. to

$$\rho'' = -\mu_1(\tau)\frac{\rho}{\rho^3}, \quad (') = \frac{d}{d\tau}. \tag{80}$$

LEMMA 7. *To reduce* (1) *to* (80) *by transformation* (8) *it is necessary and sufficient that the kernel u(t) and the factor v(t) of transformation* (8) *satisfies equations*

$$\frac{1}{2}\frac{\ddot{u}}{u}-\frac{3}{4}\left(\frac{\dot{u}}{u}\right)^2=0, \tag{81}$$

$$\ddot{v}=0, \tag{82}$$

where v and u are connected by relation (20)

$$\mu_1(\tau)=\mu(t)u^{-2}(t)v^3(t), \tag{83}$$

and the new independent variable τ(t) satisfies the third-order Kummer–Schwarz equation

$$\{r,t\}\equiv\frac{1}{2}\frac{\dddot{\tau}}{\dot{\tau}}-\frac{3}{4}\left(\frac{\ddot{\tau}}{\dot{\tau}}\right)^2=0. \tag{84}$$

Proof. The proof follows immediately from Lemma 1. ■

THEOREM 5. *In order to reduce* (1) *to* (80) *by use of transformation* (8) *it is necessary and sufficient that* (8) *takes the form of*

$$\mathbf{r}=(\alpha t+\beta)\boldsymbol{\rho},\qquad d\tau=(\alpha t+\beta)^{-2}\,dt,\qquad \tau=\frac{\alpha_1 t+\beta_1}{\alpha_2 t+\beta_2}, \tag{85}$$

where $\alpha_1\beta_2-\alpha_2\beta_1\neq 0$; *in this case* (1) *admits one-parameter Lie group* G_1 :

$$G_1: t_1=\frac{(a\alpha\beta+1)t+a\beta^2}{1-a\alpha(\alpha t+\beta)},\quad x_1=\frac{x}{1-a\alpha(\alpha t+\beta)},$$

$$y_1=\frac{y}{1-a\alpha(\alpha t+\beta)}, \tag{86}$$

$$X=(\alpha t+\beta)^2\frac{\partial}{\partial t}+\alpha(\alpha t+\beta)\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right). \tag{87}$$

Proof. The structure of transformation (85) follows immediately from Lemmas 3 and 2. Then on the basis of Lemma 1 the infinitesimal operator (16) is described by (87). Now find the finite equations of group G_1. Still as Sophus Lie (1893) has shown they may be found by means of the following expansions into infinite series

$$t_1=\exp(aXt)=\sum_{\kappa=0}^{\infty}\frac{a^\kappa}{\kappa!}X^\kappa t,$$

$$x_1=\exp(aXx)=\sum_{\kappa=0}^{\infty}\frac{a^\kappa}{\kappa!}X^\kappa x,$$

$$y_1=\exp(aXy)=\sum_{\kappa=0}^{\infty}\frac{a^\kappa}{\kappa!}X^\kappa y,$$

Since

$$Xt = (\alpha t + \beta)^2, \quad X^2 t = 2\alpha(\alpha t + \beta)^3, \quad \dots, \quad X^n t = n!\alpha^{n-1}(\alpha t + \beta)^{n+1},$$

then

$$t' = t + a(\alpha t + \beta)^2 \sum_{s=0}^{\infty} a^s \alpha^s (\alpha t + \beta)^s = t + \frac{a(\alpha t + \beta)^2}{1 - a\alpha(\alpha t + \beta)}.$$

Hence we come to the first of the formulas (86). As

$$Xx = \alpha(\alpha t + \beta)x, \quad X^2 x = 2\alpha^2(\alpha t + \beta)^2 x, \quad \dots, \quad X^n x = n!\alpha^n(\alpha t + \beta)^n x,$$

then

$$x^1 = x + a\alpha(\alpha t + \beta)x + \dots + a^n\alpha^n(\alpha t + \beta)^n x + \dots,$$

hence we come to the second of the formulas (86). The third formula (86) may be found analogously. ■

THEOREM 6. *If* (1) *is invariant under the transformation* (85), (86) *and* (87), *then the integro-differential Equation* (41) *is invariant under the transformation*

$$\mu = (\alpha t + \beta)^{-1}\mu_1, \qquad \mathrm{d}\tau = (\alpha t + \beta)^{-2}\,\mathrm{d}t, \tag{88}$$

i.e. (41) *admits one-parameter Lie group*

$$G_1 : t^1 = \frac{(a\alpha\beta + 1)t + a\beta^2}{1 - a\alpha(\alpha t + \beta)}, \qquad \mu^1 = \mu[1 - a\alpha(\alpha t + \beta)] \tag{89}$$

with the infinitesimal operator

$$X = (\alpha t + \beta)^2 \frac{\partial}{\partial t} - \alpha(\alpha t + \beta)\mu \frac{\partial}{\partial \mu}. \tag{90}$$

Proof. Necessity. Applying transformation (85) to (1) we come, according to (83), to formulas (88). It is verified directly that by substitution of (88) substituting $t \to \tau$, $\mu \to \mu_1$. Equation (41) is transformed into itself. Proceeding from (88) and (16) we get (89). Finally, we obtain the second of the (89) (the first formula (89) has been already deduced earlier) from (88) using Sophus Lie series method

$$\mu^1 = \exp(aX\mu) = \sum_{\kappa=0}^{\infty} \frac{a^\kappa}{\kappa!} X^\kappa_\mu = \mu + aX\mu = \mu[1 - a\alpha(\alpha t + \beta)]$$

as $X\mu = -\alpha(\alpha t + \beta)\mu$, $X^\kappa \mu = 0, \forall \kappa > 1$.

Thus, problem (1) is transformed into itself (invaried with respect to G_1 symmetry group (86)), if $\mu(t)$ satisfies Equation (41). ■

What happens to the integral curves of Equation (41), when it is being transformed (88)–(90)? The next lemma gives the answer to the question.

LEMMA 8. *The set of integral curves* (42)–(46) *of Equation* (41) *is invariant in total with respect to the transformation* (88)–(90), *in this case the curve families* (42) *and* (43) *turn into* (45), *the curve family* (44) *into* (46) *and the curves of the family* (45) *replace one another.*

Proof. Apply (88) to (42). We have successively

$$\mu_1 = \left(\frac{\alpha_2 t + \beta_2}{\alpha_1 t + \beta_1}\right)^{-1/2 \mp 3b_1/2\sqrt{\delta}}, \qquad \tau = -\frac{1}{\alpha(\alpha t + \beta)} + C, \tag{91}$$

C is the integration constant. Having taken

$$C = \frac{\alpha_2 - \alpha_1 \beta_1}{\alpha_1^2}, \quad \alpha_1 \beta_2 - \alpha_2 \beta_1 = -\alpha_1 = \sqrt{\delta}$$

we come to the expression

$$\alpha_1 \tau + \beta_1 = \frac{\alpha_2 t + \beta_2}{\alpha_1 t + \beta_1},$$

hence $\mu_1 = (\alpha_1 \tau + \beta_1)^{-1/2 \pm 3b_1/2\alpha_1}$, i.e. we get the curves of type (45). Apply now (88) to (45). Putting $C = -\beta/\alpha$ in (91) we come to the curves.

$$\mu_1(\tau) = \lambda(\alpha\tau + \beta)^{-1/2 \mp 3b_1/2\alpha}, \quad \lambda = (-1)^{-1/2 \pm 3b_1/2\alpha}.$$

(The value of $\lambda = 1$ corresponds to the interattracting masses and the value of $\lambda = -1$ to the inter-repulsive ones.) Finally, having transformed the curve family (44) by means of (88) we obtain the curve family (46). ■

Remark 5 It has already been mentioned that the most important part of the invariancy principles (symmetry) is to be the touchstone for testing 'candidates' to the nature laws. Due to Lemma 8 the general Eddington–Jeans law does not remain invariant with respect to transformation (89). However, this law is still of great importance for problem (1) integration.

THEOREM 7. [Radzievsky and Gelfgat, 1957]. (1) *If in problem* (1) *the mass* $\mu(t)$ *varies according to the Eddington–Jeans law* (5) *with arbitrary index* v, *then as the result of application* (88) *the transformed mass* $\mu_1(\tau)$ *varies according* (5) *with the index*

$$v_1 = \frac{2v - 3}{v - 2}; \tag{92}$$

(2) *The integration of* (1) *under* (5) *with arbitrary index* v *is reduced to the integration of* (1) *under the same law, but with the index* v *satisfying the inequality* $1 \leqslant v \leqslant 3$.

Proof. (1) By integration of the differential Equation (5) we find that $\mu = (\alpha t + \beta)^{1/(1-v)}$, hence $\alpha t + \beta = \mu^{1-v}$. Then transformation (88) becomes non-linear:

$$\mu = \mu_1^{1/(2-v)}, \qquad d\tau = \mu_1^{(2v-2)/(2-v)}\, dt. \tag{93}$$

from which we come to the equation

$$\mu_1'(\tau) = -\kappa_1 \mu_1^{(2\nu-3)/(\nu-2)}, \qquad \kappa_1 = \kappa(2-\nu). \tag{94}$$

We may get formula (92) if we put in (45)

$$-\frac{1}{2} + \frac{3b_1}{2\alpha} = \frac{1}{1-\nu}, \qquad -\frac{1}{2} - \frac{3b_1}{2\alpha} = \frac{1}{1-\nu_1}.$$

(2) Obviously, it is sufficient to prove the corresponding statement for $\nu \notin [1;3]$. Let $\nu \in (-\infty;1)$. Then $\nu_1 \in (1;2)$. If $\nu \in (3;+\infty)$, then $\nu_1 \in (2;3)$. And since $\nu = 1$ and $\nu = 3$ are the fixed points of transformation (9) the statement (2) is valid. ■

The result, proved in Theorem 7, admits considerable intensification.

THEOREM 8. *What the mass variation law admitted by problem* (1) *will be, the integration of* (1) *is reduced to the integration under Eddington–Jeans law with the index* $1 \leqslant \nu \leqslant 3$.

Proof. The statement of the theorem follows from Lemma 8 and Theorem 7. ■

6. Transformation Finite Group of Mass Variation Laws

There are constructed in this section the transformation finite groups, permitting to divide the integration problem (1) into non-crossing classes according to various mass variation laws. The ability to integrate problem (1) for one case of mass variation leads to the integration of all classes by means of known transformations.

6.1. DIHEDRAL GROUP

In our further discussion of n-order dihedral groups D_n of self-coincidence of the regular n-set-square, this D_n group has two forming elements g and f between which the following defining relations are fulfilled: $g^n = e, f^2 = e, (gf)^2 = e$. The cases of D_n dihedral group for small n are especially interesting for us. For $n = 1$ the defining relations take the following form $g = e, f^2 = e$, but they define the cyclic group C_2. Thus, $D_1 = C_2$. This is the group of self-coincidence of a polygon on the one side,

Fig. 1.

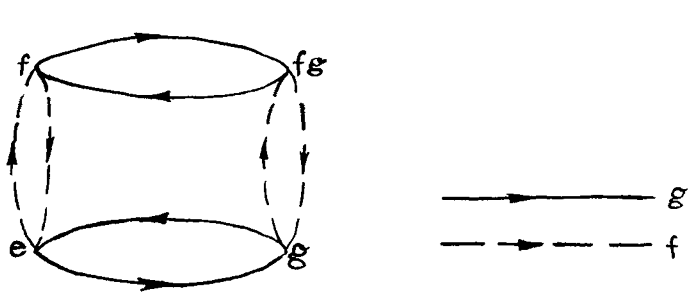

Fig. 2.

or segment. D_1 group graph has the form shown in Figure 1. Let $n = 2$, the defining relations of the D_2 group has the form $g^2 = e, f^2 = e, (gf)^2 = e$.

The D_2 group graph shows self-coincidence of *bigonal*. Here g is rotation, and f is turnover, D_2 is the fourfold group $D_2 = C_2 \times C_2$. It is the commutative (Abel) group. Its elements are e, g, f, fg. (Figure 2).

Finally, note 6-order group of self-coincidence of the regular triangular. The defining relations are $g^3 = e, f^2 = e, (gf)^2 = e$, and as D_3 elements are: e, f, g, g^2, gf, fg. The D_3 group is the direct product of the cyclic groups: $D_3 = C_2 \times C_3$. (Figure 3). (See Grossman and Magnus, 1964).

6.2. ABOUT D_3 GROUP ACTING ON THE RELATIONS (42) AND (45).

Let $\delta \neq 0$. Mass variation laws (42) may be characterized by pairs of numbers being the exponents of corresponding degree-factors.

Introduce the following designations

$$-\frac{1}{2}+\frac{3b_1}{2\sqrt{\delta}}=p, \qquad -\frac{1}{2}-\frac{3b_1}{2\sqrt{\delta}}=-1-p.$$

Consider the transformations

$$e:\mu_1=\mu, \qquad \tau=t;$$

$$g:\mu=(\alpha_2 t\times\beta_2)^{-1}\mu_1, \qquad d\tau=-\frac{\alpha_2}{\alpha_1}(\alpha_2 t+\beta_2)^{-2}dt$$

where

$$\frac{\alpha_1 t+\beta_1}{\alpha_2 t+\beta_2}=\alpha_2\tau+\beta_2, \qquad \alpha_2 t+\beta_2=(\alpha_1\tau+\beta_1)^{-1}.$$

$$\alpha_1=-\alpha_2, \qquad \alpha_1\beta_2-\alpha_2\beta_1=\sqrt{\delta}-\alpha_1;$$

$$f:\mu=(\alpha_1 t+\beta_1)^{-1}\mu_1, \qquad d\tau=-(\alpha_1 t+\beta_1)^{-2}dt,$$

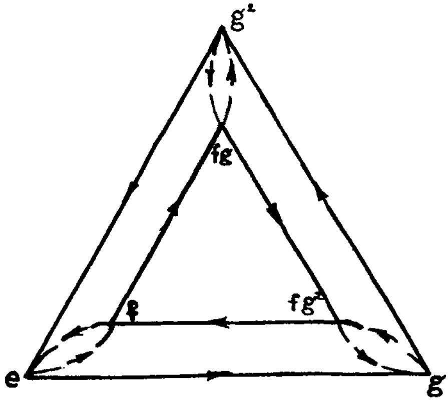

Fig. 3.

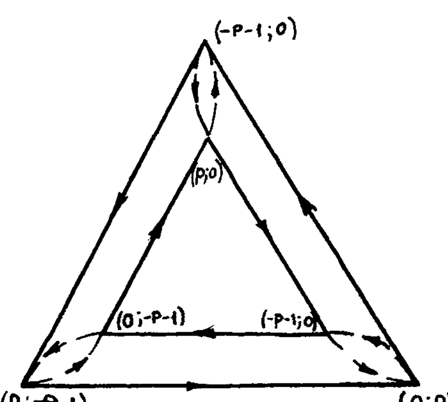

Fig. 4.

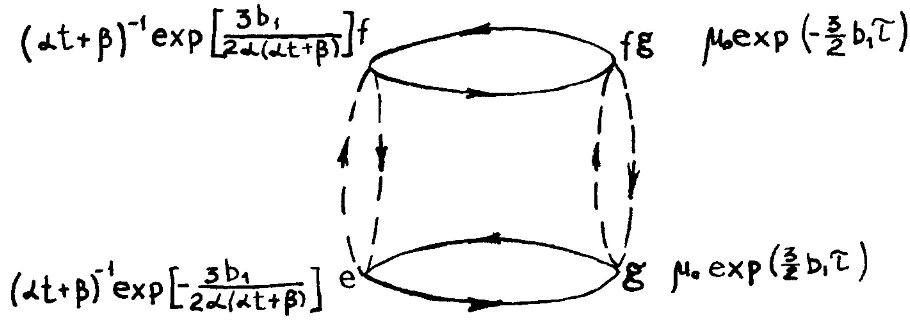

Fig. 5.

where

$$\frac{\alpha_2 t+\beta_2}{\alpha_1 t+\beta_1}=\alpha^2\tau+\beta_2, \qquad \alpha_1 t+\beta_1=(\alpha_1\tau+\beta_1)^{-1}.$$

It is directly verified that these transformations generate the D_3 group. According to D_3 group graph (Figure 4), we get the following graph for transformation mass variation laws (42) and (45).

6.3. THE FOURFOLD D_2 GROUP, ACTING ON RELATIONS (44) AND (46)

If $\delta=0$, mass variation laws take the form of (44) and (46).

Consider the transformations:

$$e:\mu_1=\mu, \qquad \tau=t;$$

$$g:\mu=(\alpha t+\beta)^{-1}\mu_1, \qquad \mathrm{d}\tau=-(\alpha t+\beta)^{-2}\mathrm{d}t,$$

where

$$\alpha t+\beta=(\alpha\tau+\beta)^{-1}; \qquad f:\mu_1=\mu, \qquad \tau=-t.$$

It is not difficult to see that the said transformations form the D_2 group. Here, according to graph D_2 (Figure 2), we obtain the following scheme for transformation of mass variation laws (44) and (46), (Figure 5).

7. Brief Review of Integrable Cases

Consider the known integrable cases of problem (1) from the group point of view.

7.1. CONSIDER THE PARTICULAR FORM OF LAWS (44) AND (46) BEING OBTAINED FOR $b_1=0$ (see also 4.1, Consequence 2)

Mass variation laws are connected by the scheme:

$$\mu=(\alpha t+\beta)^{-1}\xrightarrow{g}\mu_1=1.$$

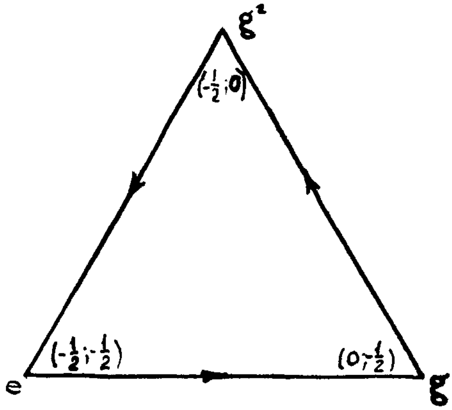

Fig. 6.

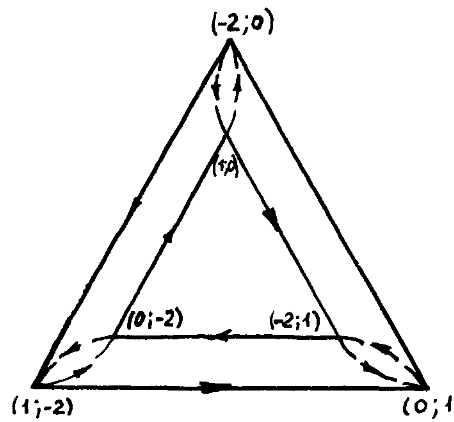

Fig. 7.

The transformation finite group is $D_1 = C_2$, the elements of which are e and $g(g^2 = e)$. The mass $\mu = 1$ is the representative of class 7.1. Integration was made by Meščerskii (1893).

7.2. Consider the particular form of laws (42) and (45) for $b_1 = 0$ (see also 4.1, Consequence 3).

Mass variation laws are connected by the scheme (Figure 6). The case $(-\frac{1}{2}; -\frac{1}{2})$ corresponds to the united Meščerskii law (4) (see also (51) and (52)). The cases $(0; -\frac{1}{2})$ and $(-\frac{1}{2}; 0)$ correspond to Meščerskii's second law (3) (see also (50)). The transformation finite group is C_3, the elements of which are e, g, g^2 $(g^3 = e)$. The indication of class 7.2 integration in elliptic functions was given by Meščerskii (1902). The actual integration of the general case was made by McMillan (1925). As it follows from the scheme in Figure 6, it is sufficient to integrate for any representative $(0; -\frac{1}{2})$ or $(-\frac{1}{2}; 0)$ of the given class.

7.3. Consider, finally the relations (55)–(57) (see also 4.1, Consequence 4).

Mass variation laws are connected by the scheme in Figure 7. Gelfgat (1959) showed that in cases $(1; 0)$, $(0; -2)$ problem (1) admits uniformed solution in Airy functions (that is in modified Bessel functions of index $\frac{1}{3}$) from an auxiliary parameter. The reference to the indicated cases may be found in Duboshin's works as well (1978). The connection of cases $(1; 0)$ and $(0; -2)$ with motions in resisting medium was considered by Radzievsky and Gelfgat (1957). The integrability of the case $(1; -2)$ was considered by Gelfgat (1968), Niţă (1958, 1973) investigating the stationary problem of the motion of satellite-particle in the attraction field of the Earth-globe under the action of rate proportional resistance force of the uniform atmosphere, has come to problem (1) under the law $\mu(t) = (1 + 2\lambda t)(1 + \lambda t)^{-2}$, which he has solved approximately using Meščerskii's first law.

The other cases of integrability of Gylden–Meščerskii problem are not known.

8. Conclusion

Point at some possible directions of Gylden–Meščerskii problem investigations. After the classification due to the law character $\mu(t)$ has been made the further work in this direction should consist of the proof of integrability or non-integrability of the representatives of the appropriate classes.

It is known that the great investigation cycle was connected to the qualitative analysis of the motion trajectories (see in particular, Armellini, 1916; Stepanov, 1930, etc.) Duboshin (1928, 1930) investigated for the arbitrary mass variation law the behaviour of motion trajectory which was earlier made by Armellini for the case of monotonous mass variation. The use of the mass variation laws stated in the present work will permit us to give a physically realizable motion picture.

Finally, point to the necessity of improvement of numerical methods, adapted to problem (1) solution.

A brief summary of the results contained in the present paper is given by the author (1980a, b).

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