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Linear Spin-Zero Quantum Fields in External Gravitational and Scalar Fields

II. Generally Covariant Perturbation Theory*

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Abstract. The quantum theory of both linear, and interacting fields on curved space-times is discussed. It is argued that generic curved space-time situations force the adoption of the algebraic approach to quantum field theory: and a suitable formalism is presented for handling an arbitrary quasi-free state in an arbitrary globally hyperbolic space-time.

For the interacting case, these quasi-free states are taken as suitable starting points, in terms of which expectation values of field operator products may be calculated to arbitrary order in perturbation theory. The formal treatment of interacting fields in perturbation theory is reduced to a treatment of "free" quantum fields interacting with external sources.

Central to the approach is the so-called two-current operator, which characterises the effect of external sources in terms of purely algebraic (i.e. representation free) properties of the source-free theory.

The paper ends with a set of "Feynman rules" which seems particularly appropriate to curved space-times in that it takes care of those aspects of the problem which are specific to curved space-times (and independent of interaction). Heuristically, the scheme calculates "in-in" rather than "in-out" matrix elements. Renormalization problems are discussed but not treated.

Introduction

0.1. Motivation

There has recently been some interest in the problem of self, or mutually *interacting* quantum fields in curved space-times (see [1] and references therein). The value of this work is two-fold. Firstly, it is important to know just how the many recent results on *linear* quantum fields (see Sect. 0.2.) in curved space-times get modified in the more realistic case of interaction. Secondly, Einstein's (and other) theories

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of gravity themselves can be viewed as special cases of the problem (see Sect. 5).

The purpose of this paper is to present a formal perturbation theory scheme which is especially tailored for curved space-times: There are several problems which are specific to curved space-times and which have nothing particularly to do with interaction. The study of linear fields has already taught us several lessons on how to tackle these problems, and we feel it is worth having a scheme which incorporates these lessons right from the start.

Our work falls naturally into two parts. The first, and preliminary part summarizes precisely those lessons. Thus, the paper begins by giving a clear statement of what it means to quantize a *linear* field equation in a generic curved space-time. In other words, we begin our discussion of perturbation theory with a suitable treatment of the zero-order case! As example, we choose the covariant Klein–Gordon equation interacting with a fixed external scalar field V.

$$(g^{\mu\nu}\nabla_\mu\partial_\nu + m^2 + V)\hat{\varphi} = 0 \tag{0.1}$$

(all fields are taken to be C^∞ as in [2], hereafter referred to as I)

The second part presents a version of perturbation theory—based on an idea of P. Hajicek [3]—which really does seem to be particularly suited to curved space-times. And the paper culminates with a statement of the corresponding set of "Feynman rules" for calculating expectation values for $\lambda\varphi^4$ theory

$$(g^{\mu\nu}\nabla_\mu\partial_\nu + m^2 + V)\hat{\varphi} = -\frac{\lambda}{3!}\hat{\varphi}^3 \tag{0.2}$$

0.2. Linear Fields (Sects. 1 and 2)

The physics of linear quantum fields in curved space-times has been intensively studied over the last few years [4–8] and many fascinating results have been obtained (see e.g. [9]).

At a more conceptual level, contemplation of the "generic curved space-time" has forced us to consider rather carefully certain aspects of quantum field theory which are often ignored in flat space-time. In particular, the existence of a unique vacuum state and its corresponding preferred Hilbert space representation are only meaningful concepts for stationary situations [2]. In general, our space-times are not stationary—or they may even be stationary in two different senses [10]. To single out one state and call it a "vacuum" can then often lead to confusion.

For these reasons, the algebraic approach to quantum field theory [11–14] which is often something of a luxury for much of flat space-time physics, becomes rather more of a necessity in curved space-time contexts. Especially important to us is the very general algebraic concept of *state* which frees us from the need to represent our field operators on some fixed Hilbert space. Since, typically, interesting states in curved space-times do not lie as vector (or density matrix) states in the same representation (in the old language, one "vacuum" can consist of an infinite number of particles in the representation corresponding to another "vacuum") this freedom is essential for us to achieve the clarification of putting *all states on an equal footing.*

In Sects. 1 and 2, we explain the above ideas in more detail and develop a formalism which allows us to handle efficiently an "arbitrary" state in an "arbitary" space-time. More precisely, we discuss the set of *quasi-free* states on the set of *globally hyperbolic* space-times. Roughly speaking, global hyperbolicity is equivalent to the existence of a "choice of time". It is also the condition which guarantees that the Cauchy problem is well posed. These matters are discussed further in Sect. 1. The class of quasi-free states includes all the well-known "frequency-splitting" states and is discussed further in Sect. 2.

Alternative discussions of quantum field theory in the generic case—not always in agreement with the present purely algebraic point of view—may be found in refs. [15–18].

0.3. Interacting Fields (Sects. 3 and 4)

In Sects. 3 and 4 we give a *heuristic* treatment of non-linear fields in curved space-times. Our approach is strongly influenced by recent work of P. Hajicek [3]. The original development of this theory [3] proceeded in two stages: first calculating in flat space-time, and then using covariance arguments to guess the result for curved space-times. The present paper gives a short and self-contained reproduction and generalization of Hajicek's principal results within a "manifestly generally covariant" framework.

Very roughly speaking, the difference between Hajicek's approach and other recent work on interacting fields in curved space-times (Birrell and Taylor [1], see also [19]) is that it yields a perturbation theory for "in-in" rather than "in-out" matrix elements of products of fields. Such an "in-in" approach seems particularly appropriate in gravitational contexts where, typically, (collapsing stars, expanding universes etc.) we have only one asymptotic regime.

The intention, then, is to provide (in a set of "Feynman rules") a formalism with the following kind of flexibility:

(a) One is free to choose (from the set of all quasi-free states) an arbitrary state of the field in the asymptotic regime (i.e. at "$-\infty$" where gravitational fields are weak).

(b) For each such choice, the rules allow one to calculate the resulting expectation values of correlations between fields in interesting regions of the space-time (i.e. where gravitational fields are strong).

Following Schwinger [20–21] the treatment of an interacting field theory such as (0.1) in perturbation theory can be reduced to a simpler problem involving classical external sources. This is explained in Sect. 4. In the case of equation (0.2), we need to study

$$(g^{\mu\nu}\nabla_\mu\partial_\nu + m^2 + V)\hat{\varphi} = J \tag{0.3}$$

where J is our external source. This equation is studied in Sect. 3, where we derive the so-called *two-current operator* (cf. Hajicek's "two-current functional" [3]). This two-current operator codes, in a convenient algebraic form information about correlations between fields in the presence of sources, given an arbitrary (algebraic) state of the field to the past of these sources. The derivation, definition, and interpretation of this two-current operator makes essential use of practically all the concepts and formalism developed in Sects. 1 and 2. It is then but an easy step (in Sect. 4) to our promised Feynman rules.

1. The Classical Theory

Note: (1) As in the previous paper [2], we use Hawking and Ellis [22] (H.E.) especially chapters 1 and 6 as a reference throughout—except that we choose signature $(+ - - -)$. (2) All space-times are assumed to be space and time orientable.

1.1. Global Hyperbolicity, Choices of Time and Space-Time Splits

Given a space-time $(\mathscr{M}, g)$; we define a *choice of time* to be a function

$\ell : \mathscr{M} \to \mathbb{R}$

satisfying
(1) ℓ is C^∞ with $d\ell$ everywhere time-like.
(2) (implied by (1) up to a sign) ℓ increases along every future directed non-spacelike curve.
(3) Along any inextendible non-spacelike curve, ℓ takes all values in $(-\infty, \infty)$
The crucial result, due (with slightly different definitions) to Geroch [23] is

Theorem: Given a space-time $(\mathscr{M}, g)$; equivalent are:
(a) $(\mathscr{M}, g)$ is globally hyperbolic
(b) There exists a (global) Cauchy surface in $(\mathscr{M}, g)$
(c) There exists a choice of time on $(\mathscr{M}, g)$
A *proof* (using H.E. definitions) follows from H.E. Props 6.6.3 and 6.6.8 and appealing to the smoothing procedure of Seifert [24] mentioned in H.E. Prop 6.6.8.

Thanks to this theorem, we need not give the (rather technical) definition of global hyperbolicity. Rather, we may think equivalently, of *space-times admitting a choice of time.*

When a choice of time exists, there will in general be many. Choosing one of them, the $\ell =$ const. surfaces $\mathscr{C}(t)$ are then smooth spacelike (global) Cauchy surfaces. In fact, (2) of "choice of time" above alone would imply they were (H.E.) partial Cauchy surfaces. (2) and (3) would imply they were (H.E.) global Cauchy surfaces.

To complete a splitting of our space-time into space and time, we augment our "choice of time" with a choice of time-like vector field Y (such Ys will exist thanks to time-orientability). (2) of *choice of time* guarantees that the integral curves of such a Y cut every $\mathscr{C}(t)$ exactly once. So we induce for each t a diffeomorphism

$$\delta(t) : \mathscr{C}(t) \to \mathscr{C}(0)$$

defined by identifying points cut by the same integral curve of Y. We shall refer to a choice of time together with such a Y as a *space-time split.* Given such a space-time split, we may then realize $\mathscr{M}$ as a product manifold:

$$\begin{aligned} &\mathscr{M} \to \mathbb{R} \times \mathscr{C}(0) \\ &x \mapsto (\ell(x), \delta(\ell(x))x) \end{aligned} \tag{1.1}$$

Now, at each point on $\mathscr{M}$ we have the unit future-pointing normal N (in local coordinates $= N^i\partial_i$) and the induced Riemannian metric ${}^3g^{ij}$ for the $\mathscr{C}(t)$ which passes through that point. Choosing local coordinates x^i on $\mathscr{C}(0)$, we get from (1.1) above coordinates (t, x^i) on $\mathscr{M}$.

Defining the lapse and shift functions (α, β^i) [25] (which we will think of as time-dependent functions on $\mathscr{C}(0)$) via:

$$\frac{\partial}{\partial t} = \alpha N + \beta^i \partial_i \tag{1.2}$$

then, we can show that the metric takes the form

$$ {}^4g^{\text{upper}} = \begin{pmatrix} \dfrac{1}{\alpha^2} & -\dfrac{\beta^j}{\alpha^2} \\ -\dfrac{\beta^i}{\alpha^2} & {}^4g^{ij} \end{pmatrix} \quad {}^4g_{\text{lower}} = \begin{pmatrix} \alpha^2 - \beta^i\beta_i & -\beta_j \\ -\beta_i & -{}^3g_{ij} \end{pmatrix} \tag{1.3.}$$

where

$$ {}^4g^{ij} = -{}^3g^{ij} + \frac{\beta^i\beta^j}{\alpha^2} $$

Also, we have $\sqrt{-{}^4g} = \alpha\sqrt{{}^3g}$.

1.2. Leray's Theorem

For completeness, we give again the fundamental result on existence and uniqueness of solutions to our equation (0.1) [26–28]

Theorem. *Let* $(\mathscr{M},\ g)$ *be an oriented globally hyperbolic space-time,* $\mathscr{C}$ *some Cauchy surface–unit future-pointing normal* $N(\mathscr{C}$ *. Then the Cauchy data*

$$\Phi \in C_0^\infty(\mathscr{C}) + C_0^\infty(\mathscr{C})$$

given by

$$\Phi = \begin{pmatrix} f \\ p \end{pmatrix} \quad f = \varphi|_{\mathscr{C}} \quad p = N(\mathscr{C})\varphi|_{\mathscr{C}}$$

defines a unique solution in $C^\infty(\mathscr{M})$ *having compact support on every other Cauchy surface. Furthermore, the solution has support in* $J^+(\operatorname{supp}\Phi) \bigcup J^-(\operatorname{supp}\Phi)$*—the union of the causal future and the causal past of the support of the Cauchy data.*

We can summarize the results of Leray's Theorem by referring to the class S of C^∞ solutions with compact support on Cauchy surfaces. S is equipped with a symplectic form σ:

$$\sigma(\varphi_1, \varphi_2) = \int_{\mathscr{C}} \varphi_1 \overleftrightarrow{N(\mathscr{C})} \varphi_2 d\eta(\mathscr{C}) \tag{1.4}$$

where $\mathscr{C}$ is some Cauchy surface, unit normal $N(\mathscr{C})$, volume element $d\eta(\mathscr{C}) = \sqrt{{}^3g}d^3x$.

That this is independent of $\mathscr{C}$ follows easily from an application of Gauss' theorem to the conserved current

$$j_\mu = \varphi_1 \overleftrightarrow{\partial}_\mu \varphi_2 \tag{1.5}$$

1.3. Classical Green's Functions

In practice, the solution to the Cauchy problem will be given in terms of classical Green's functions [26–29]. Define first the advanced and retarded Green's functions Δ^A, Δ^R satisfying

$$\begin{aligned} L_1 \Delta^{A,R}(x,x') &= \delta(x,x') \\ \Delta^A(x,x') &= 0 \quad (x > x') \\ \Delta^R(x,x') &= 0 \quad (x < x') \end{aligned} \tag{1.6}$$

where the subscript i on a differential operator indicates action on the ith variable, where $>(<)$ signifies "to the future (past) of" and $L\varphi = 0$ is an abbreviation for (0.1).

Note

(1) $\Delta^A(x,x') = \Delta^R(x',x)$ (1.7)

(2) In the sequel we shall use "3-smeared and 4-smeared distributions" e.g.

$$\Delta(x,f) = \int_{\mathscr{C}} \Delta(x,y) f(y) d\eta(\mathscr{C}), \quad f \in C_0^\infty(\mathscr{C})$$

$$\Delta(x,F) = \int_{\mathscr{M}} \Delta(x,y) F(y) \sqrt{-{}^4g}\, d^4x, \quad F \in C_0^\infty(\mathscr{M})$$

Now, we define the *Jordan–Pauli Green's function*

$$\Delta(x,x') = \Delta^A(x,x') - \Delta^R(x,x') \tag{1.8}$$

which is easily seen to satisfy (0.2), to be antisymmetric, and to provide a solution to the Cauchy problem through:

$$\begin{aligned} \varphi(x) &= \int_{x' \in \mathscr{C}} \{ f(x') N(\mathscr{C})_2 \Delta(x,x') - p(x') \Delta(x,x') \} d\eta(\mathscr{C}) \\ &= -(N(\mathscr{C})_1 \Delta)(f,x) + \Delta(p,x) \end{aligned} \tag{1.9}$$

Note also for later use, the following important special case:

$$\Delta(F_1, F_2) = \int_{y \in \mathscr{C}} \Delta(y, F_1) \overleftrightarrow{N(\mathscr{C})}_1 \Delta(y, F_2) d\eta(\mathscr{C}) \tag{1.10}$$

1.4. Classical Dynamics

For each choice of Cauchy surface $\mathscr{C}$, define the *linear phase space* $(D(\mathscr{C}), \sigma_{\mathscr{C}})$

where

$$D(\mathscr{C}) = C_0^\infty(\mathscr{C}) + C_0^\infty(\mathscr{C}) \tag{1.11}$$

and $\sigma_{\mathscr{C}}$ is the symplectic form

$$\sigma_{\mathscr{C}}(\Phi_1, \Phi_2) = \int_{\mathscr{C}} (f_1 p_2 - p_1 f_2) d\eta(\mathscr{C}) \tag{1.12}$$

where $\Phi = \begin{pmatrix} f \\ p \end{pmatrix} \in D(\mathscr{C})$.

We can then specify dynamics in a manner independent of any choice of coordinates (cf. Kuchař [30]) by giving for each pair $\mathscr{C}_1, \mathscr{C}_2$ of Cauchy surfaces the symplectic map

$$\mathscr{T}(\mathscr{C}_2, \mathscr{C}_1): (D(\mathscr{C}_1), \sigma) \to (D(\mathscr{C}_2), \sigma) \tag{1.13}$$

defined by associating Cauchy data corresponding to the same solution.

To make contact with the traditional canonical formalism, we make a space-time split (Sect. 1.1.). The diffeomorphisms $\delta(t): \mathscr{C}(t) \to \mathscr{C}(0)$ then allow us to identify all the surfaces $\mathscr{C}(t)$ with some initial surface $\mathscr{C}(0)$ say. To identify the linear phase-spaces at each time, there are several possibilities: One convenient one is the map

$$\begin{aligned} &X(\delta(t)): (D(\mathscr{C}(0)), \sigma) \to (D(\mathscr{C}(t)), \sigma) \\ &(f, p) \mapsto \left(\delta(t)^* f, \left(\frac{d\delta(t)^* \eta(0)}{d\eta(t)} \right) \delta(t)^* p \right) \end{aligned} \tag{1.14}$$

In other words, we choose Cauchy data $(f_t(x), \pi_t(x))$ at time t where

$$\pi_t(x) = \left(\left(\frac{{}^3 g(t)}{{}^3 g(0)} \right)^{1/2} p_t \right)(x)$$

(*Note*: Our definition differs slightly from the usual one: our π is a scalar, the usual one is the scalar density $\sqrt{{}^3 g(0)}\pi$).

We can now view dynamics as the two-parameter family of symplectics ("Bogolubov transformations")

$$\mathscr{T}'(t_2, t_1) = X^{-1}(\delta(t_2)) \circ \mathscr{T}(\mathscr{C}(t_2), \mathscr{C}(t_1)) \circ X(\delta(t_1)) \tag{1.15}$$

on the fixed phase-space $D(\mathscr{C}(0), \sigma) \cdot \mathscr{T}'(t_2, t_1)$ may be represented in a straight-forward way as a "matrix-integral" operator using (1.9). Finally, the time evolution $\mathscr{T}'(t_2, t_1)$ is generated in the sense of classical mechanics by the Hamiltonian

$$\begin{aligned} H(f, \pi)(t) &= \frac{1}{2} \int_{\mathscr{C}(0)} \sqrt{{}^3 g(0)} d^3 x \alpha \left(\frac{\pi^2}{R} + R\, {}^3 g^{ij} \partial_i f \partial_j f + R(m^2 + V) f^2 \right) \\ &\quad + \int \sqrt{{}^3 g(0)} \pi \beta^i \partial_i f \\ &= \tfrac{1}{2} \sigma(\Phi, \boldsymbol{h}(t)\Phi) \end{aligned} \tag{1.16}$$

where

$$\boldsymbol{h}(t)=\begin{pmatrix} -\beta^i\partial_i & -\dfrac{\alpha}{R} \\ -\partial^i(\alpha R)\partial_i+\alpha R(m^2-\Delta(\mathscr{C})+V) & -(\nabla_i\beta^i+\beta^i\partial_i) \end{pmatrix}$$

where $\Delta(\mathscr{C})$ is the Laplace–Beltrami operator for $(\mathscr{C}, {}^3g)$, ∇_i denotes the covariant derivative for $(\mathscr{C}, {}^3g)$ and R denotes $\left(\dfrac{{}^3g(t)}{{}^3g(0)}\right)^{1/2}$

The Hamiltonian equations (first order form equations) can be written

$$\left(\frac{d}{dt}\right)_1 \mathscr{T}(t_2, t_1)\Phi\Big|_{t_2=t_1} = -\boldsymbol{h}(t)\Phi \tag{1.17}$$

2. Quantization

2.1. Canonical Quantization

To quantize our equation in an arbitrary curved background, it turns out that the traditional Hilbert space formulation is not adequate and a more general algebraic approach is more appropriate. To motivate the use of an algebraic approach, we begin in this section with a heuristic discussion of canonical quantization. We shall have to take a similar canonical approach when we come to deal with external sources in Sect. 3.

We begin by introducing "3-smeared" quantum fields on $\mathscr{C}(0)$:

$$R(\Phi)=\int_{\mathscr{C}(0)}(\hat{\varphi}(x)\pi(x)-\hat{\pi}(x)f(x))d\eta(\mathscr{C})=\sigma(\hat{\varphi},\hat{\pi},f,\pi);$$
$$\Phi=\begin{pmatrix} f \\ \pi \end{pmatrix}\in D(\mathscr{C}(0)) \tag{2.1}$$

and impose the usual commutation relations

$$[R(\Phi_1), R(\Phi_2)] = i\sigma(\Phi_1, \Phi_2) \tag{2.2}$$

Equivalently, writing $W(\Phi)=e^{iR(\Phi)}$, we have the Weyl relations

$$W(\Phi_1)W(\Phi_2)=\exp\left(-\frac{i\sigma(\Phi_1,\Phi_2)}{2}\right)W(\Phi_1+\Phi_2) \tag{2.3}$$

Proceeding heuristically, the quantized Hamiltonian must satisfy (using (1.16), (1.17), (2.1), (2.2))

$$[\hat{H}(t), R(\Phi)] = -iR(\boldsymbol{h}(t)\Phi) \tag{2.4}$$

We now impose the Heisenberg-picture evolution

$$R(\Phi)\mapsto U(t_2,t_1)R(\Phi)U(t_1,t_2) \tag{2.5}$$

where the "unitary propagator" $U(t_2, t_1) = Te^{-i\int_{t_1}^{t_2}\hat{H}(t)dt}$

Using equations (1.17), (2.4); this reduces to

$$R(\Phi) \mapsto R(\mathscr{T}(t_2, t_1)\Phi) \tag{2.6}$$

and we similarly get

$$W(\Phi) \mapsto W(\mathscr{T}(t_2, t_1)\Phi) \tag{2.7}$$

We shall *not* give a rigorous mathematical meaning to equation (2.5). Recall that, in the stationary case, everything can be given mathematical meaning by a roundabout route (see I for details). One takes as starting points equations (2.3) and (2.6). First, one defines the algebra $\mathscr{W}(D(0), \sigma)$ generated by the W's in (2.3) (see e.g. [31–32]). Then (2.6) defines a one-parameter group of automorphisms of this algebra generated by $\alpha((t_2 - t_1), 0)$ where

$$\alpha(t_2, t_1) : W(\Phi) \mapsto W(\mathscr{T}(t_2, t_1)\Phi) \tag{2.8}$$

One then seeks a vacuum state and its corresponding representation. In this representation, we can define the implementing unitary group $U(t)$ and hence finally $\hat{H}(t)$ such that $U(t) = e^{-i\hat{H}(t)}$.

In the non-stationary case, everything goes through up to equation (2.8). (Of course α is no longer a group.) But, there is no analogous procedure beyond that. In general, it is unfruitful and often impossible to find a representation for which $\alpha(t_2, t_1)$ is implemented for all t. (Typically, one has creation of an infinite number of particles etc.). Thus one gives up hope of assigning any mathematical meaning to $Te^{-i\int_{t_1}^{t_2}\hat{H}(t)dt}$. One similarly gives up hope of defining the $\hat{H}(t)$'s as positive operators all on the same Hilbert space.

Fortunately, the algebraic formalism [11–14] is just what we need for making sense of the situation. We are still able to define the Weyl algebra and quantum automorphisms are still completely defined by (2.8) in terms of the classical time-evolution $\mathscr{T}(t_2, t_1)$. Thanks to the general concept of *state* (see Sect. 2.3) available in the algebraic formalism, there is no need to choose a Hilbert space representation. *The theory is completely fixed by equations* (2.3) *and* (2.7). We shall continue our discussion of the algebraic formalism in Sect. 2.3.

2.2. Covariant Formulations

We can summarize the content of equations (2.3) and (2.7) by considering the Weyl algebra over the symplectic space of classical solutions (S, σ) (see (1.4)) generated by the single equation

$$W(\varphi_1)W(\varphi_2) = \exp\left(-\frac{i\sigma(\varphi_1, \varphi_2)}{2}\right)W(\varphi_1 + \varphi_2); \quad \varphi \in S \tag{2.9}$$

For, if we were to take this as starting point and then define $W(\Phi_t)$, $\Phi_t \in (D(\mathscr{C}(0)), \sigma)$ as $W(X(\delta(t)^{-1})\Phi)$ where Φ are Cauchy data of φ on $\mathscr{C}(t)$ (see (1.14)), we would recover (2.3) and (2.7).

Finally, we define the covariant "4-smeared quantum field"

$$W(F) = W(\Delta(F, \cdot)); \quad F \in C_0^\infty(\mathscr{M}) \tag{2.10}$$

which, in virtue of (1.10) and (2.9), satisfies

$$W(F_1)W(F_2) = \exp\left(-\frac{i\Delta(F_1, F_2)}{2}\right)W(F_1 + F_2) \tag{2.11}$$

Formally,

$$W(F) = \exp i \int_{\mathscr{M}} \hat{\varphi}(x)F(x)\sqrt{-{}^4g}\,d^4x \tag{2.12}$$

where $\hat{\varphi}(x)$ is the usual quantum field.

2.3. *Algebraic States*

The algebraic concept of state is more general than that of vector (or density matrix) state. A state ω is defined as a positive linear functional on the Weyl algebra (equivalently the algebra generated by $W(\varphi)$'s, $W(F)$'s, or $W(\Phi)$'s). (Positivity corresponds to $\omega(A^*A) \geq 0$ for A in the algebra.) Roughly speaking, specifying a state corresponds to directly specifying the expectation values of all possible products of fields (cf. Wightman functions [33]). In the canonical approach, ($W(\Phi)$'s, $\Phi \in D(\mathscr{C}(0))$) we must, of course, also specify the time-evolution on states by the dual action of (2.8)

$$\omega_{t_2}(W(\Phi)) = \omega_{t_1}(W(\mathscr{T}(t_2, t_1)\Phi) \tag{2.13}$$

which we can think of as an algebraic version of the Schrödinger picture.

In usual flat space-time physics, it is usual to augment the strict C^* algebra framework: On the one hand, there are certain "observables" e.g. Hamiltonian, generators of groups etc. that are not in the C^* algebra. "Dually", it is often useful to choose a representation of our algebra, and focus attention on the set of vector states (or, more generally, density matrix states) arising in this representation. In fact one has the "vacuum representation" mentioned in Sect. 2.1. which is designed to represent the Hamiltonian as a positive operator. The advantage of this is that states can be labelled by their energy: There is something rather special about saying the vacuum state has zero-energy which one tends to miss if one just writes down the two-point correlation function

$$\omega(\varphi(x)\varphi(y)) = \int \frac{d^3p}{p_0} e^{-ip_u(x^u - y^u)}, \quad p_0 = (p^2 + m^2)^{1/2}$$

etc. Certainly, to some-one brought up on flat space-time physics, this picture with Fock spaces, "particles", Hamiltonians etc. looks more familiar. Nevertheless, we know that, strictly speaking, it is inessential: The specification of a state by giving all its n-point correlation functions contains all physical information.

Now for quantum theory on globally *stationary* space-times, we can still mimic the vacuum construction referred to above (see I).

However, in the generic case where we have no symmetries, the usefulness of global observables such as the Hamiltonian diminishes and—as we pointed out in Sect. 2.1—we cannot hope for the luxury of a preferred representation. For this reason, the "generic space-time" forces us to adopt the algebraic approach.

2.4. Quasi-Free States and Quantum Green's Functions

There is a particular class of states which will play a special rôle in our treatment of perturbation theory, namely the *quasi-free* states[1] [34–36]. Let s be a positive bilinear form on the space of classical solutions S, satisfying

$$|\sigma(\varphi, \psi)|^2 \leqq 4s(\varphi, \varphi)s(\psi, \psi) \tag{2.14}$$

and let ℓ be any linear functional on S. Then we define the quasi-free state $\omega_{s,\ell}$ via

$$\omega_{s,\ell}(W(\Phi)) = \exp - \left\{ \frac{s(\varphi, \varphi)}{2} + i\ell(\varphi) \right\} \tag{2.15}$$

(condition (2.14) is needed to ensure the positivity of $\omega_{s,\ell}$).

We shall often consider the case $\ell = 0$, whereupon we write ω_s instead of $\omega_{s,0}$.

This class of states is important for several reasons. Firstly, it is mathematically simple. Secondly, it includes all the so-called "frequency-splitting vacuums" usually considered in work on quantum theory in curved space-times (see e.g. [4, 16]). It also includes the vacuum states $\omega(W(\Phi)) = e^{-\|K\Phi\|^2/2}$ which we constructed in I [2] for stationary space-times. Note also that (considered as states on $W(D(\mathscr{C}(0), \sigma))$ this class is invariant under the time evolution "Bogolubov transformations" (2.13).

We develop some formalism for handling these states efficiently:
Given an s satisfying (2.14), we define the "positive and negative frequency Green's functions" corresponding to a choice of s

$$\Delta_s^+(F, G) = s(\Delta(\cdot, F), \Delta(\cdot, G)) + \frac{i}{2}\Delta(F, G)$$

$$\Delta_s^-(F, G) = s(\Delta(\cdot, F), \Delta(\cdot, G)) - \frac{i}{2}\Delta(F, G) \tag{2.16}$$

We also define the "Feynman" and "AntiFeynman" Green's functions[2]

$$\Delta_s^F(F, G) = \Delta_s^+(F, G) - i\Delta^A(F, G)$$
$$\Delta_s^{\mathcal{A}}(F, G) = \Delta_s^+(F, G) + i\Delta^R(F, G) \tag{2.17}$$

Finally, we define the "mean field value"

$$\langle \varphi \rangle_\ell(F) = \ell(\Delta(F, \cdot)) \tag{2.18}$$

We have from (2.10), (2.12), (2.15), (2.16), (2.17), (2.18)

$$\omega_{s,\ell}(\varphi(x)) = \langle \varphi \rangle_\ell(x) \tag{2.19}$$

---

1 One might wish to adjoin those states which arise as density-matrix states in the GNS representation of each $\omega_{s,\ell}$. To keep the exposition clear, we do not refer to these in the main text.

2 Note our definition of Feynman propagator which corresponds heuristically to $\langle \text{in} | T(\varphi(x)\varphi(y)) | \text{in} \rangle$ differs from another possible definition [1, 19] as $\langle \text{out} | T(\varphi(x)\varphi(y)) | \text{in} \rangle$

$$\omega_s(\varphi(x)\varphi(y)) = \Delta_s^+(x,y)$$
$$\omega_s(T(\varphi(x)\varphi(y))) = \Delta_s^F(x,y)$$
$$\omega_s(\perp(\varphi(x)\varphi(y))) = \Delta_s^{\perp}(x,y) \tag{2.20}$$

where T is the time-ordered product (later times to the left), $\perp$ the anti-time-ordered product (later times to the right).

When ℓ is non-zero, (2.20) represents the "truncated expectation values"

$$\omega_s(\varphi(x)\varphi(y)) = \omega_{s,\ell}(\varphi(x)\varphi(y)) - \langle\varphi\rangle_\ell(x)\langle\varphi\rangle_\ell(y) \tag{2.21}$$

3. External Sources

3.1. Canonical Quantization

In this section, we follow closely the heuristic approach of Sect. 2.1. to quantize the equation

$$(g^{\mu\nu}\nabla_\mu\partial_\nu + m^2 + V)\hat{\varphi} = J \tag{0.3}$$

where $J \in C_0^\infty(\mathscr{M})$

After a space-time split, we write the classical canonical Hamiltonian

$$H_J(t) = \tfrac{1}{2}\sigma(\Phi, \boldsymbol{h}(t)\Phi) - \sigma(\Phi, \boldsymbol{j}) \tag{3.1}$$

corresponding to the first-order form

$$\frac{d}{dt}\Phi = -\boldsymbol{h}(t)\Phi + \boldsymbol{j}(t) \tag{3.2}$$

where $\boldsymbol{h}(t)$ is given by (1.16) and

$$\boldsymbol{j}(t) = \begin{pmatrix} 0 \\ \alpha\left(\dfrac{{}^3g(t)}{{}^3g(0)}\right)^{1/2} J(t) \end{pmatrix} \tag{3.3}$$

We thus take the quantum Hamiltonian to be

$$\hat{H}_J(t) = \hat{H}_0(t) - R(\boldsymbol{j}) \tag{3.4}$$

Then, we have the following formula for the unitary propagator in the presence of the source

$$Te^{-i\int_{t_1}^{t_2}\hat{H}_J(t)dt} = \exp\frac{i}{2}\Delta^R(J',J')U(t_2,t_1)W(J') \tag{3.5}$$

where $U(t_2,t_1)$ is the unitary propagator in the absence of a source (2.5). W is the "4-smeared" Weyl operator ((2.10), (2.12)) for the source-free theory and J' is equal to the external source between the $t=t_1$, and $t=t_2$ surfaces and zero elsewhere. (see Fig. 1).

The form of our formula shows the independence from the particular space-time split chosen. To *prove* it, however, it is convenient to choose such a split, whereupon we make the following identifications in (3.5):

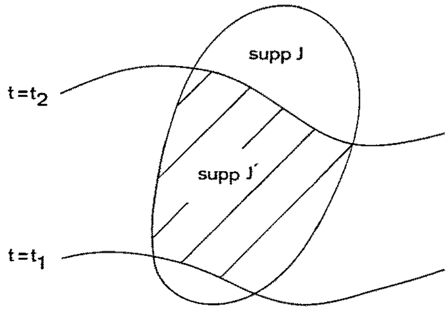

Fig. 1.

$$\Delta^R(J',J') = -\int_{t_1}^{t_2} \sigma(\mathscr{T}'(t_1,t')j(t')dt', \int_{t_1}^{t'} \mathscr{T}'(t_1,t'')j(t'')dt'') \tag{3.6}$$

$$W(J') = W\left(\int_{t_1}^{t_2} \mathscr{T}'(t_1,t')j(t')dt'\right) \tag{3.7}$$

which easily follow on using (1.9), (1.10), (2.10), (3.3). (3.5) now easily follows by differentiating both sides with respect to t and using the relations

$$\frac{d}{dt}W(\Psi(t)) = \left[\frac{i\sigma(\dot{\Psi}(t),\Psi(t))}{2} + iR(\dot{\Psi}(t))\right]W(\Psi(t)) \tag{3.8}$$

(where $\Psi(t)$ is an arbitrary function of t) and

$$U(t_2,t_1)R(\Phi) = R(\mathscr{T}'(t_2,t_1)\Phi)U(t_2,t_1) \tag{3.9}$$

which follow from (2.3) and (2.5) respectively.

3.2. The Two-Current Operator

Now consider the operator

$$\mathcal{O}(J_2,J_1) = (Te^{-i\int_{t_1}^{t_2}\hat{H}_{J_2}(t)dt})^{-1}(Te^{-i\int_{t_1}^{t_2}\hat{H}_{J_1}(t)dt}) \tag{3.10}$$

which corresponds to propagating forwards in the presence of source J_1, and then backwards in the presence of J_2. (We now assume our $t=t_1$ and $t=t_2$ surfaces are chosen to the past and future respectively of the supports of J_1 and J_2).

Using (3.5), (2.11) and (1.8), we easily obtain the formula for this *two-current operator*

$$\mathcal{O}(J_2,J_1) = \exp\frac{i}{2}\Delta^R(J_1-J_2,J_1+J_2)W(J_1-J_2) \tag{3.11}$$

In the next section, we shall use this operator as a basic tool for treating self-interacting fields in perturbation theory. We conclude this section with some comments about formula (3.11).

1. Formula (3.5) and its derivation are both quite formal, and, as we discussed in Sect. 2.1., cannot, in general, be given a mathematical meaning.
2. Formula (3.11), on the other hand, is quite remarkable: Although its derivation

was formal, it can immediately be given a mathematical meaning within the algebraic framework: It is an element of the Weyl algebra!

3. The *physical* meaning of our two current operator can be understood as follows: Suppose for a moment that (3.5) *does* make sense—i.e. that we can choose some representation in which $U(t_2, t_1)$ are implemented. Then, for any two vector states, $|a\rangle, |b\rangle$, we define the amplitude (cf. Hajicek [3])

$$\langle b|a\rangle^J = \langle b|\, T e^{-i\int_{t_1}^{t_2} H_J(t)dt} |a\rangle$$

We then have a formal identity

$$\langle a|\mathcal{O}(J_2, J_1)|a\rangle = \sum_n \langle a|n\rangle^{J_2} \langle n|a\rangle^{J_1}$$

where n is a complete set of states. This latter is Hajicek's [3] "two current functional": the amplitude for $|a\rangle$ to persist after propagating forwards under J_1 and backwards under J_2.

We can now drop our assumption about representations and implementability and returning to the algebraic framework, *define* the ω-persistence amplitude to be $\omega(\mathcal{O}(J_2, J_1))$ where ω is a state in the algebraic sense.

In particular, for the quasi-free states $\omega_{s,\ell}$ introduced in Sect. 2.4, we have (by (2.10), (2.12), (2.15), (2.16), (2.17), (2.18), (2.19), (2.20)

$$\begin{aligned}\omega_{s,\ell}(\mathcal{O}(J_2, J_1)) = \exp -\tfrac{1}{2}\{\Delta^F(J_1, J_1) - 2\Delta^+(J_2, J_1) + \Delta^d(J_2, J_2)\} \\ \cdot \exp i\langle\varphi\rangle_\ell(J_1 - J_2) \qquad (3.12)\end{aligned}$$

4. Finally, note that formula (3.11) could have been derived—up to a phase—by purely algebraic methods: Consider the symplectic transformation on classical Cauchy data at time t, caused by propagating forwards under J_1 and backwards under J_2. It is easy to see that this induces an automorphism of the Weyl algebra over such Cauchy data which is implemented by $W(J_1 - J_2)$. Note that this automorphism is inner: there is no need to choose a representation. Unfortunately, this approach is incapable of fixing the phase (which in this context is important as it can be a functional of J's) and it seems we are forced to derive (3.11) by the heuristic methods given above.

4. Perturbation Theory for Interacting Fields

4.1. Use of Sources

In this section, we sketch how our two-current operator can be used to develop a "generally covariant perturbation theory" for interacting fields on fixed curved space-times. For sake of definiteness, we shall illustrate our method with $\lambda\varphi^4$ theory, i.e. the equation

$$(g^{\mu\nu}\nabla_\mu\partial_\nu + m^2 + V)\hat{\varphi} = -\frac{\lambda\hat{\varphi}^3}{3!} + J \qquad (4.1)$$

Here, we have included also an external source $J \in C_0^\infty(\mathcal{M})$. Now, for *any* such equation (keep in mind (0.3) and (4.1)), the two current operator $\mathcal{O}(J_2, J_1)$ ((3.10) generalizes in an obvious way for (4.1)) is a convenient device for recording all the information of the source-free theory. Indeed, knowledge of the expectation value of the two-current operator in a given state suffices to completely specify that state for the source-free theory. In fact, the natural object to calculate is

$$\langle y_1 \dots y_m ; x_1 \dots x_n \rangle_\omega = \omega(T_x L_y(\varphi(y_1) \dots \varphi(y_m)\varphi(x_1) \dots \varphi(x_n))) \tag{4.2}$$

where all $\varphi(y)$'s are to the left of all $\varphi(x)$'s; T_x signifies re-ordering the $\varphi(x)$'s in order of increasing times: later times to the left; and L_y does the same job for $\varphi(y)$'s but with later times to the right.

We easily have (from (3.10))

$$\begin{aligned}\langle y_1 \dots y_m ; x_1 \dots x_n \rangle = (-i)^n (i)^m \frac{\delta}{\delta J_1(x_1)} \cdots \frac{\delta}{\delta J_1(x_n)} \frac{\delta}{\delta J_2(y_1)} \cdots \frac{\delta}{\delta J_2(y_m)} \\ \cdot \omega(\mathcal{O}(J_2, J_1))\big|_{J_1 = J_2 = 0}\end{aligned} \tag{4.3}$$

(where the limits t_1 and t_2 in $\mathcal{O}(J_2, J_1)$ are to the past and future of all $(x_1 \dots x_n, y_1 \dots y_m)$. As illustration of this equation, the reader may easily check how equations (2.19), (2.20) follow from (4.3).

4.2. Perturbation Theory

Now, our goal is to have a perturbation theory for determining a state of the self-interacting theory. Given a quasi-free state $\omega_{s,\ell}$[3] on the free theory, we seek a state $\omega'_{s,\ell}$ on the interacting theory satisfying in some sense (for an arbitrary space-time split)

$$\lim_{t \to -\infty} \{\omega'_{s,\ell}(\varphi_t(\mathbf{x}_1) \dots \varphi_t(\mathbf{x}_n)) - \omega_{s,\ell}(\varphi_t(\mathbf{x}_1) \dots \varphi_t(\mathbf{x}_n))\} = 0 \tag{4.4}$$

where $\mathbf{x}_1 \dots \mathbf{x}_n$ are points in $\mathscr{C}(0)$.

Of course, we cannot take equation (4.4) seriously (Haag's theorem and all that! [33]). It is to be understood in the spirit of perturbation theory where one still needs to renormalize later. The developments of this section are all entirely formal, culminating in a suitable set of "Feynman rules". We stop short of discussing any renormalization procedure.

To understand the meaning of this formal equation, note first that if our system was stationary for the given coordinates, and $\omega_{s,\ell}$ time-translational invariant, we could dispense with the limit, and impose equality at each time. In general, $\omega_{s,\ell}$ is to be interpreted as an "in" state, and is set equal to $\omega_{s,\ell}$ at $-\infty$ (in the sense of the Schrödinger picture).

Following the strategy of Sect. 4.1., we clearly have that the expectation value of the two current operator for the interacting system is given in terms of the

3 See footnote 1: It should be a straightforward matter to extend our perturbation theory to vector states in the GNS representation of any given ω_s —just as in usual perturbation theory it suffices to calculate vacuum expectation values to extract information about many particle states.

$\omega_{s,\ell}$ expectation value for the free two-current operator by

$$\omega(\mathcal{O}_{\varphi^4}(J_2, J_1)) = \exp i \int \sqrt{-{}^4g}\, d^4x \left\{ \left(\frac{\delta}{\delta J_2} \right)^4 - \left(\frac{\delta}{\delta J_1} \right)^4 \right\} \omega(\mathcal{O}(J_2, J_1)) \tag{4.5}$$

Formulae (4.2), (4.3), (4.4), (4.5), (3.12) lead to the following "Feynman rules" for $\langle y_1 \dots y_m, x_1 \dots x_n \rangle_{\omega'_{s,\ell}}$ (to understand how "Feynman rules" follow from (4.5), cf. [37, 38]).

(For simplicity, we assume $\ell = 0$; it is easy to generalize our rules for non zero ℓ)

1. draw all possible diagrams with endpoints $x_1 \dots x_n ; y_1 \dots y_m$ and with 4 lines meeting at each internal vertex.
2. Label all internal vertices with all possible mixtures of $x, x', x'' \dots\dots y, y', y'' \dots\dots$
3. "Propagators" are assigned to internal lines as follows:

x ——— x'	$\Delta_s^F(x, x')$
x ——— y	$\Delta_s^+(y, x)$
y ——— y'	$\Delta_s^{\text{Ⅎ}}(y, y')$

4. There is a factor of $-i\lambda$ for each x vertex and a factor of $+i\lambda$ for each y vertex.
5. Integrate over internal vertices.
6. Finally, divide by symmetry factors (where in recognizing a symmetry, lines corresponding to different propagators are regarded as different).

5. Discussion

Perhaps the most important goal for the study of interacting quantum fields on curved space-times is to improve our understanding of results on the renormalizability (or otherwise) of gravity itself.

While the question of renormalizability of Einstein's theory (with matter) seems to have been settled without doubt in the negative [39, 40], the specifically "curved background aspect" of the problem has never been seriously considered. An inspection of 't Hooft and Veltman's article' [39] for example shows that, even though they adopt the so-called "background field" method, they are forced—at crucial points in the argument—to fall back on flat-space-time methods (e.g. dimensional regularization). These flat space-time methods combine with general covariance arguments to yield the desired results. But still, it would be more satisfactory to have a complete renormalization procedure (including regularization procedure) which worked directly on an arbitrary curved space-time. Birrell and Taylor [1] and others have made some first steps in this direction, but, as they point out, a crucial step in their treatment also involves appeal to flat space-time renormalization theory combined with general covariance arguments. It thus seems fair to say that the extension of renormalization theory to apply to curved backgrounds (and, indeed other non-translationally invariant systems) remains an unsolved problem. From this point of view, our work constitutes only a small initial step. As we mentioned in Sect. 4, our Feynman rules are only formal and the *main problem*, that of short-distance divergencies, will be the same as in any other scheme: We do not expect the renormalizability question for our "in-in" scheme to differ substantially from that of the "in-out" scheme [1] (see also [3]).

The purpose of our work was to provide a scheme in which this *main* problem remains the *only* problem:

In other words, a scheme in which those problems which are common to all (including free) curved space-time QFT's are automatically taken care of. Thus we are left free to concentrate on the problems caused by interaction.

In particular, our "in-in" scheme has the following advantages[4].

(1) By focussing attention on expectation values (rather than matrix elements) it is closer to the interesting physical questions.

(2) It needs only one asymptotic regime—as in many important gravitational contexts.

(3) It fits in naturally with the algebraic approach—with the advantage of viewing all states on an equal footing and eliminating confusion about "vacuums".

(4) It automatically takes care of the "infinite particle creation" divergencies which are the characteristic feature of curved space-time backgrounds.

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4 The crucial advantage is really the algebraic approach: it is probably not too difficult to make algebraic sense also of the "in-out" scheme. Even when there is "infinite particle creation", the in and out states could be defined algebraically. One expects them to be "locally quasi-equivalent" [12], and this should suffice to make sense of "$\langle \mathrm{out} | \varphi(x_1) \dots \varphi(x_n) | \mathrm{in} \rangle$".

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