THREE-DIMENSIONAL PERIODIC ORBITS ABOUT THE TRIANGULAR EQUILIBRIUM POINTS OF THE RESTRICTED PROBLEM OF THREE BODIES

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Abstract. The third-order parametric expansions given by Buck in 1920 for the three-dimensional periodic solutions about the triangular equilibrium points of the restricted Problem are improved by fourth-order terms. The corresponding family of periodic orbits, which are symmetrical w.r.t. the (x, y) plane, is computed numerically for $\mu = 0.00095$. It is found that the family emanating from L_4 terminates at the other triangular point L_5 while it bifurcates with the family of three-dimensional periodic orbits originating at the collinear equilibrium point L_3. This family consists of stable and unstable members. A second family of nonsymmetric three-dimensional periodic orbits is found to bifurcate from the previous one. It is also determined numerically until a collision orbit is encountered with the computations.

1. Introduction

In the circular Restricted three-body Problem the infinitesimal periodic solutions about the positions of equilibrium are continued to families of periodic solutions (Moulton, 1920).

In the planar case of the problem, these families have been computed both for the collinear points (e.g. Henon, 1965) and the triangular points (Goodrich, 1966, Deprit et al., 1967) and their termination has been determined.

The families of three dimensional periodic solutions about the collinear equilibrium points have been studied for large ($\mu = 0.4$, Bray and Goudas, 1967) and small ($\mu = 0.00095$, Zagouras and Kazantzis, 1979) values of the mass parameter.

The three-dimensional periodic oscillations about the triangular equilibrium points have been studied by series expansions (Buck, 1920; Heppenheimer, 1973; Erdi, 1978) which are valid for small values of the orbital parameters used in each contribution, i.e., they are valid in the vicinity of the equilibrium points.

In this article, the above periodic motion is approximated via a fourth-order parameteric expansion w.r.t. an orbital parameter. The fourth-order terms improve the similar expansions given by Buck.

This motion is continued numerically to a family of periodic orbits

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which are symmetrical w.r.t. the plane of the two primaries. It turns out that this family, originating at L_4 intersects the family L^e_{3v} of three-dimensional periodic orbits originating at the collinear equilibrium point L_3. It is terminated at the second triangular point L_5.

The Liapunov stability of each periodic solution is examined. It is found that the first part of the family consists of stable orbits while the second of unstable orbits. At the member of the family where the stability character changes, a second family of nonsymmetric periodic solutions is found to intersect the first one. This family is computed until a collision orbit is encountered during the numerical computations. The results thus obtained are presented in Tables and Figures.

2. Parametric Expansions

In a rotating, barycentric, dimensionless coordinate system with the smaller primary on the positive Ox-axis the differential equations of motion for the circular three-dimensional Restricted three-body Problem are

$$\ddot{X} - 2\dot{Y} = X - \frac{(1-\mu)(X+\mu)}{r_1^3} - \frac{\mu(X+\mu-1)}{r_2^3} ,$$

$$\ddot{Y} + 2\dot{X} = Y - \frac{(1-\mu)Y}{r_1^3} - \frac{\mu Y}{r_2^3} , \tag{1}$$

$$\ddot{Z} = - \frac{(1-\mu)Z}{r_1^3} - \frac{\mu Z}{r_2^3} ,$$

where,

$$r_1^2 = (X+\mu)^2 + Y^2 + Z^2, \qquad r_2^2 = (X+\mu-1)^2 + Y^2 + Z^2.$$

The positions of the two triangular equilibrium points are

$$X_0 = \frac{1}{2} - \mu, \qquad Y_0 = \pm \frac{\sqrt{3}}{2} , \quad Z_0 = 0. \tag{2}$$

The positive sign corresponds to the triangular equilibrium point L_4 and the negative to L_5. What follows refers to the motion about the point L_4 with the understanding that by changing the sign of Y_0 the corresponding expressions for the point L_5 are obtained.

The origin is transferred to L_4 by means of the transformation

$$X = x + X_0, \qquad Y = y + Y_0, \qquad Z = z. \tag{3}$$

After the transformation is made the right-hand side of the equations are expanded as power series up to fourth order terms in x, y and z. Provided that the conditions

$$|x + \sqrt{3}y + x^2 + y^2 + z^2| < 1, \quad |-x + 3\sqrt{y} + x^2 + y^2 + z^2| < 1$$

are satisfied, the equations of motion, about L_4, take the form

$$x'' - 2(1+\sigma)y' = (1+\sigma)^2\left[\frac{3}{4}x + \frac{3\sqrt{3}}{4}\rho y + \frac{21}{16}\rho x^2 - \frac{3\sqrt{3}}{8}xy - \frac{33}{16}\rho y^2 + \right.$$

$$\left. + \frac{3}{4}\rho z^2 - \frac{3}{8}xz^2 - \frac{15\sqrt{3}}{8}\rho yz^2 - \frac{15}{16}\rho z^4\right],$$

$$y'' + 2(1+\sigma)x' = (1+\sigma)^2\left[\frac{3\sqrt{3}}{4}\rho x + \frac{9}{4}y - \frac{3\sqrt{3}}{16}x^2 - \frac{33}{8}\rho xy - \frac{9\sqrt{3}}{16} + \right. \tag{4}$$

$$\left. + \frac{3\sqrt{3}}{4}z^2 + \frac{15\sqrt{3}}{8}xz^2 - \frac{33}{8}yz^2 - \frac{15\sqrt{3}}{16}z^4\right],$$

$$z'' = -(1+\sigma)^2\left[z - \frac{3}{2}\rho xz - \frac{3\sqrt{3}}{2}yz - \frac{3}{2}z^3\right]$$

where $\rho = 1-2\mu$. The prime denotes differentiation with respect to the new independent variable of time τ, introduced by the relation

$$t = (1+\sigma) = \tau. \tag{5}$$

We seek periodic solutions of Equations (4) in the form of fourth order expansions in powers of a small parameter ε:

$$x(\tau) = \sum_{\nu=1}^{4} x_\nu(\tau)\,\varepsilon^\nu,$$

$$y(\tau) = \sum_{\nu=1}^{4} y_\nu(\tau)\,\varepsilon^\nu, \tag{6}$$

$$z(\tau) = \sum_{\nu=1}^{4} z_\nu(\tau)\,\varepsilon^\nu.$$

We introduce the relation

$$\sigma = \sum_{\nu=1}^{4} a_\nu\,\varepsilon^\nu \tag{7}$$

where the parameters a_ν are to be determined so as to preclude secular terms.

The series (6) and (7) are substituted into Equations (4) and the coefficients of the same powers of ε are equated. The resulting equations are:

Coefficients of ε:

$$\left(D^2 - \frac{3}{4}\right) x_1 - \left(2D + \frac{3\sqrt{3}}{4}\rho\right) y_1 = 0,$$

$$\left(2D - \frac{3\sqrt{3}}{4}\rho\right) x_1 + \left(D^2 - \frac{9}{4}\right) y_1 = 0, \qquad (8)$$

$$(D^2+1)z_1 = 0. \text{ (The operator D stands for } \frac{d}{d\tau} = ()'\text{)}$$

The only solution of the system of the three Equations (8), which is periodic for every value of the mass parameter μ, is

$$x_1(\tau) = 0, \quad y_1(\tau) = 0, \quad z_1(\tau) = c \sin \tau, \qquad (9)$$

where we have put, without loss of generality, $z_1(0) = 0$, $z_1'(0) = c$.

Coefficients of ε^2:

$$\left(D^2 - \frac{3}{4}\right) x_2 - \left(2D + \frac{3\sqrt{3}}{4}\rho\right) y_2 = \frac{3}{4}\rho\, c^2 \sin^2 \tau,$$

$$\left(2D - \frac{3\sqrt{3}}{4}\rho\right) x_2 + \left(D^2 - \frac{9}{4}\right) y_2 = \frac{3\sqrt{3}}{4} c^2 \sin^2 \tau, \qquad (10)$$

$$(D^2+1)\, z_2 = -2a_1 c \sin \tau.$$

Equations (10) are the result of the substitution of the solution (9) into the original equations of order ε^2. These equations admit the periodic solution

$$x_2(\tau) = A \cos 2\tau + B \sin 2\tau,$$

$$y_2(\tau) = \Gamma + \Delta \cos 2\tau + E \sin 2\tau, \qquad (11)$$

$$z_2(\tau) = 0,$$

with $t = \tau$, $a_1 = 0$, and

$$A = \frac{8c^2\rho}{R}, \quad B = \frac{8\sqrt{3}\, c^2}{R}, \quad \Gamma = -\frac{\sqrt{3}}{6} c^2,$$

$$\Delta = \frac{\sqrt{3}\, c^2(19-3\rho^2)}{2R}, \quad E = -\frac{8c^2\rho}{R}, \quad R = 64+36\mu(1-\mu). \qquad (12)$$

Coefficients of ε^3:

$$(D^2-\tfrac{3}{4})x_3 - \left(2D + \frac{3\sqrt{3}}{4}\rho\right)y_3 = 0,$$

$$\left(2D - \frac{3\sqrt{3}}{4}\rho\right)x_3 + (D^2 + \tfrac{9}{4}) = 0, \qquad (13)$$

$$(D^2 + 1)\, z_3 = -2a_2 z_1 + \frac{3\rho}{2}\, x_2 z_1 + \frac{3\sqrt{3}}{2}\, y_2 z_1 + \frac{3}{2}\, z_1^3 \, .$$

After the substitution of the solution (9) and (11) into the right-hand side of Equations (13), these equations admit the periodic solution

$$x_3(\tau) = 0, \qquad y_3(\tau) = 0$$
$$z_3(\tau) = H \sin \tau + \theta \sin 3\tau \qquad (14)$$

where

$$t = (1+\alpha_2\varepsilon^2)\tau$$

and

$$H = -\frac{9\mu(1-\mu)c^3}{R}, \qquad \theta = \frac{3\mu(1-\mu)c^3}{R}, \qquad a_2 = \frac{3(1-\rho^2)}{R}\,. \qquad (15)$$

Coefficients of ε^4:

$$(D^2 - \tfrac{3}{4})\, x_4 - \left(2D + \frac{3\sqrt{3}}{4}\rho\right) y_4 = 2a_2 y_2' + \frac{3\sqrt{3}}{2}\,\rho\, a_2 y_2 + \frac{3}{2}a_2 x_2 +$$
$$+ \frac{21}{16}\,\rho x_2^2 - \frac{3\sqrt{3}}{8}\, x_2 y_2 - \frac{33}{16}\,\rho y_2^2 + \frac{3\rho}{2} z_1 z_3 - \frac{3}{8} x_2 z_1^2 -$$
$$- \frac{15\sqrt{3}}{8}\,\rho y_2 z_1^2 - \frac{15}{16}\,\rho z_1^4 + \frac{3}{2}\,\rho a_2 z_1^2, \qquad (16)$$

$$\left(2D - \frac{3\sqrt{3}}{4}\,\rho\right) x_4 + (D^2 - \tfrac{9}{4}) y_4 = -2a_2 x_2' + \frac{3\sqrt{3}}{2}\,\rho\, a_2 x_2 + \frac{9}{2}\, a_2 y_2 -$$
$$- \frac{3\sqrt{3}}{16}\, x_2^2 - \frac{33}{8}\rho x_2 y_2 - \frac{9\sqrt{3}}{16}\, y_2^2 + \frac{3\sqrt{3}}{2}\, z_1 z_3 +$$
$$+ \frac{15\sqrt{3}}{8}\, x_2 z_1^2 - \frac{33}{8}\, y_2 z_1^2 - \frac{15\sqrt{3}}{16}\, z_1^4 + \frac{3\sqrt{3}}{2}\, a_2 z_1^2,$$

$$(D^2 + 1)\, z_4 = -2a_3 z_1 .$$

With the substitution of x_i, y_i, z_i, i = 1, 2, 3 into the right-hand side of (16), the above equations are written as:

$$(D^2 - \frac{3}{4})\, x_4 - \left(2D + \frac{3\sqrt{3}}{4}\rho\right) y_4 = f_4(\tau),$$
$$\left(2D - \frac{3\sqrt{3}}{4}\rho\right) x_4 + (D^2 - \frac{9}{4})\, y_4 = g_4(\tau), \qquad (17)$$
$$(D^2 + 1)\, z_4 = -\, 2a_3 c \sin\tau ,$$

where

$$f_4(\tau) = A_1 + A_2 \sin 2\tau + A_3 \cos 2\tau + A_4 \sin 4\tau + A_5 \cos 4\tau, \qquad (18)$$
$$g_4(\tau) = B_1 + B_2 \sin 2\tau + B_3 \cos 2\tau + B_4 \sin 4\tau + B_5 \cos 4\tau,$$

(The coefficients A_i, B_i, i=1, ..., 5 are given in Appendix I).

A periodic solution of Equations (17) is

$$x_4(\tau) = K + \Lambda \sin 2\tau + M \cos 2\tau + N \sin 4\tau + \Xi \cos 4\tau,$$
$$y_4(\tau) = \Pi + P \sin 2\tau + \Sigma \cos 2\tau + T \sin 4\tau + \Phi \cos 4\tau, \qquad (19)$$
$$z_4(\tau) = 0,$$

$t = (1 + \alpha_2\varepsilon^2)\tau$, $a_3 = 0$.

(The coefficients of Equations (19) are given in Appendix II).

Thus, a fourth-order approximation of periodic solutions in the vicinity of the triangular equilibrium points has been obtained:

$$x(\tau) = x_2(\tau)\varepsilon^2 + x_4(\tau)\varepsilon^4 ,$$
$$y(\tau) = y_2(\tau)\varepsilon^2 + y_4(\tau)\varepsilon^4 , \qquad (20)$$
$$z(\tau) = z_1(\tau)\varepsilon + z_3(\tau)\varepsilon^3 ,$$
$$t = (1 + \alpha_2\varepsilon^2)\tau.$$

3. Numerical Results

The periodic functions (20) satisfy the differential equations of motion to a satisfactory degree of accuracy for small values of ε. Thus, the first small part of the family emanating from the triangular equilibrium point, L_4, is obtained. To continue the family we constructed a predictor corrector algorithm based on the numerical integration of the equations of motion and equations of variation. (For details, see Zagouras, 1982.)

Applying the above algorithm, we determined numerically this family of periodic solutions. We call this family $L^e_{4\nu}$ or $L^e_{5\nu}$. It starts with infinitesimal oscillations about the triangular equilibrium point L_4, terminates with infinitesimal oscillations about the other triangular point L_5 and is being intersected at a point by the family $L^e_{3\nu}$ emanating from the collinear equilibrium point L_3 (Zagouras and Kazantzis, 1979).

We present this family in tabular and graphical forms. In Table I we give the initial conditions, the period, the stability parameters and the Jacobi constant of half the family $L^e_{4\nu}$. If in the values of y_0 and $\dot{x}_0$ a negative sign is added everywhere, while the values of the other paramters in Table I are not changed, then we can easily obtain the other half of the $L^e_{4\nu}$. Those periodic orbits included in Table I for which p, q are real and $|p| < 2$, $|q| < 2$, are stable in the linear sense (Liapunov stability).

In Figure 1 the characteristic curve of the family $L^e_{4\nu}$, is given in four orthogonal projections (a) on the (x_0, y_0) plane, (b) on the $(x_0, \dot{y}_0)$ plane, (c) on the $(y_0, \dot{y}_0)$ plane and (d) on the $(y_0, \dot{z}_0)$ plane, where (x_0, y_0, z_0) and $(\dot{x}_0, \dot{y}_0, \dot{z}_0)$ are the vectors of initial position and velocity. In Figure 1b are shown: the family m of planar periodic orbits around both primaries, the family $L^e_{3\nu}$ of three-dimensional periodic orbits originating at the collinear equilibrium point L_3 and the three-dimensional family $L^e_{4\nu}$. In this diagram the two triangular points are represented by the same point. It is seen how the family $L^e_{4\nu}$ starts from L_4, intersects the family $L^e_{3\nu}$ and terminates, retreating the same curve (in the $(x_0, \dot{y}_0)$ projection), at the other triangular point L_5.

The family is stable from the origin, at L_4, until the point B where x_0 reaches a local maximum. The stable segments of $L^e_{4\nu}$ are marked on the diagrams by bold face lines.

Representative orbits, members of the family $L^e_{4\nu}$, are given in Figure 2.

TABLE I

The Family $L^{e}_{4\nu}$ for $\mu = 0.00095$

x_0	y_0	$x_0\dot{x}_0$	$\dot{y}_0$	$\dot{z}_0$	T	p	q	C	No
0.500295	0.865304	0.004335	-0.002506	0.1	6.28319	-1.7524	-1.999	2.98903	1
0.504012	0.863136	0.017429	-0.010178	0.2	6.28323	-1.7573	-1.999	2.95864	2
0.510135	0.859518	0.039569	-0.023487	0.3	6.28329	-1.7652	-1.999	2.90693	3
0.519070	0.854133	0.073453	-0.044639	0.4058	6.28338	-1.7767	-1.999	2.82699	4
0.530899	0.846807	0.122105	-0.076556	0.5174	6.28351	-1.7918	-1.999	2.71058	5
0.542157	0.839624	0.174497	-0.112683	0.6104	6.28364	-1.8064	-1.999	2.58332	6
0.551528	0.833483	0.225707	-0.149369	0.6844	6.28377	-1.8192	-1.999	1.45740	7
0.563516	0.825408	0.313346	-0.213962	0.7844	6.28398	-1.8384	-1.999	2.23982	8
0.569721	0.821104	0.391581	-0.271780	0.8524	6.28415	-1.8530	-1.999	2.04528	9
0.570929	0.820140	0.419270	-0.292034	0.8727	6.28421	-1.8578	-1.999	1.97639	10
0.569937	0.821067	0.537338	-0.373086	0.9384	6.28443	-1.8760	-2.0000	1.69055	11
0.524218	0.851112	0.851253	-0.524580	0.999972	6.28484	-1.9143	-2.0000	0.0999292	12
0.380380	0.924611	1.23637	-0.509125	0.9413	6.28511	-1.9493	-2.0000	0.325216	13
0.227321	0.973758	1.45975	-0.341366	0.8663	6.28521	-1.9660	-2.0000	0.001316	14
-0.198588	0.980316	1.66393	0.336364	0.7163	6.28531	-1.9866	-2.0000	-0.395193	15
-0.717407	0.697286	1.25172	1.28690	0.6063	6.28535	-1.9968	-2.0000	-0.590042	16
-0.956435	0.294091	0.535641	1.74124	0.5700	6.28536	-1.9996	-2.0000	-0.642842	17
-0.995492	0.101447	0.185145	1.81605	0.56466	6.28536	-1.9999	-2.0000	-0.650198	18
-0.999823	0.040677	0.074253	1.82405	0.56452	6.28539	-2.0000	-2.0000	-0.650386	19
-1.000542	0.019630	0.035825	1.82505	0.56472	6.28535	-2.0000	-2.0000	-0.651347	20

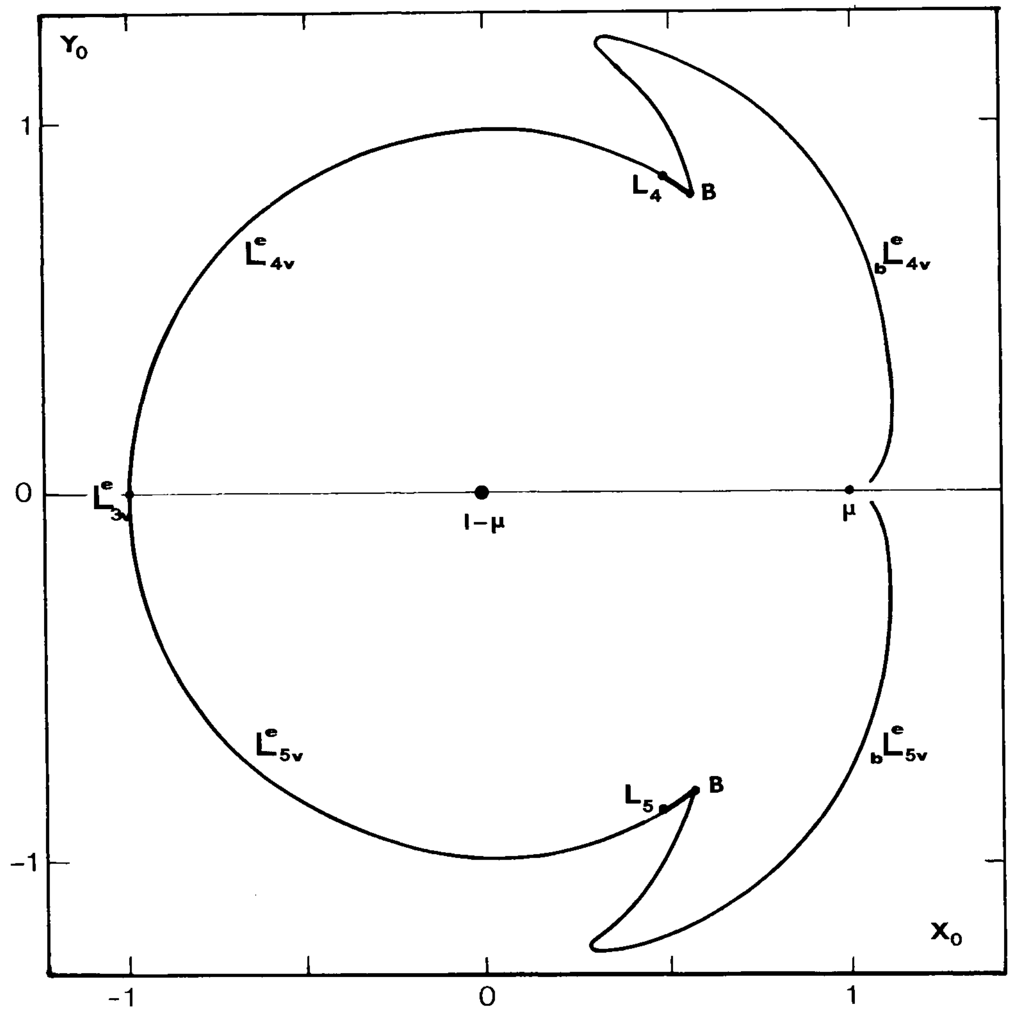

Fig. 1a. The characteristic curves of the families $L^{e}_{4\nu}$ and ${}_{b}L^{e}_{4\nu}$. Stable parts are indicated by thick lines:
(a) Projection in the (x_0, y_0) plane.

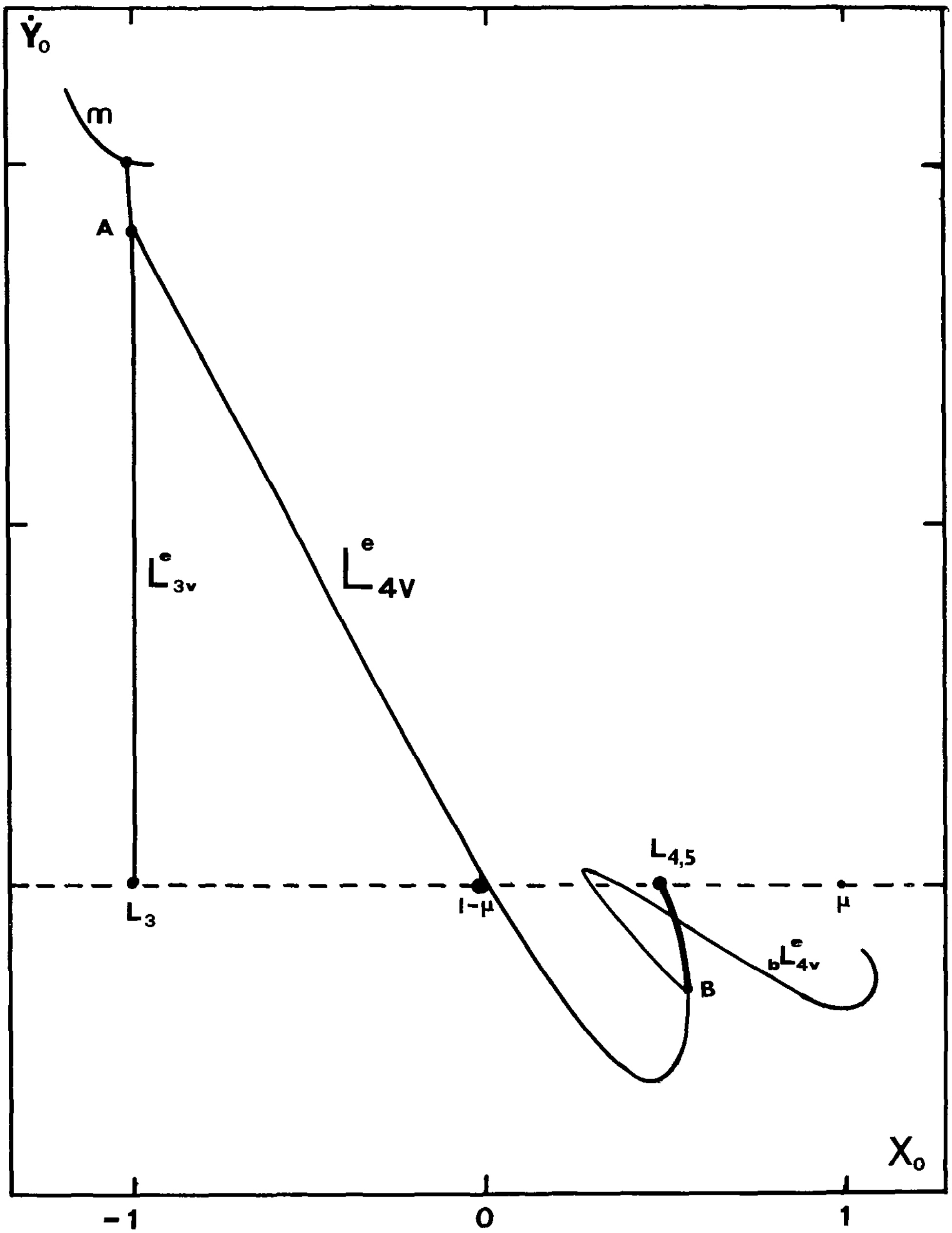

Fig. 1b. The characteristic curves of the families $L^e_{4\nu}$ and ${}_bL^e_{4\nu}$. Stable parts are indicated by thick lines:
(b) Projection in the $(x_0, \dot{y}_0)$ plane.

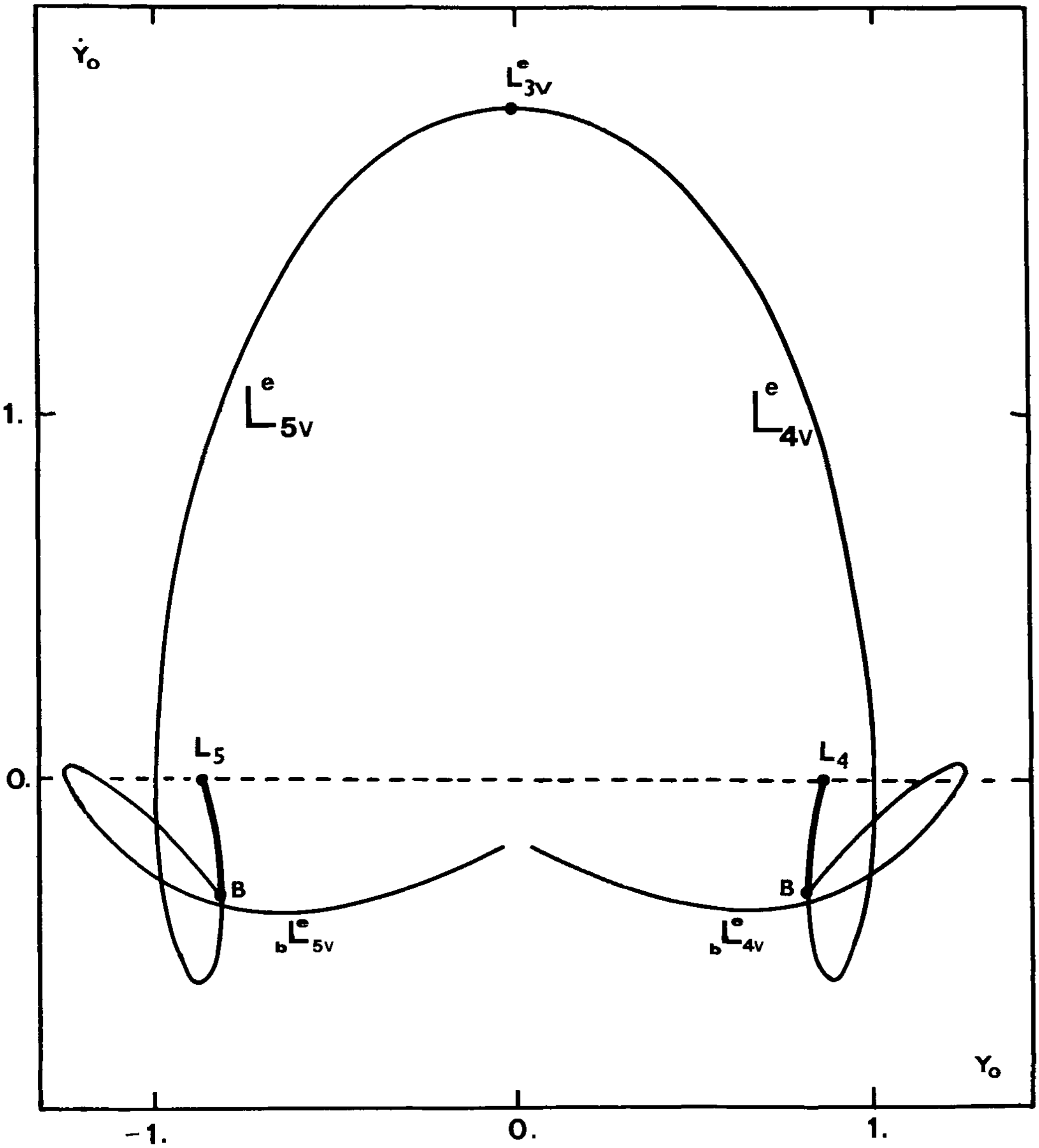

Fig. 1c The characteristic curves of the families $L^e_{4\nu}$ and ${}_bL^e_{4\nu}$. Stable parts are indicated by thick lines: (c) Projection in the $(y_0, \dot{y}_0)$ plane.

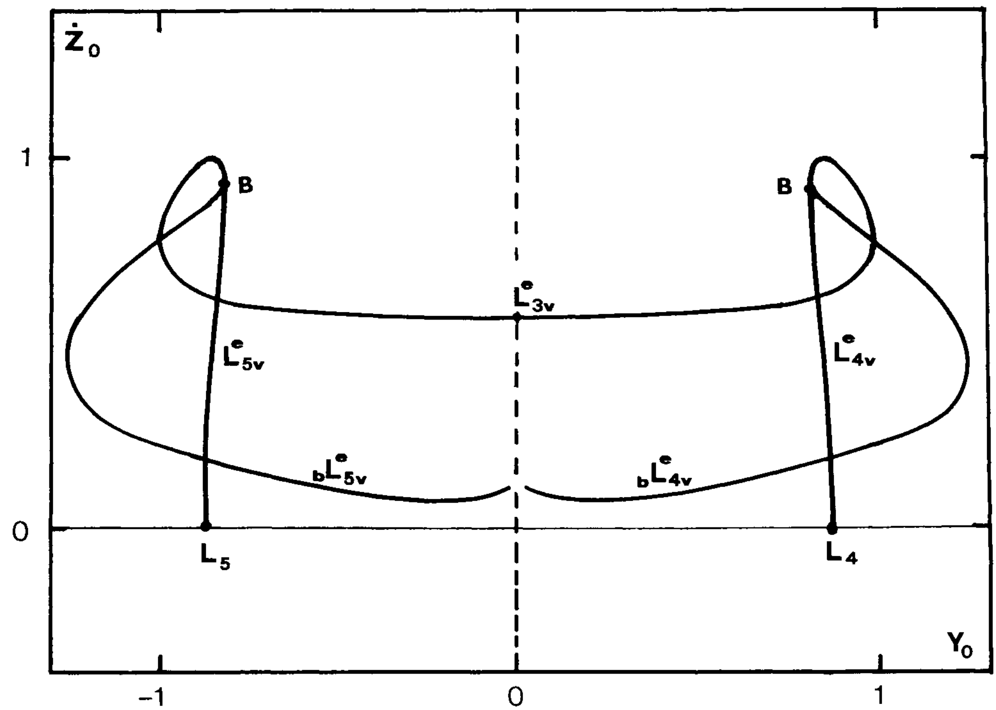

Fig. 1d. The characteristic curves of the families $L^e_{4\nu}$ and $_bL^e_{4\nu}$. Stable parts are indicated by thick lines:
(d) Projection in the $(y_0, \dot{z}_0)$ plane.

3.1. *A New Family of Nonsymmetric Periodic Solutions*

Between the periodic solutions of the family $L^e_{4\nu}$ which correspond to the 10th and 11th entries of Table I, there exists a solution for whcich $|q| = 2$. This means that the 6×6 monodromy matrix has two more eigenvalues equal to unity, that is, there is a second direction, except the tangent to the family, along which periodicity is preserved and hence a second family of periodic orbits intersects the first one at this point. Indeed, there exists such a family consisting of periodic orbits which are nonsymmetric w.r.t. any plane or axis. We continued numerically this family which we call $_bL^e_{4\nu}$. The left subscript b indicates that $_bL^e_{4\nu}$ is a bifurcation family of $L^e_{4\nu}$. We found that $_bL^e_{4\nu}$ evolves from orbits having the shape of a curved nonsymmetric eight to orbits having the shape of a loop about the small body μ. We stopped

TABLE II

The Family ${}_{b}L^{e}_{4v}$ for $\mu = 0.00095$

x_0	y_0	$\dot{x}_0$	$\dot{y}_0$	$\dot{z}_0$	T	p	q	C	No
0.567798	0.833391	0.448296	-0.299282	0.872701	6.28424	-1.8598	-2.0000	1.94714	1
0.520855	0.959561	0.595075	-0.260689	0.7940	6.28411	-1.8616	-2.0000	1.97063	2
0.431508	1.09991	0.727315	-0.161130	0.6930	6.28362	-1.8702	-2.0000	2.05296	3
0.336659	1.19929	0.788295	-0.045781	0.5930	6.28261	-1.8921	-2.0001	2.18161	4
0.286172	1.25155	0.789794	0.031867	0.4930	6.28064	-1.9369	-2.0006	2.33783	5
0.338893	1.25768	0.756903	0.023502	0.3910	6.27649	-1.9953	-2.0257	2.50532	6
0.528835	1.19465	0.692449	-0.090601	0.2930	6.26661	-1.9966	-2.1709	2.66373	7
0.768102	1.04117	0.578688	-0.236438	0.2190	6.24505	-1.9913	-2.4121	2.78070	8
1.01406	0.727252	0.359595	-0.349142	0.1500	6.16531	-1.9464	-3.1240	2.88632	9
1.11173	0.240242	0.073365	-0.271423	0.0850	5.42762	-1.8349	-7.2777	2.96985	10
1.09095	0.101529	0.003350	-0.217117	0.07777	4.49685	1.9346	-10.098	2.98322	11
1.06672	0.042252	-0.046840	-0.188118	0.08795	3.72389	1.8021	-11.196	2.98817	12
1.05329	0.025103	-0.075606	-0.176118	0.10003	3.37135	1.4480	-10.952	2.98986	13
1.05115	0.023008	-0.080492	-0.174406	0.102515	3.31848	1.3937	-10.847	2.99010	14

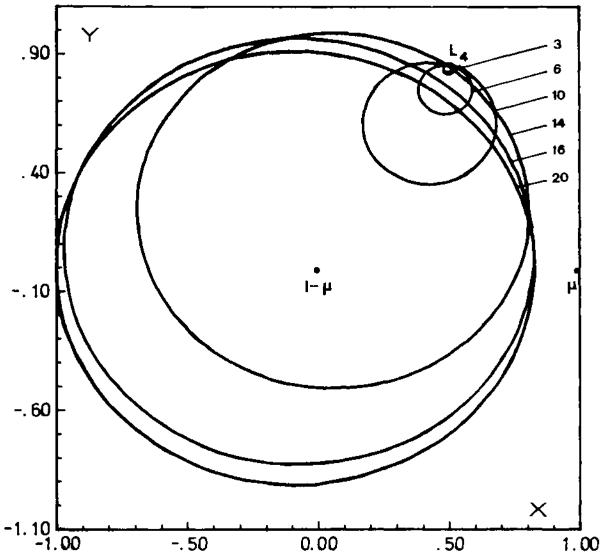

Fig. 2a

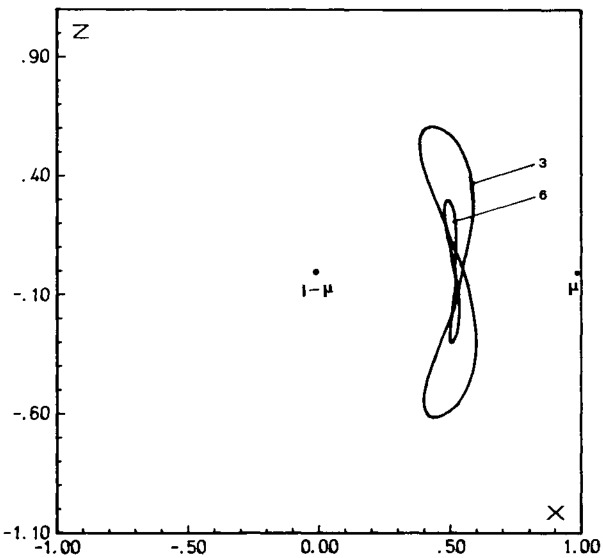

Fig. 2b

Fig. 2. Typical orbits of the family $L^e_{4\nu}$. The number following each curve indicates the corresponding serial number in Table I. Projections in the (x, y) and (x, z) planes.

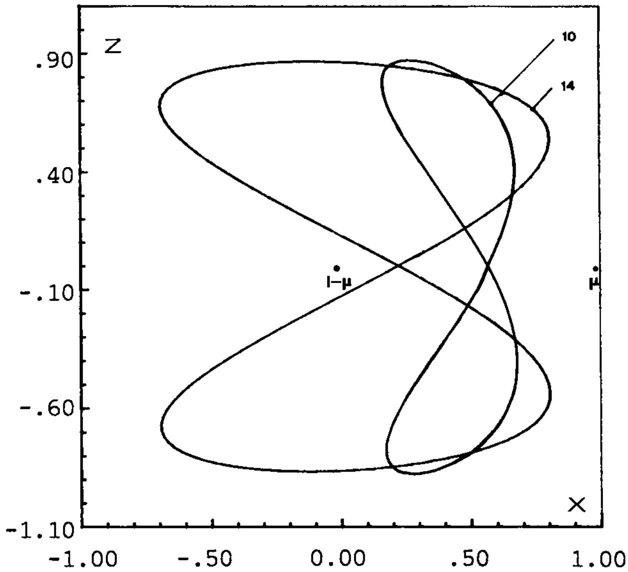
Z
.90
.40
-.10
-.60
-1.10
-1.00
-.50
0.00
.50
1.00
X
10
14
1-μ
μ

Fig. 2c

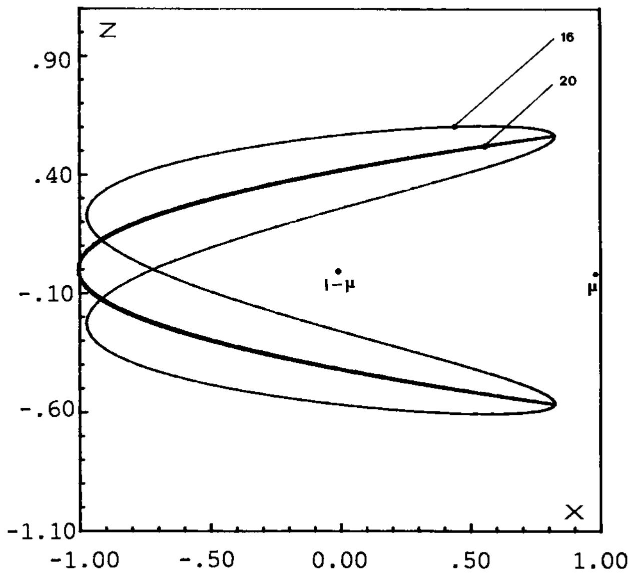
Z
.90
.40
-.10
-.60
-1.10
-1.00
-.50
0.00
.50
1.00
X
16
20
1-μ
μ

Fig. 2d

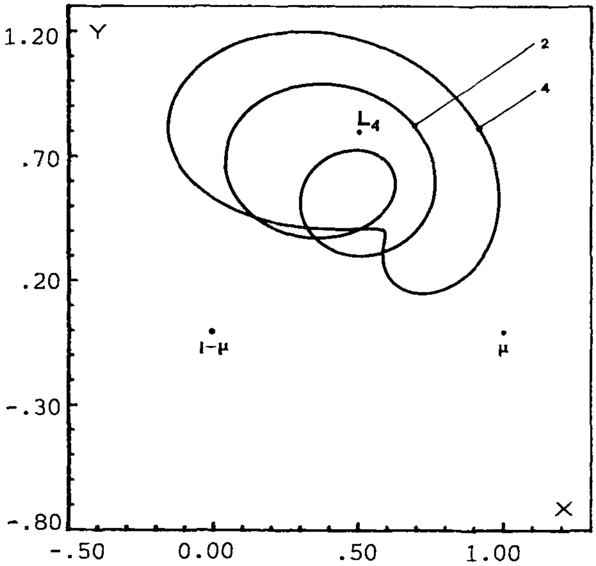

Fig. 3a

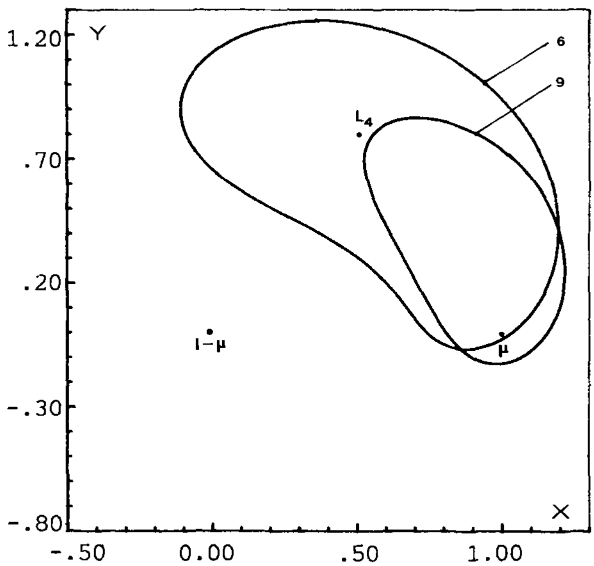

Fig. 3b

Fig. 3 Typical orbits of the family ${}_{b}L^{e}_{4\nu}$. The number following each curve indicates the corresponding serial number in Table II. Projections in the (x, y) and (x, z) planes.

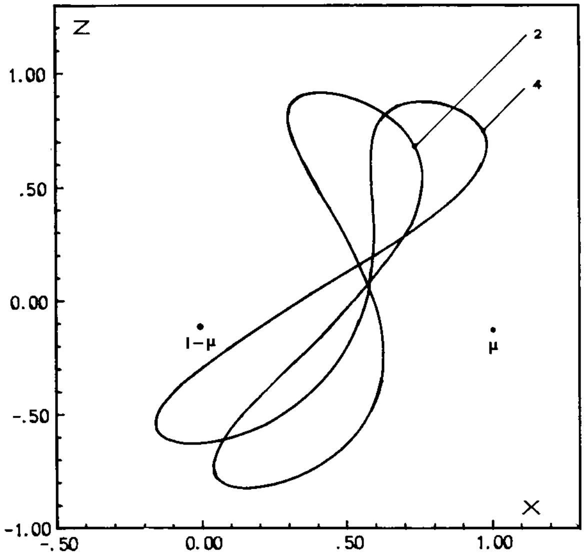
Z
1.00
.50
0.00
-.50
-1.00
-.50
0.00
.50
1.00
X
2
4
1-μ
μ

Fig. 3c

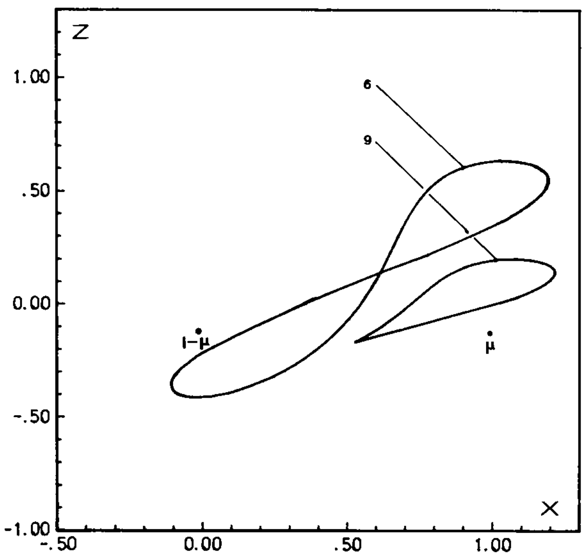
Z
1.00
.50
0.00
-.50
-1.00
-.50
0.00
.50
1.00
X
6
9
1-μ
μ

Fig. 3d

The numerical computation of this family at a point which is very close to a collision orbit since a regularization program of the Restricted problem in three dimensions is not available for the moment.

No stable orbits exist along the family. Numerical data for this family are included in Table II. In Figure 1 the family ${}_bL^e_{4\nu}$ is presented in four orthogonal projections. In Figure 3 selected periodic solutions of this family are drawn in orthogonal projections.

4. Comments

(1) The 'out of plane' infinitesimal periodic oscillations about the triangular equilibrium points L_4 and L_5 are continued to a family of periodic orbits symmetrical w.r.t. the plane of motion of the two more massive bodies. This is a new kind of symmetry concerning the three-dimensional periodic solutions of the Restricted Problem.

(2) There is a similarity in the evolution of the families of planar and three-dimensional periodic orbits, emanating from the triangular equilibrium points. The family L^s_4 of short period planar periodic solutions emanating from L_4 terminates on the family b emanating from the collinear point L_3. In three dimensions, the family $L^e_{4\nu}$ which emanates from L_4 terminates on the family $L^e_{4\nu}$ emanating from L_3.

(3) The families $L^e_{1\nu}$, $L^2_{2\nu}$, $L^e_{3\nu}$ of three dimensional symmetric periodic orbits originating at the collinear equilibrium points consist of unstable orbits (Zagouras and Kazantzis, 1979). The corresponding family $L^e_{4\nu}$ originating at the triangular equilibrium points consists of a stable and an unstable part.

(4) Each one of the families of three dimensional periodic orbits bifurcated from the equilibrium points, collinear or triangular, has an intersection with another family of periodic orbits.

Acknowledgements

Thanks are due to Mr. Galanakis for his help concerning part of the computational work.

Appendix I

$$A_1 = \frac{21}{32}\rho(A^2+B^2) - \frac{3\sqrt{3}}{16}(A\Delta+BE) - \frac{33}{16}\rho\Gamma^2 - \frac{33}{32}\rho(\Delta^2+E^2) + \frac{3\rho cH}{4}$$

$$-\frac{15\sqrt{3}}{16}\rho c^2\Gamma - \frac{45}{128}\rho c^4 + \frac{15\sqrt{3}}{32}\rho c^2 A + \frac{3\sqrt{3}}{2}\rho\alpha_2\Gamma + \frac{3}{4}\rho\alpha_2 c^2 ,$$

$$A_2 = -4\alpha_2\Delta - \frac{3\sqrt{3}}{8}B\Gamma - \frac{33}{8}\rho\Gamma E - \frac{3}{16}Bc^2 - \frac{15\sqrt{3}}{16}\rho c^2 E + \frac{3}{2}\alpha_2 B + \frac{3\sqrt{3}}{2}\rho\alpha_2 E,$$

$$A_3 = 4\alpha_2 E - \frac{3\sqrt{3}}{8}A\Gamma - \frac{33}{8}\rho\Gamma\Delta - \frac{3\rho}{4}cH + \frac{3\rho}{4}c\theta - \frac{3}{16}c^2 A - \frac{15\sqrt{3}}{16}\rho c^2\Delta$$

$$+\frac{15\sqrt{3}}{16}\rho c^2\Gamma + \frac{15}{32}\rho c^4 + \frac{3}{2}\alpha_2 A + \frac{3\sqrt{3}}{2}\rho\alpha_2\Delta - \frac{3}{4}\rho\alpha_2 c^2,$$

$$A_4 = \frac{21}{16}\rho AB - \frac{3\sqrt{3}}{16}B\Delta - \frac{3\sqrt{3}}{16}AE - \frac{33}{16}\rho\Delta E + \frac{3}{32}c^2 B + \frac{15\sqrt{3}}{32}\rho c^2 E ,$$

$$A_5 = \frac{21}{32}\rho(A^2-B^2) - \frac{3\sqrt{3}}{16}(A\Delta-BE) - \frac{33}{32}\rho(\Delta^2-E^2) - \frac{3\rho}{4}c\theta + \frac{3}{32}c^2 A$$

$$+\frac{15\sqrt{3}}{32}\rho c^2\Delta - \frac{15}{128}\rho c^4 ,$$

$$B_1 = -\frac{33}{16}\rho(A\Delta+BE) - \frac{9\sqrt{3}}{16}\Gamma^2 - \frac{9\sqrt{3}}{32}(\Delta^2+E^2) - \frac{3\sqrt{3}}{32}(A^2+B^2) + \frac{3\sqrt{3}}{4}cH$$

$$-\frac{33}{16}c^2(\Gamma-\frac{\Delta}{2}) - \frac{15\sqrt{3}}{32}c^2 A - \frac{45\sqrt{3}}{128}c^4 + \frac{9}{2}\alpha_2\Gamma + \frac{3\sqrt{3}}{4}\alpha_2 c^2,$$

$$B_2 = 4\alpha_2 A - \frac{33}{8}\rho B\Gamma - \frac{9\sqrt{3}}{8}\Gamma E - \frac{33}{16}c^2 E + \frac{15\sqrt{3}}{16}c^2 B + \frac{3\sqrt{3}}{2}\rho\alpha_2 B + \frac{9}{2}\alpha_2 E,$$

$$B_3 = -4\alpha_2 B - \frac{33}{8}\rho A\Gamma - \frac{9\sqrt{3}}{8}\Gamma\Delta - \frac{3\sqrt{3}}{4}cH + \frac{3\sqrt{3}}{4}c\theta - \frac{33}{16}c^2(\Delta-\Gamma) + \frac{15\sqrt{3}}{16}c^2 A$$

$$+\frac{15\sqrt{3}}{32}c^4 + \frac{3\sqrt{3}}{2}\rho\alpha_2 A + \frac{9}{2}\alpha_2\Delta - \frac{3\sqrt{3}}{4}\alpha_2 c^2 ,$$

$$B_4 = -\frac{33}{16}\rho(B\Delta+AE) - \frac{9\sqrt{3}}{16}\Delta E - \frac{3\sqrt{3}}{16}AB + \frac{33}{32}c^2 E - \frac{15\sqrt{3}}{32}c^2 B,$$

$$B_5 = -\frac{33}{16}\rho(A\Delta-BE) - \frac{9\sqrt{3}}{32}(\Delta^2-E^2) - \frac{3\sqrt{3}}{32}(A^2-B^2) - \frac{3\sqrt{3}}{4}c\theta + \frac{33}{32}c^2\Delta$$

$$-\frac{15\sqrt{3}}{32}c^2 A - \frac{15\sqrt{3}}{128}c^4.$$

Appendix II

$$K = \frac{4}{27\mu(1-\mu)} \left(\frac{3\sqrt{3}}{4} \rho B_1 - \frac{9}{4} A_1\right) ,$$

$$\Lambda = \frac{4}{48+27\mu(1-\mu)} \left(-\frac{25}{4} A_2 - 4B_3 + \frac{3\sqrt{3}}{4} \rho B_2\right) ,$$

$$M = \frac{4}{48+27\mu(1-\mu)} \left(-\frac{25}{4} A_3 + 4B_2 + \frac{3\sqrt{3}}{4} \rho B_3\right) ,$$

$$N = \frac{4}{960+27\mu(1-\mu)} \left(-\frac{73}{4} A_4 - 8B_5 + \frac{3\sqrt{3}}{4} \rho B_4\right) ,$$

$$\Xi = \frac{4}{960+27\mu(1-\mu)} \left(-\frac{73}{4} A_5 + 8B_4 + \frac{3\sqrt{3}}{4} \rho B_5\right) ,$$

$$\Pi = \frac{4}{27\mu(1-\mu)} \left(-\frac{3}{4} B_1 + \frac{3\sqrt{3}}{4} \rho A_1\right) ,$$

$$P = \frac{4}{48+27\mu(1-\mu)} \left(-\frac{19}{4} B_2 - 4A_3 + \frac{3\sqrt{3}}{4} \rho A_2\right) ,$$

$$\Sigma = \frac{4}{48+27\mu(1-\mu)} \left(-\frac{19}{4} B_3 + 4A_2 + \frac{3\sqrt{3}}{4} \rho A_3\right) ,$$

$$T = \frac{4}{960+27\mu(1-\mu)} \left(-\frac{67}{4} B_4 - 8A_5 + \frac{3\sqrt{3}}{4} \rho A_4\right) ,$$

$$\Phi = \frac{4}{960+27\mu(1-\mu)} \left(-\frac{67}{4} B_5 + 8A_4 + \frac{3\sqrt{3}}{4} \rho A_5\right) .$$

References

Bray, T.A. and Goudas, C.L.: 1967, Adv. Astron. Astrophys. 5, 71.
Buck, T.: 1920, in Moulton's Periodic Orbits, Johnson Repr. Co.
Deprit, A. Henrard, J., Palmore, J., and Price, J.: 1967, Monthly Notices Roy. Astron. Soc. 137, 311.
Goodrich, E.: 1966, Astron. J. 71, 88.
Érdi, B.: 1978, Celest. Mech. 18, 141.
Hénon, M.: 1965, Ann. Astrophys. 28, 992.
Heppenheimer, T.A.: 1973, Celest. Mech. 7, 177.
Moulton: 1920, Periodic Orbits, Johnson Repr. Co., London.
Zagouras, C.G.: 1982, Post doctoral dissertation, University of Patras, Greece.
Zagouras, C.G. and Kazantzis, P.G.: 1979, Astrophys. Space Sci. 61, 389.