ON THE CONTINUATION OF PERIODIC ORBITS FROM THE PLANAR TO THE THREE-DIMENSIONAL GENERAL THREE-BODY PROBLEM

M. MICHALODIMITRAKIS

Department of Theoretical Mechanics, University of Thessaloniki, Thessaloniki, Greece

(Received 22 February, 1978)

Abstract. It is shown that the vertical critical orbits of the general planar problem of three bodies can be used as starting points for finding monoparametric families of three-dimensional periodic orbits. Several numerical examples are given.

I. Introduction

A systematic way of finding three-dimensional periodic orbits for the general three-body problem seems, as far as we know, to be absent. Henon (1973) proposed that the vertical critical orbits of the planar circular restricted three-body problem can be used as starting points for finding three-dimensional periodic orbits of the restricted problem. Based on this proposal, a number of three-dimensional families of periodic orbits which bifurcate from vertical critical orbits of the planar circular restricted problem have been computed (Zagouras, 1977; Zagouras and Markellos, 1977; Zagouras *et al.*, 1978; Markellos, 1977; Markellos and Halioulias, 1977; Michalodimitrakis, 1978). Here we propose that an analogous result holds for the planar general problem. The vertical critical orbits of the planar general problem of three bodies can also be used as starting points for finding three-dimensional periodic orbits. Starting from the vertical critical orbits which are symmetric with respect to the x-axis we can find three-dimensional periodic orbits of the general problem which are symmetric with respect to the xz-plane or to the x-axis or both the xz-plane and x-axis of a suitably defined rotating frame of reference. This rotating frame reduces to the usual rotating frame of the restricted problem when the mass of the third body is equal to zero. Several numerical examples of such a continuation are given.

A second systematic way of finding three-dimensional periodic orbits for the general three-body problem is by analytic continuation, with respect to the small mass m_3, of the periodic orbits of the three-dimensional circular restricted problem. The possibility of such a continuation is a direct consequence of the work of Katopodis (1979).

2. The Lagrangian Function in a Rotating Frame

We consider three bodies P_1, P_2, P_3 with masses m_1, m_2, m_3 respectively, moving in the space under their mutual gravitational attraction. We consider the system isolated and, without loss of generality, we take the center of mass O of the system at the

Celestial Mechanics **19** (1979) 263–277. 0008–8714/79/0193–0263$02.25

origin of an inertial reference system $OXYZ$ whose Z-axis is parallel to the constant angular momentum vector L of the system.

We consider now a new, rotating, frame $Gxyz$ such that:

(a) Its origin coincides with the center of mass G of P_1 and P_2.

(b) Its z-axis is always parallel to the inertial Z-axis.

(c) Its xz-plane contains always P_1 and P_2.

Let x_i, y_i, z_i ($i = 1, 2, 3$) be the Cartesian coordinates of the bodies with respect to the rotating frame and ϑ the angle between the rotating and the inertial x-axes.

Selecting the normalization conditions

$$m_1 + m_2 + m_3 = 1, \qquad K^2 = 1, \qquad \dot{\vartheta}(0) = 1, \tag{1}$$

where K^2 is the gravitational constant, we express the Lagrangian $L=T-V$ of the system as a function of the variables x_i, y_i, z_i, ϑ and their derivatives and we find

$$L = \tfrac{1}{2}M_1(\dot{x}_1^2 + x_1^2\dot{\vartheta}^2) + \tfrac{1}{2}M_2[\dot{x}_3^2 + \dot{y}_3^2 + (x_3^2 + y_3^2)\dot{\vartheta}^2 + 2(x_3\dot{y}_3 - y_3\dot{x}_3)\dot{\vartheta}] +$$
$$+ \tfrac{1}{2}(M_1\dot{z}_1^2 + M_2\dot{z}_3^2) + \frac{m_1m_2}{r_{12}} + \frac{m_1m_3}{r_{13}} + \frac{m_2m_3}{r_{23}}, \tag{2}$$

where

$$M_1 = m_1(1 + \rho), \qquad M_2 = m_3(1 - m_3), \qquad \rho = m_1/m_2, \tag{3}$$
$$r_{ij}^2 = (x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2.$$

The corresponding formulas for the components of the angular momentum vector with respect to the rotating axes, are:

$$L_x = M_2(y_3\dot{z}_3 - z_3\dot{y}_3) - (M_2z_3x_3 + M_1z_1x_1)\dot{\vartheta} = 0, \tag{4}$$
$$L_y = M_1(z_1\dot{x}_1 - x_1\dot{z}_1) + M_2(z_3\dot{x}_3 - \dot{z}_3x_3) - M_2z_3y_3\dot{\vartheta} = 0, \tag{5}$$
$$L_z = M_2(x_3\dot{y}_3 - y_3\dot{x}_3) + [M_2(x_3^2 + y_3^2) + M_1x_1^2]\dot{\vartheta} = \text{const} \equiv P_\vartheta. \tag{6}$$

From (4), (5), (6) we find

$$z_1 = -\frac{M_2}{M_1x_1\dot{\vartheta}}[(\dot{y}_3 + \dot{\vartheta}x_3)z_3 - y_3\dot{z}_3], \tag{7}$$

$$\dot{z}_1 = -\frac{M_2}{M_1x_1^2\dot{\vartheta}}[Rz_3 - (y_3\dot{x}_1 - x_3x_1\dot{\vartheta})\dot{z}_3], \tag{8}$$

$$\dot{\vartheta} = \frac{P_\vartheta - M_2(x_3\dot{y}_3 - y_3\dot{x}_3)}{M_1x_1^2 + M_2(x_3^2 + y_3^2)}, \tag{9}$$

where

$$R = y_3x_1\dot{\vartheta}^2 + (x_3\dot{x}_1 - x_1\dot{x}_3)\dot{\vartheta} + \dot{x}_1\dot{y}_3.$$

The relations (7) and (8) can be used as the equation of motion for z_1, while relation (9) can be used to eliminate completely the angle ϑ from the equations of motion. Thus, the relative motion of the bodies P_1, P_2, P_3 described by the coordinates x_i, y_i, z_i ($i = 1, 2, 3$), can be studied independently of the motion of the system $Gxyz$ itself (Wintner, 1941). We select x_1, z_1, x_3, y_3, z_3, ϑ as independent generalized coordinates.

3. Symmetry Properties of the Motion

With the aid of (7), (8), (9) one can verify that the Lagrangian (2) remains invariant under each of the following transformations:

$$x_1 \to x_1, \quad x_3 \to x_3, \quad y_3 \to -y_3, \quad z_3 \to z_3, \quad t \to -t, \quad \vartheta \to -\vartheta, \tag{10}$$

$$x_1 \to x_1, \quad x_3 \to x_3, \quad y_3 \to -y_3, \quad z_3 \to -z_3, \quad t \to -t, \quad \vartheta \to -\vartheta, \tag{11}$$

$$x_1 \to -x_1, \quad x_3 \to -x_3, \quad y_3 \to y_3, \quad z_3 \to z_3, \quad t \to -t, \quad \vartheta \to -\vartheta, \tag{12}$$

$$x_1 \to -x_1, \quad x_3 \to -x_3, \quad y_3 \to \dot{y}_3, \quad z_3 \to -z_3, \quad t \to -t, \quad \vartheta \to -\vartheta. \tag{13}$$

Transformation (10) represents a reflection of the motion of P_1, P_2, P_3 on the xz-plane, transformation (11) a reflection on the x-axis, transformation (12) a reflection on the yz-plane and transformation (13) a reflection on the y-axis. The invariance of the Lagrangian under each of the above transformations means that for a given motion $x_1(t)$, $x_3(t)$, $y_3(t)$, $z_3(t)$ relative to the rotating frame, the mirror motions with respect to the xz-plane, the x-axis, the yz-plane and the y-axis respectively, are possible motions for P_1, P_2, P_3.

From the above symmetries we conclude that:

(a) If a motion has two perpendicular crossings with the xz-plane, then it will be closed because of the symmetry with respect to the xz-plane. We remind that P_1 and P_2 move always on the xz-plane. Consequently, the moment P_3 crosses perpendicularly the xz-plane, the velocities of P_1 and P_2 must be zero. Such a closed motion is a periodic motion symmetric with respect to the xz-plane. Therefore, a three-dimensional periodic orbit which is symmetric with respect to the xz-plane has initial conditions of the form

$$\begin{aligned} &x_1(0) = x_{10}, \quad x_3(0) = x_{30}, \quad y_3(0) = 0, \quad z_3(0) = z_{30} \\ &\dot{x}_1(0) = 0, \quad \dot{x}_3(0) = 0, \quad \dot{y}_3(0) = \dot{y}_{30}, \quad \dot{z}_3(0) = 0 \end{aligned} \tag{14}$$

and at the moment $t = \tau$ of the next perpendicular crossing satisfies the (periodicity) conditions

$$\dot{x}_1(\tau) = 0, \quad \dot{x}_3(\tau) = 0, \quad y_3(\tau) = 0, \quad \dot{z}_3(\tau) = 0. \tag{15}$$

The period of the orbit is equal to $T = 2\tau$.

(b) If a motion has two perpendicular crossings with the x-axis, then it will be closed because of the symmetry with respect to the x-axis. Such a closed motion is a periodic orbit symmetric with respect to the x-axis. Therefore, a three-dimensional periodic orbit which is symmetric with respect to the x-axis has initial conditions of the form

$$\begin{aligned} &x_1(0) = x_{10}, \quad x_3(0) = x_{30}, \quad y_3(0) = 0, \quad z_3(0) = 0 \\ &\dot{x}_1(0) = 0, \quad \dot{x}_3(0) = 0, \quad \dot{y}_3(0) = \dot{y}_{30}, \quad \dot{z}_3(0) = \dot{z}_{30} \end{aligned} \tag{16}$$

and at the moment $t = \tau$ of the next perpendicular crossing satisfies the (periodicity) conditions

$$\dot{x}_1(\tau) = 0, \quad \dot{x}_3(\tau) = 0, \quad y_3(\tau) = 0, \quad z_3(\tau) = 0. \tag{17}$$

The period of the orbit is equal to $T = 2\tau$.

(c) In an analogous way we can find periodic orbits symmetric with respect to the yz-plane or to the y-axis.

By combining the symmetries (a) and (b) we find a periodic orbit which is symmetric both with respect to the xz-plane and the x-axis, and as a consequence with respect to the xy-plane. Therefore, a three-dimensional periodic orbit which is symmetric both with respect to the xz-plane and the x-axis has initial conditions of the form

$$\begin{aligned} &x_1(0) = x_{10}, \quad x_3(0) = x_{30}, \quad y_3(0) = 0, \quad z_3(0) = z_{30} \\ &\dot{x}_1(0) = 0, \quad \dot{x}_3(0) = 0, \quad \dot{y}_3(0) = \dot{y}_{30}, \quad \dot{z}_3(0) = 0 \end{aligned} \tag{18}$$

or of the form

$$\begin{aligned} &x_1(0) = x_{10}, \quad x_3(0) = x_{30}, \quad y_3(0) = 0, \quad z_3(0) = 0 \\ &\dot{x}_1(0) = 0, \quad \dot{x}_3(0) = 0, \quad \dot{y}_3(0) = \dot{y}_{30}, \quad \dot{z}_3(0) = \dot{z}_{30} \end{aligned} \tag{19}$$

and at the moment $t = \tau$ of the next perpendicular crossing satisfies the (periodicity) conditions

$$\dot{x}_1(\tau) = 0, \quad \dot{x}_3(\tau) = 0, \quad y_3(0) = 0, \quad z_3(\tau) = 0 \tag{20}$$

or the (periodicity) conditions

$$\dot{x}_1(\tau) = 0, \quad \dot{x}_3(\tau) = 0, \quad y_3(\tau) = 0, \quad \dot{z}_3(\tau) = 0. \tag{21}$$

The period of the orbit is equal to $T = 4\tau$.

In an analogous way we can combine the symmetries with respect to the yz-plane and the y-axis.

4. Continuation of Vertical Critical Planar Orbits to the Three-Dimensional Case

We consider the case of quasi-planar motion. In this case we can set

$$\begin{aligned} &\frac{1}{r_{ij}} \simeq \frac{1}{\rho_{ij}}\left[1 - \frac{(z_i - z_j)^2}{2\rho_{ij}^2}\right], \\ &\rho_{ij}^2 = (x_i - x_j)^2 + (y_i - y_j)^2. \end{aligned} \tag{22}$$

With the approximation (22) the Lagrangian (2) of the three-dimensional motion can be written as

$$L = L_p + L_v, \tag{23}$$

where L_p depends on the plane variables $x_i, y_i, \dot{x}_i, \dot{y}_i, \dot{\vartheta}$ only and L_v depends on all the variables,

$$\begin{aligned} L_p = &\tfrac{1}{2}M_1(\dot{x}_1^2 + x_1^2\dot{\vartheta}^2) + \tfrac{1}{2}M_2[\dot{x}_3^2 + \dot{y}_3^2 + (x_3^2 + y_3^2)\dot{\vartheta}^2 + 2\dot{\vartheta}(x_3\dot{y}_3 - y_3\dot{x}_3)] + \\ &+ \frac{m_1 m_2}{\rho_{12}} + \frac{m_1 m_3}{\rho_{13}} + \frac{m_2 m_3}{\rho_{23}}, \end{aligned} \tag{24}$$

$$L_v = \tfrac{1}{2}(M_1\dot{z}_1^2 + M_2\dot{z}_3^2) - \frac{m_1m_2}{2}\frac{(z_1 - z_2)^2}{\rho_{12}^3} - \frac{m_1m_3}{2}\frac{(z_1 - z_3)^2}{\rho_{13}^3} - \frac{m_2m_3}{2}\frac{(z_2 - z_3)^2}{\rho_{23}^3}. \tag{25}$$

With the aid of (7), (8), and (25) we find that the differential equation for z_3 is

$$\ddot{z} = Az_3 + B\dot{z}_3 \tag{26}$$

where

$$A = -\frac{1}{1 - m_3}\left[\frac{m_1}{\rho_{13}^3} + \frac{m_2}{\rho_{23}^3} + \frac{m_1M_2}{M_1}\left(\frac{1}{\rho_{13}^3} - \frac{1}{\rho_{23}^3}\right)\frac{(\dot{y}_3 + \dot{\vartheta}x_3)}{x_1\dot{\vartheta}}\right] \tag{27}$$

$$B = \frac{m_1m_3M_2}{M_1}\left(\frac{1}{\rho_{23}^3} - \frac{1}{\rho_{13}^3}\right)\frac{y_3}{x_1\dot{\vartheta}}. \tag{28}$$

A periodic orbit (of period T)

$$\begin{aligned} x_1(t) &= x_{100}(t), & x_3(t) &= x_{300}(t), & y_3(t) &= y_{300}(t) \\ \dot{x}_1(t) &= \dot{x}_{100}(t), & \dot{x}_3(t) &= \dot{x}_{300}(t), & \dot{y}_3(t) &= \dot{y}_{300}(t) \end{aligned} \tag{29}$$

of the general planar problem can be considered as a particular solution of the differential equations of motion resulting from the Lagrangian (23), for $z_3(t) \equiv 0$. Let $\xi_1(t)$, $\xi_2(t)$, $\xi_3(t)$, $\xi(t)$, $\eta_1(t)$, $\eta_2(t)$, $\eta_3(t)$, $\eta(t)$ be perturbations to the planar orbit (29). Then, the perturbed solution can be written as

$$\begin{aligned} x_1 &= x_{100} + \xi_1, & x_3 &= x_{300} + \xi_2, & y_3 &= y_{300} + \xi_3, & z_3 &= \xi \\ \dot{x}_1 &= \dot{x}_{100} + \eta_1, & \dot{x}_3 &= \dot{x}_{300} + \eta_2, & \dot{y}_3 &= \dot{y}_{300} + \eta_3, & \dot{z}_3 &= \eta. \end{aligned}$$

Substituting (30) in the differential equations of motion and linearizing with respect to the perturbations we find a system of linear differential equations with periodic coefficients. The linearized Equation (26) has the form

$$\ddot{\xi} = A^*(t)\xi + B^*(t)\eta \tag{31}$$

where $A^*(t)$ and $B^*(t)$ are given periodic functions of time, of period T, because they result from the substitution of the periodic functions (29) into the expressions (27) and (28). We observe that (31) involves only the vertical perturbations ξ, η. The remaining differential equations involve only the perturbations ξ_i, η_i ($i = 1, 2, 3$) in the plane and they are, in fact, the system of variational equations considered in the usual investigation of the stability in the plane.

Suppose we give to the planar periodic orbit (30) a vertical initial perturbation such that the resulting vertical perturbation $\xi(t)$, $\eta(t)$ is periodic in time with period T. Then we get, in the linear approximation, a quasi-planar three-dimensional periodic orbit. The time evolution of a vertical perturbation is governed by Equation (31) which can be written in the form

$$\begin{pmatrix} \dot{\xi} \\ \dot{\eta} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ A^* & B^* \end{pmatrix}\begin{pmatrix} \xi \\ \eta \end{pmatrix}. \tag{32}$$

The solution of (32) is of the form

$$\begin{pmatrix} \xi(t) \\ \eta(t) \end{pmatrix} = \Phi(t) \begin{pmatrix} \xi(0) \\ \eta(0) \end{pmatrix}, \tag{33}$$

and the fundamental matrix $\Phi(t)$ can be written in the form

$$\Phi(t) = \begin{pmatrix} \xi^1(t) & \xi^2(t) \\ \eta^1(t) & \eta^2(t) \end{pmatrix}, \tag{34}$$

where $\xi^1(t)$, $\eta^1(t)$ and $\xi^2(t)$, $\eta^2(t)$ are two independent solutions of (32). In order to have a periodic solution of (32), of period T', the following condition must hold

$$\begin{pmatrix} \xi(0) \\ \eta(0) \end{pmatrix} = \Phi(T') \begin{pmatrix} \xi(0) \\ \eta(0) \end{pmatrix}. \tag{35}$$

It can be shown (Henon, 1973; Delibaltas, 1976), that for a planar periodic orbit (of the restricted or the general problem) which is symmetric with respect to the x-axis the vertical stability index α_v is equal to

$$\alpha_v = \frac{\xi^1(\tau)\eta^2(\tau) + \xi^2(\tau)\eta^1(\tau)}{\xi^1(\tau)\eta^2(\tau) - \xi^2(\tau)\eta^1(\tau)}, \tag{36}$$

where τ is the moment of the second (besides the initial) perpendicular crossing of the planar orbit with the x-axis. If $|a_v| = 1$, the periodic orbit is called 'vertical critical'.

Let us first establish the connection between the vertical stability index and the vertical continuation of the plane periodic orbit.

Suppose that the quasi-plane periodic orbit resulting from the vertical perturbation of the plane periodic orbit, is symmetric with respect to the xz-plane. We have shown (Section 3) that for such an orbit the following conditions must be satisfied

$$z_3(0) = \xi(0) \neq 0, \qquad \dot{z}_3(0) = \eta(0) = 0, \qquad z_3(\tau) = \xi(\tau) \neq 0,$$
$$\dot{z}_3(\tau) = \eta(\tau) = 0. \tag{37}$$

Substituting (37) into (33) we get

$$\begin{pmatrix} \xi(\tau) \\ 0 \end{pmatrix} = \Phi(\tau) \begin{pmatrix} \xi(0) \\ 0 \end{pmatrix} \tag{38}$$

From (38) and (34) we conclude that, $\eta^1(\tau) = 0$. Then, (36) gives $\alpha_v = +1$ which means that the plane orbit is a vertical critical orbit.

Suppose now that the above quasi-planar periodic orbit is symmetric with respect to the x-axis. For such an orbit the following conditions must be satisfied (Section 3)

$$\xi(0) = 0, \qquad \eta(0) \neq 0, \qquad \xi(\tau) = 0, \qquad \eta(\tau) \neq 0. \tag{39}$$

From (39), (33), and (34) we conclude that $\xi^2(\tau) = 0$. Then, (36) gives $\alpha_v = +1$ i.e. the plane periodic orbit is a vertical critical orbit.

Finally, suppose that the quasi-plane periodic orbit is symmetric both with respect to the xz-plane and the x-axis. For such an orbit the following conditions must be satisfied (Section 3).

$$\xi(0) \neq 0, \qquad \eta(0) = 0, \qquad \xi(\tau) = 0, \qquad \eta(\tau) \neq 0, \tag{40}$$

or

$$\xi(0) = 0, \qquad \eta(0) \neq 0, \qquad \xi(\tau) \neq 0, \qquad \eta(\tau) = 0. \tag{41}$$

From (40), (33), (34) or (41), (33), (34) we conclude that $\xi^1(\tau) = 0$ or $\eta^2(\tau) = 0$, respectively. Then, (36) gives $\alpha_v = -1$ in both cases i.e. the planar periodic orbit is a vertical critical orbit.

From the above we conclude that a symmetric (with respect to the x-axis) planar periodic orbit which can be vertically continued to a symmetric quasi-planar periodic orbit, must be a vertical critical orbit.

Given a vertical critical planar periodic orbit which is symmetric with respect to the x-axis, we distinguish the following cases:

(1) $\alpha_v = +1$. Then (36) gives $\eta^1(\tau) = 0$ or $\xi^2(\tau) = 0$.

(a) Let $\eta^1(\tau) = 0$. If the initial perturbation is of the form

$$\xi(0) = \varepsilon, \qquad \eta(0) = 0 \tag{42}$$

i.e. the perturbed orbit starts perpendicularly from the xz-plane, the perturbation at the next crossing ($t = \tau$) is

$$\begin{pmatrix} \xi(\tau) \\ \eta(\tau) \end{pmatrix} = \begin{pmatrix} \xi^1(\tau) & \xi^2(\tau) \\ 0 & \eta^2(\tau) \end{pmatrix} \begin{pmatrix} \varepsilon \\ 0 \end{pmatrix}$$

or

$$\xi(\tau) = \varepsilon\xi^1(\tau) \neq 0, \qquad \eta(\tau) = 0. \tag{43}$$

For the planar periodic component of the motion we have

$$\dot{x}_1(\tau) = 0, \qquad \dot{x}_3(\tau) = 0, \qquad \dot{y}_3(\tau) \neq 0, \qquad y_3(\tau) = 0. \tag{44}$$

From (43) and (44) we conclude that for $t = \tau$ the perturbed orbit crosses the xz-plane perpendicularly. Therefore the perturbed orbit will be a periodic orbit, of period $T' = 2\tau = T$, symmetric with respect to the xz-plane.

(b) Let $\xi^2(\tau) = 0$. If the initial perturbation is of the form

$$\xi(0) = 0, \qquad \eta(0) = \varepsilon, \tag{45}$$

i.e. the perturbed orbit starts perpendicularly from the x-axis, the perturbation at the next crossing ($t = \tau$) is

$$\begin{pmatrix} \xi(\tau) \\ \eta(\tau) \end{pmatrix} = \begin{pmatrix} \xi^1(\tau) & 0 \\ \eta^1(\tau) & \eta^2(\tau) \end{pmatrix} \begin{pmatrix} 0 \\ \varepsilon \end{pmatrix},$$

or

$$\xi(\tau) = 0, \qquad \eta(\tau) = \varepsilon\eta^2(\tau) \neq 0. \tag{46}$$

From (46) and (44) we conclude that for $t = \tau$ the perturbed orbit crosses the x-axis perpendicularly. Therefore the perturbed orbit will be a periodic orbit, of period $T' = 2\tau = T$, symmetric with respect to the x-axis.

(2) $\alpha_v = -1$. Then (36) gives $\xi^1(\tau) = 0$ or $\eta^2(\tau) = 0$.

(*a*) Let $\xi^1(\tau) = 0$. If the initial perturbation is of the form (43) i.e. the perturbed orbit starts perpendicularly from the xz-plane, the perturbation at the next crossing $(t = \tau)$ is

$$\begin{pmatrix} \xi(\tau) \\ \eta(\tau) \end{pmatrix} = \begin{pmatrix} 0 & \xi^2(\tau) \\ \eta^1(\tau) & \eta^2(\tau) \end{pmatrix} \begin{pmatrix} \varepsilon \\ 0 \end{pmatrix}$$

or

$$\xi(\tau) = 0, \qquad \eta(\tau) = \varepsilon\eta^1(\tau) \neq 0. \tag{47}$$

From (47) and (44) we conclude that for $t = \tau$ the perturbed orbit crosses the x-axis perpendicularly. Therefore, the perturbed orbit will be a periodic orbit, of period $T' = 4\tau = 2T$, symmetric both with respect to the xz-plane and x-axis.

(b) Let $\eta^2(\tau) = 0$. If the initial perturbation is of the form (45) i.e., the perturbed orbit starts perpendicularly from the x-axis, the perturbation at the next crossing $(t = \tau)$ is

$$\xi(\tau) = \varepsilon\xi^2(\tau) \neq 0, \qquad \eta(\tau) = 0. \tag{48}$$

From (48) and (44) we conclude that for $t = \tau$ the perturbed orbit crosses xz-plane perpendicularly. Therefore, the perturbed orbit will be a periodic orbit of period $T' = 4\tau = 2T$, symmetric both with respect to the xz-plane and x-axis. This case is equivalent to the case $\xi^1(\tau) = 0$ with the initial $(t = 0)$ and final $(t = \tau)$ crossings exchanged.

From the above we conclude that if we give suitable initial vertical perturbations to a vertical critical orbit of the planar general three-body problem, we can find, in the linear approximation, initial conditions for three-dimensional periodic orbits. The form of the initial conditions for these quasi-planar periodic orbits as well as their symmetry and period, depend on the symmetry and the stability character of the critical orbit.

The above linear argument does not constitute a mathematical proof of the possibility of the vertical continuation of the vertical critical orbits. It does not prove the existence of families of periodic orbits which bifurcate from the vertical critical orbits. However, it gives us a way to find, by numerical integration of the full non-linear equations of motion, the above mentioned bifurcating families. Indeed, we give two such families, as a numerical evidence for the existence of the bifurcating families.

5. Numerical Results

In the following we shall describe two families of three-dimensional periodic orbits which are symmetric with respect to the xz plane and bifurcate from vertical critical orbits which are symmetric with respect to the x axis.

Starting from the vertical critical orbits of the restricted ($m_3 = 0$) problem (Henon, 1973) we can find, by continuation with respect to m_3, families of vertical critical orbits of the general ($m_3 \neq 0$) problem along which m_3 varies. As an example, we computed a family $c1v$ of vertical critical orbits (with $m_1 = m_2$) starting from the critical orbit $c1v$ (using Henon's notation) of the circular restricted problem with $\mu = 0.5$. The initial conditions v.s m_3 diagrams for this family are shown in Figure 1, and the initial conditions of some of its orbits are given in the Table I. We note that this family extends up to $m_3 = 1$. Figure 2 shows the form of its orbits. We observe that the periodic orbits are approximately ellipses centered at the origin and having the y axis as major axis. The orbits of the family $c1v$ are symmetric with respect to the x axis.

We selected the vertical critical orbits No. 2 and No. 3 of the family $c1v$ and computed the three-dimensional families C1 and C2 which bifurcate from the orbit No. 2 and the orbit No. 3, respectively.

*Family C*1. Its orbits are symmetric with respect to the xz plane. Figures 3, 4, and 5 show the projections of some orbits of this family on the xy, xz, and yz planes, respectively. We observe that the periodic orbits are approximately, plane ellipses centered at the origin and having the y axis as major axis, i.e. the plane of these ellipses is perpendicular to the xz plane. As we proceed along the family, the inclination of the orbits relative to the xy plane, as well as their size, increases until it reaches

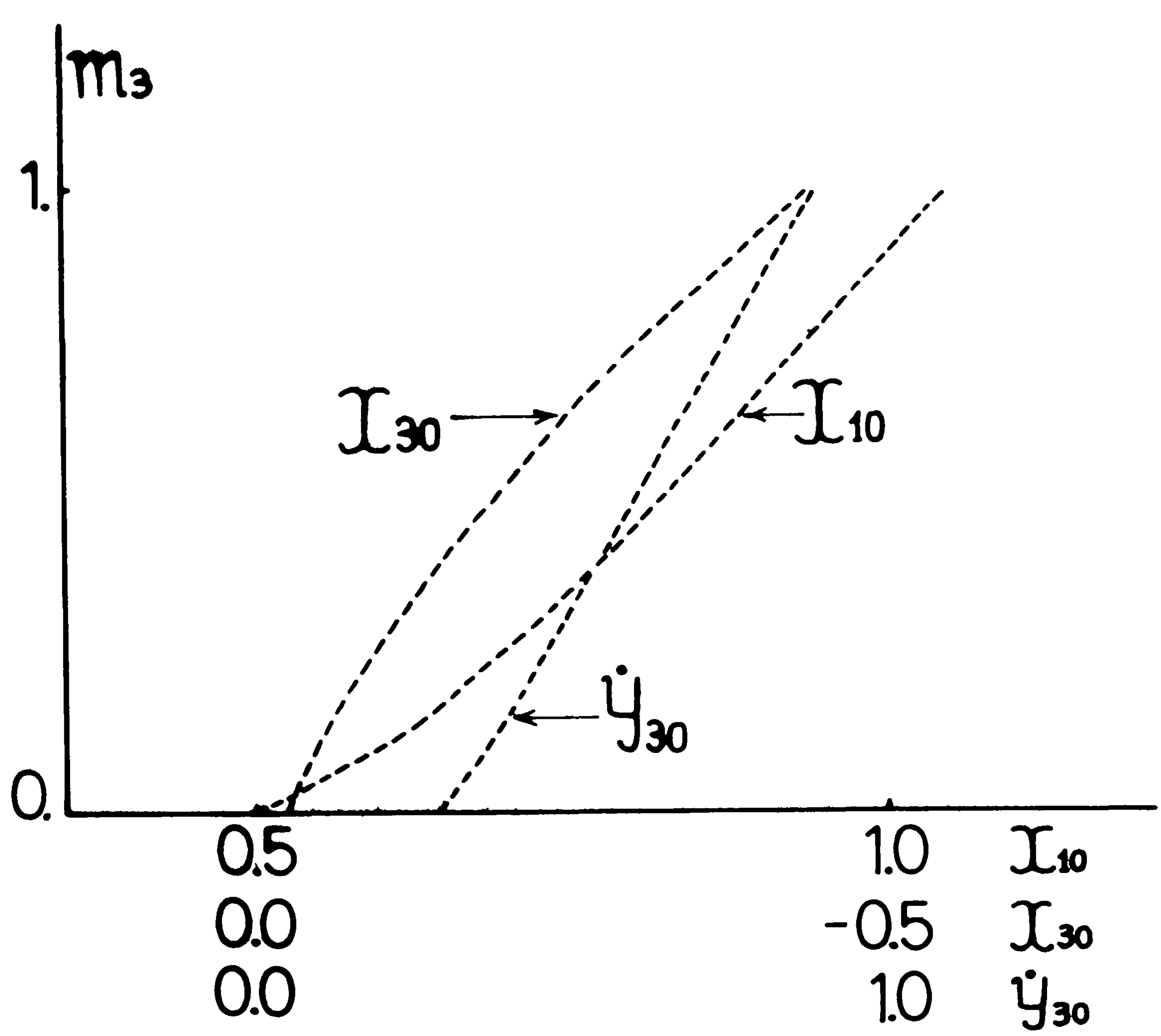

Fig. 1. The initial conditions v.s m_3 diagram for the critical family $c1v$.

TABLE I
Initial conditions for the vertical critical family $c1v$

	x_{10}	x_{30}	y_{30}	$T/2$	m_3
1	0.500000	−0.023766	0.294193	1.114	0.000
2	0.501132	−0.023988	0.295006	1.118	0.001
3	0.593566	−0.048432	0.366764	1.444	0.100
4	0.663664	−0.077347	0.429751	1.712	0.200
5	0.721985	−0.110135	0.488911	1.947	0.300
6	0.773790	−0.146785	0.546706	2.166	0.400
7	0.822153	−0.187433	0.604404	2.377	0.500
8	0.869170	−0.232222	0.662608	2.586	0.600
9	0.916362	−0.281145	0.721387	2.798	0.700
10	0.964765	−0.333882	0.780308	2.014	0.800
11	1.014878	−0.389700	0.838526	3.236	0.900
12	1.067003	−0.448897	0.897899	3.461	1.000

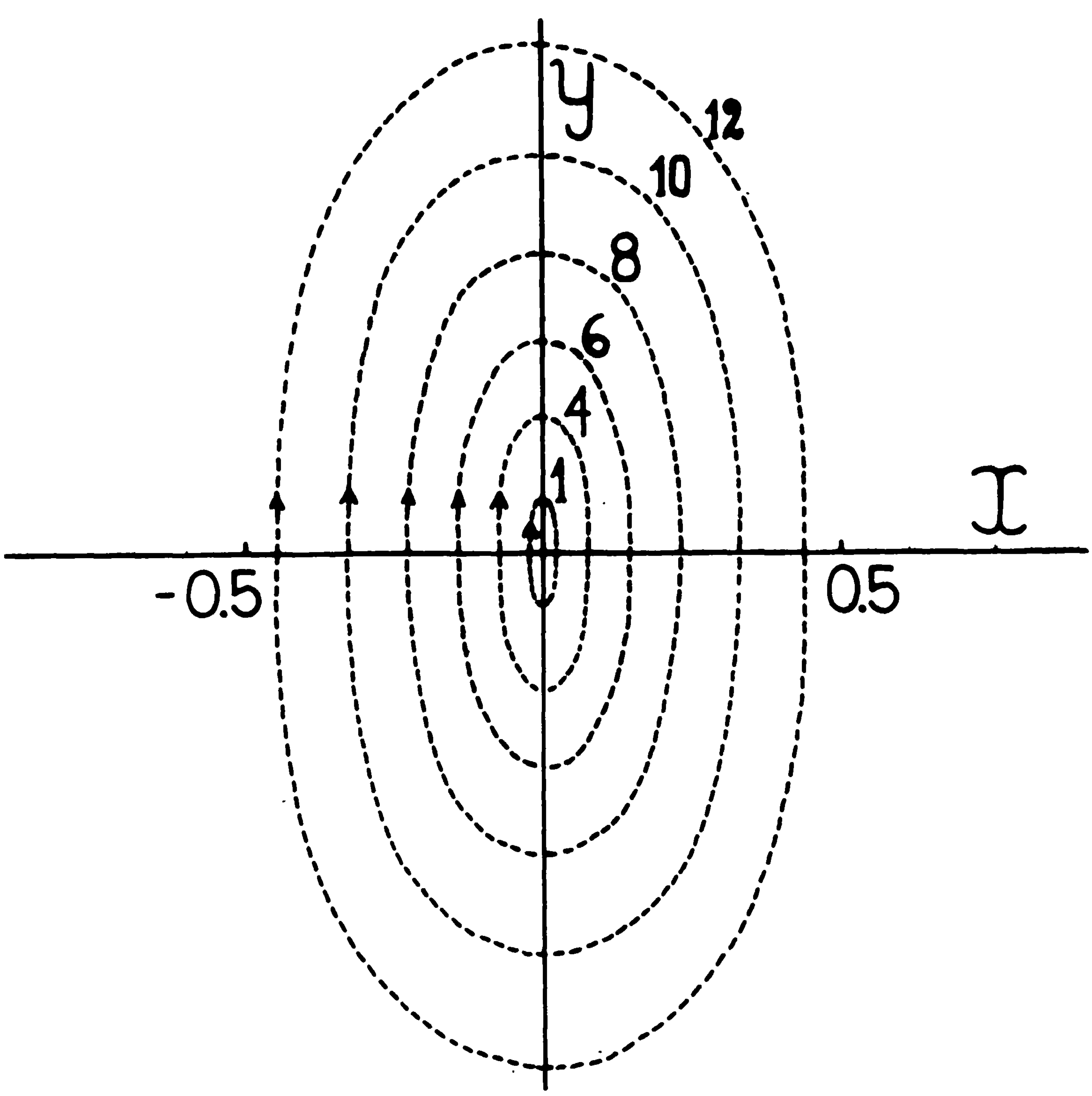

Fig. 2. Periodic orbits of the critical family $c1c$. Only the orbit of m_3 is shown. (The numeral nea each orbit denotes the position of the orbit in the corresponding table of initial conditions.)

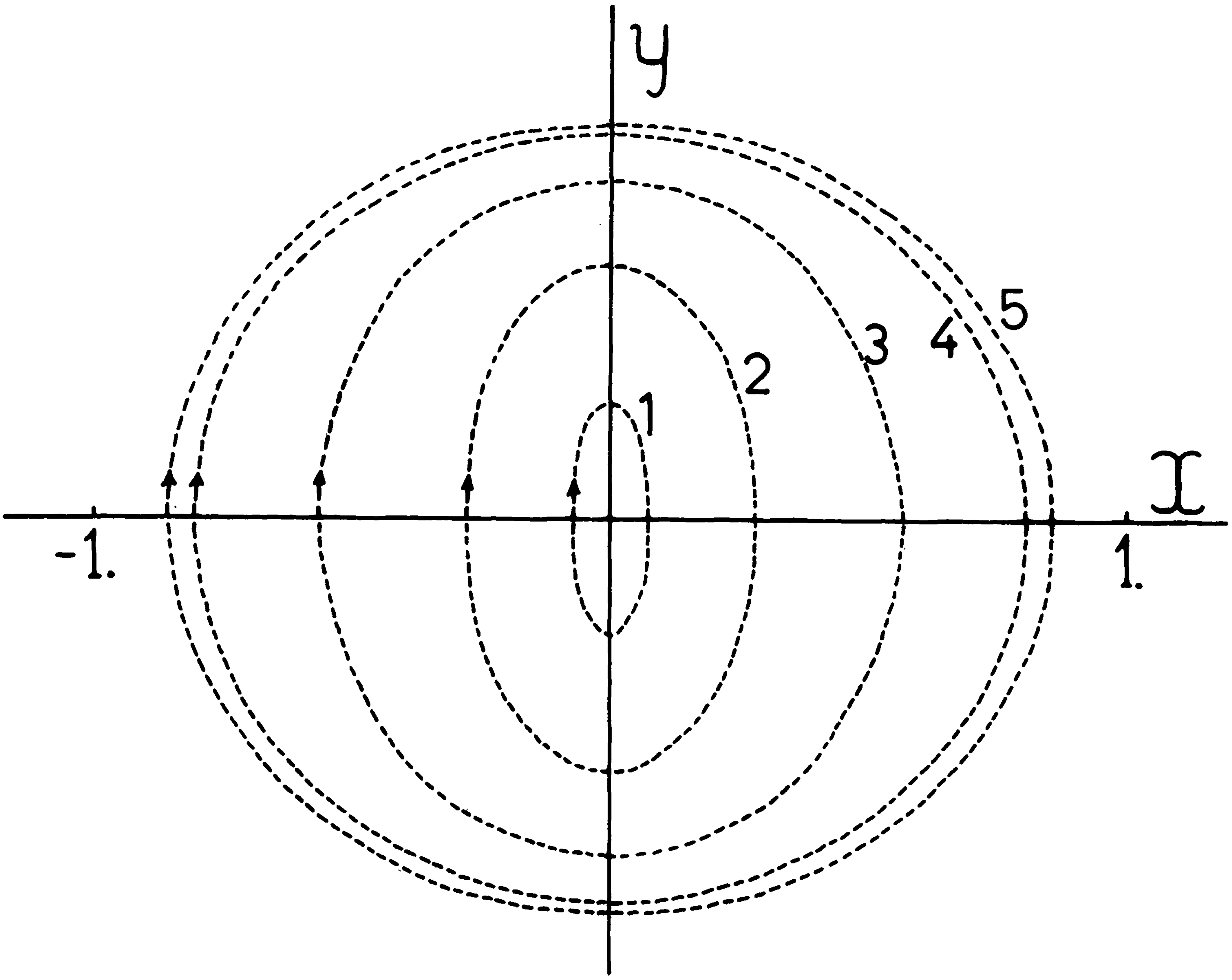

Fig. 3. Projections of some three-dimensional periodic orbits of the family C1 on the xy plane.

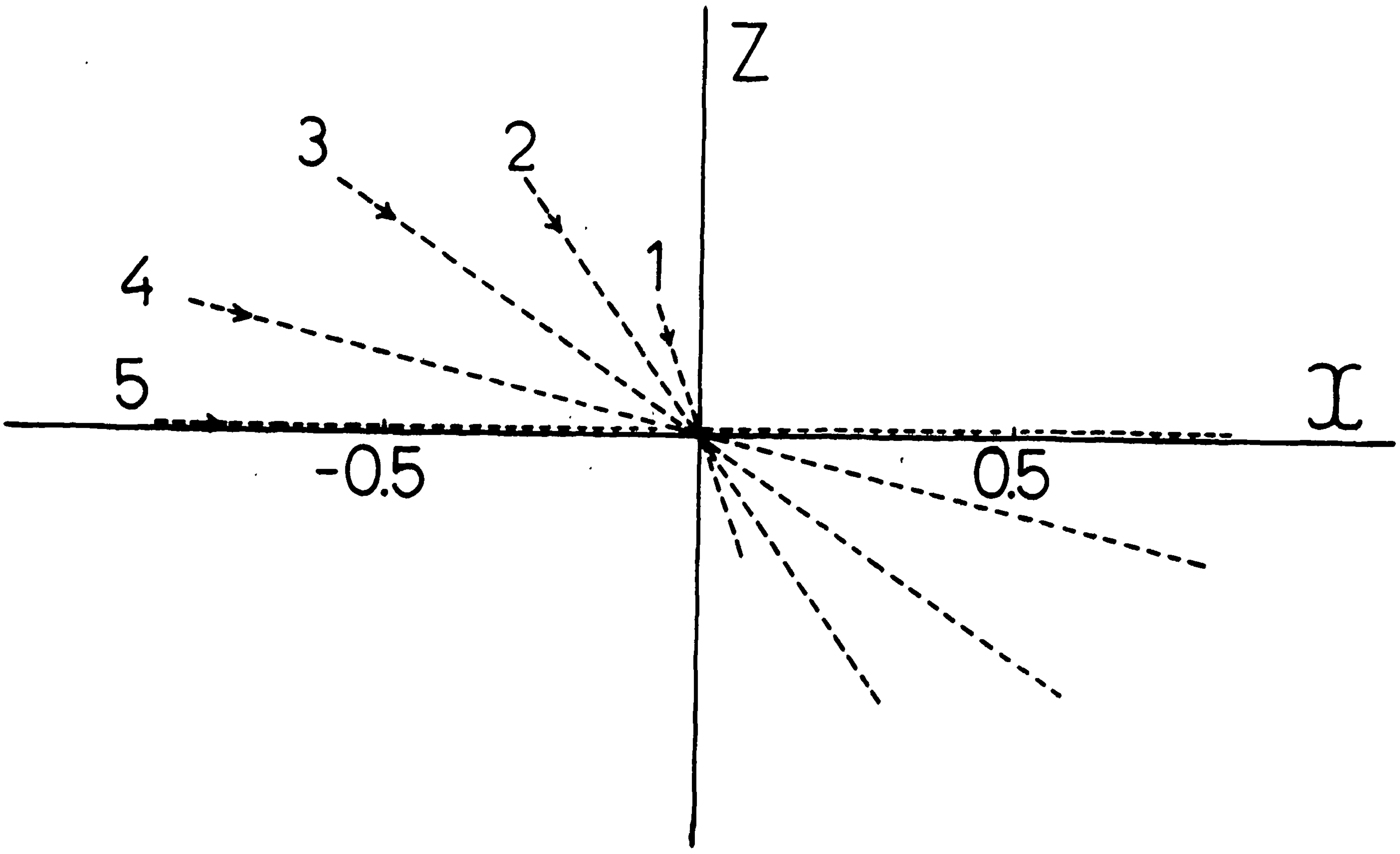

Fig. 4. Projections of some three-dimensional periodic orbits of the family C1 on the xz plane.

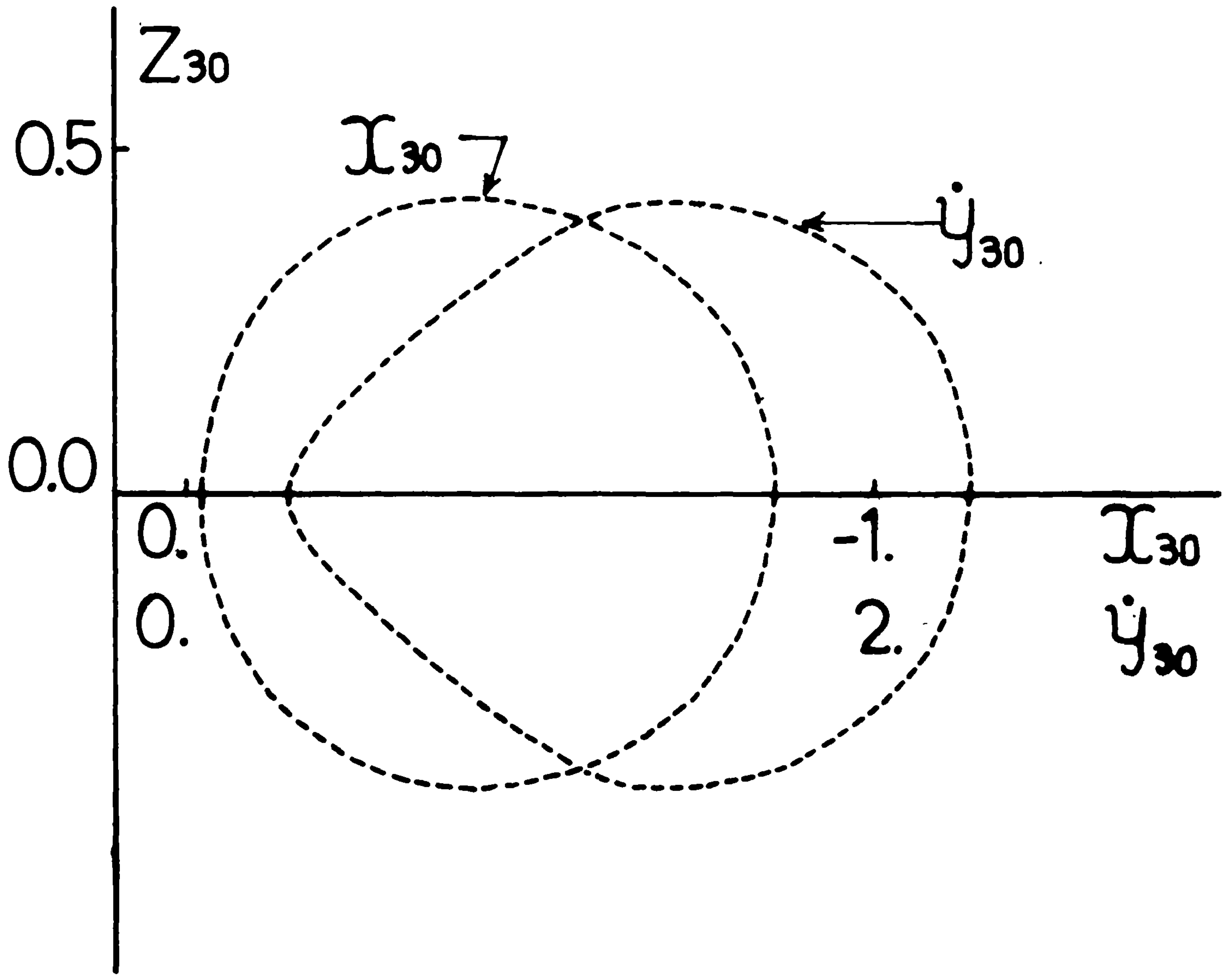

Fig. 5. Projections of the characteristic curve of the family C1 on the $x_{30}z_{30}$ and $y_{30}z_{30}$ planes.

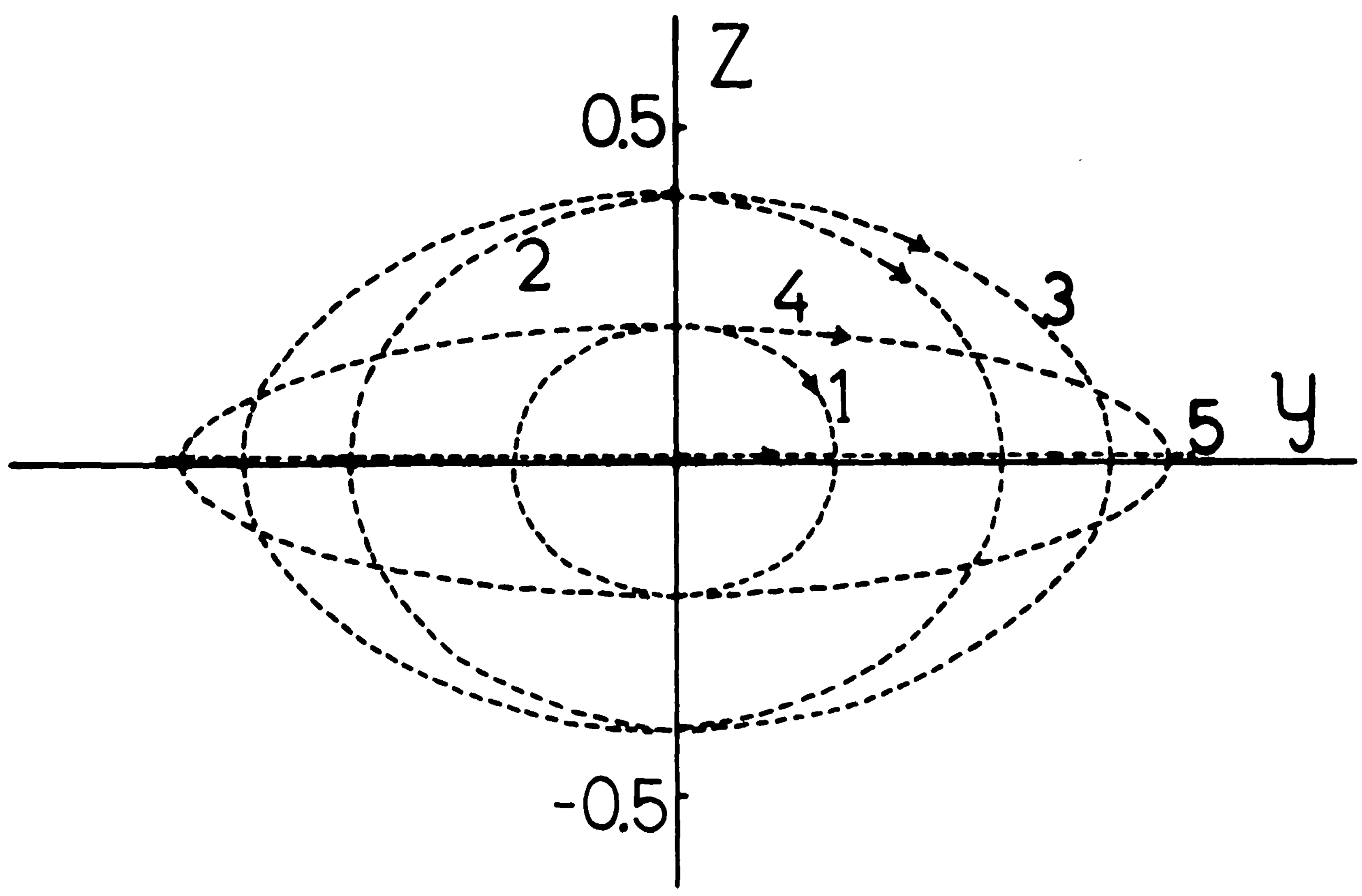

Fig. 6. Projections of some three-dimensional periodic orbits of the family C1 on the yz plane.

TABLE II
Initial conditions for the three-dimensional family C1

	x_{10}	x_{30}	y_{30}	z_{30}	$T/2$
1	0.501132	−0.023994	0.295064	0.002000	1.118
2	0.501068	−0.036418	0.410596	0.100000	1.145
3	0.500908	−0.069965	0.621538	0.200000	1.219
4	0.500715	−0.133997	0.847972	0.300000	1.319
5	0.500502	−0.282009	1.176341	0.400000	1.400
6	0.500378	−0.420000	1.442490	0.424391	1.390
7	0.500252	−0.567938	1.725582	0.400000	1.345
8	0.500097	−0.730920	2.040981	0.300000	1.283
9	0.500015	−0.807280	2.190076	0.200000	1.253
10	0.499970	−0.846535	2.267004	0.100000	1.238
11	0.499956	−0.858828	2.291129	0.002000	1.233

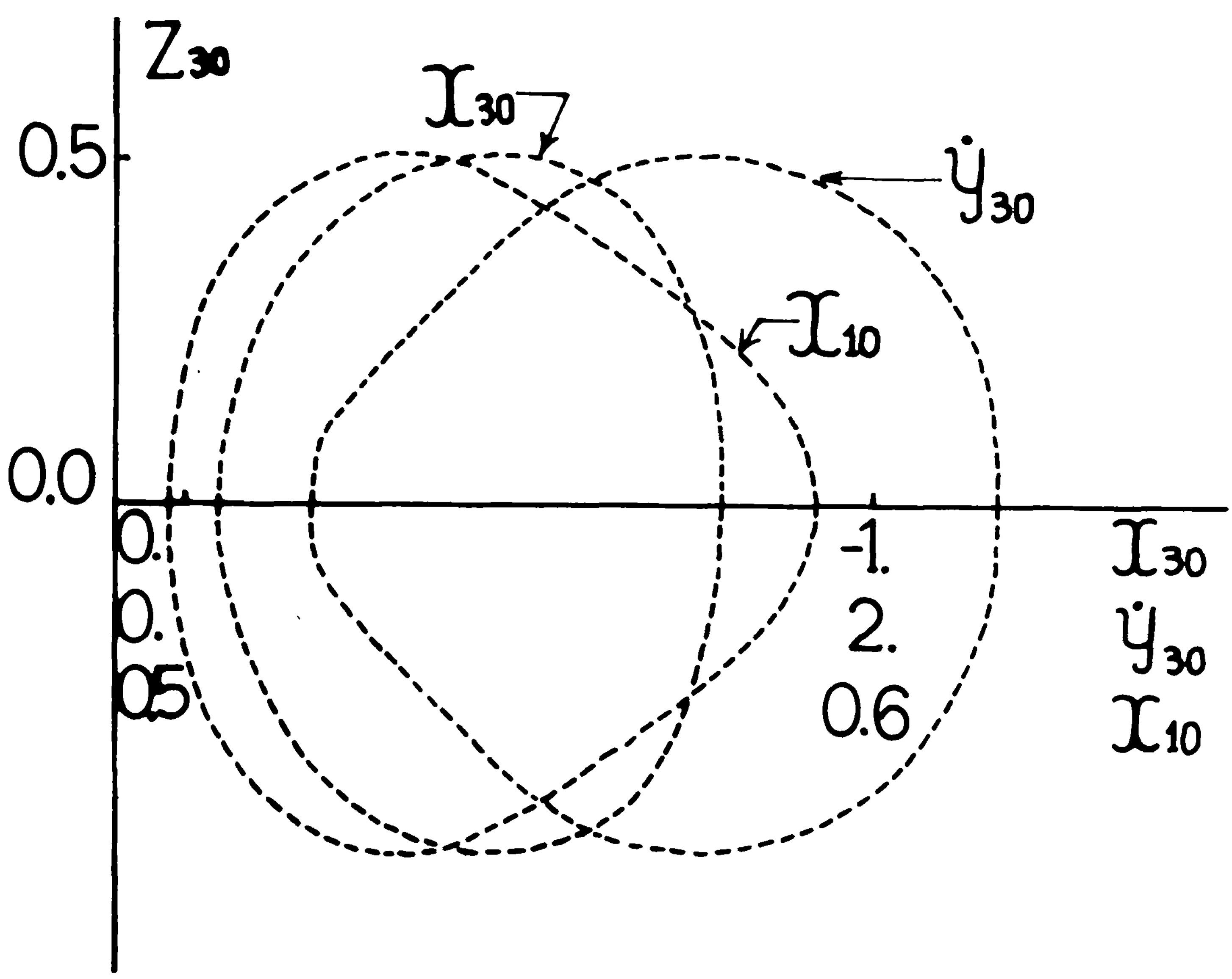

Fig. 7. Projections of the characteristic curve of the family C2 on the $x_{10}z_{30}$, $x_{30}z_{30}$, $\dot{y}_{30}z_{30}$ planes.

a maximum. Then the inclination decreases (while the size still increases) until it reaches again the zero value corresponding to a plane orbit. This plane orbit is a vertical critical orbit which encircles both primaries and belongs to the family of vertical critical orbits which originate from the vertical critical orbit $m2v$ of the circular restricted problem with $\mu = 0.5$. In this way, starting from a small ellipse around the origin (on the xy plane) we arrived, via three-dimensional periodic orbits, to a big ellipse around both primaries (on the xy plane).

TABLE III
Initial conditions for the three-dimensional family C2

	x_{10}	x_{30}	y_{30}	130	$T/2$
1	0.593565	−0.048436	0.366792	0.002000	1.444
2	0.590617	−0.058393	0.429963	0.100000	1.459
3	0.582512	−0.085092	0.574143	0.200000	1.502
4	0.571139	−0.129306	0.750066	0.300000	1.567
5	0.558265	−0.204806	0.963228	0.400000	1.630
6	0.539941	−0.382990	1.369412	0.500000	1.611
7	0.533273	−0.455000	1.532086	0.509008	1.566
8	0.509410	−0.680000	2.082627	0.395855	1.338
9	0.503410	−0.734097	2.230461	0.300000	1.263
10	0.500176	−0.765985	2.321680	0.200000	1.213
11	0.498638	−0.782855	2.371304	0.100000	1.185
12	0.498185	−0.788177	2.387164	0.002000	1.176

The characteristic curve of the family C1 is a closed curve in the four-dimensional space $(x_{10}, x_{30}, y_{30}, z_{30})$ of initial conditions. Figure 6 shows the projections of the characteristic curve on the $x_{30}z_{30}$ and $y_{30}z_{30}$ planes (along the family we have $x_{10} \simeq 0.5$). We observe that these projections are symmetric with respect to the z_{30} axis.

The initial conditions of some orbits of the family C1 are given in Table II.

Family C2. The behaviour of this family is qualitatively similar to that of C1. Figure 7 shows the projections of its characteristic curve on the $x_{10}z_{30}$, $x_{30}z_{30}$ and $\dot{y}_{30}z_{30}$ planes and Table ıII gives the initial conditions of some of its orbits.

The behaviour of the families C1 and C2 is qualitatively similar to that of the family $c1v$ of the three-dimensional circular restricted problem which bifurcates from the plane vertical critical orbit $c1v$ (Michalodimitrakis, 1978).

A more systematic numerical study of the three-dimensional general problem of three-bodies is now taking place at the University of Thessaloniki.

Appendix: Transformation Equations from the Inertial to the Rotating Coordinate System

Let x_i, y_i, z_i and X_i, Y_i, Z_i $(i = 1, 2, 3)$ be the Cartesian coordinates of the particles relative to the rotating and the inertial coordinate system respectively. Let also ϑ be the angle between the rotating and the inertial x axes.

The transformation equations from the inertial to the rotating coordinate system are:

$$x_i = (X_i - X_G)\cos\vartheta + (Y_i - Y_G)\sin\vartheta$$

$$y_i = -(X_i - X_G)\sin\vartheta + (Y_i - Y_G)\cos\vartheta$$

$$z_i = Z_i - Z_G$$

$$\dot{x}_i = (\dot{X}_i - \dot{X}_G)\cos\vartheta + (\dot{Y}_i - \dot{Y}_G)\sin\vartheta + y_i\dot{\vartheta}$$
$$\dot{y}_i = -(\dot{X}_i - \dot{X}_G)\sin\vartheta + (\dot{Y}_i - \dot{Y}_G)\cos\vartheta - x_i\dot{\vartheta}$$
$$\dot{z}_i = \dot{Z}_i - \dot{Z}_G$$

where

$$X_G = \frac{m_1 X_1 + m_2 X_2}{m_1 + m_2}, \qquad Y_G = \frac{m_1 Y_1 + m_2 Y_2}{m_1 + m_2}, \qquad Z_G = \frac{m_1 Z_1 + m_2 Z_2}{m_1 + m_2},$$

$$\dot{\vartheta} = -\frac{1}{\rho(Y_1 - Y_2)}[(\dot{X}_1 - \dot{X}_2)\rho - (X_1 - X_2)\dot{\rho}],$$

$$\rho^2 = (X_1 - X_2)^2 + (Y_1 - Y_2)^2, \qquad \cos\vartheta = (X_1 - X_2)/\rho, \quad \sin\vartheta = (Y_1 - Y_2)/\rho.$$

(the dot over a letter denotes differentiation with respect to time).

References

Delibaltas, P.: 1976, *Astrophys. Space Sci.* **45**, 000.

Hénon, M.: 1973, *Astron. Astrophys.* **28**, 415.

Katopodis, K.: 1979, *Celest. Mech.* **19**, 43.

Markellos, V. V.: 1977, *Monthly Notices Roy. Astron. Soc.* **180**, 103.

Markellos, V. V. and Halioulias, A. A.: 1977, *Astrophys. Space Sci.* **46**, 183.

Michalodimitrakis, M.: 1978, *Astron. Astrophys.* **64**, 83.

Wintner, A.: 1941, *The Analytical Foundations of Celestial Mechanics*, Princeton Univ. Press, Princeton Univ. Press, Princeton, New Jersey, Section 323–326, pp. 370–374.

Zagouras, C.: 1977, *Astrophys. Space Sci.* **50**, 383.

Zagouras, C. and Markellos, V.: 1977, *Astron. Astrophys.* **59**, 79.

Zagouras, C. and Kalogeropoulou, M.: 1978, to be published in *Astron. Astrophys.*