PERIODIC ORBITS AROUND AN OBLATE SPHEROID

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Abstract. The general properties of certain differential systems are used to prove the existence of periodic orbits for a particle around an oblate spheroid.

In a fixed frame, there are periodic orbits only for $i = 0$ and i near $\pi/2$. Furthermore, the generating orbits are circles.

In a rotating frame, there are three families of orbits: first a family of periodic orbits in the vicinity of the critical inclination; secondly a family of periodic orbits in the equatorial plane with $0 < e < 1$; thirdly a family of periodic orbits for any value of the inclination if $e = 0$.

1. Introduction

The present paper aims to apply the general properties of nearly-integrable differential systems in order to demonstrate the existence of classes of periodic solutions for the motion of a particle which gravitates around an oblate spheroid.

The potential is limited to the J_2 term. The equations of the satellite's motion form a differential system which is close to an integrable system. The properties of periodic solutions for such a system are described by Roseau (1966), Haag (1948) and others. By applying these properties we find all the periodic solutions in the vicinity of the periodic solutions of the integrable systems. Among them are some solutions found by other authors with different methods, e.g. MacMillan (1920) and more recently Delmas (1978), Kammeyer (1976).

The main interest of the described method is probably the fact that the formalism is the same for all classes of orbits and a great part of calculations has to be done only once.

2. Equations and Method

The method we use was described in previous papers by Stellmacher (1976; 1977; 1979). We summarize the main points.

In a cartesian frame, the equations of the motion are given by the autonomous system

$$\dot{x} = f(x) + J_2 g(x) . \tag{1}$$

Equation (1) defines a sixth order system; x, $f(x)$ and $g(x)$ are vectors with six components; $f(x)$ and $g(x)$ have successive continuous derivatives with respect to x. J_2 is a small quantity which is approximatively 10^{-3} in the Earth's case. For $J_2 = 0$, system (1) is reduced to

$$\dot{x} = f(x) . \tag{2}$$

System (2) has a family of elliptical solutions depending on six parameters ρ_i, $i=1\ldots6$; the elliptical elements for instance. Note that the period T of these solutions, is a function of one parameter ρ_j, *i.e.*, the semimajor axis of the ellipse.

The question is: Does (1) possess a (or several) periodic solution which tends to *a* (or several) solution of (2) for $J_2 \to 0$.

System (1) is an autonomous system, its periodic solution, if any, has a period T' close to T; we put $T'=T/(1+J_2\delta)$, where δ is a continuous function of J_2 expanded in a power series $\delta=\delta_0+J_2\delta_1+\cdots$. System (1) can now be written

$$\begin{aligned}\dot{x} &= (1+J_2\delta)f(x)+J_2\bar{g}(x,J_2)\\ \bar{g}(x,J_2) &= g(x)-\delta f(x)\,.\end{aligned}\tag{1a}$$

We define as main system associated with (1a), the system

$$\dot{x}=(1+J_2\delta)f(x)\,.\tag{2a}$$

System (2a) has a family of periodic solutions, with period T', depending on six parameters. Let $z(t,\rho_i)$ be this family of solutions; ρ_i can be arbitrarily chosen.

The variational equation is

$$\dot{y}=(1+J_2\delta)Q(t)y\,.\tag{3}$$

$Q(t)$ is the matrix 6×6 of elements $(\partial f_j/\partial x_i)_{x=z}$. System (3) is a linear differential system with periodic coefficients. It has a set of independent solutions $\varphi^i=\partial z/\partial\rho_i$, $i=1\ldots6$. Let Φ be the matrix of elements φ^i.

For $\rho_i\neq\rho_j$, φ^i are periodic solutions with periodT'. If $\rho_i=\rho_j$, $z(nt'),\rho_i)$ is solution with period T with regard to t' of the system $\mathrm{d}x/\mathrm{d}t'=f(x)$; $t'=(1+J_2\delta)t$ and $n=2\pi/T$ is a function of ρ_j; $z(n(t'-\gamma),\rho_i)=z(l,\rho_i)$ is also solution of this system.

$$\begin{aligned}\varphi^j &= \frac{\delta z}{\delta\rho_j}+\frac{\partial z}{\partial l}\times\frac{\partial l}{\partial n}\times\frac{\partial n}{\partial\rho_j}=\frac{\partial z}{\partial\rho_j}-\frac{l}{n^2}\frac{\partial n}{\partial\rho_j}\frac{\partial z}{\partial\gamma}\\ \varphi^j &= \bar{\varphi}+Kl\varphi^{j-1}\,,\end{aligned}$$

with $K=(-1/n^2)(\partial n/\partial\rho_j)$ and $\rho^{j-1}=\gamma$. δ means the explicit derivation and φ^j is a solution of $\mathrm{d}y/\mathrm{d}t'=Q(t')y$. With variable t, φ^j is solution of (3). $\bar{\varphi}$ is a periodic function with period T', $l=n(t'-\gamma)=n(1+J_2\delta)t-n\gamma=n't-n\gamma$.

Remark. The parameters ρ_i can be ordered in any way. If the semi major axis carries the index j, we only admit that γ carries the index $j-1$.

The adjoint system to (3)

$$\dot{y}=-(1+J_2\delta)Q(t)^*y\tag{4}$$

has a set of six solutions ψ^i; among them five solutions are periodic with period T', one solution is $\psi^{j-1}=\bar{\psi}-Kl\psi^j$. $\bar{\psi}$ is a periodic function with period T'. Let Ψ be the matrix of elements ψ^i; ψ^j is the column vector associated with the column vector φ^j in the transformation $\Psi=(\Phi^{-1})^*$.

A necessaty condition for which (1a) has a periodic solution which tends to a particular solution of (2a) for $J_2 \to 0$ is: (Roseau, 1966)

$$\sigma_i(\rho_k) = \int_0^{T'} \psi^i \bar{g}(z, 0)\, \mathrm{d}t = 0, \quad i \neq j-1\,. \tag{5}$$

Since $-\psi^i f(z) = c_i = cste$, Equation (5) can be written as

$$\sigma_i(\rho_k) = c_i \delta_0 T' + \int_0^{T'} \psi^i g(t)\, \mathrm{d}t = 0\,, \quad i \neq j-1$$

if $i = j-1$:

$$\sigma_{j-1}(\rho_k) = c_{j-1}\delta_0 T' + \int_0^{T'} [\bar{\psi} g(z) + n'K \int \psi^j g(z)\, \mathrm{d}t]\, \mathrm{d}t = 0\,. \tag{5a}$$

Proof. $z\{n[(1+J_2\delta)t - \gamma]\}$ is solution with period T' of system (2a).$\bar{\varphi} = \delta z/\delta\rho_j$ is solution of the differential system

$$\dot{\bar{\varphi}} = (1 + J_2\delta)\left[Q(t)\bar{\varphi} + \left(\frac{\delta f}{\delta \rho_j}\right)_{x=z}\right]$$

it is:

$$\bar{\varphi}(t) = \sum_{i=1}^{6}\left[A_i + \int \psi^i (\delta f/\delta\rho_j)\, \mathrm{d}t\right]\varphi^i\,.$$

Evaluating the integrals:

$$\varphi^{j-1}\int \psi^{j-1}\frac{\delta f}{\delta\rho_j}\mathrm{d}t + \varphi^j \int \psi^j \frac{\delta f}{\delta \rho_j}\mathrm{d}t,$$

by part, and writing the condition of periodicity: $\bar{\varphi}(T') - \bar{\varphi}(0) = 0$, we have

$$0 = \sum_{\substack{i\neq j\\ i\neq j-1}}\left[\varphi^i(0)\int_0^{T'} \psi^i \frac{\delta f}{\delta\rho_j}\mathrm{d}t\right] + \bar{\varphi}(0)\int_0^{T'} \psi^j \frac{\delta f}{\delta\rho_j}\mathrm{d}t +$$

$$+ \varphi^{j-1}(0)\left[2\pi K A_j + \int_0^{T'}\left(\bar{\psi}\frac{\delta f}{\delta\rho_j} + n'K\int \psi^j \frac{\delta f}{\delta \rho_j}\mathrm{d}t\right)\mathrm{d}t\right] = 0\,.$$

Functions φ are linearly independent, their coefficients, in the last equation, have to be equal to zero, then for $i \neq j-1$:

$$\int_0^{T'} \psi^i \frac{\delta f}{\delta\rho_j}\mathrm{d}t = 0 \quad \text{and} \quad \int_0^{T'} \bar{\psi}\frac{\delta f}{\delta\rho_j}\mathrm{d}t = \frac{T}{n}\frac{\partial n}{\partial \rho_j}A_j\,.$$

We have $f(z,\rho_j)=p(\rho_j)\,k(z)$, so the last two equations give respectively:

$$\int_0^{T'} \psi^i f(z)\,\mathrm{d}t=\int_0^{T'} c_i\,\mathrm{d}t=0\,,\quad i\neq j-1$$

and

$$\int_0^{T'} \bar{\psi} f(z)\,\mathrm{d}t=\frac{T}{n}\frac{\partial n}{\partial \rho_j}A_j\frac{p(\rho_j)}{\partial p/\partial \rho_j}\,,$$

it follows that $c_i=0$ for $i\neq j-1$.

The system $\dot{y}=(1+J_2\delta)Q(t)y+h(t)$ with $h(t)=\bar{g}(z,0)=-\delta_0 f(z)+g(z)$, must have a periodic solution with period T', it is:

$$y(t)=\sum_{i=1}^{6}\left[B_i+\int \psi^i h(t)\,\mathrm{d}t\right]\varphi^i\,.$$

From the condition of periodicity, and with $c_i=0$ for $i\neq j-1$, we derive:

$$\int_0^{T'} \psi^i \bar{g}(z,0)\,\mathrm{d}t=\int_0^{T'} \psi^i g(z)\,\mathrm{d}t=0\,,\quad i\neq j-1 \tag{5}$$

and

$$-\delta_0\int_0^{T'} \bar{\psi} f(z)\,\mathrm{d}t+\int_0^{T'}\left[\bar{\psi} g(z)+n'K\int \psi^i g(z)\,\mathrm{d}t\right]\mathrm{d}t=0\,. \tag{5a}$$

These are Equations (5) and (5a) with

$$c_{j-1}=-\frac{A_j}{n}\frac{\partial n}{\partial \rho_j}\frac{p(\rho_j)}{\partial p/\partial \rho_j}\,.$$

Equation (5a) determines δ_0.

3. Periodic Solutions in a Fixed Frame

3.1. THE GENERAL CASE: $\mathrm{i}\neq 0$, $e\neq 0$.

System (1a) is explicitly given

$$f_i=x_{i+3}\,,\qquad f_{i+3}=-(\mu x_i/r^3)\,,\qquad \bar{g}_i=-\delta x_{i+3}\,,\quad i=1,2,3\,.$$

$$\bar{g}_4=\mu\left[\frac{\delta x_1}{r^3}+\frac{3}{2}R_0^2\frac{x_1}{r^5}\left(-1+\frac{5x_3^2}{r^2}\right)\right]$$

$$\bar{g}_5 = \mu\left[\frac{\delta x_2}{r^3} + \frac{3}{2}R_0^2\frac{x_2}{r^5}\left(-1+\frac{5x_3^2}{r^2}\right)\right] \tag{6}$$

$$\bar{g}_6 = \mu\left[\frac{\delta x_3}{r^3} + \frac{3}{2}R_0^2\frac{x_3}{r^5}\left(-3+\frac{5x_3^2}{r^2}\right)\right]$$

where $\mu = kM$,

k = gravitational constant,

M = mass of the planet,

R_0 = Equatorial radius of the planet, and

r = distance from the particle to the planet's center.

The periodic solution of (2a) can also be written explicitly, but it depends on six parameters ρ_i which are in this order:

γ = instant of perigee passage,

a = semi major axis,

e = eccentricity,

ω = longitude of the perigee in the orbital plane,

Ω = longitude of the ascending node of the orbit,

i = inclination of the orbit.

We get

$$z_1 = Ak_1 - Bk_2\,,$$

$$z_2 = Ak_3 - Bk_4\,,$$

$$z_3 = Ak_5 + Bk_6\,,$$

$$z_{i+3} = \dot{z}_i(1+J_2\delta)^{-1}\,, \quad i = 1, 2, 3$$

with $A = a(\cos u - e)$, $B = a(1-e^2)^{1/2}\sin u$, $u - e\sin u = n't - n\gamma = l$; $n' = n(1+J_2\delta)$. k_i, $i = 1 \ldots 6$ are the well known functions of ω, Ω, i.

$$\sum_{i=0}^{2} k_{2i+1} = \sum_{i=1}^{3} k_{2i} = 1$$

$$k_1k_2 + k_3k_4 - k_5k_6 = 0 \quad \text{and} \quad n^2a^3 = \mu\,.$$

The fundamental matrix Φ, solution of system (3) can be calculated easily; we have: $\varphi^1 = \partial z/\partial\gamma$; $\varphi^2 = \partial z/\partial a = \bar{\varphi} + (3/2na)l\varphi^1$; $\varphi^3 = \partial z/\partial e$ etc....

In order to obtain the fundamental matrix Ψ, solution of system (4), and finally Equations (5), we must calculate the inverse of the matrix Φ. In general, this operation is rather complicated, but in the present case there exist some properties which allow large simplifications. The 6×6 matrix $Q(t)$ has the form:

$$Q(t)=\begin{pmatrix} 0 & I \\ A(t) & 0 \end{pmatrix}.$$

I and $A(t)$ are matrices 3×3. I is the identity matrix $A(t)=A^*(t)$ with

$$J=\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

we get $\phi^* J\phi = L$. L is a constant matrix; then $\phi^{-1}=L^{-1}\phi^* J$ and $\Psi=(\phi^{-1})^*$ (Battin, 1964; Broucke, 1970). It can be seen (Broucke, 1970) that the matrix L is the matrix with 'Lagrange Brackets' hence $L^{-1}=-P$ where P is the matrix with 'Poisson Parenthesis'. Following the classical definition (Chazy, 1953) we have

$$L_{ik}=\frac{n}{n'}\sum_{j=1}^{3}\left(\frac{\partial z_j}{\partial\rho_i}\frac{\partial \dot z_j}{\partial\rho_k}-\frac{\partial z_j}{\partial\rho_k}\frac{\partial \dot z_j}{\partial\rho_i}\right),$$
$$1\leqslant i\leqslant 6;\qquad 1\leqslant k\leqslant 6.$$

The elliptical elements ρ_i are used in the previous order, we successively obtain L and L^{-1}

$$L^{-1}=\begin{pmatrix}
0 & \frac{2}{n^2a} & \frac{1-e^2}{n^2a^2e} & 0 & 0 & 0 \\
-\frac{2}{n^2a} & 0 & 0 & 0 & 0 & 0 \\
-\frac{1-e^2}{n^2a^2e} & 0 & 0 & -\frac{(1-e^2)^{1/2}}{na^2e} & 0 & 0 \\
0 & 0 & \frac{(1-e^2)^{1/2}}{na^2e} & 0 & 0 & \frac{-\cot g\, i}{na^2(1-e^2)^{1/2}} \\
0 & 0 & 0 & 0 & 0 & \frac{1}{na^2(1-e^2)^{1/2}\sin i} \\
0 & 0 & 0 & \frac{\cot g\, i}{na^2(1-e^2)^{1/2}} & -\frac{1}{na^2(1-e^2)^{1/2}\sin i} & 0
\end{pmatrix}$$

Now, it is easy to obtain the matrices Φ^{-1} and Ψ.

Equation (5a) leads to

$$\sigma_1=\delta_0+\frac{3}{2}\left(\frac{R_0}{a}\right)^2(1-e^2)^{-3/2}(\tfrac{3}{2}\sin^2 i-1)=0.$$

Equations (5) can be calculated for $i=2\ldots 6$; we find

$$\sigma_2=\sigma_3=\sigma_6\equiv 0,$$

$$\sigma_4 = \frac{T'}{(1-e^2)^2}(-1+5\cos^2 i) = 0\,,$$

$$\sigma_5 = \frac{T'\cos i}{(1-e^2)^{3/2}} = 0\,.$$

It is clear that this system of six equations has no solution thus, in the general case, there are no periodic solutions for system (1a).

3.2. Equatorial case: $i = 0$; $e \neq 0$

System (1a) is reduced to a fourth order system and the solution $z(t, \rho_i)$ of (2a) depends only on the four parameters: γ, a, e, ω. We can construct the matrices L^{-1} and Ψ as in Section 3.1 we get

$$L^{-1} = \begin{pmatrix} 0 & \dfrac{2}{n^2 a} & \dfrac{1-e^2}{n^2 a^2 e} & 0 \\ -\dfrac{2}{n^2 a} & 0 & 0 & 0 \\ -\dfrac{(1-e^2)}{n^2 a^2 e} & 0 & 0 & -\dfrac{(1-e^2)^{1/2}}{n a^2 e} \\ 0 & 0 & \dfrac{(1-e^2)^{1/2}}{n a^2 e} & 0 \end{pmatrix}.$$

Equations (5a) and (5) determine four relations between the parameters ρ_i

$$\sigma_1 = \delta_0 - \frac{3}{2}\left(\frac{R_0}{a}\right)^2 (1-e^2)^{-3/2} = 0\,,$$

$$\sigma_2 = \sigma_3 \equiv 0\,,$$

$$\sigma_4 = \frac{T'}{(1-e^2)^2} \neq 0\,.$$

No periodic solution, even in the equatorial plane, does exist here.

3.3 Circular case: $e = 0$; $i \neq 0$

The periodic solution of Equation (2a) is

$$z(t, \rho_i) \begin{cases} z_1 = a(\cos\Omega \cos l - \sin\Omega \sin l \cos i) \\ z_2 = a(\sin\Omega \cos l + \cos\Omega \sin l \cos i) \\ z_3 = a \sin l \sin i \\ z_{i+3} = \dfrac{n}{n'}\dot{z}_i\,, \quad \mathrm{i} = 1, 2, 3\,, \end{cases}$$

where $l = n't - n\gamma$.

The matrix Φ, solution of Equation (3), is made up by four solutions $\varphi^1 = \partial z/\partial\gamma$, $\varphi^2 = \partial z/\partial a$, $\varphi^5 = \partial z/\partial\Omega$, $\varphi^6 = \partial z/\partial i$ and two solutions φ^3 and φ^4 with period $T'/2$ which are obtained from solution φ^3 of the general case (Section 3.1).

We calculate the matrices L and L^{-1}. It is

$$L^{-1} = \begin{pmatrix} 0 & \frac{2}{n^2 a} & 0 & 0 & 0 & \frac{\cot g\, i}{n^2 a^2} \\ -\frac{2}{n^2 a} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{na^2} & 0 & 0 \\ 0 & 0 & \frac{1}{na^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{na^2 \sin i} \\ -\frac{\cot g\, i}{n^2 a^2} & 0 & 0 & 0 & -\frac{1}{na^2 \sin i} & 0 \end{pmatrix}.$$

Thus the matrix $\Psi = (\phi^{-1})^* = (L^{-1}\phi^* J)^*$ is known.

The six Equations (5) lead to:

$$\sigma_1 = \delta_0 + 3\left(\frac{R_0}{a}\right)^2 (-\tfrac{3}{2} + 2\sin^2 i) = 0\,,$$

$$\sigma_2 = \sigma_3 = \sigma_4 = \sigma_6 \equiv 0\,,$$

$$\sigma_5 = -\frac{3}{2}\left(\frac{R_0}{a}\right)^2 T' \cos i = 0\,.$$

This system has for solution $i = \pi/2$; $\delta_0 = -\frac{3}{2}(R_0/a)^2$.

So, for each given value of T, (or a), (1a) has a family of periodic orbits of first kind, with period

$$T' \simeq \frac{T}{1 - \frac{3}{2}(R_0/a)^2 J_2}\,.$$

These orbits tend to circles in the plane $i = \pi/2$, for $J_2 \to 0$. The initial position on these generating circles is arbitrary.

3.4. Circular Equatorial Case: $i = 0$; $e = 0$.

Solution of system (2a) is a family of circles lying in the equatorial plane. The matrix Φ is made of two vectors $\varphi^1 = \partial z/\partial\gamma$, $\varphi^2 = \partial z/\partial a$ and two vectors φ^3 and φ^4 having the period $T'/2$ (Stellmacher, 1976; 1977). In this simple case, the matrix $(\phi^{-1})^*$ can be obtained directly from the matrix Φ. We get

$$(\phi^{-1})^* \begin{pmatrix} \frac{1}{na}(-\sin l - 3l\cos l) & 2\cos l & \frac{1}{a}\cos 2l & \frac{1}{a}\sin 2l \\ \frac{1}{na}(\cos l - 3l\sin l) & 2\sin l & \frac{1}{a}\sin 2l & -\frac{1}{a}\cos 2l \\ \frac{1}{n^2a}(2\cos l + 3l\sin l) & -\frac{2}{n}\sin l & -\frac{1}{2na}\sin 2l & -\frac{1}{2na}(3-\cos 2l) \\ \frac{1}{n^2a}(2\sin l - 3l\cos l) & \frac{2}{n}\cos l & \frac{1}{2na}(3+\cos 2l) & \frac{1}{2na}\sin 2l \end{pmatrix}.$$

The four Equations (5) and (5a) lead to:

$$\sigma_1 = \delta_0 - 3\left(\frac{R_0}{a}\right)^2 = 0\,, \qquad \sigma_2 = \sigma_3 = \sigma_4 \equiv 0\,.$$

So, there exist periodic solutions, of first kind with period

$$T' \simeq \frac{T}{1 + 3J_2(R_0/a)^2}\,.$$

in the equatorial plane.

For $J_2 \to 0$, these orbits tend to circles with radius a, and the initial position on these circles is arbitrary.

4. Periodic Solution in a Rotating Frame

4.1. The general case $i \neq 0$; $e \neq 0$.

We choose as rotating plane, the meridian plane ZOX (Figure 1), which contains the particle S and its projection on the equatorial plane. Because of the integral

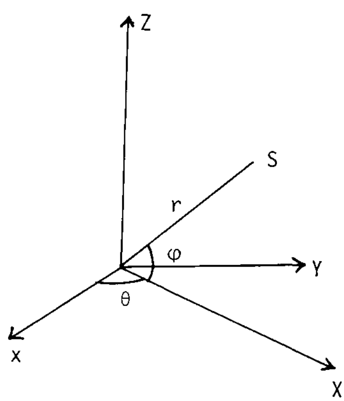

Fig. 1

$\dot{\theta}r^2 \cos^2 \varphi = C$, the equations with variables X, Z and θ can be separated. We get

$$\ddot{X} = -\frac{\mu X}{r^3} + \frac{C^2}{X^3} - \tfrac{3}{2}\mu J_2 R_0^2 \frac{X}{r^5}\left(1 - \frac{5Z^2}{r^2}\right),$$

$$\ddot{Z} = -\frac{\mu Z}{r^3} - \tfrac{3}{2}\mu J_2 R_0^2 \frac{Z}{r^5}\left(3 - \frac{5Z^2}{r^2}\right), \tag{7}$$

$$\dot{\theta} = \frac{C}{X^2}.$$

The first two Equations (7) constitute an independant fourth order system. If we put $x_1 = \dot{X}$, $x_2 = \dot{Z}$, $x_3 = X$, $x_4 = Z$, this system has the form (1a) with :

$$f(x) = \begin{cases} f_1 = x_3 \\ f_2 = x_4 \\ f_3 = -\dfrac{\mu x_1}{r^3} + \dfrac{C^2}{x_1^3} \\ f_4 = -\dfrac{\mu x_2}{r^3} \end{cases}$$

$$\bar{g}(x, J_2) = \begin{cases} \bar{g}_1 = -\delta x_3 \\ \bar{g}_2 = -\delta x_4 \\ \bar{g}_3 = \delta\left(\dfrac{\mu x_1}{r^3} - \dfrac{C^2}{x_1^3}\right) - \tfrac{3}{2}\mu \dfrac{R_0^2}{r^5} x_1\left(1 - \dfrac{5x_2^2}{r^2}\right) \\ \bar{g}_4 = \delta \dfrac{\mu x_2}{r^3} - \tfrac{3}{2}\mu \dfrac{R_0^2}{r^5} x^2\left(3 - \dfrac{5x_2^2}{r^2}\right). \end{cases}$$

The associated system (2a) has a family of periodic solutions with period T', which depends on the four parameters γ, a, e, ω; i ia a function of a and e, defined by $\cos i = C/(\mu a(1-e^2))^{1/2}$. Let $z(t, \rho_i)$ be this family of solutions; we have

$$z(t, \rho_i) = \begin{cases} z_1 = (a^2(1-\cos u)^2 - z_2^2)^{1/2} \\ z_2 = Ak_5 + Bk_6 \\ z_3 = \dfrac{n}{n'}\dot{z}_1 \\ z_4 = \dfrac{n}{n'}\dot{z}_2 \quad \text{and} \quad u - e\sin u = l = n't - n\gamma. \end{cases}$$

The 4×4 matrix $Q(t)$ in Equation (3) conserves the property given in Section 3.1. The matrix Φ, solution of (3), is made up by the vectors $\varphi^1 = \partial z/\partial\gamma$, $\varphi^2 = \partial z/\partial a$, $\varphi^3 = \partial z/\partial e$, $\varphi^4 = \partial z/\partial\omega$ the matrix $\Psi = (L^{-1}\phi^* J)^*$, solution of (4), is obtained after calculation of the matrix $L = \phi^* J\Phi$. We obtain

$$L^{-1} = \begin{pmatrix} 0 & \dfrac{2}{n^2 a} & \dfrac{1-e^2}{n^2 a^2 e} & 0 \\ -\dfrac{2}{n^2 a} & 0 & 0 & 0 \\ -\dfrac{(1-e^2)}{n^2 a^2 e} & 0 & 0 & -\dfrac{(1-e^2)^{1/2}}{na^2 e} \\ 0 & 0 & \dfrac{(1-e^2)^{1/2}}{na^2 e} & 0 \end{pmatrix}$$

with Ψ, the four Equations (5) are calculated.

Equation (5a), $\sigma_1 = 0$, leads to the same equation as in Section 3.1.

$$\sigma_2 = \sigma_3 \equiv 0\,, \qquad \sigma_4 = \left(\frac{R_0}{a}\right)^2 \frac{T'}{n(1-e^2)^2}(1 - 5\cos^2 i) = 0\,.$$

This system has for solution $\cos^2 i = \frac{1}{5} \rightarrow i = i_c$ and $\delta_0 = -\frac{3}{10}(R_0/a)^2(1-e^2)^{-3/2}$. Thus, in the meridian plane XOZ, there exists a family of periodic solutions with period $T' \simeq T/(1 + J_2\delta_0)$.

For $J_2 \rightarrow 0$, the family tends to the family of elliptical solutions in the plane with inclination $i = i_c$.

4.2. The Equatorial Case: $i = 0$; $e \neq 0$

The system (1a) is reduced to a second order system. The periodic solution with period T' of system (2a) is

$$z(t, \rho_i) = \begin{cases} z_1 = a(1 - e\cos u) \\ z_2 = \dfrac{n}{n'}\dot{z}_1 \end{cases}$$

z depends on the two parameters γ and a; e is a function of a, since $C = (\mu a(1-e^2))^{1/2}$. The matrix Φ is made up by the vectors $\varphi^1 = \partial z/\partial\gamma$, $\varphi^2 = \partial z/\partial a$. In this simple case we can directly obtain the matrix $\Psi = (\phi^{-1})^*$. It is

$$\Psi = \begin{pmatrix} -\dfrac{2}{n^2 a}\varphi_2^2 & \dfrac{2}{n^2 a}\varphi_1^2 \\ \dfrac{2}{n^2 a}\varphi_1^2 & -\dfrac{2}{n^2 a}\varphi_1^1 \end{pmatrix}.$$

The two Equations (5) and (5a) are

$$\sigma_1 = \delta_0 - \frac{3}{2}\left(\frac{R_0}{a}\right)^2 (1-e^2)^{-3/2} = 0\,, \qquad \sigma_2 \equiv 0\,.$$

There exist periodic solutions on the axis OX, whose generating orbits are ellipses.

For any given value of T(or a), δ_0 and consequently T' are determinated for $0<e<1$.

4.3. The circular case: $e=0$; $i\neq 0$

The equations of the motion are the same as in Section 4.1. but the associated system (2a) has a family of periodic solutions with period T' which depends on two parameters, γ and a; i is a function of a since $C=(\mu a)^{1/2}\cos i$.

Let $z(t,\rho_i)$ be these solutions, we have

$$z(t),\rho_i)\begin{cases} z_1 = a(1-z_2^2)^{1/2} \\ z_2 = a\sin l\sin i \\ z_3 = \dfrac{n}{n'}\dot{z}_1 \\ z_4 = \dfrac{n}{n'}\dot{z}_2\,. \end{cases}$$

The matrix Φ is made up by the two vectors $\varphi^1=\partial z/\partial\gamma$, $\varphi^2=\partial z/\partial a$, and the two vectors φ^3 and φ^4 with period $T'/2$. The matrices L^{-1} and $\Psi=(L^{-1}\phi^*J)^*$ are calculated as in Section 4.1.

$$L^{-1}=\begin{pmatrix} 0 & \dfrac{2}{n^2a} & 0 & 0 \\ -\dfrac{2}{n^2a} & 0 & 0 & 0 \\ 0 & 0 & 0 & \dfrac{1}{na^2} \\ 0 & 0 & -\dfrac{1}{na^2} & 0 \end{pmatrix}.$$

The four Equations (5) and (5a) lead to

$$\sigma_1=\delta_0-\frac{3}{2}\left(\frac{R_0}{a}\right)^2(3-4\sin^2 i)=0\,,$$

$$\sigma_2=\sigma_3=\sigma_4\equiv 0\,.$$

So, for any value of the inclination i, there exists a family of periodic solutions with period T', in the meridian plane XOZ. The generating orbits are circles. For any value of T (or a), δ_0 and consequently T' are determinated for $0<i<\pi$.

4.4. Circular equatorial case: $e=0$, $i=0$

This is a trivial case; $f(z)=0$ and the two Equations (5a) and (5) are reduced to: $\sigma_1=\sigma_2\equiv 0$.

TABLE I

	Solutions in a fixed frame	Solutions in the meridian plane				
e	0	0	$0<e<1$	$0<e<1$	0	0
i	0	$\pi/2$	$i_c \simeq 63°26'$	0	$0 \leqslant i \leqslant \pi$	0
T/T'	$1+3\epsilon$*	$1-\dfrac{3\varepsilon}{2}$	$1-\dfrac{3\varepsilon}{10(1-e^2)^{-3/2}}$	$1+\dfrac{3\varepsilon}{2(1-e^2)^{-3/2}}$	$1+\dfrac{3\varepsilon}{2}(3-4\sin^2 i)$	$\sigma_1 \equiv 0$ T' undetermined

* $\varepsilon = J_2(R_0/a)^2$

5. Conclusion

The conditions for periodic solutions are summarized in Table I. Some of the periodic solutions were already obtained by different methods. Our results are not always directly comparable, but after some calculations, they are found perfectly consistent with those given by other authors. For instance, Delmas (1978) investigates the equatorial case. He proves the existence of periodic orbits in a slowly rotating frame, for $0 < e < 1$. This case can be compared with the case 4.2. We will get the same result, if the rotating plane is not the meridian plane, but a plane rotating with the velocity of the perigee. MacMillan (1922), considers the orbits in the meridian plane. He takes an earth's potential with even zonal terms and divides the orbits in two classes:

(a) the orbits for which $T' = T$ ('orbits re-entrant after one revolution'),
(b) the orbits for which $pT' = kT$, where p and k are relatively prime integers, ('orbits re-entrant after many revolutions').

He does not find the orbits for $i = i_c$, which however are mentioned by Kammayer (1976), in an other rotating frame. Further, MacMillan gives a method for construction of these orbits. He finds orbits of first class in the equatorial plane, where the generating orbits are circles; this is our case 4.4. If the orbits are inclined on the equatorial plane, there exists only one orbit for assigned values of the inclination and the mean distance; the orbits reduce to circles for vanishing oblateness. This result is consistent with our case 4.3, in which generating orbits are circles, and the condition $T' = T$ leads to sin $i = 3\frac{1}{2}/2$ (first order in J_2). For orbits with $pT' = kT$, it is only necessary to verify whether this condition is consistent with the relation $\sigma_1 = 0$ which leads to the value T'/T.

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